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DIGITAL SIGNAL PROCESSING AND CONTROL AND ESTIMATION THEORY -- POINTS OF TANGENCY, AREA OF INTERSECTION, AND PARALLEL DIRECTIONS

by

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Acknowledgements

Acknowledgements such as these are usually placed at the end of the report and usually contain thanks to colleagues for their advice and suggestions during the development of the ideas described in the manuscript. In this case, however, it seems more appropriate to begin with the acknowledgements, as the author would never have even undertaken this project without the substantial contributions of others. I am greatly indebted to Prof. Alan V. Oppenheim of M.I.T. who invited me to give a lecture on this topic (before I really knew anything about it) at the 1976 IEEE Arden House Workshop on Digital Signal Processing. The original impetus for and much of the insight in this report is due to Al, and most of the ideas have grown directly out of an intensive set of discussions we held during the five months preceding Arden House and the nine months following. Those familiar with Prof. Oppenheim's work and philosophy will see his influence in various parts of this manuscript, and those familiar with the author can readily see the substantial impact Al Oppenheim has had on my perspective and on the direction of my work. Thanks Al.

During many of our discussions Al and I were joined by Prof. James McClellan of M.I.T. Fortunately Jim has a deep understanding of both disciplines -- digital signal processing and modern estimation and control theory -- and for long periods of time he not only provided useful insights but he also served as an interpreter between Oppenheim and myself.

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DIGITAL SIGNAL PROCESSING AND CONTROL AND ESTIMATION THEORY -- POINTS OF TANGENCY, AREAS OF INTERSECTION, AND PARALLEL DIRECTIONS

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Abstract

The purpose of this report is to explore a number of current research directions in the fields of digital signal processing and modern control and estimation theory. We examine topics such as stability theory, linear prediction and parameter identification, system synthesis and implementation, two-dimensional filtering, decentralized control and estimation, image processing, and nonlinear system theory, in order to uncover some of the basic similarities and differences in the goals, techniques, and philosophy of the two disciplines. An extensive bibliography is included in the hope that it will allow the interested reader to delve more deeply into some of these interconnections than is possible in this survey.

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I am particularly in Jim's debt for his valuable insights into the topics discussed in Sections B and D of this report.

As the research behind this report was begun, Dr. Wolfgang Mecklenbräuker of Philips Research Laboratory was visiting M.I.T. I would like to express my deep appreciation to Wolfgang for his numerous suggestions in many of the areas explored in this report and in particular for his major contribution to my knowledge concerning the topic discussed in Section A.

I have also benefited from numerous discussions with many other colleagues. My conversations with the attendees to the Arden House Workshop were of great value to me, and I would particularly like to thank Dr. James W. Cooley, Dr. Ronald E. Crochiere, Prof. Bradley W. Dickinson, Dr. Daniel E. Dudgeon, Dr. Michael P. Ekstrom, Prof. Alfred Fettweis, Dr. Stephan Horvath, Prof. Leland B. Jackson, Dr. James F. Kaiser, Dr. John Makhoul, Prof. Russell M. Mersereau, Prof. Martin Morf, Prof. Thomas Parks, Dr. Lawrence Rabiner, Dr. Michael Sablatash, Prof. Ronald W. Schafer, Prof. Hans A. Schuessler, Prof. Kenneth Steiglitz, and Dr. R. Viswanathan. My thanks also go to many of my M.I.T. colleagues who aided the development of these ideas. Special thanks go to Mr. David Chan for his contributions to Sections C and D and to Prof. Nils R. Sandell for listening to and talking about this stuff every day as we drove in, to and away from M.I.T. That fact that Nils is still talking to me hopefully means that there is something of interest in what follows.

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Introduction: Point of View, Goals, and Overview

This report has grown out a series of discussions over the past year between the author and Prof. Alan V. Oppenheim of M.I.T. These talks were motivated by a mutual belief that there were enough similarities and differences in our philosophies, goals, and analytical techniques to indicate that a concerted effort to understand these better might lead to some useful interaction and collaboration. In addition, it became clear after a short while that one could not accomplish this by trying to understand the two fields in the abstract. Rather, we felt that it was best to examine several specific topics in detail in order to develop this understanding, and it is out of this study that this report has emerged.

Thus the goal of this report is to explore several directions of current research in the fields of digital signal processing and modern control and estimation theory. Our examination will in general not be result-oriented. Instead, we are most interested in understanding the goals of the research and the methods and approach used. Understanding the goals may help us to see why the techniques used in the two disciplines differ. Inspecting the methods and approaches may allow one to see areas in which concepts in one field may be usefully applied in the other. The report undoubtedly has a control-oriented flavor, since it reflects the author's background and also since the original purpose of this study was to present a control-theorist's point of view at the 1976 Arden House Workshop on Digital Signal Processing. However, an effort

has been made to explore avenues in both disciplines in order to encourage researchers in the two fields to continue along these lines.

It is hoped that the above comments will help explain the spirit in which this report has been written. In reading through the report, the reader may find many comments that are either partially or totally unsubstantiated or that are much too black and white. These points have been included in keeping with the speculative nature of the study. However, we have attempted to provide background for our speculation and have limited these comments to questions which we feel represent exciting opportunities for interaction and collaboration. Clearly these issues must be studied at a far deeper level than is possible in this initial survey-oriented effort. Also, we have not been so presumptuous as to attempt to define the two fields (although some may feel we come dangerously close), since we feel that a valid mutual understanding can and will grow out of closer examination of the directions we describe. To this end, we have included an extensive bibliography which should help the interested reader to make inroads into the various areas.

The following is an annotated list of the topics considered in the following sections. Sections are denoted by capital letters, and, for ease of reference, the bibliography is coded similarly (e.g., [A-21] is the 21st reference for Section A -- Stability Analysis). Due to variations in the author's expertise, maturity of the subject areas, and nature of the questions, the sections vary greatly in depth and style. Some sections are very specific, while others are more philosophical and speculative.

A. Stability Analysis -- In this section we discuss methods used in both disciplines for the study of stability characteristics of systems. In digital signal processing one is primarily concerned with the possibility of limit cycles caused by the effects of finite arithmetic in digital filters. In control theory, one is often concerned with determining conditions for stability of feedback systems. The techniques used in the two disciplines have many similarities. Lyapunov theory, frequency domain methods, and the concept of passivity are widely used by researchers in both fields. We speculate on a potential research topic -- the effects of finite arithmetic on digitally implemented feedback control systems.

B. Parameter Identification, Linear Prediction, Least Squares, and Kalman Filtering -- Identification of parametric models arises in a variety of problems, from digital processing of speech to adaptive control. Using the speech problem as a focus, we explore several methods for identification. We examine the autocorrelation method for linear prediction and relate it to the determination of the time-varying weighting pattern of an optimum predictor. We also discuss the efficient Levinson algorithm and its relationship to recently developed fast algorithms for determining optimum time-varying Kalman filter gains. The covariance method for linear prediction is discussed, as are

its relationships with the Kalman filter structure of recursive least squares. Using this framework, we speculate on potential recursive methods for identifying time-varying models for speech. We also discuss the relationship between the parametric identification problem and the problem of stochastic realization. Crucial differences in the underlying assumptions are brought out, and we speculate on the utility of a stochastic realization approach for the identification of pole-zero models of speech. We also discuss other pole-zero identification techniques including recursive maximum likelihood methods, which resemble recursive least squres (and hence the covariance method) both in form and spirit.

C. Synthesis, Realization, and Implementation -- We discuss state space models and realization theory and the uses of such realizations for direct synthesis and for "indirect synthesis", in which a state space model of a process of interest allows one to apply state space methods to synthesize systems for estimation, stabilization, optimal control, etc. We also explore the key issues involved in the design of digital filters meeting certain design specifications. We discuss several filter design methods, but the major emphasis of our examination of this topic is on filter structures. Minimality -- the key concept in state space realization theory -- is only one of several issues. Sensitivity and behavior in the presence of perturbations caused

by finite arithmetic are crucial questions as well. Here we find some limitations of state space methods. All minimal structures cannot be obtained from straightforward algorithmic interpretations of different state space realizations. We speculate on some recent work indicating that state space methods may be useful in analyzing the performance of different structures, that certain factorizations of state space realizations include all structures, and that state realizations combined with an understanding of structures issues may lead to useful implementations for multivariable filters. Finally, we speculate on the possibility of designing controllers, filters, or other systems by directly taking the constraints of digital implementation into account from the start. This area contains some intriguing, potentially very useful, and extremely difficult problems.

D. Multiparameter Systems, Distributed Processes, and Random Fields -- We explore a number of the issues that arise in studying systems defined with two or more independent variables. We see that the issues of recursion, causality, and the sequencing of the required computations for a filter become extremely complicated in this setting. We find that a precedence relation among the computations exists and is of the same form and spirit as the precedence relation arising in multi-decision-

maker control problems. A number of relationships with one-dimensional concepts are explored. Specifically a multidimensional system can be made into a (often quite complex) one-dimensional system by totally ordering the computations in a way that is compatible with the precedence relation. We also discuss the possibility of transforming distributed or multivariable systems to scalar, multidimensional systems, and we speculate on the utility of such an approach. The algebraic difficulties that arise in multidimensional problems lead to complications in areas such as stability analysis and spectral factorization, and we also point out that similar algebraic problems arise in considering lumped-distributed systems, certain time-varying systems, and specific classes of nonlinear systems. A number of design methods are discussed, and many of these are closely related or in fact rely on one-dimensional methods. We also describe a number of state space models for multidimensional systems, and we run into many of the same difficulties -- causality, nonfactorizability, etc. We speculate on the utility of state models for stability and roundoff noise analysis and for multidimensional recursive Kalman filtering. We discuss a number of statistical and probabilistic approaches to multidimensional filtering and analyze their utility in the context of the problem of image processing. We also speculate on the utility of the two-dimensional stochastic framework for the

consideration of space-time and decentralized control problems. This section offers some of the most exciting and difficult potential research directions.

E. Some Issues in Nonlinear System Analysis: Homomorphic Filtering, Bilinear Systems, and Algebraic System Theory -- There has been substantial work in both disciplines in analyzing and synthesizing nonlinear dynamic systems that possess certain types of algebraic structure. We consider the work in digital signal processing on homomorphic systems and filter design, and we relate this to some work on state space models that possess related algebraic properties.

Finally, we make some concluding remarks, summing up our feelings about the relationship of the two fields and the possibility of increased interaction. From the point of view of Prof. Oppenheim and the author, this study has been a success, since we are convinced of the benefit of such interaction. This report will be a success if we can convince others.

A. Stability Analysis

Of all of the topics that we have investigated, it is in this area that we have found some of the clearest areas of intersection and interaction between the disciplines. In the field of digital signal processing, stability issues arise when one considers the consequences of finite word length in digital filters. Two problems arise (not mentioning the effects due to finite accuracy in filter coefficients [A-12,C-1]). On the one hand, a digital filter necessarily has finite range, and thus overflows can occur, while on the other, one is inevitably faced with the problem of numerical quantization -- roundoff or truncation. Since the filter has finite range (it is after all a finite-state machine) the question of the state of the filter growing without bound is irrelevant. However, the nonlinearites in the filter, introduced by whatever form of finite arithmetic is used, can cause zero-input limit cycles and can also lead to discrepancies between the ideal and actual response of the filter to certain inputs. Following the discussions in [A-3,15], the typical situation with which one is concerned is depicted in Figure A.1. The filter is described (in state-variable form) by equations of the form

$$\begin{aligned} x(n+1) &= A\chi(n) + Bu(n) \\ y(n) &= Cx(n) \\ \chi(n) &= N(x(n)) \end{aligned} \tag{A.1}$$

where N is a nonlinear, memoryless function that accounts for the effects of overflow and quantization. If these effects were not present -- i.e. if N were the identity function -- equation (A.1) would reduce to a linear equation. If one assumes that this associated linear system is designed to meet certain

specifications, one would like to know how the nonlinearity N affects overall performance. In particular, one important question is: assuming that the linear system is asymptotically stable, can the nonlinear system (A.1) sustain undriven oscillations, and will its response to inputs deviate significantly from the response of the linear system? We will make a few remarks about this question in a moment. We refer the reader to the survey papers [A-3,5] and to the references for more detailed descriptions of known results.

In control theory the question of system stability has long played a central role in the design and analysis of feedback systems. Following [A-42], a typical feedback system, depicted in Figure A.2, is described by the functional equations

$$\begin{aligned} e_1 &= u_1 - y_2 \quad , \quad & e_2 &= u_2 + y_1 \\ y_1 &= G_1 e_1 \quad , \quad & y_2 &= G_2 e_2 \end{aligned} \tag{A.2}$$

where u_1, u_2, e_1, e_2, y_1, and y_2 are <u>functions</u> (of time -- discrete or continuous) and G_1 and G_2 are operators (possibly nonlinear) describing the dynamics of the forward and feedback paths, respectively. In control theory one is interested either in the analysis or the synthesis of such systems. In the synthesis problem one is given an open loop system G_1 and is asked to define a feedback system (A.2) such that the overall system has certain desirable stability properties. In the case of stability analysis, with which we are most concerned here, one may be interested either in the driven or the undriven (u_i=0) characteristics. In the driven case one wishes to determine, for example [A-42], if bounded inputs lead to bounded outputs and if the input-output relationship is continuous -- i.e. if small changes in the u's lead to

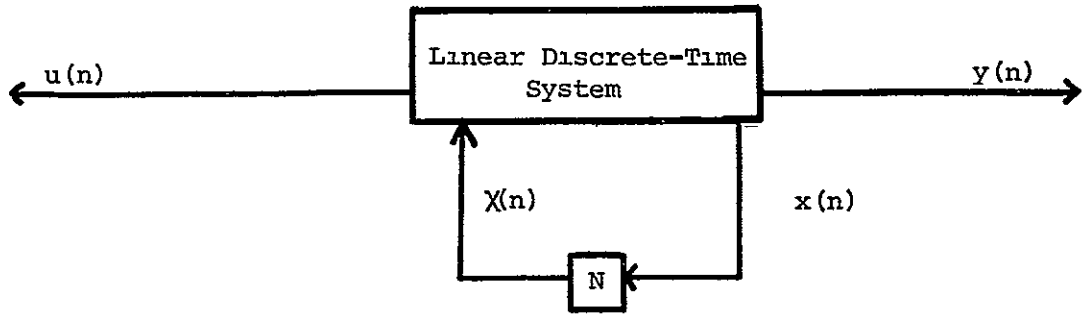

Figure A.1: Illustrating a Digital Filter with Quantization and Saturation Nonlinearities

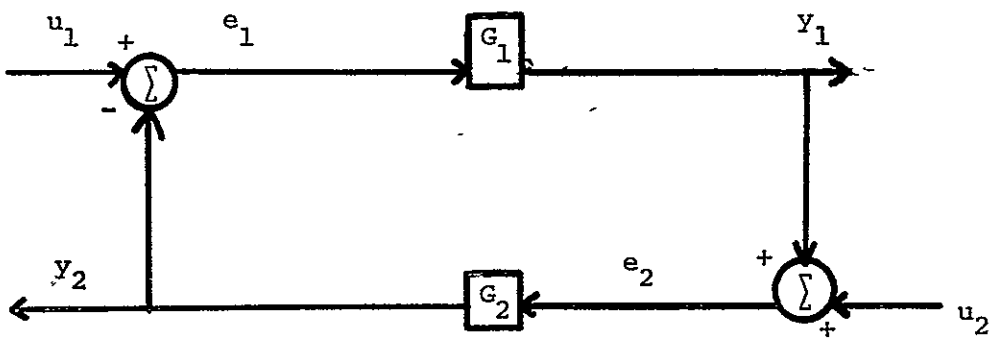

Figure A.2: Illustrating a Typical Feedback Control System

small changes in the y's. In the undriven case, one wishes to determine if the system response decays, remains bounded, or diverges when the only perturbing influences are initial conditions. Again, the literature in this area is quite extensive, and we refer the reader to the texts [A-42,44,47], the survey paper [A-43], and to the references for more on these problems.

From the above descriptions one gets a clear indication about some of the similarities and differences in the two topics[1]. In both areas one wants the answers to some qualitative questions -- is the system stable; is it asymptotically stable; is the system continuous [A-42] or does it exhibit "jumps" when one makes small changes in the inputs [A-32,43]. In addition, one often wants some quantitative answers. In digital filter design one is often interested in determining bounds on the magnitudes of limit cycles and in finding out how many bits one needs to keep the magnitudes of such oscillations within tolerable limits. In the study of feedback control systems one is interested in measures of stability as provided by quantities such as damping ratios and eigenvalues (poles). In addition, one is often interested in the shapes of these modes -- i.e. in determining the state eigenvector corresponding to a particular eigenvalue.[2]

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[1] One of the most trivial of these is the fact that control theorists put minus signs in their feedback loops, while there are none in the nonlinear digital filter of Figure 1. The reader should be careful to make the proper changes of sign in switching between results.

[2] This is of interest, for example, in the design of stability augmentation systems for aircraft. In this case one is quite interested in the shape of modes such as "Dutch roll", which involves both the bank and sideslip angles of the aircraft [A-71].

In addition to the similar goals of the two problem areas, as we shall see, people in each area have obtained results by drawing from very similar bags of mathematical tricks. However, there are differences between the methods used and results obtained in the two areas. In the analysis of digital filters the work has been characterized by the study of systems containing quite specific nonlinearities. In addition, much of the work has dealt with specific filter structures. In particular, second-order filters have received a great deal of attention [A-2,3,11,15,18,31] since more complex filters can be built out of series - parallel interconnections of such sections. Also, the class of wave digital filters [A-6,7,8,9,10] have been studied in some detail. Studies in these areas have yielded extremely detailed descriptions of regions of stability in parameter space (see, for example, [A-3]) and numerous upper and lower bounds on limit cycle magnitudes (see [A-3,4,20,26,31, 35,56,59,60,63]).

In control theory, on the other hand, the recent trend has been in the development of rather general theories, concepts, and techniques for stability analysis. A number of rather powerful mathematical techniques have been developed, but there has not been as much attention paid to obtaining tight bounds for specific problems. In addition, problems involving limit cycles have not received nearly as much attention in recent years as issues such as bounded-input, bounded-output stability and global asymptotic stability (although there clearly is a relationship between these issues and limit cycles)

In the rest of this section, we briefly discuss the relationship between some of the results in the two fields. Our aim here is to point out areas in

which researchers have used similar techniques, obtained similar results, or relied on similar concepts.

A.1 The Use of Lyapunov Theory

The technique of constructing Lyapunov functions to prove the stability of dynamical systems has been used by researchers in both fields. The basic ideas behind Lyapunov theory are the following (see [A-47,48,52,64] for details and further discussions): consider the dynamical system

$$x(k+1) = f(x(k)),\ f(0)=0 \tag{A.3}$$

(where x is a vector). Suppose we can find a function V(x) such that V(0)=0 and the first difference along solutions satisfies

$$\Delta V(x) \triangleq V(f(x)) - V(x) \leq W(x) \leq 0 \tag{A.4}$$

Such a function is called a <u>Lyapunov function</u>. If this function has some additional properties, we can prove stability or instability of (A.3). Examples are (see [A-47,48] for proofs):

<u>Theorem A.1</u>: Suppose V is such that

(i) It is positive definite -- i.e. these exists a continuous, nondecreasing scalar function α, such that $\alpha(0)=0$ and

$$V(x) \geq \alpha(|x|)>0 \qquad \forall\, x\neq 0 \tag{A.5}$$

(ii) $$\alpha(|x|)\to\infty \text{ when } |x|\to\infty \tag{A.6}$$

(iii) ΔV is negative definite -- i.e. there exists a continuous, nondecreasing scalar function γ, such that

$$\Delta V(x) \leq -\gamma(|x|)<0 \tag{A.7}$$

Then all solutions of (A.3) converge to 0.

In this result, we can think of V as an "energy" function, and (A.5), (A.6) essentially state the intuitive idea that the larger the system state, the more energy that is stored in it. With this interpretation, the theorem states that if the system dissipates energy (equation (A.6)), the state will converge to 0. If we allow ourselves to consider "energies" which can take on negative values, we can get instability results, such as

Theorem A.2: Suppose V satisfies (A.4) and suppose there exists an x_0 such that $V(x_0)<0$. Then the system is not asymptotically stable in the large since the solution starting at x_0 does not converge to 0.

The point here is that since energy decreases, once we arrive at a negative energy state, we can never reach the zero energy state.

As mentioned earlier, Lyapunov stability has been used by many researchers. A crucial advantage of Lyapunov-type results is that the hypotheses for results such as Theorems A.1 and A.2 can be checked using the function V and f only -- i.e. one does not have to construct explicit solutions to difference or differential equations. However, the major problem with the theory is the difficulty in finding Lyapunov functions in general. For linear systems, however, a theory exists, and one can always find a quadratic Lyapunov function

$$V(x) = x'Qx \tag{A.8}$$

that will determine if the system is asymptotically stable (in fact a constructive procedure using the Lyapunov equation [A-47,48] can be used). For nonlinear

systems the construction of Lyapunov functions is much more difficult (see [A-47,48] for several techniques).

With respect to the limit cycle problem, Willson [A-2,13] has utilized Lyapunov functions (and essentially Theorem 1) to determine conditions under which second order digital filters will not have overflow limit cycles and will respond to "small" inputs in a manner that is asymptotically close to the ideal response. Parker and Hess [A-26] and Johnson and Lack [A-59,60] have used Lyapunov functions to obtain bounds on the magnitude of limit cycles. In each of these the Lyapunov function used was a quadratic form which in fact proved asymptotic stability for the ideal linear system. In Willson's work [A-13], he was able to show that his results were in some sense tight by constructing counterexamples when his condition was violated. In [A-26,59,60] the bounds are not as good as others that have been found, and, as Parker and Hess state, this may be due to the difficulty of determining which quadratic Lyapunov function to use. As pointed out by Claasen, et.al., [A-3], it appears to be difficult to find appropriate Lyapunov functions for the discontinuous nonlinearities that characterize quantization (see Appendix 1 for an example of the type of result that one can find).

There is a class of digital filters -- wave digital filters (WDF) [A-6,7, 8,9,10] -- for which one can use Lyapunov techniques to prove stability. Such filters have been developed by Fettweis so that they possess many of the properties of classical analog filters. Motivated by these analogies, Fettweis [A-8] defines the notion of "instantaneous pseudopower", which is a particular quadratic form in the state of the WDF. By defining the notion of

"pseudopassivity" of such a filter, Fettweis introduces (in a very natural way for this setting) the notion of dissipativeness. With this framework, the pseudopower becomes a natural candidate for a Lyapunov function, and in [A-10], Fettweis and Meerkötter are able to apply standard Lyapunov arguments to obtain quite reasonable conditions on numerical operations that guarantee the asymptotic stability of pseudopassive WDF's. The introduction of the concept of dissipativeness in the study of stability is an often-used idea (see the note of Desoer [A-36]), and a number of important stability results have as their basis (at least from some points of view) some notion of passivity. We will have a bit more to say about this in the next subsection. We note here that the use of passivity concepts and the tools of Lyapunov theory appear to be of some value in the development of new digital filter structures that behave well in the presence of quantization. As an example, we refer the reader to the recent paper [A-11] in which a new second order filter structure is developed and analyzed using pseudopower-Lyapunov arguments.

Lyapunov concepts have found numerous applications in control theory. Detailed studies of their use in system analysis are described in the important paper of Kalman and Bertram [A-48] and the texts [A-47], [A-52], and [A-64]. As mentioned earlier the construction of quadratic Lyapunov equations for linear systems is well understood and is described in detail in these texts. The key result in this area is the following:

<u>Theorem A.3</u>: Consider the discrete-time system

$$x(k+1) = Ax(k) \tag{A.9}$$

This system is asymptotically stable (i.e. all of the eigenvalues of A lie inside the unit circle in the complex plane) if and only if for <u>any</u> positive definite matrix L, the solution Q of the (discrete) <u>Lyapunov equation</u>

$$A'QA - Q = -L \tag{A.10}$$

is also positive definite. In this case the function

$$V(x) = x'Qx \tag{A.11}$$

is a Lyapunov function satisfying the hypotheses of Theorem A.1 -- i.e. it proves the asumptotic stability of (A.9).

The equation (A.10) and its continuous-time analog (see [A-47]) arise in several contexts in control theory, and we will mention it again later in a different setting. Also, note that Theorem A.3 provides a variety of choices for Lyapunov functions(we can choose any L>0 in (A.10)). Parker and Hess [A.26] obtain bounds on the magnitude of limit cycles by choosing L=I (here (A.9) represents the ideal linear model). Tighter bounds might be possible with other choices of L, but, as they mention, it is not at all clear how one would go about finding a "better" choice (other than by trial and error). We also refer the reader to the paper of Kalman and Bertram [A-48] in which they use Lyapunov techniques to bound the magnitude of solutions of difference equations perturbed by nonlinearities.

For specific applications of Lyapunov theory to linear and nonlinear systems, we refer the reader to the references or to the literature (in particular the <u>IEEE Transactions on Automatic Control</u>). In the remainder of this subsection we concentrate on another use of Lyapunov concepts -- as

intermediate steps in the development of other results in control theory. An example of this occurs in the analysis of optimal control and estimation systems [A-64,65,66,67]. Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{A.12}$$

and suppose we wish to find the control u that minimizes the cost

$$J = \sum_{i=0}^{\infty} y'(i)y(i) + u'(i)u(i) \tag{A.13}$$

This is a special case of the output regulator problem [A-66]. Here the cost (A.13) represents a tradeoff between regulation of the output (the y'y term) and the conservation of control energy (the u'u term). The following is the solution for a particular case:

<u>Theorem A.4</u>: Suppose the system (A.12) is completely controllable (any state can be reached from any other state by application of an appropriate input sequence) and completely observable (the state can be uniquely determined from knowledge of the input and output sequences). Then the optimal control in feedback form is

$$u(k) = -(R+B'KB)^{-1} B'KA\, x(k) \tag{A.14}$$

where K is the unique positive definite solution of the <u>algebraic Riccati equation</u>

$$K = A'KA+C'C - A'KB(R+B'KB)^{-1} B'KA \tag{A.15}$$

One proof of this result proceeds along the following lines. Suppose we are presently in the state x. We can then define the <u>optimal cost to go</u>, V(x), as the minimum of J in (A.13) when we start in state x. With the aid of dynamic programming methods [A-66], one can show that V has the form

$$V(x) = x'Kx \tag{A.16}$$

where K satisfies (A.15). The finiteness of V is proven using controllability, while observability guarantees that if $x \neq 0$, then y and u cannot both be identically zero and thus $J>0$. As a final important question, consider the closed loop system (A.12), (A.14). As discussed in [A-66] one can show that this system is asymptotically stable, and, in fact, the cost-to-go function V(x) is a Lyapunov function which proves this result. Observability and controllability (and somewhat weaker counterparts -- detectability and stabilizability) are important concepts in the development of this result and may others. In fact, the concept of observability allows one to prove [A-51].

<u>Theorem A.5</u>: Consider the system (A.9) and the function $V(x) = x'Qx$. Suppose

(i) $Q>0$

(ii) $V(Ax)-V(x) = x'[A'QA-Q]x \leq x'C'Cx$

(iii) The system (A.9) is observable from the output $y(k) = Cx(k)$

Then (A.9) is asymptotically stable.

Comparing Theorems A.3 and A.5, we see that we have replaced the negative definiteness of $A'QA-Q$ with negative semidefinitess and an observability condition. The intuitive idea is the following: negative definiteness makes it clear that $V(x(k))$ strictly decreases along solutions whenever $x(k)\neq 0$, and from this we can deduce asymptotic stability; negative semidefiniteness only says V does not increase. However, is it possible that V can remain stationary indefinitely at a non-zero value? The answer is no, since if it did, we would be able to conclude that $Cx(j)=0$, $j=k, k+1, k+2,\ldots$, and observability would require $x(k)=0$. Thus V must decrease (not necessarily at every single step), and we can again deduce asymptotic stability.

Thus, we see that Lyapunov concepts, when combined with ideas from the theory of state-space models, can lead to important results concerning optimal designs of controllers and estimators. See [A-64,65,66,67] for continuous time analogs of these results and dual results for estimators (the reader is also advised to examine [A-68] in which the interplay of many of these ideas is discussed).

In addition to its use in studying design methods such as the regulator problem, Lyapunov theory has been used as a framework for the development of many more explicit stability criteria (recall, Lyapunov theory in principle requires a search for an appropriate function). Examples of these are a number of the frequency domain stability criteria that have been developed in the last 10 to 15 years (see [A-1,21,22,23,24,33,37,38,39,43,44,45]. Several of these results have analogs for the limit cycle problem. For example, Tsypkin's criteria [A-33,21,2] and [A-44, p.194], which are analogs of the circle and

Popov criteria in continuous time (see [A-43,44]), have counterparts in the theory of limit cycles [A-15,16]. We note also that instability counterparts of the Tsypkin-Popov type of result have been developed from a Lyapunov point of view [A-1,39], and a thorough understanding of the basis for these results may lead to analogous results for limit cycles in digital filters.

We defer further discussion of these results to the next subsection, in which we are interested in examining the interplay among a number of stability concepts (passivity, Lyapunov, Tsypkin, frequency domain analysis, positive real functions, etc.). The key point is that many stability results can and have been derived in a number of different ways, and an examination of these various derivations reveals an interrelationship between the various methods of stability analysis. Some of the most fundamental work that has been done in this area has been accomplished by J.C. Willems [A-49,50,69], and the reader is referred to his work for a more thorough treatment of these issues and for further references.

A.2 Frequency Domain Criteria, Passivity, and Lyapunov Functions

We have already mentioned that the notion of passivity is of importance in stability theory and have seen that Fettweis and Meerkotter have been able to utilize passivity notions to study certain digital filters via Lyapunov techniques. The relationship between passivity, Lyapunov functions, and many of the frequency domain criteria of stability theory is quite deep, and in this subsection we wish to illustrate some of these ideas. The interested reader is referred to the references for more details.

In recent years the concept of passivity has become one of the fundamental

notions in the study of feedback stability. This notion, which is very much an input/output concept, is developed in detail by J.C. Willems [A-41,42,50,69]. We follow [A-42,69][3]. Let U and Y be input and output sets, respectively, and let $\mathcal{U}$ and $\mathcal{Y}$ be sets of functions from a time set T into U and Y (T may be continuous or discrete, as discussed [A-69]). Let G: $\mathcal{U}\to\mathcal{Y}$ be a dynamic system, mapping input functions $u\varepsilon U$ into output functions $Gu\varepsilon Y$ (we assume that G is a causal map [A-69]). Intuitively, stability means that small inputs lead to small outputs, and the following makes this precise.

<u>Definition A.1</u>: Let $\tilde{\mathcal{U}}$, $\tilde{\mathcal{Y}}$ be subspaces of $\mathcal{U}$ and $\mathcal{Y}$, respectively (these are our "small signals"). The system G is <u>I/O stable</u> if $u\varepsilon\tilde{\mathcal{U}}$ implies $Gu\varepsilon\tilde{\mathcal{Y}}$. Furthermore, if $\mathcal{U}$, $\mathcal{Y}$ are normed spaces, then G is <u>finite gain I/O stable</u> if there exists $K<\infty$ such that

$$||Gu|| \leq K||u|| \qquad \forall u\varepsilon\tilde{\mathcal{U}} \tag{A.17}$$

A typical example is the case T=positive integers, $\mathcal{U}=\mathcal{Y}=$ all real sequences of numbers, $\tilde{\mathcal{U}}=\tilde{\mathcal{Y}}=$ all square-summable sequences. In this case I/O stability means (y=Gu)

$$\sum_{i=1}^{\infty} u_i^2 <\infty \Rightarrow \sum_{i=1}^{\infty} y_i^2 <\infty \tag{A.18}$$

---

[3] Our development is by no means complete, as our intention is to relate several ideas and not to prove theorems. Thus, the reader is referred to the references (in particular to [A-42]) for a thorough treatment and for precise statements of the results described here (for example, we have not included a discussion of system well-posedness, which bears some similarities to the constraints on feedback paths imposed by Fettweis in his development of wave digital filters).

and finite-gain I/O stability means

$$\sum_{i=1}^{\infty} u_i^2 < \infty \Rightarrow \left(\sum_{i=1}^{\infty} y_i^2\right)^{1/2} \leq K \left(\sum_{i=1}^{\infty} u_i^2\right)^{1/2} \tag{A.19}$$

Note one property of this example. Let P_T be the operator

$$(P_T x)\ (t) = \begin{cases} x(t) & t \leq T \\ 0 & t > T \end{cases} \tag{A.20}$$

Then for any $u \varepsilon \mathcal{U}$, $y \varepsilon \mathcal{Y}$, we have $P_T u \varepsilon \tilde{\mathcal{U}}$, $P_T y \varepsilon \tilde{\mathcal{Y}}$. In this case $\mathcal{U}$, $\mathcal{Y}$ are called (causal) extensions of $\tilde{\mathcal{U}}$, $\tilde{\mathcal{Y}}$, and we assume this to be the case from now on. We now can define passive systems.

Definition A.2: Let $\mathcal{U}=\mathcal{Y}$, and assume that $\tilde{\mathcal{U}}=\tilde{\mathcal{Y}}$ is an inner product space. Then G is passive if

$$\langle P_T u, P_T G u\rangle \geq 0 \qquad \forall u \varepsilon \mathcal{U}, t \varepsilon T \tag{A.21}$$

and strictly passive if there is an $\varepsilon > 0$ such that

$$\langle P_T u,\ P_T G u\rangle \geq \varepsilon ||P_T u||^2 \tag{A.22}$$

In terms of our example, G is passive if and only if (y=Gu)

$$\sum_{i=1}^{N} u_i y_i \geq 0 \qquad \forall u_i, N \tag{A.23}$$

and is strictly passive if and only if there exists an $\varepsilon>0$ such that

$$\sum_{i=1}^{N} u_i y_i \geq \varepsilon \sum_{i=1}^{N} u_i^2 \qquad u_i, N \tag{A.24}$$

Much as with Fettweis's pseudopassive blocks, passive systems can be interconnected in feedback arrangements and remain passive. The following result is of this type, and it, in fact, is one of the cornerstones of feedback stability theory [A-69].

<u>Theorem A.6</u>: Consider the feedback system of Figure A.2 with all inputs and outputs elements of the same space $\mathcal{U}$ (for simplicity). The feedback system is strictly passive and finite gain I/O stable if

(i) G_1 is strictly passive and finite-gain input/output stable

(ii) G_2 is passive

As outlined by J.C. Willems in [A-69], there are three basic stability principles -- the one above, the <u>small loop gain theorem</u> (stability arises if the gains of G_1 and G_2 are each less than unity -- a result used in the digital filter context in [A-72] and the next result, which depends upon the following

<u>Definition A.3:</u> Same conditions on $\mathcal{U}$, $\mathcal{Y}$ as in Definition A.2. Let $a \leq b$ be given real numbers. Then G is inside (outside) the sector [a,b] if

$$<(G-aI)u,\ (G-bI)u> \ \leq 0 \ (\geq 0) \qquad \forall u \in \tilde{\mathcal{U}} \qquad (A.25)$$

It is strictly inside the sector [a,b] if there exists an $\varepsilon>0$ such that

$$<(G-aI)u,\ (G-bI)u> \ \leq -\varepsilon||u||^2 \qquad (\geq \varepsilon||u||^2) \qquad \forall u \in \tilde{\mathcal{U}} \qquad (A.26)$$

We now state a variation of Willems' third stability condition (see [A-42]).

<u>Theorem A.7:</u> Consider the feedback system of Figure A.2. This system is finite gain stable if G_2 is Lipschitz continuous -- i.e.

$$||G_2u_1-G_2u_2|| \leq K||u_1-u_2||, \qquad \forall u \in \tilde{\mathcal{U}}$$

and if for some $a \leq b > 0$, G_2 is strictly inside the sector [a,b], $I + \frac{1}{2}(a+b)G_1$ has a causal inverse on $\mathcal{U}$ (not necessarily $\tilde{\mathcal{U}}$), and G_1 satisfies:

(i) $a<0 \Rightarrow G_1$ is inside the sector $[-\frac{1}{b},-\frac{1}{a}]$ on $\tilde{\mathcal{U}}$

(ii) $a>0 \Rightarrow G_1$ is outside the sector $[-\frac{1}{a},-\frac{1}{b}]$ on $\mathcal{U}$

(iii) $a=0 \Rightarrow G_1 + \frac{1}{b} I$ is passive on $\mathcal{U}$.

As develop by J.C. Willems [A-42,69], this result leads to the circle criterion (in continuous time). Let us examine the third case in the Theorem in order to sketch the derivation of one of Tsypkin's criteria. Consider the

system in Figure A.3. Here G_2 is a memoryless nonlinearity, and we assume that f is in the sector [0,k]. We also take $\tilde{\mathcal{U}}$= all square summable sequences. The system G_1 is a linear time-invariant system characterized by the transfer function G(z), which we assume to be stable. Condition (iii) of Theorem A.7 then says that $(G_1 + \frac{1}{k})$ must be passive on $\mathcal{U}$, and, as developed in [A-42,69], this will be the case if and only if $G(z) + \frac{1}{k}$ is <u>positive real</u>:

$$\mathrm{Re}(G(e^{jw})) + \frac{1}{k} \geq 0 \qquad \forall w \in [0,2\pi) \tag{A.27}$$

which is precisely Tsypkin's condition [A-33]. The fact that $1 + \frac{k}{2} G$ is invertible can be obtained by analogy with the continuous time results in [A-42, Chapter 5] (in fact, this result is a simple consequence of the Nyquist criterion when we observe that G is stable and take (A.27) into account).

Consider the feedback system in Figure A.2. It is clear that the input-output behavior of this system is the same as that for the system in Figure A.4, where M and N are operators (not necessarily causal). As discussed in [A-42,44], one can often find appropriate multipliers so that the modified forward and feedback systems satisfy the criteria of Theorem A.7. This is in fact the basis for Popov's criterion [A-37], for its generalizations [A-38,39,40,42,43,44,45], and for Tsypkin's discrete-time version [A-23,44].

Consider a nonlinear feedback system as in Figure A.3 but in continuous-time (i.e. replace G(z) with G(s)), and again suppose f is strictly inside the sector [0,k]. Using the multipliers

$$N=I, \; M = \frac{1}{1+\alpha s}$$

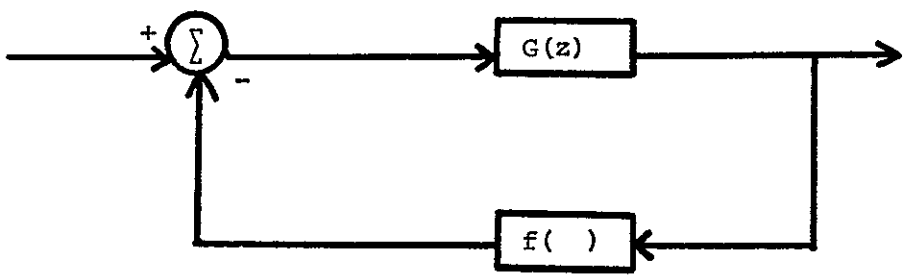

Figure A.3: Linear System with memoryless nonlinear feedback

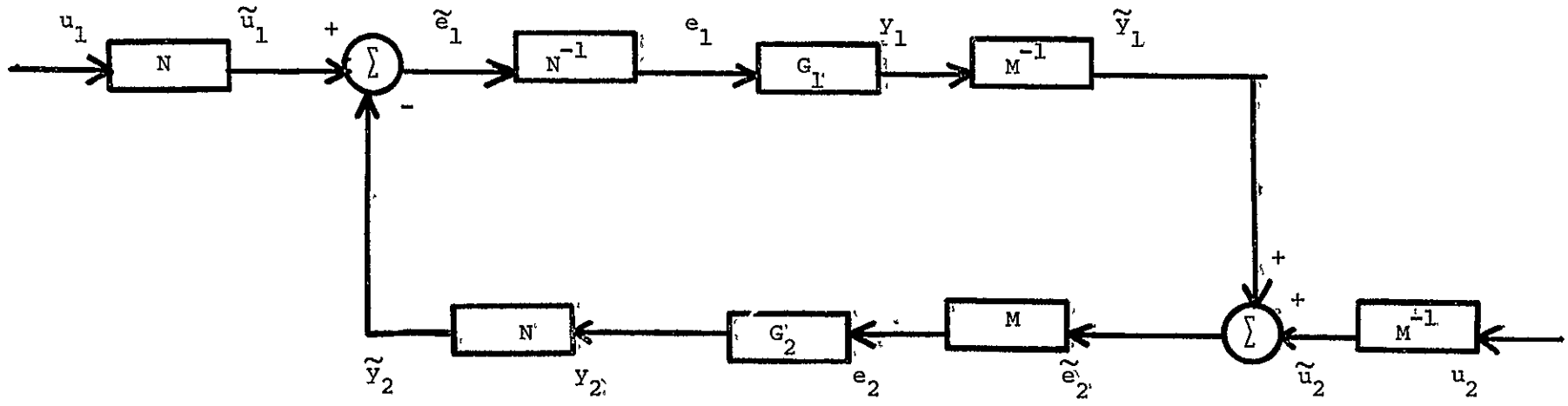

Figure A.4: A Feedback System with Multipliers

we can show that the feedback path is also strictly inside the sector [0,k] and hence the modified forward loop must satisfy a passivity condition. Specifically, we obtain Popov's condition (see [A-38]) that the feedback system is finite gain I/O stable if G is stable (all poles in the left-hand plane) and if $(1+\alpha s)G(s) + \frac{1}{k}$ is positive real for some $\alpha \geq 0$ -- i.e. if

$$\mathrm{Re}[(1+\alpha jw)G(jw)] + \frac{1}{k} \geq 0 \qquad \forall w$$

To obtain Tsypkin's result [A-23,43], we must in addition assume that f is nondecreasing. In this case, the discrete-time system is finite gain I/O stable if there exists $\alpha \geq 0$ such that

$$\mathrm{Re}[(1+\alpha(1-e^{-jw}))G(e^{jw})] + \frac{1}{k} \geq 0 \qquad w \in [0,2\pi) \tag{A.28}$$

As mentioned earlier, a number of extensions of Popov's criterion in continuous-time are available, and we refer the reader to [A-42,44,45] and in particular to [A-38]. As we shall see, some of the results on digital filter limit cycles resemble Tsypkin-type criteria.

Sector nonlinearity characteristics play a major role in the study of digital filter limit cycles (see in particular [A-15]). Specifically, consider the roundoff quantizer in Figure A.5. This function is inside the sector [0,2] (see [A-3,15] for other quantizers and their sector characteristics).

Using simply the sector nature of a nonlinearity, Claasen, et.al. [A-15] prove the following

Theorem A.8: Consider the feedback system of Figure A.3, where f is in the sector [0,k]. Then limit cycles of period N are absent if

$$\mathrm{Re}(G(e^{j2\pi\ell/N})) + \frac{1}{k} > 0 \tag{A.29}$$

for $\ell=0,1,\ldots,N-1$.

If one also takes the nondecreasing nature of f into account, we obtain [A-15]:

Theorem A.9: If f is inside the sector [0,k] and also is nondecreasing, then limit cycles of period N are absent from the system of Figure A.3 if there exist $\alpha_p \geq 0$ such that

$$\mathrm{Re}\left\{\left[1 + \sum_{p=1}^{N-1} \alpha_p (1-e^{j2\pi\ell p/N})\right] G(e^{j2\pi\ell/N})\right\} + \frac{1}{k} > 0 \tag{A.30}$$

If we take α_{N-1} to be the only nonzero α_p, we obtain the condition derived by Barkin [A-16] which is quite similar to Tsypkin's criterion (A.28). Note also the relationship between (A.29) and (A.27). The proofs given in [A-15] rely heavily on the passivity relations (A.29),(A.30). Theorem A.8 then follows from an application of Parseval's theorem in order to contradict

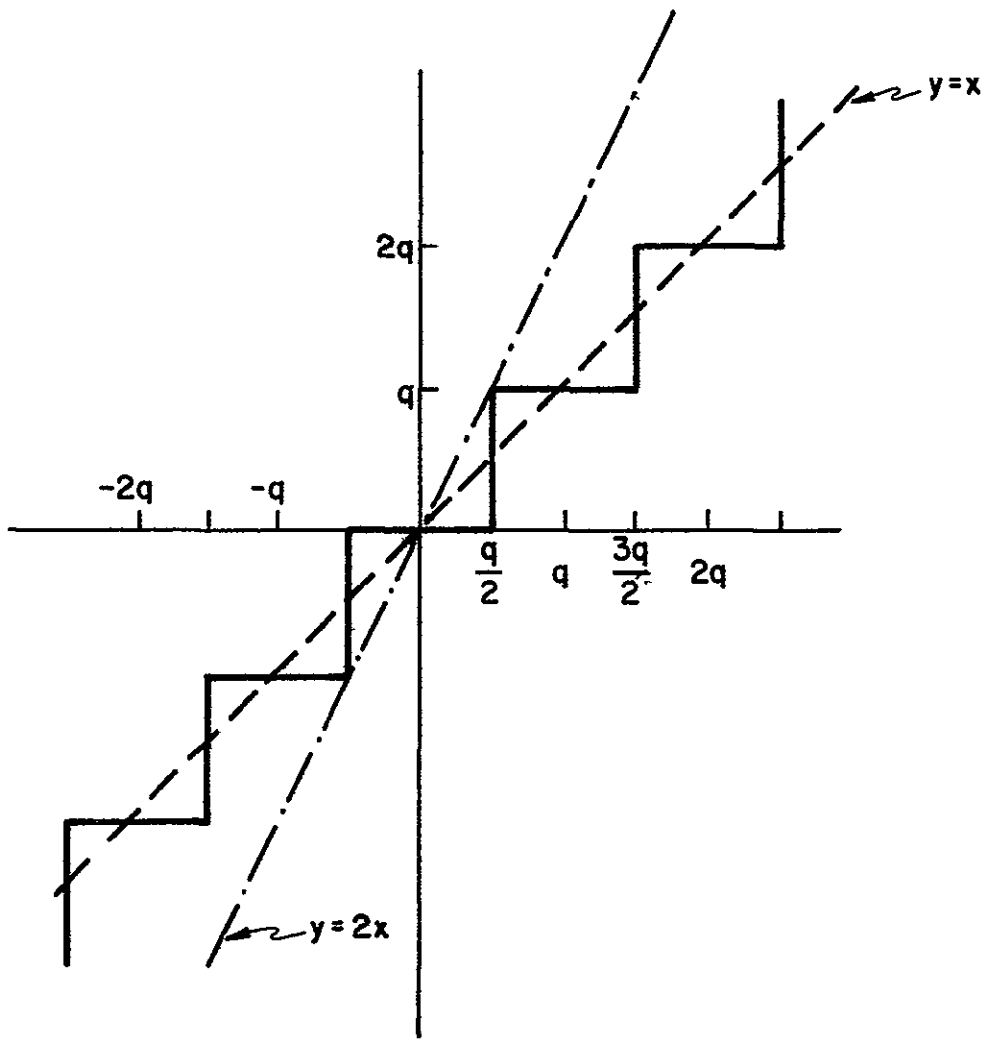

Figure A.5: A Roundoff Quantizer

the existence of a limit cycle of period N. This last step involves the assumed periodicity in a crucial way, but the application of Parseval and the use of the positive real relationship (A.29) is very reminiscent of stability arguments in feedback control theory [A-42]. In the proof of Theorem A.9, the monotonicity of f is used in conjunction with a version of the rearrangement inequality [A-40,42].

Theorem A.10: Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers that are similarly ordered -i.e.

$$x_n \leq x_m \Rightarrow y_n \leq y_m \tag{A.31}$$

Then if π is any permutation

$$\sum_n x_n y_n \geq \sum_n x_{\pi(n)} y_n \tag{A.32}$$

Corollary [A-40]: If f is a monotone function, then for any sequence $\{x_n\}$ and any permutation π

$$\sum_n f(x_n)\,[x_n - x_{\pi(n)}] \geq 0 \tag{A.33}$$

We note that Theorem A.9 bears some resemblance to the multiplier-type results of Popov and Tsypkin. In addition, Willems and Brockett [A-40,42] utilize the rearrangement inequality to obtain a general multiplier stability result for discrete-time systems with single monotone nonlinearities. A thorough understanding of the relationships among these results would be extremely useful, as it might lead to new results on nonexistence of limit cycles. In addition, Claasen, et.al. [A-15] have developed a further improvement over (A.30) if f is in addition antisymmetric ($f(-x) = -f(x)$), and have devised linear programming techniques to search for the coefficients α_p in (A.30). This algorithmic concept may prove to be of use in developing search techniques for other, more complex multipliers. Also, Cook [A-70] has recently reported several criteria for the absence of limit cycles in continuous time systems. His results bear a strong relationship to those of Claasen, et.al., [A-15]. In particular, passivity conditions and Parseval's theorem are used in very similar ways in the two papers.

We now turn our attention to the relationship between input/output concepts and questions of internal-stability, (i.e. the response to initial conditions). Intuitively, if we have an internal, state space representation of a system with specific input/output behavior G (wirh $G(0)=0$), we clearly cannot deduce asymptotic stability from input/output stability without some conditions on the state space realization. For example, the map $G\equiv 0$ is input/output stable but the realizations

$$\dot{x}(t) = x(t), \qquad y(t) = x(t) \tag{A.34}$$

and

$$\dot{x}(t) = x(t) + u(t), \qquad y(t)=0$$

are clearly not asymptotically stable. In the first case the state space has an unstable mode, but if we start at $x(0)=0$ (as we would to realize G), we can never excite this mode. Hence, I/O stability can tell us nothing about it. In the second case, we can excite the mode but we cannot observe it. These are precisely the difficulties that can arise; however, if one imposes certain controllability and observability conditions on the realization, one can deduce asymptotic stability from I/O stability. Thus, controllability and observability play a crucial role in translating from I/O results to Lyapunov-type stability results. For a precise statement of the relationship between the two, see [A-49,69].

Having established the above relationship, it is natural to discuss the generation of Lyapunov functions (which deal with internal stability) for systems satisfying some type of passivity condition. Some of the most important work in this area is that of J.C. Willems [A-49,50,69]. In [A-49,69], Willems discusses the generation of Lyapunov functions for I/O stable systems. For passive systems he defines the notions of available and required energy as the solution of certain variational problems. If one then has a state space realization satisfying certain controllability and observability conditions, one can use these functions as Lyapunov functions. This very general, physically motivated theory is further developed in [A-50]. Dissipative systems and the associated notions of storage function (an internal variable) and supply rate (input/output quantity) are defined, and, much as with Fettweis' pseudopassivity, dissipative systems have many appealing properties (such as preservation under interconnections). We refer the reader to [A-50,69] for details of topics such as the

construction of storage functions and their use as Lyapunov functions.

As mentioned at the end of the preceding subsection, many frequency domain results can be derived with Lyapunov-type arguments. We have also seen in this subsection that many of these results can be derived via passivity arguments. Clearly the two are related, and the crucial result that leads to this relationship is the Kalman-Yacubovich-Popov lemma [A-61,62,69], which relates the positive realness of certain transfer functions to the existence of solutions to particular matrix equalities and inequalities. Kalman [A-62] utilized this result to obtain a Lyapunov-type proof of the Popov criterion, and Szego [A-61] (see also the discussion at the end of [A-33]) used a discrete-time version to obtain a Lyapunov-theoretic proof of Tsypkin's criterion plus several extensions when the derivative of the nonlinearity is bounded. In addition, several other researchers [A-1,38,39] have utilized similar ideas to relate positive real functions to the existence of certain Lyapunov functions. It is beyond the scope of this paper to discuss this problem in depth, but we refer the reader to the references, since this area of research provides a number of insights into the relationships among various stability concepts. In addition, these results provide examples of nonlinear problems for which there exist constructive procedures for Lyapunov functions. We also note that the positive real lemma plays a crucial role in several other problem areas including the stochastic realization and spectral factorization problem [B-21] and the study of algebraic Riccati equations [A-67].

Finally, we note that many of these passivity-Lyapunov results have instability counterparts (e.g., see [A-1,39]). We refer the reader to the detailed development in [A-39] in which a Lyapunov-theoretic methodology for generating

instability results is described. Such results may be useful in developing sufficient conditions for the existence of non-zero, undriven solutions such as limit cycles.

In this section we have considered some of the aspects of stability theory that we feel deserve the attention of researchers in both disciplines. We have not, of course, been able to consider all of the possible topics that one might investigate. For example, the "jump phenomenon" in which small changes in input lead to large changes in output is of interest in digital filter theory [A-32] and also has been considered in feedback control theory [A-42,43], where the concept of feedback system continuity is studied. In addition, Claasen, et.al. [A-31] have introduced the concept of accessible limit cycles, and its relationship to concepts of controllability and also to the structure of the state transition function of the filter are intriguing questions. We also have not discussed the use of describing functions in digital filter analysis. There have been several attempts in this area (see [A-5,29]), but none of these has proven to be too successful (see comments in [A-30]). Except for the work of Parker and Hess [A-26] and Kalman and Bestram [A-48], we have not spoken about bounds on the magnitudes of responses. In the digital filtering area these exist a number of results [A-31,35,56], the latter two of which use an idea of Bestram's [A-58] as a starting point. In control theory, the notion of I/O gain [A-42,44] is directly tied to response magnitude bounds, although it is not clear how tight these would be in any particular case. Finally, in this section, we have not discussed stability criteria for systems with multiple nonlinearities. There do exist some results in this area for digital filters (see [A-3,15]), and

on the other side, the general framework allows one to adapt results such as Theorem A.7 to the multivariable case with little difficulty (hence one can readily obtain matrix versions of Tsypkin's criterion involving positive real matrices). Also, the techniques of Lyapunov theory should be of some use in obtaining stability results much like those in [A-2] for filters of higher order than the second order section.

As we have seen many of the results in the two disciplines involve the use of very similar mathematical tools. On the other hand, the perspectives and goals of researchers in the two fields are somewhat different. The development of a mutual understanding of these perspectives and goals can only benefit researchers in both fields and is in fact absolutely crucial for the successful study of certain problems. For example, in the implementation of digital control systems one must come to grips with problems introduced by quantization. Digital controller limit cycles at frequencies near the resonances of the plant being controlled can lead to serious problems. In addition, the use of a digital filter in a feedback control loop creates new quantization analysis problems. Recall that limit cycles can occur only in recursive (infinite impulse response) filters, while that do not occur in nonrecursive (finite impulse response) filters. However, if a nonrecursive filter is used in a feedback control system, quantization errors it produces <u>can</u> lead to limit cycles of the closed-loop system [A-72]. How can one analyze this situation, and how does one take quantization effects into account in digital control system design? Questions such as these await further investigation.

B. Parameter Identification, Linear Prediction, Least Squares, and Kalman Filtering

A problem of great importance in many disciplines is the determination of the parameters of a model given observations of the physical process being modeled. In control theory this problem is often called the system identification problem, and it arises in many contexts. The reader is referred to the special issue of the IEEE Transactions on Automatic Control [B-15] and to the survey paper of Åström and Eykhoff [B-16] for detailed discussions and numerous references in this problem area. One of the most important applications of identification methods is adaptive estimation and control. Consider the situation depicted in Figure B.1. Here we have a physical process that is to be controlled or whose state is to be estimated. Many of the most widely used estimation and control techniques are based on a dynamic model (transfer function, state space description, etc.) for the system under consideration. Hence it is necessary to obtain an appropriate model in order to apply these techniques. Often, one can perform tests on the process before designing the system and can apply an identification procedure to determine the system. On the other hand, there are many occasions in which the values of certain system parameters cannot be determined a priori or are known to vary during system operation. In such cases, one may often design a controller or estimator "which depends explicitly on these parameters. In this manner we can adjust the parameters on-line as we perform real time parameter identification. A number of methods of this type exist, and, in addition to the two survey references [B-15,16], we refer the reader to [B-80,81,98] for other examples.

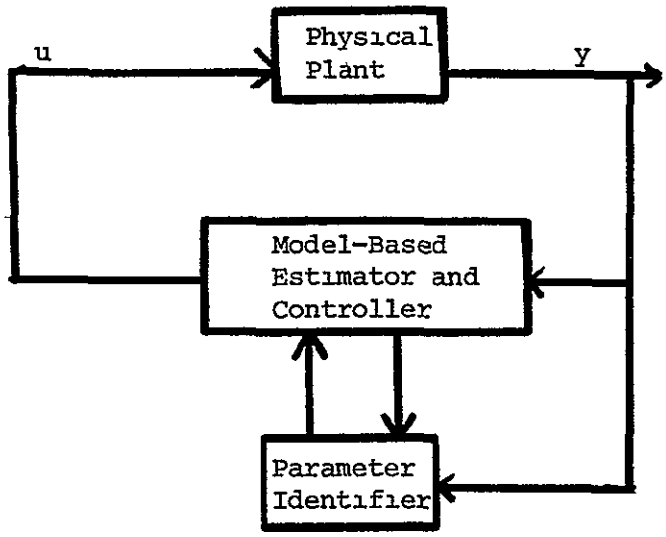

Figure B.1: Conceptual Diagram of an Adaptive Estimator-Controller Utilizing On-Line Parameter Identification

The last of these, [B-98] is of interest, as it consists of a variety of adaptive control techniques all applied to the control of the F-8C aircraft,[1] and thus provides some insight into the similarities, differences, advantages, and disadvantages of the various techniques.

A little thought about the identification problem makes it clear that there are several issues. Before one can apply parameter identification schemes, one must have a parametric model, and the determination of the appropriate structure for such a model is a complex question in itself. We will not consider this issue in much detail in this paper, and we refer the reader to the references for details (see several of the papers in [B-15] on canonical forms and identifiability; also see the work of Rissanen and Ljung [B-79]).

Parameter identification problems also arise in several digital signal processing applications. Several examples of such problems are given in the special issue of the Proceedings of the IEEE [B-99], and these include (see [B-26]) seismic signal processing and the analysis, coding, and synthesis of speech. This latter application has received a great deal of attention in the past few years [B-24-26,28-30,44-55,69-71,74], and we will use this problem as a basis for our discussion of the identification question. We follow the work Atal [B-48], Atal and Schroeder [B-70], Markel and Gray [B-44], and Makhoul [B-26]. Our presentation is necessarily brief and intuitive, and the reader is referred to these references for details.

As discussed in [B-44] a popular and widely accepted model for a discretized speech signal $\{y(k)\}$ is as the output of a linear system, which, over short

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[1] All of these projects were sponsored by NASA Langley. This "fly-by-wire" adaptive control program is still in its evolutionary stages, and new methods and concepts are still being developed.

enough intervals of time, can be considered to be time-invariant

$$y(z) = G(z)U(z) \tag{B.1}$$

where G represents the overall transfer function and U(z) is the z-transform of the input, which is often taken as a periodic pulse train (whose period is the pitch period) for voiced sounds and as white noise for unvoiced sounds. In addition, a common assumption is that G is an all-pole filter

$$G(z) = \frac{1}{1 + \sum_{k=1}^{p} a_z z^{-k}} \tag{B.2}$$

This assumption has been justified in the literature under most conditions, although strong nasal sounds require zeroes [B-44]. Note that under condition (B.2), equation (B.1) represents an autoregressive (AR) process

$$y(k) + a_1 y(k-1) + \ldots + a_p y(k-p) = u(k) \tag{B.3}$$

The problem now is to determine the coefficients $a_1, \ldots, a_p$. Having these coefficients, one is in a position to solve a number of speech analysis and communication problems. For example, one can use the model (B.2) to estimate formant frequencies and bandwidths, where the formants are the resonances of the vocal tract [B-55]. In addition, one can use the model (B.3) for efficient coding, transmission, and synthesis of speech [B-70]. The basic idea here is the following: as the model (B.1)-(B.3) indicates, the speech signal y(k) contains highly redundant information, and a straightforward transmission of the signal will require high channel capacity for

accurate reconstruction of speech. On the other hand, rearranging terms in (B.3)

$$y(k) = - \sum_{i=1}^{p} a_i y(k-i) \quad + u(k) \tag{B.4}$$

we see that (B.4) represents a predictor, in which

$$\hat{y}(k) = - \sum_{i=1}^{p} a_i y(k-i) \tag{B.5}$$

is the one-step predicted estimate of y. As discussed in [B-70], one often (and, in particular, in the speech problem) requires far fewer bits to code the prediction error u than the original signal y. Thus, one arrives at an efficient transmission scheme (linear predictive coding --LPC): given y, estimate the a_i, compute u, transmit the a_i and u. At the receiver, we then can use (B.4) to reconstruct y (of course, one must confront problems of quantization, and we refer the reader to the references (e.g., [B-119]) for discussions of this problem). An alternative interpretation of this procedure is the following: gives y, estimate G in (B.2), pass y through the inverse, all zero (moving average -- MA) filter 1/G(z), transmit the coefficients in G and the output of the inverse filter. At the receiver, we then pass the received signal through G to recover y (thus this procedure is causal <u>and</u> causally invertible).

The question remains as to how one estimates the a_i. The most widely used technique in the literature is linear prediction. Using the interpretation of $1 - \frac{1}{G(z)}$ as a one-step predictor for the signal y, we wish to choose the coefficients $a_1,\ldots,a_p$ to minimize the sum of squares of the prediction errors[2]

$$J = \sum_n e^2(n) \tag{B.6}$$

$$e(n) = y(n)-\hat{y}(n)$$

Here we assume that we are given $y(0),\ldots,y(N-1)$. Also, the range of n in the definition of J can be chosen in different manners, and we will see in the following subsections that different choices can lead to different results and to different interpretations. A number of these interpretations are given in [B-26,44], and we will discuss several of these as we investigate this problem somewhat more deeply. Specifically, in the next two subsections we consider two linear prediction methods -- the autocorrelation and covariance methods -- and we relate them to several statistical notions of importance in control and estimation applications. Following this, we will discuss several other identification methods and their relationship to the speech problem.

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[2] We note that one can modify the linear prediction formulation in order to take into account the quasi-periodic nature of speech for voiced sounds. We refer the reader to [B-70] in which such a procedure is developed in which one also obtains an estimate of the pitch period. An alternative approach to this problem is to solve the linear prediction problem as outlined in the next two subsections, pass the speech through the inverse filter, and analyze the resulting signal to determine the pitch [B-25,44]. Recently, Steiglitz and Dickinson [B-100] have described a method for improving pole estimation by completely avoiding that part of a voiced speech signal that is driven by glottal excitation.

Before beginning these investigations, let us carry out the minimization required in linear prediction. Taking the first derivative of J with respect to the a_i and setting these equal to zero, we obtain the <u>normal equations</u>

$$\sum_{i=1}^{p} a_i c_{ik} = -c_{0k} , \qquad k=1,\ldots,p \tag{B.7}$$

where

$$c_{ik} = \sum_{n} y(n-i)y(n-k) \tag{B.8}$$

These equations are typical of the types of equations that arise in linear, least-squares problems, and their efficient solution has been the topic of many research efforts. This issue is the central focus in the next two subsections.

B.1 The Autocorrelation Method, Kalman Filtering for Stationary Process, and Fast Algorithms

Suppose we consider minimizing the sum-squared error in (B.6) over the infinite interval, $-\infty<n<\infty$. Here, we define $y(n)=0$ for $n<0$, $n\geq N$. In this case, we find that

$$c_{ij} = \sum_{n=0}^{N-1-|i-j|} s(n)s(n+|i-j|) \triangleq r(|i-j|) \tag{B.9}$$

and the normal equations become

$$\mathbf{T}\mathbf{a} = \mathbf{c} \tag{B.10}$$

where $a' = (a_1,\ldots,a_p)$, $c' = (-r(1), -r(2),\ldots,-r(p))$, and T is a symmetric Toeplitz matrix [B-37,84,91] (i.e. the ijth element depends only on $|i-j|$):

$$T = \begin{bmatrix} r(0) & r(1) & \ldots & r(p-1) \\ r(1) & r(0) & \ldots & r(p-2) \\ r(2) & r(1) & \ldots & r(p-3) \\ \vdots & & & \\ r(p-1) & r(p-2) & . & r(0) \end{bmatrix} \tag{B.11}$$

Before we consider the solution of (B.10), let us derive equations of the very same form from a probabilistic point of view (here we follow [B-26]). Suppose that y is a stationary random process, and, instead of (B.6), we are interested in minimizing

$$\tilde{J} = E(e^2(n)) \tag{B.12}$$

where e and $\hat{y}$ are defined as before (although they now are random processes themselves). Differentiating (B.12) as before, we obtain the normal equations

$$\tilde{T}a = \tilde{c}$$

where $\tilde{c}' = (-R(1),-R(2),\ldots,R(p))$, $\tilde{T}$ is the symmetric Toeplitz matrix whose ijth element is $R(|i-j|)$, and $R(i)$ is the autocorrelation

$$R(i) = E(y(n)y(n+i)) \tag{B.14}$$

Examining (B.9)-(B.14), we see that the two formulations are strikingly similar, and, one can view (B.9) as a method for estimating the autocorrelation of

an ergodic, stationary process [B-26] (if we normalize (B.9) appropiately). This statistical point of view is extremely usefu in order to obtain certain insights into the approach and also in order to allow us to connect this method with certain recent results in linear estimation theory (see [B-44] for several other interpretation of this method).

The solution of equations such as (B.10) and (B.13) has been the subject of a great deal of attention in the mathematical, statistical, and engineering literature [B-4,7,26,34,35,36,37,50,72,84,91,94,95,96]. An efficient algorithm was proposed by Levinson [B-34], improved upon by Durbin [B-94], and studied in the speech processing context by several authors, including Itakura and Saito [B-50](a version of this algorithm is given later in this subsection). As discussed in [B-26,44], the method essentially consists of solving forward and backward prediction problems of increasing size in a recursive manner and is known to be extremely efficient. That is, the algorithm computes the coefficients $a(1|i),\ldots,a(i|i)$ for the best prediction of $y(n)$ based on $y(n-1),\ldots,y(n-i)$ and the coefficients $b(1|i),\ldots,b(i|i)$ for the best prediction of $y(n-i-1)$ based on $y(n-i),\ldots,y(n-1)$. The algorithm iterates on i. As a part of this algorithm, one computes the prediction error (for both forward and backward prediction), and thus one can determine when to stop based on the size of this quantity. Also, we must compute a coefficient k_i, which is known as the <u>partial correlation coefficient</u> between the forward and backward prediction errors (see [B-26,44,50]). We will mention this quantity again at the end of this subsection.

Let us now examine what this algorithm means from a statistical point of

view. The first stage of the algorithm produces $a(1|1)$ and $b(1|1)$, which are the coefficients of the best one-step predictors

$$\begin{aligned} \hat{y}(1) &= -a(1|1)y(0) \\ \hat{y}(0) &= -b(1|1)y(1) \end{aligned} \tag{B.15}$$

At thenext stage, we have $a(1|2)$, $a(2|2)$, $b(1|2)$, $b(2|2)$

$$\begin{aligned} \hat{y}(2) &= -a(1|2)y(1) - a(2|2)y(0) \\ \hat{y}(0) &= -b(1|2)y(1) - b(2|2)y(2) \end{aligned} \tag{B.16}$$

Continuing, we find that after i steps we have the predictors

$$\hat{y}(i) = -\sum_{j=1}^{i} a(j|i)y(i-j) \tag{B.17}$$

$$\hat{y}(0) = -\sum_{j=1}^{i} b(j|i)y(j) \tag{B.18}$$

Thus, we can think of the linear prediction solution as providing us with the <u>time-varying</u> coefficients of the weighting pattern of the optimal one-step predictor (B.17) or of the optimal initial time smoother (B.18). Note that these coefficients <u>are</u>, in general, time varying in the following sense: from (B.17), we see that $a(j|i)$ is the coefficient that multiplies the data point that occurs j units of time before the one whose value we wish to predict. If the filter were time-invariant, this would not depend on i. The reason

for the time-varying nature of the predictor coefficients is that, although the y's are a stationary process, the mechanism of prediction is time-varying when one bases the prediction on only a finite set of data (recall that the time-invariant Wiener filter assumes an infinite record of observations).

What does this mean as far as all-pole modeling via linear prediction goes? The answer to that is not much. In the all-pole modeling problem, we are equivalently only interested in designing a FIR filter -- i.e. a prediction filter that produces the best estimate of $y(n)$ gives the "data window" $y(n-1),\ldots,y(n-p)$. The coefficients of such a filter are <u>precisely</u> $a(1|p),\ldots,a(p|p)$, and it doesn't matter (except from a computational point of view) that these coefficients were generated as part of a time-varying filter weighting pattern.

On the other hand, the time-varying weighting pattern interpretation is extremely important from a statistical point of view, especially if one wishes to design recursive predictors that are capable of incorporating all past measurements and not just a data window. Clearly one inefficient way to do this is to implement a nonrecursive filter that stores all past data, $y(0),\ldots,y(n-1)$, multiplies by the appropriate $a(i|n)$, and combines to form $\hat{y}(n)$. This requires gorwing memory and is hardly appealing. How can one avoid such difficulties? An answer that is popular in state-space control and estimation theory arises if y has a <u>Markovian representation</u>[3]

$$\begin{aligned} x(k+1) &= Ax(k) + w(k) \\ y(k) &= c'x(k) \end{aligned} \tag{B.19}$$

[3] We will briefly discuss the problem of finding such a representation later in this section.

where x is a random n-vector (x(0) is assumed to be zero mean), A is a constant nxn matrix, c is a constant n-vector, and w is a zero-mean uncorrelated sequence (uncorrelated with x(0)) with

$$E(w(k)w'(k)) = Q \tag{B.20}$$

The correlation coefficients of y can be computed from the equations

$$E(y(k)y(j)) = c'E(x(k)x'(j))c \tag{B.21}$$

$$E(x(k)x'(j)) = \begin{cases} A^{k-j}P(j) & k \geq j \\ [E(x(j)x'(k))]' & k < j \end{cases} \tag{B.22}$$

where P is the covariance of x, which satisfies

$$P(j+1) = AP(j)A' + Q \tag{B.23}$$

Note that in general E(y(k)y(j)) will not depend on (k-j) alone. This will occur if and only if A is a stable matrix and P=P(0) satisfies the Lyapunov equation

$$APA' - P = -Q \tag{B.24}$$

(in which case both x and y are stationary).

Suppose now that (B.24) holds and that

$$R(|i-j|) = c'A^{|i-j|}Pc \tag{B.25}$$

where the R(i) are the quantities defined in (B.13)-(B.14). We now wish to

design an optimal predictor for estimating (recursively) $y(n)$ given $y(0),\ldots,y(n-1)$. This is a standard state-space estimation problem [A-65] and the solution is the Kalman filter (which actually produces a prediction for the vector $x(n)$):

$$\begin{aligned} \hat{y}(n) &= c'\hat{x}(n) \\ \hat{x}(n) &= A\hat{x}(n-1) + AK(n-1)\gamma(n-1) \\ \gamma(n-1) &= y(n-1) - \hat{y}(n-1) \\ \hat{x}(0) &= 0 \end{aligned} \tag{B.26}$$

where the time-varying gain satisfies[4]

$$K(n) = \frac{P(n|n-1)c}{c'P(n|n-1)c} \tag{B.27}$$

Here $P(n|n-1)$ is the covariance of the prediction error $x(n) - \hat{x}(n)$,

$$P(n+1|n) = AP(n|n-1)A' + Q - \frac{AP(n|n-1)cc'P(n|n-1)A'}{c'P(n|n-1)c} \tag{B.28}$$

Let us make a few comments about these equations. Note that the <u>filter innovations</u> $\gamma(n)$ is precisely the prediction error, and its covariance is $c'P(n|n-1)c$, which is nothing more than (B.12). Also, recall that in the all-pole framework, we could alternatively view the prediction filter as specifying an inverse filter, which took the y's as inputs and produced the

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[4]Note that we require $c'P(n|n-1)c \neq 0$. As discussed in [B-67], this requires the positivity of the covariance $R(i)$, which is clearly related to the statement that $y(n)$ is not a deterministic functions of $y(0),\ldots,y(n-1)$ for any n.

uncorrelated sequence of prediction errors as the output. In the context of the Kalman filter, the analogous filter is the <u>innovations representation</u> (see representation IR-1 of [B-67]), in which we view the output of (B.26) as being $\gamma(n)$. Finally, note that one can compute the predictor coefficients $a(j|i)$ as the weighting pattern of the filter:

$$a(1|1) = c'AK(0)$$

$$a(1|2) = -c'AK(1) \qquad a(2|2) = -c'A^2K(0) + c'AK(1)c'AK(0) \tag{B.29}$$

$$\vdots$$

The Kalman filter and innovations representations have been the subjects of a great deal of research in the last 15 years, and the technique described above has been studied in discrete and continuous time[5], for multiple output systems, for time-varying systems, and for systems in which the actual observations are noisy versions of the y's

$$z(n) = y(n) + v(n) \tag{B.30}$$

We refer the reader to the many references on this subject, including [A-65], [B-7,58,67].

Examining (B.26)-(B.28), we see that the computation of the recursive filter coefficients requires the solution of the (discrete time) <u>Riccati equation</u> (B.28). If x is an n-vector, then (using the fact that P is symmetric),

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[5] We note that in continuous time one has a somewhat more difficult time --i.e. we don't consider "one-step" prediction and in fact run into difficulties if we assume we observe y as opposed to a noise-corrupted version. We refer the reader to [B-67] and to the references therein for more on this problem.

(B.28) represents $\frac{n(n+1)}{2}$ equations. For reasonably large values of n, this can be an extreme computational load, especially given that all that is needed for the filter is the gain matrix K, which in the scalar output case is an n-vector. Also, if there are m outputs, K is nxm, and, as is often the case, the number of parameters in K is much smaller than the number in P(i.e. m is substantially smaller than n). Thus, the question of computing K without P arises quite naturally, and this issue -- in both continuous and discrete time, in stationary and in some nonstationary cases -- has been the subject of numerous papers in the recent past [B-1-8,23,39,40,56,60,64-66,72,73,77].

It is not our intention here to discuss these techniques in detail. What we do want to do is to point out that the underlying concepts that have led to these "fast algorithms" (at least in the stationary case) are the same as those that lead to the Levinson algorithm. For some historical and mathematical perspective on this subject, we refer the reader to [B-4,7,63, and 66]. In particular, the extension of the Levinson algorithm to the multivariable case is discussed in these papers (see also references [B-35,36]). In this case, the matrix T in (B.10) or (B.12) is block-Toeplitz, and the extension to this case is decidedly nontrivial (for other methods for handling equations involving block-Toeplitz matrices, we refer the reader to [B-37,56,84,91,95,96]). Also, in [B-4], the derivation of the Levinson type algorithms and Kalman gain equations in discrete <u>and</u> continuous time are shown (in the stationary case) to rely on the simultaneous solution of forward and backward filtering problems (thus introducing a "backward innovation process," representing backward prediction errors). It is also shown that both continuous and discrete algorithms

are obtainable from the Bellman-Krein formulas [B-4,7,42,43,64,65,66], which describe the evolution of the weighting pattern of the optimal estimator of a stationary process. From this, one can obtain the Levinson algorithms (and its continuous analog) and some well-known relationships with orthogonal polynomials [B-4,41]. If one knows that the process y has a Markovian representation, one can then take the Levinson-type equations together with the state space representation and obtain fast algorithms for the Kalman gain. An excellent treatment of this is given in [B-4], and it is recommended that the reader compare the discrete-time results here to those in [B-26,44] in order to see the relationship between the linear prediction equations and the version of the Levinson algorithm derived in [B-4]. For a thorough historical perspective, we recommend the survey paper [B-7].

In this paper we will limit ourselves to a brief outline of one of the derivations in [B-4]. Let $y(n)$ be a vector stationary, zero mean process with covariance

$$R(t-s) = E(y(t)y(s)') \tag{B.31}$$

We observe the process[6]

$$z(n) = y(n) + w(n) \tag{B.32}$$

where w is a zero mean, uncorrelated process, uncorrelated with y, with covariance

$$E(w(n)w(n)')=I \tag{B.33}$$

Let $\hat{y}(t|\bar{r})$ denote the wide sense conditional mean of $y(t)$ given $z(0),\ldots,z(r)$, then [B-4]

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[6] As before, one can take $w=0$ if R is positive definite. Lindquist discusses this in [B-4].

$$\hat{y}(t|r) = \sum_{s=0}^{r} G_r(t,s)z(s) \tag{B.34}$$

where the weighting pattern is defined by

$$G_r(t,s) = E[\tilde{y}(t|r)\tilde{y}(s|r)'] = G_r(s,t)' \tag{B.35}$$

(here $\tilde{y}(i|j)$ is the estimation error $y(i) - \hat{y}(i|j)$. Also, the G_r satisfy the (Toeplitz) equations

$$\begin{aligned} G_r(t,s) + \sum_{i=0}^{r} G_r(t,i)R(i-s) &= R(t-s) \\ G_r(t,s) + \sum_{i=0}^{r} R(t-i)G_r(i,s) &= R(t-s) \end{aligned} \tag{B.36}$$

Note that $\hat{y}(t|t-1)$ is the one-step prediction estimate, and from (B.34) we can identify (in the scalar case)

$$G_{t-1}(t,s) = -a(t-s|t) \tag{B.37}$$

Comparing (B.36),(B.37), we see that we have similar equations (the first term on the left-hand side of (B.36) comes from the presence of $\dot{w}$, but these equations can also be obtained when $w=0$ if we can write $R=\tilde{R} + \varepsilon I$ for some positive semidefinite $\tilde{R}$ -- see [B-9]). Also, as pointed out in [B-4], the Toeplitz equations are the counterparts of certain Fredholm resolvent equations that arise in the continuous case [B-64,65].

Lindquist's derivation of the fast algorithms for computing $G_{t-1}(t,s)$ (one-step prediction) and $G_t(t,s)$ (filtering estimate) begins with the Bellman-Krein formulas[7]

$$G_{r+1}(t,s) = G_r(t,s) - G_{r+1}(t,r+1)G_r(r+1,s)$$

$$G_{r+1}(t,s) = G_r(t,s) - G_r(t,r+1)G_{r+1}(r+1,s)$$

We next define the "backwards weighting pattern"

$$G_r^*(t,s) = G_r(r-t,r-s) \tag{B.38}$$

and the matrix polynomials

$$\phi_t(z) = I - \sum_{s=1}^{t} z^s G_{t-1}^*(s-1,-1) \tag{B.39}$$

$$\phi_t^*(z) = z^t \left[I - \sum_{s=1}^{t} z^{-s} G_{t-1}(s-1,-1) \right] \tag{B.40}$$

As pointed out in [B-4], in the scalar case these polynomials are related to the Szegö polynomials. Also, if we let $\phi_{t,i}$ denote the coefficient of z^i in ϕ (similarly for ϕ^*), and if we use (B.34), (B.35), (B.38)-(B.40) we obtain the prediction and smoothing equations

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[7] We note that the existence of two such formulas is related to the existence of both a one-step prediction and a filtering estimate, which is in clear distinction to the continuous-time case, in which we only have one such formula and filter. Indeed, the discrete time problem leads to a number of different types of innovations representations (see discussion in [B-67]) and also leads to more complex equations to be solved for the weighting pattern and gain. We refer the reader to [B-4,67, 101,102] for more on the differences between the continuous and discrete time cases.

$$\hat{y}(t|t-1) = -\sum_{i=0}^{t-1} \phi'_{t,t-i}\, z(i) \tag{B.41}$$

$$\hat{y}(-1|t-1) = -\sum_{i=0}^{t-1} \left(\phi^*_{t,t-i-1}\right)' z(i) \tag{B.42}$$

Thus, if we can recursively compute ϕ_t, ϕ^*_t, we can recursively solve for the weighting pattern of the desired predictor. Utilizing the Bellman-Krein equations, Lindquist derives these recursions, which yield the multivariable Levinson equations:

$$\phi_{t+1}(z) = \phi_t(z) - z\phi^*_t(z)\Gamma^*_t \quad , \qquad \phi_0(z)=I \tag{B.43}$$

$$\phi^*_{t+1}(z) = z\phi^*_t(z) - \phi_t(z)\Gamma_t \quad , \qquad \phi^*_0(z)=I \tag{B.44}$$

$$R_{t+1} = R_t - \left(\Gamma^*_t\right)' R^*_t \Gamma^*_t \tag{B.45}$$

$$R^*_{t+1} = R^*_t - \Gamma'_t R_t \Gamma_t \tag{B.46}$$

$$S_t = R(t+1) - \sum_{i=0}^{t-1} R(t-i) G_{t-1}(i,-1) \tag{B.47}$$

$$\left(\Gamma^*_t\right)' \left(R^*_t\right)' = S_t = R_t \Gamma_t \tag{B.48}$$

Here, R_t plays the role of forward prediction error, R^*_t is the backwards error, and Γ_t, Γ^*_t are the multidimensional analogs of the partial correlation coefficient introduced earlier. These relationships can be seen much more

easily if one looks at the scalar case and uses the following special relationships that hold in this case (note that these include the fact that in the scalar case the forward and backward predictors are essentially the same -- a statement that is not true in the vector case);

$$\phi_t(z) = z^t \phi_t^*(z^{-1}), \qquad \Gamma_t^* = \Gamma_t \tag{B.49}$$

Then, the algorithm becomes

$$\phi_{t+1}^*(z) = z\phi_t^*(z) - z^t \phi_t^*(z^{-1})\Gamma_t \tag{B.50}$$

$$R_{t+1} = R_t(1-\Gamma_t^2) \tag{B.51}$$

$$\Gamma_t = \left[R(t+1) - \sum_{i=0}^{t-1} R(t-i) G_{t-1}(i,-1) \right] / R_t \tag{B.52}$$

and the comparisons with the usual Levinson equations (equations (38a)-(38d) in [B-26]) are clear.

Following this development, Lindquist next considers the case in which the y's have a Markovian representation. Using the algorithm (B.43)-(B.48), he is able to obtain a fast algorithm for the Kalman gain. For the details of this derivation, we refer the reader to [B-4].

Finally, we note that there are numerous physical and mathematical relationships between fast algorithm that have been derived in a number of disciplines. As discussed in [B-26,44], the auxiliary variable k_i in the scalar Levinson algorithm[8] has an interpretation as a reflection coefficient,

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[8] We note that in the multivariable case, the k_i have two matrix counterparts (Γ_t^* and Γ_t in [B-4]) which in general coincide only in the scalar case. This is due to the fact that the covariance matrix R is only block Toeplitz. This also leads to the differences between the forward and backward predictors, which in turn leads to an increase in computational complexity in the vector case).

and this fact has been utilized in speech processing, in which these coefficients specify certain parameters in an acoustic model of the speech process [B-26,44]. In addition Casti and Tse [B-40] Kailath [B-6,7] and Sidhu and Casti [B-11] have shown that the fast Kalman gain algorithms are closely related to the work of certain astrophysicists, in particular Chandrasekhar [B-38], who devised algorithms for solving finite time Wiener-Hopf equations arising in radiative transfer. Also, relationships between linear filtering and scattering theory have been brought to light in the recent papers [B-77,101,102]. And finally, for a good overview of some of the mathematical relationships, we refer the reader to Genin and Kamp [D-145]. These ideas are of interest in that seeing these algorithms from several perspectives allows us to gain insight into their properties, potentials, and limitations.

B.2 The Covariance Method, Recursive Least Squares Identification, and Kalman Filters

Consider again the normal equations (B.7),(B.8). We now consider the range of n to be only as large as the actual data allows -- i.e., in equation (B.3) we will require that k, k-1,...,k-p all be within the range 0,...,N-1. This leads to the following range for n

$$p \leq n \leq N-1 \tag{B.53}$$

Note that in this case the normal equations become

$$Sa = -d \tag{B.54}$$

where $d' = (c_{01}, c_{02}, \ldots, c_{0p})$, and S is the symmetric matrix whose ijth element is c_{ij}. Note that c_{ij} is <u>not</u> in general a function of i-j, and thus S is not Toeplitz.

We note that this method also has several interpretations. As discussed by Makhoul [B-26], one can obtain equations of identical form as the linear least squares predictor for a nonstationary process. In addition, as discussed in [B-44], if one makes a Gaussian assumption, then the covariance method produces the conditional maximum likelihood estimate of a, given $y(0),\ldots,y(p-1)$. We refer the reader to [B-44] for several other interpretations of the covariance method.

Turning to the solution of (B.54), we find that the fast methods described in the preceding section do not carry over quite so nicely, since S is not Toeplitz. In [B-44], however, a method analogous to the Levinson routine, in that it iterates on the order of the predictor filter and computes forward and backward predictors simultaneously, is described. This method is not nearly as efficient as in the autocorrelation case, and this can be traced to the fact that (B.49) does not hold in this case (even for the one-dimensional problem). As discussed in [B-44], the solution to the autocorrelation and covariance equations can be viewed as performing a Cholesky decomposition, or equivalently a Gram-Schmidt orthogonalization, of T and S. In the Toeplitz case, very fast algorithms exist for Cholesky decomposition (see the previous section and [B-37]), while this procedure is somewhat slower for symmetric, non-Toeplitz matrices. Recently, however, Morf, et.al. [B-71] have obtained fast algorithms for the covariance method by exploiting the fact that, although S is not Toeplitz, it is the <u>product</u> of Toeplitz matrices (see equations (B.56)-(B.59)). We refer the reader to [B-71] for the details of several algorithms that essentially involve embedding the original scalar prediction problem into a multidimensional one to which the fast vector Levinson algorithm can be applied.

Let us take a look at the covariance method from a slightly different point of view. Recall that the algorithm mentioned above and the one in the preceding subsection involve recursions on the order of the filter given a fixed set of data. Suppose now we consider a recursion for updating coefficients of a fixed order filter given more and more data. To do this, we refer to the survey paper [B-16], where the covariance method, termed the "least squares method" is discussed [9]. Given the data $y(0),\ldots,y(N-1)$, the covariance method attempts to find a least squares fit to the equation

$$L_{N-1}a = f_{N-1} \tag{B.55}$$

where

$$L_{N-1} = \begin{bmatrix} -y(p-1) & -y(p-2) & \ldots & -y(0) \\ -y(p) & -y(p-1) & \ldots & -y(1) \\ -y(p+1) & -y(p) & \ldots & -y(2) \\ \vdots & & & \vdots \\ -y(n-2) & -y(N-3) & \ldots & -y(N-p-1) \end{bmatrix} \tag{B.56}$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}, \qquad f_{N-1} = \begin{bmatrix} y(p) \\ y(p+1) \\ \vdots \\ y(N-1) \end{bmatrix} \tag{B.57}$$

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[9] In this survey paper the autocorrelation method -- called the "correlation method" -- is also discussed and is compared to least squares

The least -squares solution is given by

$$\left(L'_{N-1}\ L_{N-1}\right)\hat{a} = L'_{N-1}\ f_{N-1}\ , \tag{B.58}$$

which can be seen to be identical to (B.54). Thus, the covariance method computes

$$\hat{a}(N-1) = \left(L'_{N-1}\ L_{N-1}\right)^{-1} L'_{N-1}\ f_{N-1} \tag{B.59}$$

Suppose we have $\hat{a}(N-1)$ and we now obtain the new data point $y(N)$. We would like to update our estimate to $\hat{a}(N)$ in a manner more efficient than re-solving (B.58) from scratch. Following standard recursive least squares (RLS) procedures [B-16], we note that incorporation of $y(N)$ into (B.55) adds a new equation -- i.e. it adds a last row to L_{N-1}

$$\ell'(N) = (-y(N-1),\ -y(N-2),\ldots,-y(N-p)) \tag{B.57}$$

and a last element, $y(N)$, to f_{N-1}. Thus, (B.55) takes the form

$$\begin{bmatrix} L_{N-1} \\ \ell'(N) \end{bmatrix} a = \begin{bmatrix} f_{N-1} \\ y(N) \end{bmatrix} \tag{B.58}$$

and (B.59) becomes

$$\hat{a}(N) = \left[L'_{N-1}L_{N-1} + \ell(N)\ell'(N)\right]^{-1}\left[L'_{N-1}f_{N-1} + \ell(N)y(N)\right] \tag{B.59}$$

With the aid of the matrix inversion lemma [A-65], we can rewrite (B.59)

$$\hat{a}(N) = \hat{a}(N-1) + K(N)[y(N)-\ell'(N)a(N-1)] \tag{B.60}$$

where

$$K(N) = \frac{P(N-1)\ell(N)}{1+\ell'(N)P(N-1)\ell(N)} \tag{B.61}$$

and

$$P(N) = (L_N'L_N)^{-1} = P(N-1) - \frac{P(N-1)\ell(N)\ell'(N)P(N-1)}{1+\ell'(N)P(N-1)\ell(N)} \tag{B.62}$$

Examining these equations, we see that they represent a Kalman filter (see [B-17]). In fact, referring to [B-24,47], we see that these are precisely the Kalman filter equations used by Melsa, et.al. in speech processing. Specifically, they consider the dynamic equations

$$a(k+1) = a(k) \tag{B.63}$$

$$y(k) = z'(k)a(k) + v(k) \tag{B.64}$$

where

$$z'(k) = -(y(k-1),y(k-2),\ldots,y(k-p))' \tag{B.65}$$

and $v(k)$ is a zero-mean, white process with

$$E(v^2(k)) = \Psi \tag{B.66}$$

If Ψ is set to 1, we obtain the solution to the covariance equations. Also, in this formulation, P(N) has the interpretation as the covariance of the estimation error $a-\hat{a}(N)$.

Let us note some of the properties of the recursive solution (B.60)-(B.62). Examining (B.60), we see that the increment in our estimate $\hat{a}$ is proportional to the error (innovations) in predicting the latest value of y using preceding values and our previous estimate of a. This suggests that a monitoring of the residuals

$$r(N) = y(N) - \ell'(N)\hat{a}(N-1) \tag{B.67}$$

can be used to help detect abrupt changes in the predictor coefficients[10] or the presence of glottal excitation in voiced sounds. In this manner one may be able to improve upon the estimation of a. Whether such a procedure would be of value is a matter for future study. We only note here that such techniques have been developed and have been successfully applied to a variety of problems including the detection of arrhythmias in electrocardiograms [B-103,104]. Also, it is possible to make the filter more responsive to changes in the coefficients by using one of several methods available for adjusting Kalman filters [A-65]. These include exponentially age-weighting old data in favor of the more recent pieces of information or the modeling of a as a slowly-varying Markov process

$$a(k+1) = Aa(k) + w(k) \tag{B.68}$$

where A is a stable matrix, and w is zero mean white noise with covariance Q. In this case, equation (B.60)-(B.62) become

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[10] We note that Bergland [B-124] has suggested monitoring the residuals of a linear predictor in order to determine when to update the estimates of the predictor coefficients.

$$\hat{a}(N) = A\hat{a}(N-1) + K(N)[y(n) - \ell'(N)A\hat{a}(N-1] \tag{B.69}$$

$$K(N) = \frac{P(N|N-1)\ell(N)}{1+\ell'(N)P(N|N-1)\ell(N)} \tag{B.70}$$

$$P(N|N-1) = AP(N-1|N-1)A' + Q \tag{B.71}$$

$$P(N|N) = P(N|N-1) - \frac{P(N|N-1)\ell(N)\ell'(N)P(N|N-1)}{1+\ell'(N)P(N|N-1)\ell(N)} \tag{B.72}$$

Again, the utility of such a procedure is not clear, and further thought and experimentation is necessary.

Let us now consider the computational complexity of (B.60)-(B.62). First note that one does not have to compute the correlation coefficients (elements of S in (B.54)). However, one does have to calculate K(N) at every stage, and if one solves for the gain from the Riccati equation (B.62), one has on the order of p^2 multiplications per stage. However, Morf, et.al. [B-71] and Morf and Ljung [B-120] have exploited the structure of the equations to obtain fast algorithms for the direct computation of K. Combined with the fast algorithm mentioned earlier, one now has efficient recursive procedures for the covariance method as one increases either the order p of the predictor or the number N of data points (or both simultaneously). The most efficient procedure is to use p=1 and process the data points successively. At the end of this procedure, one can then increase p until an acceptable prediction error is obtained. We refer the reader to [B-71,120] for details.

We also note that Gibson, et.al. [B-47] have proposed a filter of the same structure as (B.60)-(B-62) but that requires far fewer multiplications per stage (the order of p). This procedure is based on stochastic approximation methods and replaces (B.61)-(B.62) with

$$K(N) = \frac{g\ell(N)}{100+\ell'(N)\,\ell(N)} \tag{B.73}$$

where g is a gain to be determined by experimentation (see[B-47])[11]. We refer the reader to [B-24,47] for details and experimental results.

Finally, we turn to one final note concerning the relative merits of the autocorrelation and covariance methods. As pointed out by Makhoul [B-115], the autocorrelation method offers the advantage of guaranteering the stability of the resulting all-pole filter; however, the fact that the method relies on setting y(i)=0 outside the available range of data leads to spectral distortion. The covariance method, on the other hand, avoids the distortion problem by not considering points outside the given range, but it need not lead to a stable filter. As stability for these methods is guaranteed if and only if all of the reflection coefficients have magnitude less than one [B-30,115], a number of modified covariance-type methods that have this property have been devised. We refer the reader to [B-115] for a discussion of the relative merits of several methods and a new fast algorithm. We also note that Morf, et.al. [B-71] point out that if one considers a hybrid method -- we define $y(i)=0$, $N+1\leq i\leq N+p$ but do not use $y(j)$, $J<0$ -- we can guarantee the

[11] In [B-47] the gain K(N) is calculated in a slightly different way because of the inclusion of quantization effects.

stability of the resulting filter and can still obtain fast algorithms (due to the product of Toeplitz form of the covariance matrix).

B.3 Design of a Predictor as a Stochastic Realization Problem

A problem that has attracted a great deal of attention in the control and estimation literature is the stochastic realization problem [B-7,11,13, 15,20,21,22,37,63,67,72,85,90,105]. Briefly stated, (a special version of) the stochastic realization problem asks the following: given a stationary Gaussian random process y (taken as a scalar here for simplicity[12]) with correlation function R(n), find a Markovian representation

$$\begin{aligned} x(n+1) &= Ax(n) + w(n) \\ y(n) &= c'x(n) \end{aligned} \tag{B.74}$$

where w is a zero mean white noise process with covariance Q. Referring to (B.19)-(B.25), we see that this is equivalent to finding a factorization of R of the form

$$R(i) = c'A^i b \tag{B.75}$$

where

$$\begin{aligned} b &= Pc \\ APA'-P &= -Q \end{aligned} \tag{B.76}$$

Examining (B.75), (B.76), we see that the algorithm falls naturally into two pieces: (1) find a triple (A,b,c) satisfying (B.75); (2) find P and Q satisfying (B.76). One of the best-known studies of this problem is that of

[12] We note that the various algorithms discussed in this section have been extended -- in most cases nontrivially -- to the vector case.

Faurre [B-21,57,85]. As he pointed out, the first step of the algorithm is simply the well-known deterministic realization problem when one is given the "weighting pattern" R(0), R(1), R(2),... . This problem has been widely studied in the literature [B-9,10,11,12,13,14,72,106,107], and we will make a few comments about this aspect of the problem in a few moments. Before discussing the numerical aspects of the first step or the details of the second, let us see what the first part yields in the frequency domain (here we follow [B-63]). Let us define the power spectral density

$$S_y(z) = \sum_{i=-\infty}^{+\infty} R(i) z^{-i} \tag{B.77}$$

Then, using the fact that $R(-i) = R(i)$, we see that the factorization (B.75) yields[13]

$$S_y(z) = c'(zI-A)^{-1}zb + c'(z^{-1}I-A)^{-1}Ab \tag{B.78}$$

Noting the form of (B.78), and defining

$$\alpha(z) = \det(zI-A) \tag{B.79}$$

we see that the first step in the algorithm yields[14]

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[13] If we had realized $\frac{1}{2}$ R(0),R(1),R(2),..., instead of R(0),R(1),R(2),..., we would have a more symmetrical version of (B.78) (see [B-63]). Note that equality of (B.77) and (B.78) is as formal power series.

[14] Note the assumption that we <u>can</u> factor R as in (B.75) implies (and is implied by) the fact that $S_y(z)$ is a rational function.

$$S_y(z) = \frac{p(z)}{\alpha(z)\alpha(z^{-1})} \tag{B.80}$$

That is, we have obtained a factorization of the denominator of S_y. If we can also factor the numerator

$$S_y(z) = \frac{\mu\beta(z)\beta(z^{-1})}{\alpha(z)\alpha(z^{-1})} \qquad (\mu>0) \tag{B.81}$$

we will have determined the desired transfer function[15]

$$G(z) = \frac{\beta(z)}{\alpha(z)} \tag{B.82}$$

which, when driven by white noise with spectrum $\mu^{1/2}$, yields the spectrum $S_y(z)$. It is clear from (B.74) that it is this second part of the spectral factorization that is accomplished by the second step of the stochastic realization algorithm. Finally, note that the model (B.82) contains <u>both</u> poles and zeroes (it is an autoregressive-moving-average (ARMA) model).

There are several methods for performing the second step of the algorithm. Faurre [B-21,85] showed that (B.76) could be solved for any P inside a given range

$$P_* \leq P \leq P^* \tag{B.83}$$

(here inequality is in the matrix sense), and he identified the smallest such covariance, P_*, as that arising from an <u>innovations representation</u> of y -- i.e.

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[15] We choose β and α to consist of those poles and zeroes of $S_y(z)$ that lie within the unit circle. This will guarantee the stability of G and of its inverse (see [B-63]).

a Kalman filter (see Gevers-Kailath [B-67] for a full description). This representation is of the form

$$\begin{aligned} \xi(n+1) &= A\xi(n) + K\varepsilon(n+1) \\ y(n) &= c'\xi(n) \end{aligned} \tag{B.84}$$

where ε is an innovations process with covariance

$$R_\varepsilon = c'b - c'P_*c \tag{B.86}$$

and P_* is the solution of the <u>algebraic Riccati equation</u>

$$P_* = AP_*A' + \frac{A[b-P_*c][b-P_*c]'A'}{c'b-c'P_*c} \tag{B.87}$$

Then the Kalman gain is given by

$$K = \frac{[b-P_*c]}{c'b-c'P_*c} \tag{B.88}$$

Comparing this with (B.26), we see several differences. First of all, in (B.26) we had an equation of the form

$$\begin{aligned} \hat{x}(n+1) &= A\hat{x}(n) + AK\varepsilon(n) \\ y(n) &= c'\hat{x}(n) + \varepsilon(n) \end{aligned} \tag{B.89}$$

The differences between (B.84),(B.89) can be explained by noting that (B.84) represents a representation based on the filtered estimate of $x(n)$ (given $y(0),\ldots,y(n)$) and (B.89) is the one-step predicted estimate of $x(n)$ (given $y(0),\ldots,y(n-1)$). We also note that it is easy to pass from one of these representation to the other (see [B-67]).

Thus, examining (B.84)-(B.88), we see that the second step of the algorithm consists of solving the equations defining a steady-state Kalman filter, and again the most difficult step is solving for the covariance -- in this case P_*- from the nonlinear equation (B.87). However, note that P_* itself is not needed in (B.84). All we really need are R_ε and K. Thus, an alternative procedure is to use the "fast algorithms," as described in Subsection B.1 (see [B-63,69] for the development of this idea). These will produce the time-varying hi tories of K and R_ε. If we let the transients (due to the finite data with which the filter must work to produce an estimate) die out, we will obtain K and R_ε. We note that although this approach involves solving for K and R_ε recursively (in time), this procedure may be much faster than direct solution of (B.86)-(B.88).

Before turning to an alternative approach, let us note that once we have K, we have in fact determined the optimal recursive predictor or filter (i.e. comparing (B.26) and (B.89), we can readily turn the innovations representation into a one-step predictor). Note also that this model is causal and causally invertible [B-67,69] and hence the method can be interpreted as an inverse filter approach to the identification of $G(z)$ -- i.e. we have equivalently determined the optimal predictor or a whitening filter. Also, as mentioned

before, this method allows for zeroes in the model. A method of this type was proposed in [B-69]. Actually, in that reference it was proposed that one might benefit from the use of the time-varying innovations representation (before it reaches steady-state). We refer the reader to [B-69,72] for more on the time-varying problem. We will have more to say about the numerical aspects of the steady-state algorithm in a moment.

There is an alternative approach to the Kalman filter method for finding a factorization of the numerator of $S_y(z)$. Examining (B.80), suppose we pass the process y through the all zero filter $\alpha(z)$. The resulting process η has power spectral density $p(z)$ -- i.e. it is finitely correlated (moving average (MA) process. Given its correlation function $p(z)$, one wishes to factor it

$$\begin{aligned} p(z) &= \sum_{i=-m}^{m} p_i z^i = \left(\sum_{i=0}^{m} \beta_i z^i\right)\left(\sum_{i=0}^{m} \beta_i z^{-i}\right) \\ &= \beta(z)\beta(z^{-1}) \end{aligned} \tag{B.90}$$

As described in [B-11,13,37,56], this is equivalent to obtaining a factorization of the infinite symmetric Toeplitz matrix (with finitely many nonzero diagonals)

$$P = \begin{bmatrix} p_0 & p_1 & \cdots & p_m & 0 & 0 & \cdots \\ p_1 & p_0 & \cdots & p_{m-1} & p_m & 0 & \cdots \\ \vdots & & & & & & \\ p_m & p_{m-1} & \cdot & p_2 & p_1 & p_0 & \cdots \\ 0 & p_m & \cdots & p_3 & p_2 & p_1 & \cdots \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \end{bmatrix} \tag{B.91}$$

into the product of an upper triangular matrix and its transpose. Recursive procedures for this are discussed in [B-37], and clearly the Levinson-type algorithm can be used in this scalar case. As the recursion proceeds, certain of the elements of the Cholesky factor converge to the desired β_1 (see [B-13]). Clearly an alternative to this procedure is to find the innovations representation of η using the fast algorithms described earlier. This method is closely related to the "fast Cholesky" algorithms, and the reader is referred to [B-63] for details (see also [B-72]). For a detailed discussion and new results on the use of the Riccati equation for spectral factorization, we refer the reader to [B-112].

Let us now turn to the numerical aspects of this two-stage procedure. We concentrate here on the first stage -- i.e. the computation of the factorization (B.75). The algorithms of Rissanen [B-11] and Ho [B-106] are based on examination of the Hankel matrix

$$H_N = \begin{bmatrix} R(0) & R(1) & R(2) & \cdots & R(N-1) \\ R(1) & R(2) & R(3) & \cdots & R(N) \\ \vdots & & & & \\ R(N-1) & R(N) & R(N+1) & \cdots & R(2N-2) \end{bmatrix} \tag{B.92}$$

It is well-known [B-107] (see also Subsection C.1) that R admits a factorization (B.75) if and only if there is some integer n such that

$$\text{rank } H_N \leq n \qquad \forall N \tag{B.93}$$

Ho's original algorithm yielded a minimal realization (i.e. dim A in (B.75) is a small as possible) if a bound n was known in advance. A far more critical question (from a practical point of view) is the partial realization question. Here we take into account that we only have available a finite number of correlations R(0), R(1),...,R(N-1), and one would like to obtain the minimal factorization that matches these. One can use Ho's algorithm for this, but it is not recursive -- i.e. if we incorporate R(N), we must re-solve the whole problem. Fortunately, Rissanen [B-11] and Dickinson, et.al. [B-9] have developed efficient, recursive procedures (the latter of which is based on the Berlekamp-Massey algorithm [B-10], which was developed for the scalar case). We note that these algorithms essentially solve the Pade approximation problem, and we refer the reader to the references for details.

Thus, efficient algorithms exist for spectral factorization and one would expect good results if the process y truly has a Markovian representation and if one has the exact values of the correlations. This points out a conceptual difference between linear prediction and the above stochastic realization procedure. In linear prediction, no pretense is made about exactly matching a model. All that is wanted is a least-squares fit, and thus one would expect this procedure to be relatively robust when one uses a finite record of real data to generate an estimate of the correlation function which is then used in the linear prediction procedure. On the other hand, it can easily be seen that an infinitessimal perturbation of H_N in (B.92) can make it have full rank. In this case, the partial realization procedures -- which in essence are looking to match a model exactly -- will yield a system of extremely high dimension.

Thus, it appears that these algorithms are inherently sensitive to errors in estimates of the correlation coefficients. In addition, if y has no Markovian representation, the linear prediction approach will still work fine, but the partial realization procedures, which are based on exact model matching, may very well run astray as it tries to fit the data "too closely".

Does this mean that the above procedure is of no use in identifying parameters in a speech model? The answer to that is perhaps not. What is needed is a modification of the first step of the stochastic realization algorithm. As the version described here stands, it is too sensitive and in fact, DeJong [B-108] has shown that these methods are numerically unstable in that the inexact minimal realization supplied by these algorithms, as implemented on a finite wordlength computer, may not be a "numerical neighbor" of the sequence $\{R(i)\}$ that is to be factored. A great deal of the difficulty is due to the illposedness of the problem of finding the rank of the Hankel matrix. By rephrasing the algorithm in terms of the ε-rank -- the least rank of all systems within an "ε-neighborhood" of the given sequence -- De Jong obtains a slower algorithm that is similar to Rissanen's but is numerically stable. This approach is extremely appealing for two reasons: (1) We can, within this framework, seek minimal realizations in the ε-neighborhood of a sequence $\{R(i)\}$ that itself is <u>not</u> realizable by a finite dimensional system; (2) We can seek the "nearest" reduced-order realization of given dimension of a given system. These two properties may help overcome some of the sensitivity problems with the two step procedure.

In addition to the work of De Jong, a number of other methods have been proposed for "approximate" Pade approximations, and any of these could be used as the first step in the algorithm. McDonough and Huggins [B-113] propose to approximate a time function f(t) by a sum of (possibly complex) exponentials

$$f_a(t) = \sum_{i=1}^{N} A\, e^{s_i t}$$

They study numerical methods for the iterative determination of the A_i and s_i that minimize

$$\int_0^T e^2(t)dt$$

where e is the <u>signal error</u>

$$e(t) = f(t) - f_a(t)$$

One needs iterations, as this is a nonlinear problem. This is clearly closely related to the discrete-time problem of finding $\{A,b,c\}$ with A nxn (n fixed) to minimize some function of the error

$$e(i) = R(i) - cA^i b$$

Some effort has been put into this problem in the recent past [B-19,75,110], and one possibility, of course, is the all-pole approximations -- e.g. we might perform linear prediction with the R(i) as the observed signal (regarded as the impulse response of some filter). This would require computing the correlation of R(i), or, in other words, the correlation of the correlation of

the $y(i)$! Note that the all-pole assumption for $R(i)$ would <u>not</u> necessarily lead to an all-pole model for $G(z)$ in (B.82).

Another possible method has been proposed by Burrus and Parks [B-114]. They consider approximating

$$r(z) = R(0) + R(1)z^{-1} + R(2)z^{-2} + \ldots.$$

by

$$G(z) = \frac{a_0+a_1z^{-1}+\ldots+a_{N-1}z^{-N+1}}{1+b_1z^{-1}+\ldots+b_{M-1}z^{-M+1}} = \frac{a(z)}{b(z)}$$

In addition to specifying some exact realizability conditions on $\{R(i)\}$ (which can easily be reduced to Hankel matrix conditions and statements), they suggest the following: we would like

$$r(z) \simeq \frac{a(z)}{b(z)}$$

Multiplying by $b(z)$, we obtain

$$b(z)r(z) \simeq a(z)$$

and if we attempt to minimize some norm on the difference between these quantities (called the <u>equation error</u>), we can obtain <u>linear</u> approximation algorithms. We refer the reader to [B-114] for details.

We close by noting that initial results ([B-109],[B-111]) utilizing the two-step procedure indicate the potential of the approach. In particular, the work at IRIA [B-109] has produced good results for the design of whitening

(inverse) filters. Given this limited success and the previous discussion, it appears that the utility of the two-step stochastic realization procedures merits further investigation.

B.4 Some Other Issues in System Identification

It is appropriate to mention several other identification procedures. Recall that in Subsection B.2 we saw that the covariance method was equivalent to a Kalman filter when we recursively update our estimates of the predictor coefficients. As discussed in [B-17], several other recursive identification schemes can also be considered as Kalman filter-type algorithms. One of these is the instrumental variables approach, which bears some similarity to the least squares algorithm and which, in fact, leads to Toeplitz equations in the stationary case [B-91]. In that reference it is pointed out how one can devise the Toeplitz Yule-Walker equations to determine the poles (AR part) in an ARMA model.[16] This is essence requires knowledge of the order of the MA part and thus is much more apt to lead to the sensitivity problems that one confronts in using a technique that is based on the assumption that the data obeys certain constraints (as in the first step of the stochastic realization algorithm of the preceding subsection). In addition, we no longer are guaranteed that the solution to the Yule-Walker equations leads to a stable inverse filter.

The methods of least squares (covariance) and instrumental variables, as described in [B-17] are used for all pole models of the noise (prediction error)/

[16] This method is similar in spirit to the Burrus-Parks generalized-Pade-equation-error approach for the determination of the denominator of a pole-zero model [B-114].

output behavior (i.e. for AR models). However, both the usual least squares and the instrumental variables can be easily modified for the identification of input zeroes -- i.e. consider the model

$$y(k+1) + a_1 y(k) + \ldots + a_p y(k-p+1) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m) + \varepsilon(k) \tag{B.94}$$

where we measure both the y's and u's (here $\varepsilon(k)$ is the driving noise, or equivalently, the "equation error"). In this case, let

$$\theta' = (-a_1, \ldots, -a_p, b_0, \ldots, b_m) \tag{B.95}$$

$$\phi'(k) = (y(k), \ldots, y(k-p+1), u(k), \ldots, u(k-m)) \tag{B.96}$$

Then the recursive least squares procedure reduces to a Kalman filter for the system

$$\begin{aligned} \theta(k+1) &= \theta(k) \\ y(k+1) &= \phi'(k)\theta(k) + \varepsilon(k) \end{aligned} \tag{B.97}$$

Although this input model is not of interest in the speech problem, it <u>is</u> of great importance in control applications in which one is interested in manipulating the system via the input u. We refer the reader to [B-17] for the analogous development for the instrumental variables method.

There are two other algorithms in [B-17] that are of interest. These methods allow zeroes both in the input/output response <u>and</u> in the noise/output

response -- i.e. they can be used to identify ARMA models. Both of these algorithms are recursive (in the data), approximate maximum likelihood methods, and both methods are of the Kalman filter type. The second of these (RML2 in [B-17]) is discussed in detail in [B-18]. The first of these, RML1 is, in some sense, an approximation to RML2, and we outline the basic idea. Consider the ARMA model

$$y(k+1) + a_1 y(k) + \ldots + a_p y(k-P+1) = e(k) + c_1 e(k-1) + \ldots + c_q e(k-q) \tag{B.98}$$

We can rewrite (B.98) as

$$\begin{aligned} \theta(k+1) &= \theta(k) \\ y(k+1) &= \phi'(k)\theta(k) + e(k) \end{aligned} \tag{B.99}$$

where

$$\begin{aligned} \theta' &= (-a_1, \ldots, -a_p, c_1, \ldots, c_q) \\ \phi'(k) &= (y(k), \ldots, y(k-p+1), e(k-1), \ldots, e(k-q)) \end{aligned} \tag{B.100}$$

Having ϕ, one could again devise a Kalman-filter structure for the estimate $\hat{\theta}$. However, the noises, e, are not known. As suggested in [B-17,82,83] a natural approximation is to replace e(j) in (B.100) by its <u>estimated</u> value -- i.e. the <u>residual</u>

$$\varepsilon(j) = y(j+1) - \hat{\phi}'(j)\hat{\theta}(j) \tag{B.101}$$

If we do this, we obtain the following recursive scheme:

$$\hat{\theta}(j+1) = \hat{\theta}(j) + K(j+1)\varepsilon(j) \tag{B.102}$$

$$K(j+1) = \frac{P(j)\hat{\phi}(j)}{1+\hat{\phi}'(j)P(j)\hat{\phi}(j)} \tag{B.103}$$

$$P(j+1) = P(j) = \frac{P(j)\hat{\phi}(j)\hat{\phi}(j)'P(j)}{1+\hat{\phi}'(j)P(j)\hat{\phi}(j)} \tag{B.104}$$

$$\hat{\phi}(j)' = (y(j),\ldots,y(j-p+1),\ \varepsilon(j-1),\ldots,\varepsilon(j-q)) \tag{B.105}$$

We refer the reader to [B-17] for a detailed description of this and the other algorithms. In addition, uniqueness of stationary points and the stability of these algorithms is considered in detail in this reference. In particular, it is shown that RLS is stable and has a unique solution, that RML1 and RML2 have unique solutions for ARMA models, that RML2 always converges, and that RML1 converges for MA models, for first-order ARMA models, but that it may diverge in higher-order cases (an example is given). The reader is referred to [B-17] for details, further references, and for many insights into the characteristics of these identification procedures. Also, we refer the reader to [B-120] for fast on-line algorithms for these identification schemes. These methods are analogous to that mentioned earlier for the covariance method.

We note that the methods described above and in the preceding subsection in principle allow one to identify poles as well as zeroes. In addition, several other methods for zero modelling have been described in the literature [B-26,68,

100,113,114,123]. The method in [B-68] is based on cepstral analysis. Let $Y(z)$ be the z-transform of a signal y, which we wish to model as the response of an ARMA model:

$$Y(z) = \frac{N(z)}{D(z)} \tag{B.106}$$

Usual linear prediction (with care taken to avoid the zeroes [B-26,68,100]) will identify D. Suppose now we define the complex ceptrum $\hat{y}(n)$ so that

$$\hat{Y}(z) = \log Y(z)$$

Then the z transform of $n\hat{y}(n)$ is

$$-z \frac{d\hat{Y}(z)}{dz} = -z \frac{D(z)N'(z)-N(z)D'(z)}{N(z)D(z)} \tag{B.107}$$

and thus linear prediction on $n\hat{y}(n)$ will identify the zeroes (and the poles) of y. We refer the reader to [B-26,68] for more on this technique. In addition, the generalized Pade methods in [B-113,114,123] can also be used for pole-zero modeling directly (as well as for the first step of the two-step procedure of the preceding section). Also Atashroo and Boll [B-21] have suggested a multi-step procedure in which one performs linear prediction to obtain the poles, inverse filters to obtain a finitely correlated sequence, uses linear prediction again to obtain a high-order all-pole model of this sequence, and then performs a third linear prediction to obtain a lower order all-zero inverse of the all-pole model.

One further issue that we have not discussed is the determination of an

appropriate order for the parametric model to be identified. Clearly, as we allow more and more free parameters, we can get a better and better fit, but one would expect a diminishing return beyond a certain number of parameters. Åström and Eykhoff [B-16] propose one test criterion, while Akaike [B-32,92; B-15, p.716] (see also [B-26]) proposed an information-theoretic criterion which provides a direct tradeoff between the value of the log-likelihood function and the number of free parameters in the model. Recently, Rissanen and Ljung [B-79] have obtained a related criterion that incorporates the assumed model structure (as well as the number of parameters).

In this section we have examined a number of aspects of the identification-estimation problem, and we have pointed out a number of similarities between the goals and techniques of the two disciplines. We have also seen some of the differences, but others have not been discussed. In particular, in this section we have treated identification for identification's sake. As pointed out in [B-16] in control system design, identification is often simply a means toward the goal of efficient control. Thus, in many control applications, the value of identification is not measured by the accuracy of the parameter estimates, but rather by the performance of the overall system. This is discussed somewhat in [B-17] and also in the study of "self-tuning regulators" [B-80,81]. In addition, in control one has several types of identification problems, since one has the opportunity to excite the system through inputs. One finds somewhat different problems if the system is operating open loop, in a time-invariant closed-loop mode, or in an adaptive closed loop mode. We refer the reader

to [B-15,17] for more on this subject and for further references. Finally, in the control context, one often deals with systems for which one is interested in determining system structure as well as in identifying the parameters of a model. The issues involved here are complex and are discussed in [B-15,16].

On the digital filtering side, one is often interested in the accuracy of the parameter estimates. This is of importance, for example, if one is attempting to design an all-pole filter that matches a given impulse response in a least squares sense, or if one is attempting to estimate formants from an all-pole speech model. On the other hand, for linear predictive coding, the accuracy of the parameters may be of secondary interest, while the primary concern is more efficient coding of speech data. In this case, accuracy is of importance only in so far as it makes the coding scheme more efficient. In this regard, a very important question involves the quantization of the predictor specifications -- that is, what is the most efficient method for transmitting the specifications of the all-pole model. As discussed in [B-119], the reflection coefficients (from which one can construct the filter) offer the most efficient parametrization from a quantization point of view.

We note that the linear prediction approach appears to be particularly well-suited to the speech problem. The all pole model is a good one in many cases (from a physical point of view), the algorithms are fast, the intermediate variables in the algorithm (i.e. the partial correlation coefficients) have useful physical interpretations, the linear prediction procedure tends to match the spectral envelope, etc. (see [B-26] for many of the properties of linear prediction and [B-116] for some of its statistical properties). Finally and

above all, linear prediction has been proven in practice to work well on speech signals, and further work is needed before one can say with confidence that any of the other techniques described in this section can improve upon this performance.

Thus, we see that there are a surprising number of relationships, similarities, and differences among the techniques and goals of researchers in both disciplines who are concerned with parameter identification. The possibilities for collaboration and interaction that will benefit all involved seem particularly abundant in this area. In particular, we have barely scratched the surface on the question of the relative merits of the various methods or the issue of precisely what problems a particular method addresses and does not address. A thorough investigation of questions such as these remains for the future.

C. Synthesis, Realization, and Implementation

In this section we consider the question of design. However, our discussion will not deal very much with design methods but rather with the question of trying to pinpoint what researchers in the two disciplines mean by "design" and what sorts of problems their techniques are equipped to handle. As we shall see, the issues considered in the two fields are often quite different, but there are many occasions in which techniques from one discipline could be of use in the other. Also, the problem of implementation confronts designers in both disciplines.

C.1 State Space Realizations and State Space Design Techniques

State space concepts and methods have a number of uses from a design point of view. Let us first take a look at realization theory [A-64,68,B-12,C-2-13]. Let us recall some of the basic concepts from realization theory (see [A-64, B-12, C-2] for details and for further references). We will follow [B-12] and will state several results in the continuous-time framework, but analogous results hold for the discrete-time problem (see the last part of Subsection B.3). We are interested in time-varying linear system representations of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad x(t_0)=x_0 \\ y(t) &= C(t)x(t) \end{aligned} \tag{C.1}$$

where $x(t)\varepsilon R^n$, $u(t)\varepsilon R^m$, $y(t)\varepsilon R^p$, and A,B,C are matrices of appropriate dimension. From an input-output point of view, the system (C.1) is equivalent to the representation

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau \qquad (C.2)$$

where Φ is the nxn state-transition matrix

$$\dot{\Phi}(t,\sigma) = A(t)\Phi(t,\sigma), \qquad \Phi(\sigma,\sigma)=I \qquad (C.3)$$

and the matrix

$$H(t,\tau) = C(t)\Phi(t,\tau)B(\tau) \qquad t\geq\tau \qquad (C.4)$$

is the <u>impulse response matrix</u>. As pointed out in [B-12], in many control and estimation problems, we are often interested in the weighting pattern matrix[1]

$$K(t,\tau) = C(t)\Phi(t,\tau)B(\tau) \qquad t,\tau \qquad (C.5)$$

If A,B, and C are constant, then Φ and K have particularly nice expressions:

$$\Phi(t,\tau) = e^{A(t-\tau)}, \quad K(t,\tau) = Ce^{A(t-\tau)}B, \quad \forall t,\tau \qquad (C.6)$$

and in this case, given the dependence on $t-\tau$ only, we write $K(t,0)=K(t)$, $H(t,0)=H(t)$. Also, in this case, an equivalent input-output representation is provided by the Laplace transform of H(t) -- the <u>transfer function</u>

$$G(s) = \mathcal{L}[H(t)] = C(Is-A)^{-1}B \qquad (C.7)$$

---

[1] As mentioned in [B-12,C-10], if K is real analytic in t and τ (as it is if A, B,C are constant), then (C.4), (C.5) are equivalent, since H has a unique extension to $\tau>t$. Otherwise, there can be nonunique extensions [C-10].

The realization problem, then, is to obtain a recursive description of the form (C.1) when we are given the weighting pattern, impulse response function, or transfer function. It can easily be seen that if a realization exists, then many solutions exist. For example, we obtain the same weighting pattern as (C.1) if we take $\xi=2x$ to be our state variable

$$\begin{aligned} \dot{\xi}(t) &= A(t)\xi(t) + 2B(t)u(t) \\ y(t) &= \frac{1}{2}C(t)\xi(t) \end{aligned} \tag{C.7}$$

or if we take $\eta' = (x,0)$ as our state

$$\dot{\eta}(t) = \begin{bmatrix} A(t) & 0 \\ 0 & \alpha(t) \end{bmatrix} \eta(t) + \begin{bmatrix} B(t) \\ \beta(t) \end{bmatrix} u(t) \tag{C.8}$$

$$y(t) = [C(t),\gamma(t)]\eta(t)$$

where α is arbitrary and either β or γ is identically zero. These two examples illustrate the two basic issues that arise. In the first case, ξ and x are in some sense equivalent, since they contain identical information and one can be obtained from the other via an invertible linear transformation. This is not the case in the second example, in which η carries superfluous information (from an input-output standpoint) in its last component η_{n+1}. If $\beta=0$, the input can never affect η_{n+1} (a controllability problem), while if $\gamma=0$, the output never

C.2

sees η_{n+1} directly or indirectly (since η_{n+1} is decoupled from the other state components)-- an observability problem.

Thus, one of the key issues in realization theory involves the characterization of minimal realizations -- those which contain no superfluous information in their state variables. We refer the reader to the references (see, in particular, [B-12]) for the full development of realization theory for time-invariant multivariable systems. As one might guess from the preceding paragraph, the concepts of controllability and observability are very closely tied to the minimality of a state space realization. For the sake of bravity, we state the major results only for the time-invariant case (i.e. stationary weighting pattern and constant realizations of it).

Definition C.1: A realization (time-varying or time-invariant) of a weighting pattern or transfer function is minimal if any other realization has a state vector of dimension at least as large.

Definition C.2: A constant linear system in state space form (C.1) is controllable if for every state $x \in R^n$ and any $T>0$ there exists an input function $u(t)$, $t \in [0,T]$ that drives the system from $x(0)=0$ to $x(T)=x$.[2]

Definition C.3: A constant linear system in state space form (C.2) is observable if, for any $T>0$, given $u(t)$ and $y(t)$, $t \in [0,T]$, we can uniquely determine $x(t)$ in this interval.[2]

Theorem C.1: Suppose we are given a stationary impulse response matrix $H(t)$ or its transfer function $G(s)$. This system has a state-space representation of the form (C.1) if and only if $G(s)$ is a matrix of rational functions of s, each of

---

[2] For time-varying systems, the intervals over which one tries to control or observe the system may vary with time (see [A-64,B-12,C-2,C-10]).

which is proper (degree (denom.)>degree (numer.))[3]. In this case, G(s) has a minimal, constant realization, and, in fact, a realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{C.9}$$

is minimal if and only if it is controllable and observable. In addition, <u>any</u> minimal constant realization can be obtained from a given one via an invertible linear transformation of the state variable, $\xi = Px$, or equivalently

$$(A,B,C) \longrightarrow (PAP^{-1}, PB, CP^{-1}) \tag{C.10}$$

Finally, if dim x=n, the realization (C.9) is controllable if and only if

$$\text{rank} \left[B \vdots AB \vdots \cdots \vdots A^{n-1}B \right] = n \tag{C.11}$$

and it is observable if and only if

$$\text{rank} \left[C' \vdots A'C' \vdots (A')^{n-1}C' \right] = n \tag{C.12}$$

We note that essentially the same result holds in discrete time, in which we have the (z transform) transfer function G(z) and we wish to represent it as

$$G(z) = C(Iz-A)^{-1}B \tag{C.13}$$

which is equivalent to the state space description

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{C.14}$$

---

[3] It is easy to allow deg(denom)=deg(numer.) by including a feedthrough term: $y(t) = C(t)x(t) + D(t)u(t)$. This is readily taken care of and leads to minor modifications of the results stated here.

Examining (C.7) and (C.13), we see that any algorithm that realizes the continuous time system G(s) also is a valid realization algorithm for the discrete-time system G(z) (and vice versa). We thus will turn to the discrete-time framework for a moment in order to gain some insight into the realization question.

As discussed in [A-64,B-12,106,107,C-2], there are relatively simple algorithms for obtaining controllable or observable realizations of G(z), assuming it is given in rational form (so that we can compute the least common denominator of all of the elements of G). The algorithm of Ho [B-106,107] and that of Silverman and Meadows [C-13] provide methods for extracting minimal constant realizations from the Hankel matrix (see Subsection B.3). Basically, in this approach one writes G(z) in series form

$$G(z) = \sum_{i=1}^{\infty} T_i z^{-i} \tag{C.15}$$

and we recognize that $\{T_i\}$ is the impulse response sequence. Referring to (C.13), the realization problem is equivalent to finding A,B,C so that

$$T_i = CA^{i-1}B \qquad \forall i \tag{C.16}$$

As described in [A-64,B-106,107], one can find such a factorization if and only if the ranks of the Hankel matrices

$$H_N = \begin{bmatrix} T_1 & T_2 & \cdots & T_{N-1} \\ T_2 & T_3 & \cdots & T_N \\ \vdots & \vdots & & \vdots \\ T_{N-1} & T_N & \cdots & T_{2N-3} \end{bmatrix} \tag{C.17}$$

are bounded by some integer (and then the maximal rank of the H_N is the dimension of the minimal realization of G). If G is proper rational, one can show that this is indeed the case and, given the degree of the least common multiple of the denominators of elements of G, can find a particular H_N that achieves the maximal rank. From this matrix, one can then extract the minimal realization [B-12,106,107,C-13]. However, if we are given G in the form (C.15) as opposed to in rational form, in general one cannot easily determine if G is rational (or equivalently if the ranks of H_N are bounded). In this case, the partial realization algorithms discussed in Subsection B.3 are of use. These algorithms essentially produce minimal dimension systems of the form (C.14) that match the expansion (C.15) up to some specified power of (z^{-1}) -- i.e. these systems match the impulse response out to some specified point. As mentioned earlier, these algorithms have numerical difficulties which must be overcome. However, if G is given in rational form, the algorithms of Ho-Kalman and Silverman-Meadows provide a procedure for determining minimal realizations (see also [A-64] for a procedure based on partial fraction expansions).

Thus, the realization problem can, in principle, solve certain questions related to system synthesis. The input-output description (C.2) for continuous systems or the analogous one for time-invariant, discrete-time systems

$$y(n) = \sum_{i=0}^{n-1} T_{n-i} u(i) \tag{C.18}$$

is non-recursive in nature -- i.e. equation (C.18) implies an algorithm in which at each point in time the entire past sequence of input vectors $u(0),\ldots,u(n-1)$ are multiplied by the appropriate impulse response matrices and then summed. Clearly such an approach is feasible only if the system to be implemented has a finite impulse response (FIR -- $T_i=0\ \forall i>$some integer). In general, however, (C.18) requires growing memory, and even in the FIR case, the nonrecursive implementation may require exorbitant amounts of storage. In this case, recursive implementations are called for, and the state spce realization (C.14) provides an answer to this question. In fact, the computation of minimal realizations allows one to find out the minimal amount of storage that is needed in any linear, recursive realization, and one of the most important aspects of the state-space approach is that it allows one to consider multiple input/multiple output systems and time-varying systems. It is this last point -- the ability to handle multivariable and time-varying systems -- that is one of its most important assets from a synthesis point of view.

One field in which state space realization theory has played a major role is in network synthesis for both time-invariant and time-varying circuits. A

number of papers have been written in this area (see [C-3-9] and the references therein), and, in fact in some of this work (see, for example [C-4]), realization concepts are tied together with some concepts concerning dissipative systems (see Subsection A.2) to yield useful results in network synthesis.

We will not discuss analog network synthesis further, since our major concern is with relationships with the implementation of digital filters. This topic will be looked at in some depth in the next subsection, and thus we content ourselves at present with making only a few comments. For discrete-time systems, the state-space approach tells us the minimal amount of storage -- i.e. the minimal number of delays -- that are needed to realize a given transfer function. In addition, we know how to obtain any minimal state space realization from a given one -- i.e. we apply (C.10) for any invertible P, and any recursive linear realization can be written in vector difference equation (i.e. state space) form by keeping track of all memory updates. Does this mean that state-space realization solvesthe digital filter design question? The answer to that is decidely no. As we will discuss in the next section, there are many issues besides minimal storage involved in choosing a "good" filter structure (i.e. algorithm). However, we know that one can obtain any minimal state space realization algorithm via the choice of an invertible matrix P and the application of (C.10). Does this mean that the selection of a "good" filter structure is equivalent to finding a "good" P? The answer to this is again no, and the primary reason for this is distinction between interpreting a state space realization as a description of dynamical behavior and as an algorithm. We defer the clarification of this cryptic comment until the next section.

In control theory, state-space realizations play a major role in a number of very important design problems. In these problems the part played by realization theory is indirect, in that it allows one to bring into play some powerful state-space design methods. We illustrate a few of these here. Consider the system pictured in Figure C.1. We are given an open loop pxm transfer function G(z) and we wish to design a feedback compensator that has certain properties. For example, one may wish to design a feedback system so that all of the modes of the closed loop system have time constants in a specified range. For scalar systems (p=m=1) techniques (in the frequency domain) for the solution to this problem have been available for a number of years [A-53,C-2], and frequency domain techniques for single input systems are discussed in [C-2]. However, as discussed in [C-2], if one uses a state-variable description of the system, one can obtain a solution in the general, multivariable setting. We briefly outline a method discussed in [C-14]. Let us suppose G(z) is proper, rational, and reduced (no element of G has common poles and zeroes). In this case, let us find a realization

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{C.19}$$

and we note that the poles of G(z) are precisely the eigenvalues of A if and only if (C.19) is minimal. Suppose we implement a control law of the form

$$u(k) = -Kx(k) \tag{C.20}$$

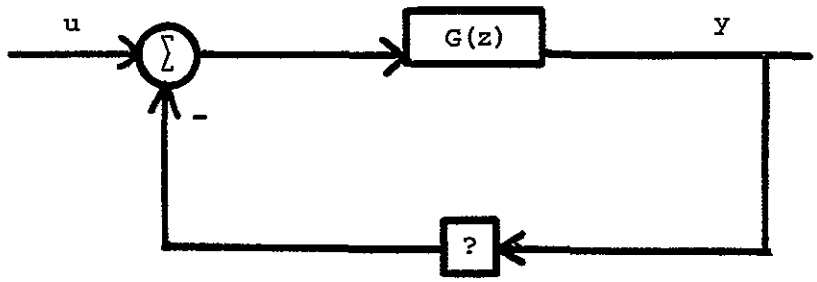

Figure C.1: Depicting a Feedback Design Problem

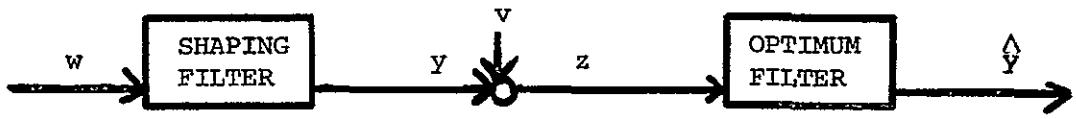

Figure C.2: Illustrating an Optimum Filtering Problem

Then the closed-loop poles are just the eigenvalues of (A-BK). As discussed in [C-14], we can find a K to place these eigenvalues wherever we want if and only if (C.19) is controllable. A constructive algorithm is given in [C-14].

Suppose we cannot implement (C.20) -- i.e. we only have u and y at our disposal. One might then consider the design of a system that estimates x from u and y. A natural structure for such an "observer". [C-14,15] is

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y(k) - C\hat{x}(k)) \tag{C.21}$$

Note that if $x(k) = \hat{x}(k)$, then $x(n) = \hat{x}(n)$, $\forall n>k$, and if one looks at the error $e(k)=x(k)-\hat{x}(k)$, we find that it obeys the equation

$$e(k+1) = (A-HC)e(k) \tag{C.22}$$

and the poles of A-HC can be placed arbitrarily if and only if (C.19) is observable.[4] If one then implements the control law

$$u(k) = -K\hat{x}(k) \tag{C.23}$$

one finds that the poles are just the eigenvalues of (A-BK) and (A-HC), and we have solved the pole placement problem. This procedure illustrates one of the crucial aspects of many state space design methods -- the solution to design problems is an <u>algorithm</u>, which, with some cases, can be implemented on a general

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[4] Note that (A,C) is observable if and only if (A',C') is controllable (see (C.11), (C.12)), and that the eigenvalues of A-HC are the same as for A'-C'H'. Thus, we can use the same algorithms for finding H as that used to find K in (C.20).

purpose computer[5].

Algorithmic, state space solutions exist to a wide variety of other problems -- decoupling ("design a feedback law so that the ith input effects only the ith output, [C-16]), invertibility ("when can we design a system that will take the output of our given system and recover the input [C-17]), etc. -- and we refer the reader to the special issue of the IEEE Transactions on Automatic Control [A-68] for an overview of the various design methods that have been developed. One important aspect of some of these techniques is that they allow one to solve quantitative optimization problems. The linear-quadratic optimal control problem is an example of this, as is the design of a Wiener filter as a steady-state Kalman filter [A-65,68,C-18]. Consider the estimation problem illustrated in Figure C.2. We have a Gaussian, stationary process y with given, rational power spectral density $\Phi_y(s)$, and we observe the signal z, which consists of the sum of y and a Gaussian white noise process v. We wish to design a causal filter that minimizes the variance of the prediction error

$$e(t) = y(t) - \hat{y}(t) \tag{C.24}$$

As discussed in [C-18-21], if we assume that we have an infinite record length on which to operate, the solution to this problem is the Wiener filter, which can be obtained by performing a certain spectral factorization. We also know, however (see [C-18] and Subsection B.2), that the Kalman filter can be used to

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[5] The problem of computer design algorithms is a very important one at present. Difficulties with ill conditioning are present in many of these, and the design of "robust" algorithms is a crucial research question in control theory. See [C-22] for references on this subject.

solve this problem. Given that y has a rational power spectral density, we can find a minimal representation ("shaping filter")

$$\begin{aligned} \dot{x}(t) &= Ax(t) + w(t) \\ y(t) &= Cx(t) \\ z(t) &= y(t) + v(t) \end{aligned} \tag{C.25}$$

Here $E(w(t)w'(\tau)) = Q\delta(t-\tau)$, $E(v(t)v'(\tau)) = R\delta(t-\tau)$, and w and v are independent. Also, (assuming stationarity) $x(0)$ is zero mean with covariance P_0 which satisfies the (continuous-time) Lyapunov equation

$$AP_0 + P_0A' = -Q \tag{C.26}$$

Then, it is well-known [C-18] that the optimal filter is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + K(t)\ [z(t)-C\hat{x}(t)] \\ \hat{x}(0) &= 0, \quad \hat{y}(t) = C\hat{x}(t) \end{aligned} \tag{C.27}$$

where

$$K(t) = P(t)C'R^{-1} \tag{C.28}$$

and P is the solution of the Riccati equation

$$\begin{aligned} \dot{P}(t) &= AP(t) + P(t)A' - P(t)C'R^{-1}CP(t) + Q \\ P(0) &= P_0 \end{aligned} \tag{C.29}$$

Equivalently, one could use one of the fast algorithms discussed in the preceding section to obtain K(t) directly.

In addition to providing a framework for the specification of designs, the state space framework allows one to analyze the performance characteristics of the overall system after it has been implemented. For example, the techniques described in Section A can be used to study the stability characteristics of the system. In addition, a subject of much interest is the sensitivity of such designs (see [C-23] and the references in [C-22]). The major emphasis here is that designs that come from state space algorithms are model-based, and deviations between true and assumed parameter values and the fact that the assumed model is often an idealization of true system behavior will inevitably lead to variations in the performance of the "optimal" design. Issues such as these have led to sensitivity studies and to the development of design methods which are adaptive (see the introduction to Section B) or inherently "robust" [C-24,25] (see also the discussion in [A-65] on the methods that are used to overcome sensitivity problems for Kalman filters).

Another analytical tool used to study system performance is <u>covariance analysis</u>. For linear systems, we consider the model

$$\begin{aligned} x(k+1) &= Ax(k) + w(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \tag{C.33}$$

where w and v are zero mean, independent white noises,

$$E(w(k)w(j)') = Q\delta_{kj} \qquad E(v(k)v(j)') = R\delta_{kj} \tag{C.34}$$

These noises may represent actual noise sources or the effects of small nonlinearities (such as quantization noise -- see the next subsection), unmodeled

Suppose we let $t \to \infty$ -- i.e. we consider the limit of an infinite record length. One can show [A-65,C-18] that the algorithms for K(t) (Riccati or the fast algorithms) will converge to

$$K = P_\infty C' R^{-1} \tag{C.30}$$

where P_∞ is the unique positive definite solution of the algebraic Riccati equation

$$AP_\infty + P A' - P_\infty C' R^{-1} C P_\infty + Q = 0 \tag{C.31}$$

Thus, the state space formulation provides several algorithms which solve the Wiener filtering spectral factorization problem to yield the optimal transfer function (from z to $\hat{y}$)

$$G(s) = C(Is - A + P_\infty C' R^{-1} C)^{-1} P_\infty C' R^{-1} \tag{C.32}$$

Thus we see that realization theory -- in providing a state space model for the system to be controlled or the signal to be estimated -- plays an important role in allowing us to utilize rather powerful state-space algorithms for the specification of designs that possess certain performance characteristics. Note that all of these algorithms lead to designs that are specified in state space (e.g. (C.27)) or transfer function (e.g. (C.32)) terms. One must then face the issue of implementation. If the system is to be implemented in digital form, the issues raised in the next subsection must be considered in evaluating the performances of the overall system.

phenomena, etc. A simple calculation yields an equation for the covariances $P(k)$ and $S(k)$ of $x(k)$ and $y(k)$, respectively (assuming $x(0)$ is zero mean with covariance $P(0)$):

$$\begin{aligned} P(k+1) &= AP(k)A' + Q \\ S(k) &= CP(k)C' + R \end{aligned} \tag{C.35}$$

If A is a stable matrix, we can evaluate the steady-state covariances P and S by solving the Lyapunov equation[6]

$$APA' - P = -Q \tag{C.36}$$

In the nonlinear case, a number of approximate methods exist (see [A-65, C-26]), and we refer the reader to [C-27] for the discussion of one widely used method based on describing functions.

As mentioned earlier, in implementing the designs that arise from state space methods, one must consider a number of issues that digital signal processors have studied in great detail. On the other hand, it is possible that some of the analysis methods mentioned above can be of use in evaluating the performance of various system implementations.

C.2 The Implementation of Digital Systems and Filters

As discussed in [C-1], the design of digital systems consists of three steps

Specification of desired properties

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[6] As mentioned in Section A, this equation appears in several problems in state space system analysis.

- Approximation or realization of these by a causal, discrete-time system
- Implementation of the system using finite precision arithmetic.

From this point of view, the methods of the preceding section deal with the first two issues. Design procedures such as pole allocation and Kalman filtering specify desired input-output behavior for feedback compensators or optimal estimation. Realization procedures clearly play an indirect role in these techniques in providing the state space models on which the design techniques are based. But what about realizations from the point of view of system synthesis and implementation? As we shall see, state space realizations can play some role in implementation, but they are far from providing the entire solution.

The digital filter design techniques we wish to consider are discussed in great detail in [C-1,28-33,D-2] (see also the many references in these texts and papers), and the major emphasis of these methods is toward the second and third tasks in digital filter design. The techniques for the second task, as described in [C-1], take as their starting point the specification of certain frequency response or impulse response characteristics. The role of the second task is then to take these specifications and produce a scalar transfer function that meets these design specifications. An excellent description of the range of available techniques for this problem is given in [C-1, Chapter 5]. We will mention several of these methods but refer the reader to this and the other references for a thorough treatment. A number of the methods that exist are based on transformation of analog filter transfer functions. One of these

is the "impulse invariance" method in which one samples a continuous-time impulse response to obtain a discrete-time impulse response. This method suffers from aliasing problems if the analog frequency response is not strictly band-limited. A somewhat more complex procedure which avoids the aliasing problem is the bilinear transformation.

$$s(z) = k \frac{1-z^{-1}}{1+z^{-1}} \tag{C.37}$$

which is an invertible transformation of the z-plane into the s-plane which maps the inside of the unit circle in the z-plane onto the open left-half plane of the s-plane (thus preserving stability). One can then transform an analog transfer function H(s) into a digital function

$$\tilde{H}(z) \triangleq H(s(z)) \tag{C.38}$$

Note also that if H is rational, so is $\tilde{H}$. Also, this transformation introduce nonlinear distortion in the frequency domain (the mapping of the unit circle in z onto the imaginary axis in s), and care must be taken in achieving a design with the desired frequency response.

In addition to these methods that yield closed form solutions, there are a number of computer-aided design methods. These include minimizing the mean-squared error between the actual frequency response and the desired response at a selected (finite) set of frequencies. Also, as mentioned in Section B, one can use linear prediction to fit an all-pole model to a desired impulse response. In addition, the discussion of the preceding section suggests that the Pade approximation-partial realization algorithms described in [B-9-12] can be used

to find least-order pole-zero transfer functions that match a certain number of terms of a desired impulse response.

There are also a number of methods used to design FIR filters. Many of these involve "windows", in which one multiplies a desired impulse response by a finite duration window. The usual rectangular window leads to the well-known Gibbs phenomenon, and the more sophisticated windows have been devised to reduced this effect. The reader is referred to [C-1] for more on windowing and computer-aided methods for FIR filter design. In addition, for a good discussion of the issues involved in the overall design problem and of the design of optimum filters that approximate a given frequency response in the Chebyshev (L_∞) sense, we refer the reader to [D-2]. As these references indicate, a number of filter design methods are algorithmic in nature (much as the state space design methods discussed in the previous subsection), and the issue of efficient numerical design procedures is of central importance.

Once an IIR or FIR filter has been determined, these still remains the major problem of implementation -- the determination of a filter structure (algorithm) that realizes the given transfer function. One factor that does enter into this design question is the number of storage elements (delays) in the filter structure. Structures that contain the minimal number of delays are called "canonic", and this is clearly the same as the concept of "minimal" realization. Of course, in dealing with single-input, single-output transfer functions, one can read off the order of a canonic structure and can construct several quite easily by simple inspection of the specified transfer function ([C-1,Chapter 4]).

The determination of the order of a canonic realization and the ability to construct several minimal realizations without much difficulty barely scratches the surface of the structures problem. That is, the question of minimizing storage -- which is essentially what the state space realization problem considers -- is just one of several problems in digital filter implementation. As pointed out in [C-1], the various filter structures available may be equivalent from an input-output viewpoint if one didn't have to worry about computation time, the complexity of the digital architecture or algorithm required to implement a given structure, the effect of finite precision in representing filter coefficients, or the effects of overflow and quantization. These are the issues that motivate much of the study of various filter structures. It is not our intention to explore all of the various filter structures and the analytical considerations associated with them. We will mention a few, however, to illustrate several key points and refer the reader to the references [C-1,28,34] and to the many papers in the IEEE Transactions on Circuits and Systems.

For FIR filters, a number of methods exist for the implementation of the finite convolution

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k) \tag{C.39}$$

(here h is the FIR). Clearly, one can directly implement the product by keeping the last N values of the input in storage. This is the so-called "direct form" realization [C-1] and requires N multiplications per stage. If one is designing a linear phase network, this number can be cut in half by using the symmetry properties of the impulse response [C-1]. Also, the convolution (C.39)

can be implemented using fast Fourier transform (FFT) techniques, and this is particularly useful when N is large in which case one might use a sectioning algorithm [C-1,32,D-2]. In using FFT techniques, one often sacrifices storage in order to gain computational efficiency -- e.g. we may take N to be a power of 2 or may use overlap sectioning methods [C-1] for efficient operation when the length of x is long.

For IIR filters, a number of filter structures have been developed. In this case, we are attempting to realize the transfer function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \tag{C.40}$$

which is equivalent to the difference equation

$$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \tag{C.41}$$

The direct implementation of equation (C.41) -- called the direct form I realization -- requires storage of the last N values of y and the last M values of u. This structure is far from minimal, as it is easily seen that the minimal number of delays is max(N,M). However, a slight modification of direct form I yields the canonic realization direct form II (see [C-1,p.150]).

By examining the transfer function (C.40), one can obtain a number of other canonic structures. For example, if H(z) is expanded in partial fraction form, we can obtain parallel form structures, while if we factor H(z) as the product of simpler transfer functions, we can obtain series or cascade structures. Let us give an example of the cascade structure. Suppose we have

$$H(z) = \frac{z^2+(b+d)z+bd}{z^2-(a+c)z+ac} = \frac{(1+bz^{-1})(1+dz^{-1})}{(1-az^{-1})(1-cz^{-1})} \tag{C.42}$$

In Figure C.3 we have realized this filter as the cascade of two first order filters in direct form II. Note that the overall filter <u>is</u> canonic.

The major questions surrounding the choice of filter structure include the consideration of computational efficiency, the effects of finite word length on filter stability and performance, and the effect of finite precision in representing filter parameters. We have already said a few words concerning computational efficiency, and refer the reader to the references for more on this issue (in particular, see [C-1,32] for detailed discussions and further references on the use of the FFT algorithm)[7]. In addition, in Section A we considered the effects of quantization and overflow on system stability. An alternative, approximate method for evaluating the effect of finite word length

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[7] An interesting question in the area of computational efficiency is the determination of filter structures that require the smallest number of delays <u>and</u> multiplies. For second order transfer functions Lueder [C-50] has shown that there are precisely 32 such structures. An intriguing related question in the state space area is the determination of a realization in which A,B, and C have as few elements as possible that are not 0,1, or -1. As far as we are aware, no work exists on this problem.

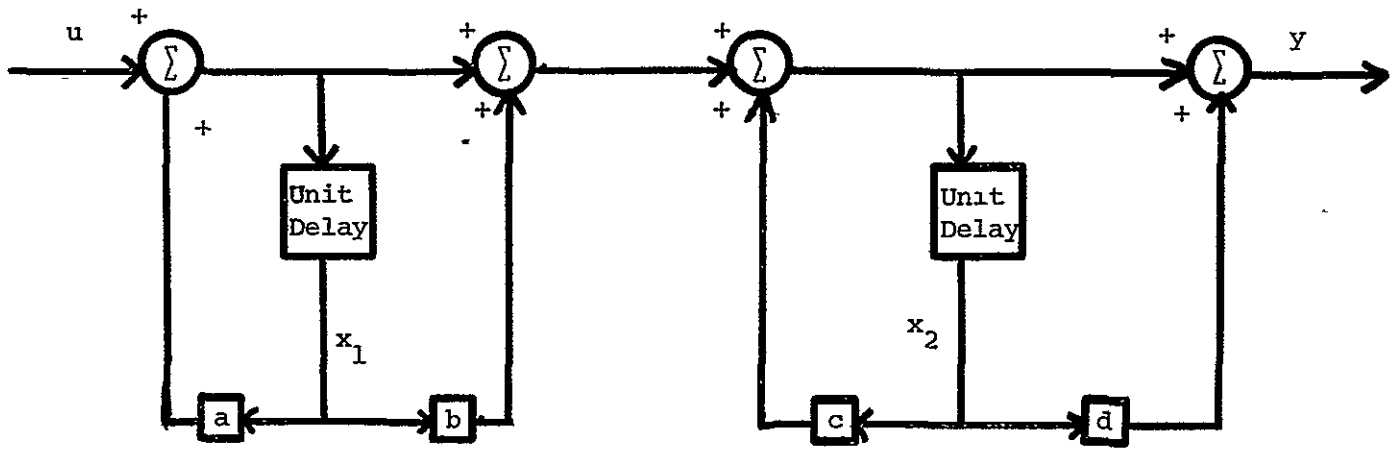

Figure C.3: A Second-Order Cascade Filter Structure

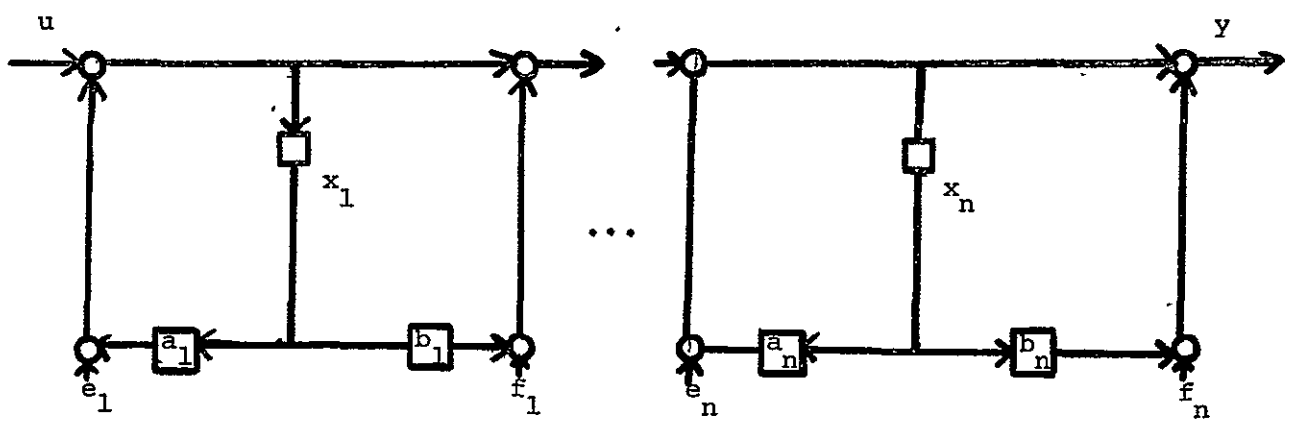

Figure C.4: An n-th Order Cascade Filter Including Quantization Noise

on system performance is to model each quantization as if it introduced noise (representing, for example, roundoff or truncation) into the system. This approach is discussed at some length in [A-3,12,C-1,33-37]. The basic idea is that whenever a quantization occurs, one replaces it by an "equivalent noise source". Then, by assuming independence of these various sources -- a rather strong and many times unjustified assumption (as the existence of periodic effects, i.e. limit cycles, indicates) -- one can in principle evaluate the overall noise power at the output, and thus can obtain a measure of the size of quantization effects.[8] As an example, consider the case [C-1] of fixed-point arithmetic and roundoff quantization (Figure A.5) in which the quantization interval q is 2^{-b} (i.e. the number of bits used to represent fractions is b).[9] In this case, the quantization error e introduced by a single multiplication falls in the bound

$$-\frac{1}{2}2^{-b} < e \leq \frac{1}{2}2^{-b} \tag{C.43}$$

If one makes the assumption that e is uniformly distributed, we find that it has

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[8] Parker and Girard [C-55] have shown how one can take the correlation in these noise sources into account. Specifically, quantization noises due to multiplication of the same signal by two different coefficients are correlated, and the correlation can be approximated by a function that depends on the coefficients. In addition, Parker and Girar point out that correlation increases as the number of bits decreases. We also refer the reader to the work of Eckhardt and Schussler [C-56] on evaluating quantization error variances.

[9] Here, we follow the standard fixed point procedure in which all numbers are represented as fractions. One can also consider noise analysis for floating point [C-1,33]. See also the work of Fettweis [C-52,53,54] in which noise analysis is performed with the aid of certain system sensitivity functions.

zero mean and variance

$$\sigma_e^2 = \frac{1}{12} \; 2^{-2b} \tag{C.44}$$

Using these assumptions, one can add independent noise sources to filter representations to account for quantization effects. For example, in the cascade example of Figure C.3, one could add one noise source following each of the four multiplications (fewer noise sources might result from a different quantization procedure -- e.g. if we add the products bx_1 and cx_2 before quantizing).

Another extremely important issue in filter design is the sensitivity of filter performance to variation in coefficients. This is quite central an issue, since one can only represent coefficients up to a finite degree of accuracy, and hence one cannot obtain filters with arbitrary pole and zero locations. As described in [C-1(Chapter 4),C-28,31], the allowable poles and zeroes and the sensitivity to variations in parameters depends quite significantly on the particular structure under consideration. For example, parallel and cascade structures are often used because of their sensitivity properties, since the perturbations in the poles are isolated from one another [C-1].

A great deal of work [C-1,28-31,33-37] has gone into developing methods for answering a variety of questions concerning various filter structures. Questions considered include: (1) the determination of the number of bits needed in a given filter structure to obtain required accuracy in overall performance both from the point of view of parameter sensitivity and quantization noise; and (2) determination of "rules of thumb" [C-33,37] for the pairing and ordering of poles and zeroes in a cascade structure in order to minimize the effects of

quantization noise. The study of questions such as these for large interconnected networks is a complex problem, and efficient algorithms are needed to evaluate overall sensitivities, effects of noise, etc. One such large-scale package involves the use of techniques for the manipulation of signal flow graphs. The use of such techniques is discussed in [C-1,28,31], and a detailed description of a computer package to perform a number of types of analysis on digital networks is contained in [C-28].

For the remainder of this section, we wish to examine the relationship of state space techniques and concepts to some of the questions in digital filter design. This discussion is a first attempt to study such relationships, and a great deal more work is needed before the issues can be thoroughly understood. Let us first examine the use of state space techniques to determine filter structures. As described in the preceding subsection, realization techniques can be used to obtain minimal realizations -- i.e. certain canonic algorithms. Consider the transfer function (C.42). In this case, state space techniques yield a variety of minimal -(in this case two-) dimensional realizations of the form

$$\begin{aligned} x(k+1) &= Fx(k) + gu(k) \\ y(k) &= h'x(k) + u(k) \end{aligned} \tag{C.45}$$

where

$$h'(zI-F)^{-1}g+1 = \frac{z^2+(b+d)z+bd}{z^2-(a+c)z+ac} \tag{C.46}$$

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} , \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} , \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{C.47}$$

Let us interpret (C.45) as an algorithm. Assume we presently have computed $x(k)$ and receive the new input $u(k)$

<u>PART #1:</u>
(a) Multiply h_1 and $x_1(k)$
(b) Multiply h_2 and $x_2(k)$
(c) Add these, together with $u(k)$ to yield $y(k)$

<u>PART #2:</u>
(a) Multiply f_{11} and $x_1(k)$
(b) Multiply f_{21} and $x_1(k)$
(c) Multiply f_{12} and $x_2(k)$
(d) Multiply f_{22} and $x_2(k)$
(e) Multiply g_1 and $u(k)$
(f) Multiply g_2 and $u(k)$
(g) Add (a),(c), and (e) to yield $x_1(k+1)$
(h) Add (b),(d), and (f) to yield $x_2(k+1)$

Clearly a number of these steps can be done in different orders, but the above steps do indicate the basic algorithm implied by (C.45). Note that in general, there are 8 multiplications and 6 additions required.

Now let us examine the cascade structure of Figure C.3, and let us interpret it as an algorithm:

(a) Multiply a and $x_1(k)$

(b) Multiply b and $x_1(k)$

(c) Multiply c and $x_2(k)$

(d) Multiply d and $x_2(k)$

(e) Add (a) and u(k)

(f) Add (b) and (e)

(g) Add (c) and (f)

(h) Add (d) and (g)

Then

$$(e) = x_1(k+1)$$

$$(g) = x_2(k+1)$$

$$(h) = y(k)$$

Note that these algorithm requires 4 multiplications and 4 additions, but this is not the most crucial difference between the two algorithms, since it is possible to obtain realizations (C.45) with some zero elements in (F,g,h). However, the crucial difference is the following: <u>if one interprets a state space realization as determining an algorithm of the type indicated, then there is no way that the cascade algorithm is of this type!</u> This is not to say that one cannot find a state-space description of the cascade realization. In fact

$$x(k+1) = \begin{bmatrix} a & 0 \\ (a+b) & c \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [(a+b),\ c+d]x(k) + u(k) \tag{C.48}$$

is such a realization. Note that if one takes into account that one doesn't have to multiply by 1 and that one multiplication is used twice, then (C.48) requires only 4 multiplications

$$ax_1(k)$$
$$(a+b)x_1(k)$$
$$ex_2(k)$$
$$(c+d)x_2(k)$$

and 5 additions.

The point made above may, at first glance, seems to be trivial, but it is not, since it points out that although any (infinite precision) algorithm can be describe dynamically in state space terms direct interpretation of a state space description as an algorithm does <u>not</u> allow one to consider all possible algorithms. That is, it is relatively easy to go from an algorithm to a state-space description -- e.g. (C.48)-- but it is not at all natural or clear how to go the other way, and hindsight is needed in order to interpret the realization

$$x(k+1) = \begin{bmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = [f_{21}, h_2]x(k) + u(k) \tag{C.49}$$

as a cascade structure with

$$a=f_{11},\ b=f_{21}-f_{11},\ c=f_{22},\ d=h_2-f_{22} \tag{C.50}$$

Recently, Chan [D-107] defined a unified framework for the consideration of all 1-D structures. Chan noted that if one viewed (C.45) as a map from present state and input to next state and present output

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \Phi \begin{bmatrix} \alpha(k) \\ u(k) \end{bmatrix}$$

then <u>any</u> filter structure can be viewed as a factorization of Φ and a change of basis on x. Specifically, consider the example (C.46), with the realization (C.48), which yields

$$\Phi = \begin{bmatrix} a & 0 & 1 \\ (a+b) & c & 1 \\ (a+b) & (c+d) & 1 \end{bmatrix}$$

Let us write $\Phi=\Phi_2\Phi_1$, where

$$\Phi_1 = \begin{bmatrix} a & 0 & 1 \\ b & c & 0 \\ 0 & d & 0 \end{bmatrix} \quad , \quad \Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then, if we interpret this factorization as an algorithm -- perform the operations indicated by Φ_1 first and then perform those specified by Φ_2 -- it is clear that we essentially have the cascade algorithm as depicted in Figure C.3. Thus Chan's technique provides a conceptual framework in which to consider structures from a state point of view. As Chan points out, it is not yet clear how one can <u>use</u> this factorization technique in an algorithmic fashion to determine useful <u>new</u>

structures. At the very least, it provides a unified framework for the consideration of questions related to realization structures.

Thus, we see that there are potential limitations to the state space framework for determining new filter structures, although the ideas of Chan may provide a conceptual unification of these subject areas. In addition to Chan's work, there appear to be several other structures-related areas in which state space concepts may play a role. Recall that state space realization techniques allow one to determine minimal realizations for systems with multiple inputs and outputs. It is possible that this fact, combined with a thorough understanding of the relationship between state-space realizations and various digital system structures will lead to the development of useful filter structures (possessing desirable storage, computational, sensitivity, and quantization characteristics) for multivariable systems. It is hoped that the preceding treatment of a simple cascade example will help expose some of the issues that need to be understood.

Also, as mentioned in the preceding subsection, the state space framework is particularly useful for the analysis of the properties of dynamical systems. Thus, it seems natural to ask if these techniques might be useful in the analysis of various filter structures. We have already discussed this question in Section A with respect to stability analysis techniques. It is also possible that state-space sensitivity techniques [C-23] could be useful in the study of the sensitivity of various digital filter structures, but this awaits further study.

Finally, let us examine the utility of state-space techniques in the analysis of the effect of quantization noise on filter performance. We do this by example, although it should be clear that this approach extends to arbitrary structures. Consider the cascade structure in Figure C.4. Here we have included quantization noise after each multiplication. A state space representation of this filter can be written down by inspection.

$$x(k+1) = Ax(k) + bu(k) + \Gamma e(k) + \Delta f(k)$$
$$y(k) = c'x(k) + u(k) + \Theta e(k) + \Psi f(k) \tag{C.51}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} (a_1+b_1) \\ \vdots \\ (a_n+b_n) \end{bmatrix}$$

$$\Theta = \Psi = (1,\ldots,1)$$

$$A = \begin{bmatrix} a_1 & 0 & \cdots\cdots & 0 \\ (a_1+b_1) & a_2 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ (a_1+b_1) & (a_2+b_2) & \cdots\cdots & a_n \end{bmatrix} \tag{C.52}$$

$$\Gamma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

$$e(k) = \begin{bmatrix} e_1(k) \\ \vdots \\ e_n(k) \end{bmatrix}, \quad f(k) = \begin{bmatrix} f_1(k) \\ \vdots \\ f_n(k) \end{bmatrix}$$

Also, the noises $e_1,\ldots,e_n$, $f_1,\ldots,f_n$ are assumed independent, identically-distributed, zero-mean white processes with variance (C.44). Then, assuming that A is a stable matrix and using the covariance analysis procedure described in Subsection C.1, we can compute the steady-state covariance $\sum$ of y[10]

$$\sum = c'Pc + \frac{n}{6} 2^{-2b} \tag{C.53}$$

where P, the covariance of x, is the solution of the Lyapunov equation

$$P = APA' + \frac{1}{12} 2^{-2b} [\Gamma\Gamma'+\Delta\Delta'] \tag{C.54}$$

Equations (C.54) and (C.53) are perfectly suited to computer implementation. Also, note that the solution of (C.54) yields the effect of noise throughout the network. The utility of an approach such as this for digital network analysis needs to be examined more carefully, but it appears that it may be computationally superior to other methods, such as those that require computing a number of partial transfer functions (from each noise source to the output -- see [C-5]). We also note that if the noise sources are correlated, as they are shown to be in [C-55], one can adapt the preceding procedure by augmenting the filter state

[10] Here we have assumed u=0. The analysis of the deviation of y from the desired value when $u \neq 0$ is identical to the above (assuming that e and f are independent of u).

equations with a shaping filter that yields the correct correlation in the error sources. We note that Parker and Girard [C-55] used Lyapunov-type equations and analysis quite similar to our development for the evaluation of output noise power due to correlated quantization errors. In addition, similar analyses have been undertaken by Hwang [C-64], Mullis and Roberts [C-65], and Sripad and Snyder [C-66, 67]. Hwang uses Lyapunov-state space equations to study the effects of possible structure transformations and state-amplitude scalings; Mullis and Roberts use a similar framework to study what they call "minimal noise realizations"; and Sripad and Snyder develop conditions under which quantization errors are in fact white, and they also use Lyapunov-type analysis to compare the performance of two different realizations. These references clearly indicate the potential benefits of this type of analysis.

Within the framework described above, one can pose a number of other questions. For example one can perform a similar noise analysis if random rounding is used.[11] Also, Schussler [C-51] has proposed a figure of merit for structures -- the required number of bits to meet given noise specifications. In terms of (C.53) and (C.54) this would mean determining b so that the resulting Σ is less than some prescribe limit. Is it possible that we can devise algorithms for the solution of such problems for this and for more general structures? In addition, in the case of floating point arithmetic, the quantization error depends on the size of the signal. Can state-space procedures for analyzing "state-dependent noise" [C-57,58] be of value here? Questions such as these await future investigation.

C.3 Direct Design Taking Digital Implementation Into Account

As discussed in the preceding subsections, design procedures in both disciplines consist of several parts -- determining the desired input/output

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[11] As Schussler [C-51] points out, one often designs filters with limit cycles, since filters without limit cycles often have poor noise behavior, and one can overcome the limit cycle problem by using randomized rounding (hence adding a bit more "noise" to the system).

behavior to be synthesized and the design of an algorithm that approximate this behavior given the constraints of digital implementation. The procedures as described up to this point treat these issues separately, but, as discussed in [D-2], it would be of value to consider overall design methods that take the discrete nature of the computer into account during the process of developing design specifications and allow the study of tradeoffs such as performance versus number of bits used. The development of full-fledged design procedures is clearly a long way off; however, in recent years some research in control and estimation theory has been aimed at developing designs that reflect the interaction of system specification and the limitations and structure of the digital system that is to be used to implement the system. We will briefly described several of these and refer the reader to [C-38-46,D-85,86,87,88,89,91,92] for details.

Consider the continuous time linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{C.55}$$

where $x \in R^n$, $u \in R^m$. Suppose we wish to control the system with a digital control system. Specifically, suppose we can observe $x(k) \stackrel{\Delta}{=} x(k\Delta)$, $k=1,2,\ldots$, and, based on these observations, we feedback a control

$$u(t) = u(k) \qquad k\Delta < t \leq (k+1)\Delta \tag{C.56}$$

In addition, suppose we wish to design the control law to minimize

$$J = \int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt \tag{C.57}$$

For a fixed value of Δ, this problem leads to an optimal discrete-time control problem [C-39] with the feedback law

$$u(k) = Gx(k) \tag{C.58}$$

Suppose this law is to be implemented in a digital system that takes τ_m seconds to perform a multiplication. Then (assuming add time is negligible), in general the control law requires

$$\Delta \geq nm\tau_m \tag{C.59}$$

Thus, it is clear that each control algorithm requires a minimum time Δ_{min} between successive samples, and the following question arises: suppose we consider a "suboptimal" control algorithm that can be implemented at a faster sampling rate than the bound for the "optimal" law in (C.59); is it possible that such a law can outperform the slower, "optimal" law. This question is answered in the affirmative in [C-38], in which a simple example is given and an indication is given that one can achieve performance improvements for a class of large-scale, "loosely-coupled" systems. One can also interpret these results as providing a method for determining the value of a faster computer, as measured by the accompanying decrease in J -- i.e. for a given control law and two possible multiplication times τ_1, $\tau_2 (\tau_1 < \tau_2)$ the cost difference $J(\tau_2) - J(\tau_1)$ can be interpreted as the amount one would be willing to pay for the faster machine. This can provide a basis for a tradeoff analysis -- the cost of a faster computer versus achievable performance improvement.

The question of devising control and estimation designs and digital architectures that are especially natural for particular applications is receiving more and more attention as available digital systems are being improved and made less expensive. Specifically, the development of microprocessor technology has led to a great increase in the design of control and estimation systems that involve a number of identical modules, parallel structures, and distributed processing [B-103,104,C-40-46]. In the area of decentralized control [D-85,86,87,88,89,91,92] one often has an extremely large and distributed system with many inputs and outputs, and one wishes to design a set of "local" controller -- i.e. a set of several control laws, each of which uses only some of the inputs and some of the outputs and is perhaps implemented on a dedicated processor. Clearly the architecture of such a system (i.e. who gets to know what) is a major design variable. Again one can interpret the difference in performance of two different architectures as a measure of how much more one would be willing to pay for one system than another. Clearly a totally centralized system would perform best, but the cost of relaying all information to and from one central location may be prohibitive.

The study of problems such as this -- i.e. the interaction of implementation and architecture issues (parallelism, decentralization) and the design of control and signal processing algorithms -- is still in its infancy, and it appears to offer an extremely promising avenue for research and for applications to problems in fields such as aircraft control [C-40,42-44] and nonlinear stochastic filtering [C-45,46]. We note that architectural issues have received a great deal of attention in the field of digital signal processing [C-28,31,47,48], and this appears to be a promising direction for future interaction and collaboration.

We also note that there has been work [C-68,69] in digital filter design aimed at developing structures and design techniques that take the constraints of finite arithmetic into account at the start. In addition, the restrictions of finite arithmetic have, in part, motivated the study of linear systems in which the vectors and scalars are all integer valued -- i.e. linear systems over rings [E-28-31]. The work to this point has been quite theoretical, and its value in allowing one to design digital controllers or filters has yet to be established.

Finally, we note that in both disciplines there is a great deal of interest in the development of fast on-line algorithms. In digital signal processing, fast Fourier Transform algorithms [C-1,59,60] have been widely used (for example, in the implementation of FIR filters). The FFT has also found use in control theory (see, for example, its use in implementing matched filters for detection of failures in dynamic systems [C-61] and in designing efficient optimal controllers for certain large interconnected systems that possess some symmetry in their structure [D-93]). In addition, motivated by the algebraic treatment of Nicholson [C-60], Willsky [C-62] has developed fast algorithms for several types of "noncommutative convolutions" that occur in certain nonlinear filtering problems (see also [C-63]). Also, all of the fast Kalman gain algorithms discussed in Section B are potentially useful in the design of efficient adaptive control systems. The implementation of systems along these lines and the development of new efficient on- and off-line procedures remains an active area of research in both disciplines.

D. Multiparameter Systems, Distributed Processes, and Random Fields

A growing interest has developed over the past few years into problems involving signals and systems that depend on more than one independent variable. In some cases one of these variables is time, and the others represent spatial dimensions -- as in the study of distributed parameter systems [D-157-159] or decentralized control [D-87,88,93,94] -- while for other problems -- such as image processing [D-4,6,7,20,21] -- none of the independent variables can be thought of as time.

This research area is rich in both potential application areas and in challenging theoretical problems. Among the areas of application are image processing, seismic signal processing, meteorology, gravity field mapping, pollution monitoring and control, and inertial navigation. On the theoretical side, there are a number of basic conceptual questions. How does one handle the processing of distributed data in an efficient manner? What properties do recursive techniques have in a setting where the recursion is in more than one dimension? Do causality and state make any sense here? What about stability? What are the tools for analyzing stochastic processes? How do we "predict" when the independent variables aren't time? What role do recursive estimation techniques play (what <u>are</u> recursive estimation techniques?)? Which concepts concerning signals and systems in one independent variable carry over to the multiparameter case? Which do not, and why don't they?

In this section we consider several problem areas involving multiparameter signals and systems in order to discuss some of the issues mentioned above in more detail.

D.1 Two Dimensional Systems and Filters

Over the past few years, a great deal of work has been done in attempting to extend one-dimensional filtering concepts to the design and analysis of systems that process data that is distributed in two-dimensional (2-D) arrays.

The consideration of 2-D systems has opened up an entirely new set of questions, and in this section we want to explore some of these design and analysis issues. For an excellent and thorough overview of 2-D digital filtering, we refer the reader to [D-3].

As in 1-D, we can define a linear shift-invariant system (LSI) that processes 2-D input arrays $x(m,n)$ to produce 2-D output arrays in a linear fashion and so that a shift in the "time" origin for the input merely induces an analogous shift in the output. Such a system has a convolutional representation, much as in 1-D

$$y(m,n) = \sum_{k,\ell=-\infty}^{+\infty} h(m-k,n-\ell)x(k,\ell) = h(m,n)*x(m,n) = x(m,n)*h(m,n) \tag{D.1}$$

Here $h(j,k)$ is the unit sample response -- i.e. the response of the system to the input

$$x(k,\ell) = \delta_{k0}\,\delta_{\ell 0} \tag{D.2}$$

(here δ_{ij} is the Kronecher delta which is nonzero and equals 1 only if $i=j$). The unit sample response is sometimes referred to as the <u>point-spread function</u> [D-4], a term used in image processing, where $h(j,k)$ has the interpretation as the observed image when the input illumination is a point source at the origin.

Again as in the 1-D case we can take z-transforms. For example, the system function of (D.1) is

$$H(z_1,z_2) = \sum_{k,\ell=-\infty}^{+\infty} h(k,\ell) z_1^{-k} z_2^{-\ell} \tag{D.3}$$

and a simple calculation transforms (D.1) into

$$Y(z_1,z_2) = H(z_1,z_2)X(z_1,z_2) \tag{D.4}$$

An important class of LSI systems arises from rational system functions

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{D.5}$$

$$\begin{aligned} A(z_1,z_2) &= \sum_{(k,\ell)\varepsilon I_1} a(k,\ell) z_1^{-k} z_2^{-\ell} \\ B(z_1,z_2) &= \sum_{(k,\ell)\varepsilon I_2} b(k,\ell) z_1^{-k} z_2^{-\ell} \end{aligned} \tag{D.6}$$

where I_1, I_2 are finite sets of pairs of integers. As a strightforward consequence of (D.4)-(D.6), we obtain a 2-D (partial) difference equation relating y and x:

$$\sum_{(k,\ell)\varepsilon I_2} b(k,\ell) y(m-k,n-\ell) = \sum_{(k,\ell)\varepsilon I_2} a(k,\ell) x(m-k,n-\ell) \tag{D.7}$$

Up to this point, the mathematical steps taken follow the 1-D steps very closely, but now we begin to see some of the conceptual as well as

mathematical difficulties that arise in the 2-D case. Let us first discuss the problem of recursion. Given the equation (D.7), we want to use it to calculate the next output given previous outputs and the input. Enbedded in this statement is the heart of one of the problems. Unlike the 1-D case, in which the index n has the interpretation of time, in the 2-D case, in general, it is not clear what "next" or "previous" mean.[1] In fact, just given (D.7) it is not clear that there is <u>any</u> definition of next or previous that will allow us to recursively compute y(m,n). Dudgeon [D-3,5,33] Pistor [D-42], and Ekstrom and Woods [D-103,119] have studied this problem in great detail, and we now briefly overview their work.

First note that in the nonrecursive (FIR) case -- i.e. when B=1, there is no problem in computing (D.7) output point by output point. There is, however, an issue concerning what part of the input must be stored at any one time. In 1-D, we just keep the most recent input points (assuming we compute y(n) sequentially), but the situation is more complex in 2-D. For example, suppose we have the "nearest neighbor" filter [D-31]:

$$I_1 = \{(-1,0),(0,0),(1,0),(0-1),(0,1)\} \tag{D.8}$$

then to compute y(m,n) we need x(m+1,n), x(m,n), x(m-1,n), x(m,n+1), x(m,n-1) Conversely, we must hold on to x(m,n) until we have computed y(m-1,n), y(m,n), y(m+1,n), y(m,n-1), y(m,n+1). Thus, depending on the order in which we compute the y's, we can have very different requirements for storing the x's. Here we get our first glimpse at the fact that the required storage depends not only

[1] Unless one of the two dimensions <u>is</u> time and we wish to process the input in real-time. We will have more to say about this later in this section.

on the degree of the filter but also on the sequencing of computations. Of course for FIR filters, as in the 1-D case, we can process inputs in blocks, using a 2-D FFT algorithm together with an appropriate method for taking care of the overlaps in the blocks. Methods along these lines exactly parallel the 1-D methods, and we refer the reader to [D-3] for further discussion and references.

Thus, we have that the right-hand side of (D.7) does not raise any insurmountable obstacles for the sequential processing of inputs (although there are several interesting questions as we've seen). The situation is far different in the recursive case ($B \neq$ constant). Since the right-hand side of (D.7) causes no difficulties, we assume that it is trivial ($A=1$) for convenience. Let us consider one of the most widely used special cases of (D.7):

$$\sum_{k=0}^{M} \sum_{\ell=0}^{N} b(k,\ell)y(m-k,n-\ell) = x(m,n) \tag{D.9}$$

Assuming that $b(0,0) \neq 0$, we have

$$y(m,n) = -\frac{1}{b(0,0)} \sum_{\substack{k=0 \\ (k,\ell)\neq(0,0)}}^{M} \sum_{\ell=0}^{N} b(k,\ell)y(m-k,n-\ell) + \frac{1}{b(0,0)} x(m,n) \tag{D.10}$$

and we immediately see that to calculate $y(m,n)$, we only need the values of outputs to the "southwest"[2]. Figures D.1-D.4 illustrate the situation.

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[2] This terminology appears to be due to Pistor [D-42]. It seems to be particularly appropriate for conveying the geometry of 2-D recursions and causality.

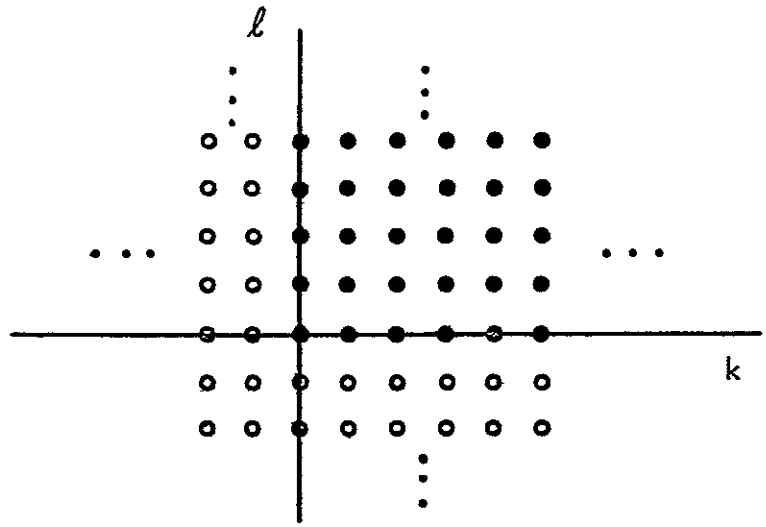

Figure D.1: Support of a First Quadrant or "Northeast" (NE) Function. (Possible nonzero locations are indicated by solid dots.)

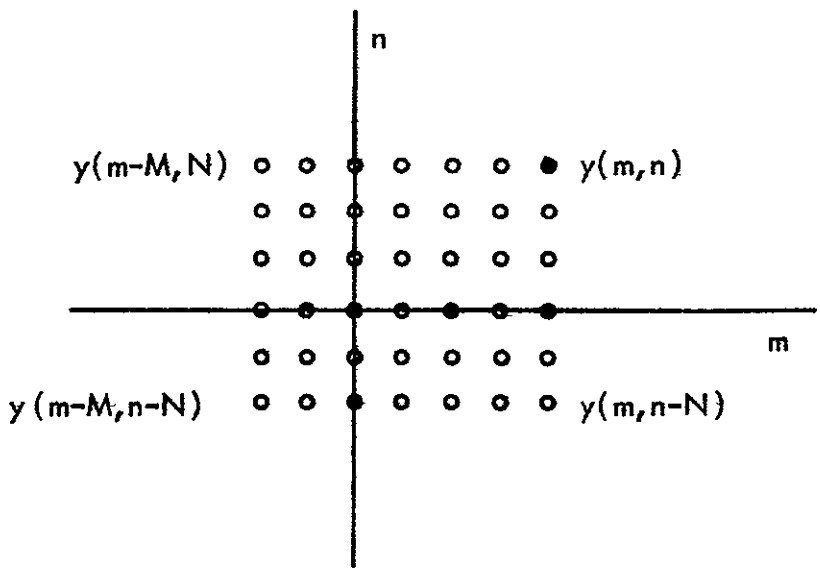

Figure D.2: Required Output Points (Open Dots) to Calculate y(m,n) for the system given by (D.9)

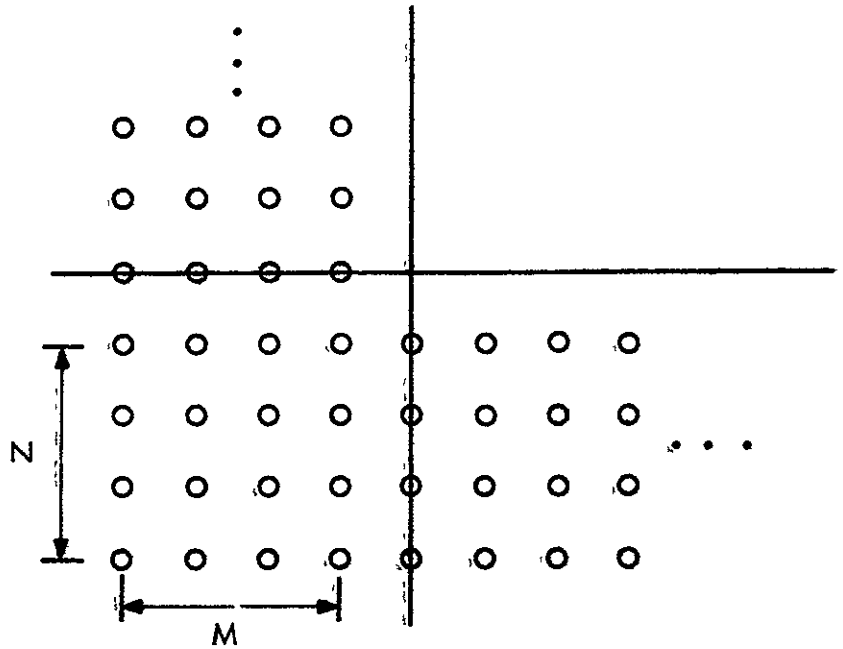

Figure D.3: Required Boundary Conditions for (D.9) in Order to Calculate NE quadrant of y.

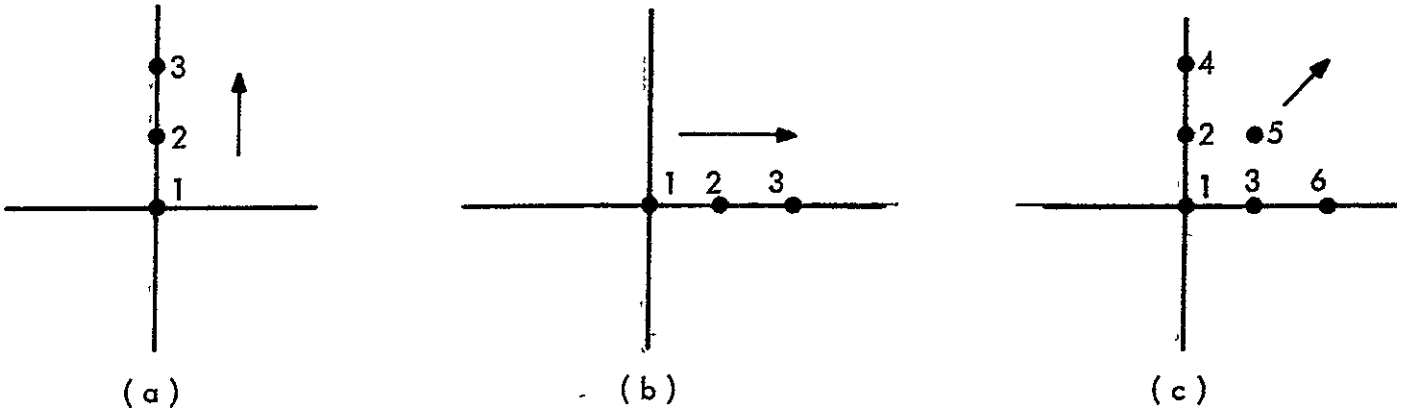

Figure D.4: Several Possible Directions of Recursion for (D.9)

In Figure D.1, we see that the support of the function $b(k,\ell)$ is in the first quadrant. We will call such a function a "northeast" (NE) function for reasons that will become clear shortly. In Figure D.2 we see the set of data points to the SW that must be stored in order to enable us to calculate $y(m,n)$. A consequence of this is seen in Figure D.3. If we are interested in calculating $y(m,n)$ in the NE quadrant, we must specify initial or boundary conditions as shown. As we calculate and store some of the output points, we can discard some of the old values, but it is clear that the amount of storage needed depends not only on M and N in (D.9) but also on the <u>range</u> of values of m and n for which we want to calculate y. If either of these ranges is infinite, the storage needed is infinite.

In addition to this consideration, we also find that the storage requirements depend on the sequencing of the recursion (we had seen this earlier in the FIR case). Several directions of recursion are indicated in Figure D.4. Here (a) depicts the north recursion, (b) is the east recursion, and (c) is a NE recursion. We can generate other directions of recursion as long as they remain within the NE quadrant. Each recursion calls for its own sequence of data accessing and discarding. The N and E recursions appear to have particularly simple sequencing rules, but the data must be processed serially. On the other hand, the NE recursion has a more complex sequencing but leads to the possibility of parallel computation, since, for example, points 4,5, and 6 can be calculated simultaneously. The possible directions for recursion and potential uses of parallel computation can be determined with the aid of a conceptual device -- the precedence relation, which partially orders points

with the rule

$(m,n) \prec (\ell,k)$ if $y(m,n)$ is needed to be calculated first in order to be able to calculate $y(\ell,k)$ (D.11)

Thus $(m,n) \prec (\ell,k)$ if $y(m,n)$ is directly needed to calculate $y(\ell,k)$ or if it is used to calculate some $y(r,s)$ that is used directly to calculate $y(\ell,k)$, etc. A discussion of this topic has been given by Chan [D-107,152]. We will come back to this issue later in this section.

Let us now return to the question of recursibility. Clearly the picture is symmetric -- i.e. we can have NW, SE, and SW recursions, with $b(k,\ell)$ restricted to be a function on the corresponding quadrant. However, as shown by Dudgeon [D-5,33], this by no means exhausts the possibilities for recursion. In addition to the one quadrant functions, we can obtain recursive difference equations with $b(k,\ell)$'s that are one-sided [D-5]. To illustrate the idea, consider the equation

$$y(m,n) = -\frac{1}{b(0,0)} \sum_{k=1}^{M} \sum_{\ell=-N}^{N} b(k,\ell)y(m-k,n-\ell) - \frac{1}{b(0,0)} \sum_{\ell=1}^{N} b(0,\ell)y(m,n-\ell) + \frac{1}{b(0,0)} x(m,n) \qquad \text{(D.12)}$$

Figure D.5 illustrates the support of such a function, while Figure D.6 indicates how points are recursed and what initial conditions are needed. Here we calculate the data points column by column, using data points to the south and to the west (not just the southwest). Hence the directions of recursion are far more limited than in the single quadrant case, since we <u>cannot</u>

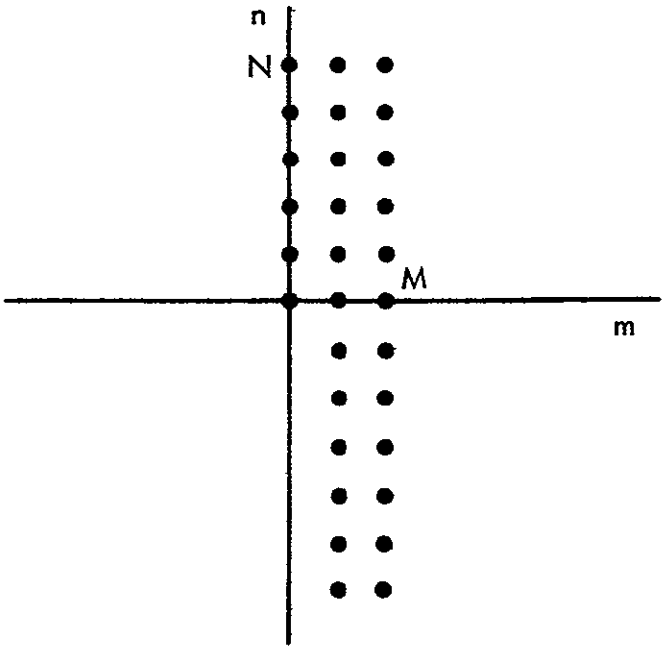

Figure D.5: Support of a One-Sided Function

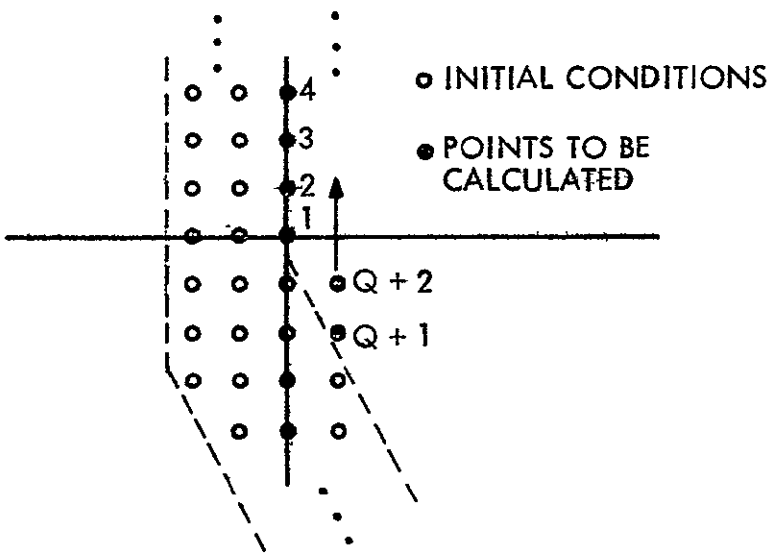

Figure D.6: Illustrating the Required Initial Conditions and the Direction of Recursion for the Filter of Equation (D.12).

move east until all of the data points required in the present column have been calculated. For more details, we refer the reader to [D-5,33,100], in which related issues are discussed, such as the rotation of the support of one-sided or one-quadrant functions to obtain recursions at various angles.

Thus, we have seen one place in which the 2-D case is more complex than in one-dimension: the notion of recursibility and some of its geometric interpretations. One can avoid many of these difficulties by sticking to nonrecursive designs, but recursive techniques offer enough potential advantages in computation time and storage to warrant further detailed study.

Let us make another connection with 1-D processing. Suppose that one of the two indices, say m, has the interpretation as time. Then one might think of y(m,n) and x(m,n) as (1-D) spatially distributed processes that evolve in time. Temporal causality might then correspond to the support of b in Figure D.5 being modified by deleting the points on the positive n axis, yielding a "strictly" one-sided function. In this case, one could define the "state" of the system, and it is clear that this "state" will be finite dimensional only if the range of n is bounded, which is precisely when the required storage for the 2-D recursion is finite. This clearly shows why the order of a 2-D filter does not specify the storage requirements by itself, but one must also know the range of m and n. Hence we see that 2-D digital filtering of scalar (or perhaps vector) variables bears some resemblance to the 1-D state space framework for multi- and possibly infinite dimensional systems that arise in multivariable and distributed system and control theory.

An intriguing question is: can this interrelationship be exploited to yield useful insights and/or results on either or both sides of the coin.

The answer is, of course, yes. Such problems arise in seismic signal processing, in which the data to be processed $x(m,n)$ varies in time (m) and also in array sensor location (n). We refer the reader to the references for more on this problem. Encouraged by this example of the successful exploitation of the 2-D, multivariable 1-D interrelationship, one can ask a number of rather speculative questions. In large-scale system theory, we often have a number of subsystems coupled together, and one is interested in efficient processing of data and control of such systems. Viewing the variables as functions of two independent parameters -- time and subsystem index -- can we obtain any insights into the control and processing of large systems with the aid of 2-D digital filtering concepts? Note that this would involve the consideration of feedback for 2-D systems,[3] a topic that, to our knowledge, has never been addressed in the digital filtering context (with good reason -- it is irrelevant for usual 2D processing problems). We have been somewhat vague about this topic, but we shall return to this large system- 2D_filter idea several times in this section, as these are a number of interesting insights and questions that can be raised. Another possible use of 2-D concepts for 1-D problems is in the analysis of time-varying 1-D systems, in which one can define a system function in two variables -- a transform variable and time. Such concepts may also have value in developing time-varying linear prediction algorithms. On the other side, in order to study questions such as stability or roundoff noise behavior for 2-D filters, is there any benefit in viewing the 2-D filter as a multivariable 1-D system? Capetenakis [D-90] has begun such an investigation for NE filters (<u>not</u> strictly one-sided). Although his results

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[3] Of course, causality constraints would have to be built in. For example the feedback from y to x would also have to involve a strictly one-sided recursion.

do not yield any new results, they are very preliminary, and this also remains as a possible direction for further work.

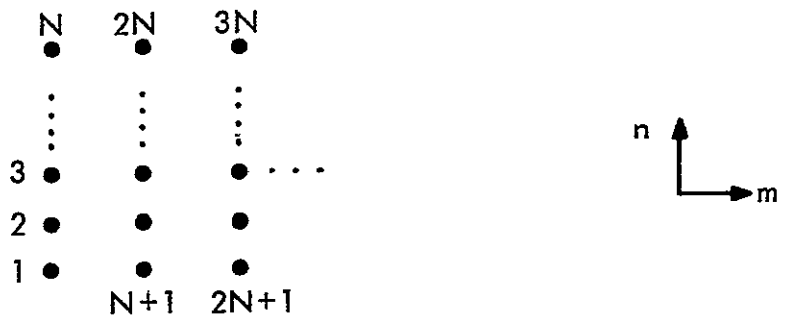
As mentioned earlier and as discussed in [D-107,152], the ability to solve a 2-D difference equation recursively leads directly to the definition of a partial order (D.11) on the part of the 2-D grid over which we wish to solve the equation. Given this partial order -- the precedence relation --, one then has some freedom in deciding how to sequence the calculations. Specifically, if we think of a sequence of calculations as determining a <u>total</u> order (denoted by $\leq$) on the part of the 2-D grid of interest, all we require is that this total order be compatible with the precedence relation. That is (m,n) is calculated before (ℓ,k) (written $(m,n)\leq(\ell,k)$) if $(m,n)\prec(\ell,k)$. Manry and Aggarwal [D-56] have studied such order relations for NE recursive filters. One of their first observations is the following: given a compatible total order $\leq$, the first quadrant can be put into a 1-1, order preserving correspondence with the nonegative integers:

$$Q(m,n) = r \iff \text{there are precisely } (r+1) \text{ points in the NE quadrant } \leq(m,n) \qquad \text{(D.13)}$$

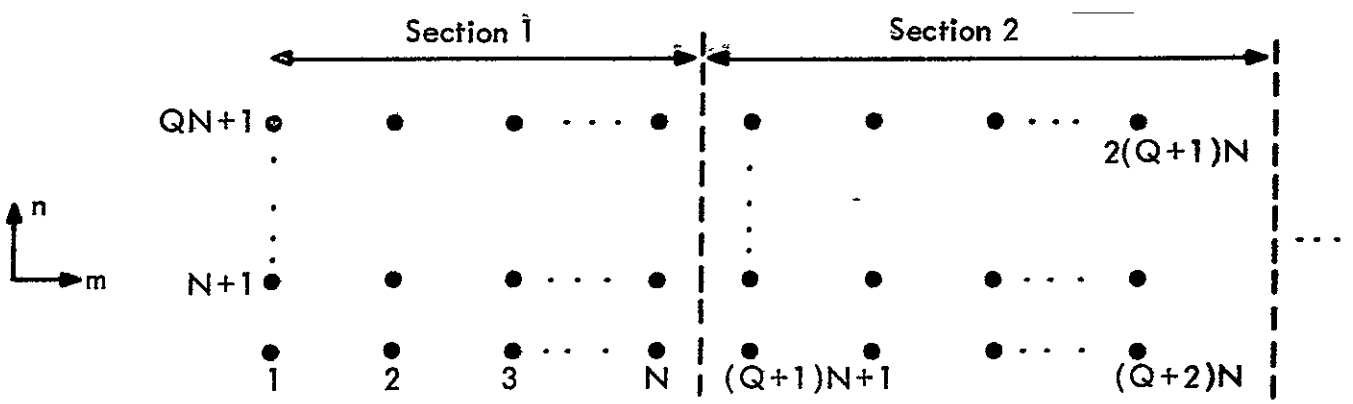
Given the function Q, one can think of (D.10) as determining a <u>1-D</u> filter, with (m,n) replaced by $Q(m,n)$, etc. Alternatively, given the ordering (D.13), we can think of processing the input $x(m,n)$ <u>with</u> a linear time-invariant 1-D filter. One finds (see also Mersereau and Dudgeon [D-3,55]) that in general <u>neither</u> of these filters -- the 1-D filter obtained from (D.10) and (D.13) or the 2-D filter obtained from a given LTI 1-D filter and (D.13)-- are shift-invariant invariant (they are, of course, both still linear).

Let us examine several orders. Manry and Aggarwal suggest the order of Figure D-4(c), since every point in the NE quadrant is mapped by (D.13) into a finite integer. The orders suggested by Figure D.4(a) or (b) are well-posed only if the desired range of one of the variables m or n is finite. In this case, we obtain several possible orders, as given in Figure D.7. In both cases, the range of values is limited in the n direction. Manry and Aggarwal suggest the section-scan of Figure D.7(b). They then show that except for effects near the bottom line or at the junctions of two-sections, the 1-D difference equation from (D.10) and (D.13) looks shift-invariant. They then show that assuming this shift-invariance holds throughout the entire region, one obtains a 1-D stable filter, and one can overlap sections in order to reduce the errors at section junctions, leaving substantial errors only at the far left and along the bottom. These errors notwithstanding, this method provides an extremely promising method for using 1-D filter design techniques to design filters to process 2-D data.

The "scan" order of Figure D.7(a) has been widely used in processing images via line by line scans [D-3,21,55,58]. Nahi [D-21,58] has used this to develop stochastic models for image processing, and the shift-variance introduced by doing 1-D processing on the scan-ordered data points causes errors along the bottom (we will have more to say about this in the next subsection). Mersereau and Dudgeon [D-3,55] point this out, noting that only periodic unit sample responses of the form $h(m,n) = h(m+1,n-N)$ can be realized exactly by a 1-D shift-invariant filter working on the scan-ordered data. They also spend a great deal of time studying this order when the data array is finite in both



(a) The "Scan" Order of Dudgeon-Mersereau [D-3,55] and Nahi, et.al.[D-21,58].



(b) The "Section-Scan" Order of Manry-Aggarwal [D-56].

Figure D.7: Two Orders for 1-D Processing of 2-D Signals.

directions, such as it is for a 2-D finite impulse response function. As they point out, in this case the Fourier transform of the 1-D scan signal is a "slice" of the 2-D Fourier transform of the original data array. Since the order is invertible, we immediately see that we can completely recover the 2-D transform from this slice (which they term a critical slice because of this property). Consequently with the aid of 1-D design methods, one can use this scan-ordering for 2-D FIR filter design: given the Fourier transform of the ideal 2-D filter, we take a critical slice, hence obtaining an ideal 1-D filter; we use 1-D design methods to determine an approximation to this ideal transfer function. We then either use this 1-D filter to process the scanned data[4] or we can invert the 1-D filter, regarded as a critical slice, to find a 2-D filter which can operate directly on the 2-D array. For details we refer the reader to [D-3,55]. We also note that a closely related result involves the recovery of 2-D images from knowledge of 1-D projections of the array. Such a technique is of great interest in biomedical applications such as tomography, and we refer the reader to [D-29] for a detailed survey of the theory and available algorithms related to this subject.

We close our discussion of 2-D orders and precedence relations by noting that these very same issues arise naturally in certain feedback control problems. Ho and Chu [D-87,88] consider optimal control problems in which one has a set of decision makers who base their decisions on certain observed data, which may be affected by the decisions of others. These decisions may be specified to be made at different points in time and/or by distinct decision makers at the

---

[4] One must be careful here to pad the 1-D finite impulse response and the scan signal with zeroes. This is necessary because the extent of the convolution of two finite 2D arrays is larger than the original arrays. In order to be able to invert ("unscan") the convolved 1-D signal to obtain the 2-D output, we must effectively scan enough zeroes at the end of each of the original 2-D arrays. See [D-55] for details.

same points in time. Ho and Chu define a precedence relation among decisions

$$j \prec i \quad \text{if the decision of } j \text{ affects the observation of } i \tag{D.14}$$

and they assume that this is a partial order -- i.e. that if $j \prec i$, we cannot have $i \prec j$ (this is precisely the condition needed for recursibility of 2D filters). Then, under a "partially nested information condition" -- if $j \prec i$, then i's observation includes knowledge of j's observation -- they solve an optimal control problem[5]. When the partial order is a total order -- i.e. when $\prec$ is really just time ordering, this is the usual optimal control problem. In the non-total order case, one can have simultaneous -- i.e. incomparable -- decision makers who do not affect each others observations.

Witsenhausen [D-93,94] has also studied this partial order and has raised issues analogous to those of Chan [D-107,152]. Witsenhausen points out that the amount of parallelism in the control system is essentially a measure of the number of incomparable decision makers (this number may vary with time). In addition, if one totally orders the set of decision makers in a way compatible with (D.14), one can then define the state evolution of the system. Hence we see that there may be many possible sets of states corresponding to different compatible total orders. In fact, using a generalization of the Nerode notion of state, Witsenhausen shows that the set of possible states forms a lattice.

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[5] When this condition is not satisfied, the problem is more difficult -- essentially information is forgotten. In this case Chu [D-88] discusses some examples in which the optimal solution can be found with the aid of the partially nested result, and he discusses some suboptimal methods. We refer the reader to [D-88] for details.

All of this is developed with certain decentralized control problems (i.e. involving incomparable decision makers) in mind, and Witsenhausen points out that it is not clear if the notion of state he has introduced will be of use in solving such problems(as it is in the classical totally ordered case). He also mentions that the a priori partial order restriction does not hold in some game theory problems, in which the sequence of future decision makers can be affected by prior decisions. The difficulties here, as with those of nonrecursible 2-D filters, are quite substantial.

An important problem in the study or design of 2-D recursive filters is stability, where, as in [D-3,16,49], we define stability as the absolute summability of the unit impulse response

$$\sum_{m,n=-\infty}^{+\infty} |h(m,n)| < \infty \tag{D.15}$$

This condition is equivalent to bounded-input/bounded-output stability. As one might expect from knowledge of the 1-D case, the stability of a filter might depend on the direction of recursion -- i.e. the equation

$$y(m+1,n) = 2y(m,n) + x(m+1,n) \tag{D.16}$$

is unstable if recursedto the east, but it is stable

$$y(m,n) = \frac{1}{2}y(m+1,n) - \frac{1}{2}x(m+1,n) \tag{D.17}$$

if we solve to the west.

Shanks, Treitel, and Justice [D-49] considered the stability of 2-D systems with rational transfer functions as in (D.5), (D.6), with b a NE function, as in (D.9). They also explicitly considered stability of the recursion in the NE direction only -- i.e. we use (D.10) to compute y(m,n) from inputs plus outputs to the SW. In this case, they obtained a direct analog of the 1-D stability result:

> A rational transfer function $H(z_1,z_2)$ as in (D.5) with b a NE function is recursively stable in the NE direction if and only if no zero of the denominator $B(z_1,z_2)$ lies in the region
>
> $$\{|z_1|\geq 1\}\cap\{|z_2|\geq 1\}$$ (D.18)

As in the 1-D case, we can use a NE b to define a SW recursion -- instead of going from (D.9) to (L.10), remove y(m-M,n-N) from the sum in (D.9); we can then recursively compute this quantity using outputs to the NE. Similarly, we can pull out the other two "corner" elements to obtain NW and SE recursions. Hence we have 4 possibilities as opposed to the 2 in 1-D, (D.16), (D.17). As in the 1-D case, we obtain different stability conditions for these four cases, which Huang [D-50] has derived. For example, for the SW recursion, we have stability if and only if no zeroes of $B(z_1,z_2)$ lie in $\{|z_1|\leq 1\}\cap\{|z_2|\leq 1\}$. Huang showed that at <u>most</u> one of the four directions of recursion can lead to a stable filter. In addition, Justice and Shanks [D-16] extended these ideas to recursions in different directions for N-D filters in which B does not necessarily have to have finite degree in $z_1,\ldots,z_N$ and $z_1^{-1},\ldots,z_N^{-1}$. We refer the interested reader to [D-16] for a detailed statement and proof of these results.

We now turn our attention to the problem of checking conditions such as (D.18). As mentioned in [D-3], the problem is complicated by the fact that the zeroes of $B(z_1,z_2)$ are <u>not</u> isolated points, but rather are surfaces. This makes the direct checking of (D.18) quite difficult (one must map $|z_1|\geq 1$ into the z_2 -plane via the implicit relation $B(z_1,z_2)=0$; we then have stability if and only if the image lies within $|z_2|\leq)$). Fortunately, a number of simplifications of the criterion (D.18) have been made. Huang [D-50] showed that (D.18) holds if and only if

$$B(z_1,\infty)\neq 0 \qquad |z_1|\geq 1 \tag{D.19}$$

$$B(z_1,z_2)\neq 0 \qquad |z_1|=1,\ |z_2|\geq 1 \tag{D.20}$$

A generalization of this type of criterion to N-dimensions has been made by Anderson and Jury [D-45].

Let us consider the computations involved in (D.19), (D.20). First we note that the test of condition (D.19) is essentially a 1-D stability test, since $B(z_1,\infty)$ is a polynomial in z_1^{-1}. On the surface, however, it appears that (D.20) requires an infinite amount of computation (again we must map $|z_1|=1$ into the z_2 -plane via $B(z_1,z_2)=0$). Fortunately, there are several finite algorithms for testing for conditions such as (D.20). Huang himself used a 2-D bilinear transformation to modify condition (D.20) in such a manner that the continuous 2-D parameter results of Ansell [D-64] could be used[6]. Ansell's test consisted of a Hermite test which checks for the positivity of the principal minors of a symmetric matrix of polynomials in <u>one</u> variable

[6] Two variable system functions arise in a variety of problems. We will return to investigate the connections among these problems at a later point in our discussion.

(this is a positive-real type of test). The positivity tests in turn can be performed using Sturm tests (we refer the reader to [D-50,54,64] for details).

Anderson and Jury [D-54] suggested another method for checking (D.20). Instead of using the bilinear transform plus the Hermite test, one can work directly with condition (D.20), using a Schur-Cohn test (see [D-125]) that replaces (D.20) with a check for the positivity of all of the minors of a certain Hermitian matrix of polynomials in one variable. Again one can use Sturm tests on the individual minors. An alternative to this approach was proposed by Maria and Fahmy [D-130] who used a modified version of the Jury table [D-125] to obtain a finite check of (D.20).

Recently, an algorithm, far simpler than these and also better suited for computer implementation, was developed by Siljak [D-27]. The key to this algorithm is a powerful result [D-122] on the positivity of polynomial matrices. This result, developed with the applications of multivariable positive real functions to networks in mind (see, for example, [D-131]), replaces the sequence of tests of positivity of principal minors with <u>two</u> tests, independent of the dimension of the matrix. Specifically, one need only test for positivity of the matrix at a single value of the independent variable and for the positivity of the determinant. We refer the reader to [D-27] for details and for further remarks on the relationship of these stability results to multivariable techniques arising in network synthesis.

We also note that a great deal of work has been done on extending tests for stability and positivity to the N-D case. Anderson and Jury [D-45] extended their use of the Schur-Cohn test to higher dimensions, but did not directly propose a finite algorithm for the positivity tests one must perform

on polynomials in (N-1) variables (which arise as principal minors from the Schur-Cohn test). Bose and Jury [D-57] developed such an algorithm in the 3-D case, in which the 2-D positivity tests reduce to tests for sign variations of single-variable polynomials defined on the unit circle in the complex plane. They also develop an extremely efficient method for computing multidimensional bilinear transformations, which allows them to develop a stability test algorithm for 3-D continuous systems. Subsequently and with the aid of results from decision algebra, Bose and Kamat [D-124] devised an algorithm for implementing Jury table calculations for an N-D stability test which involves a finite number of multivariable polynomial multiplications and a finite number of single variable polynomial factorizations (a nontrivial numerical problem). In addition, Bose [D-147,148] and Bose and Modarressi [D-118,150] have used concepts from Jury's theory of inners [D-123] for developing tests for positivity, nonnegativity, and greatest common factors of multivariable polynomials. Such tests are needed not only in multivariable stability and positivity tests, but also find applications in applying Lyapunov's direct method to test for the stability of multi-state-variable, 1-D systems.[7] We refer the reader to the references.

The issue of stability is clearly of great importance in filter design, but, as Mersereau and Dudgeon [D-3] point out, it is note enough to have a stability test. Rather, one wants a procedure for taking given frequency response characteristics and generating stable, recursive filters that possess these characteristics. One approach is to take a given transfer function and

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[7] This leads to the question of extending Lyapunov methods to systems with more than 1-D time. To do this, requires the notion of "state" for such systems. We shall discuss this problem at length later in this section.

to stabilize it by finding a stable system function that has the same magnitude function for its frequency response. In 1-D, this process can easily be accomplished by replacing poles outside the unit circle with poles at conjugate reciprocal locations. An algebraic approach to this problem does not work in 2-D, since in general we cannot factor 2-D polynomials.[8] However, another 1-D approach to performing this stabilization, involving the use of the discrete Hilbert transform techniques, has been extended to the 2-D case by Read and Treitel [D-53]. In this approach, one takes the denominator polynomial of a given rational response and calculates its log-magnitude function. Then the 2-D discrete Hilbert transform can be used to determine the minimum phase function associated with the log-magnitude function. One can then exponentiate to obtain the desired stable denominator. As Mersereau and Dudgeon [D-3] point out, one of the difficulties with this method is that the resulting denominator need not be of finite order. Read and Treitel point out that this also can be traced to the lack of a fundamental theorem of algebra.

Another approach to stabilization is to use spectral factorization to break a given system function into the product of several pieces, each of which is stable with respect to a different direction of recursion. In 1-D, the fundamental theorem of algebra allows us to write any rational $H(z)$ as

$$H(z) = H_E(z)H_W(z) \tag{D.21}$$

where H_E has all its poles inside the unit circle (and hence is stable if used to process inputs in the eastern direction) and H_W has all its poles outside the unit circle (stable to the west). Thus, in 2-D, one is tempted

[8] This is often referred to as the "absence of the fundamental theorem of algebra" for multivariable polynomials (see, for example, [D-3]).

to seek one of several such factorizations. All of these involve the use of 2-D cepstral analysis to perform the factorization, and we refer the reader to [D-33,42,100,119] for results on the existence and properties of 2-D cepstra. Following [D-100], let us recall just a few of these properties. Given a 2-D signal $s(m,n)$ and its transform $S(z_1,z_2)$, the complex cepstrum (if it exists) $\hat{s}(m,n)$ is the inverse transform of $\ln[S(z_1,z_2)]$. Thus, if we are given a rational system function $H(z_1,z_2)$ and wish to break it up into the cascade of four stable quadrant filters [D-42][9]

$$H(z_1,z_2) = H_{NE}(z_1,z_2)H_{NW}(z_1,z_2)H_{SW}(z_1,z_2)H_{SE}(z_1,z_2) \tag{D.22}$$

or two stable half-plane filters (using Dudgeon's one-sided functions) [D-5,33]

$$H(z_1,z_2) = H_E(z_1,z_2)H_W(z_1,z_2) \tag{D.23}$$

this can be accomplished by <u>additively</u> decomposing $\hat{h}(m,n)$ into the "corresponding pieces." Thus, we will in principle have developed the desired spectral factorization algorithm once we determine the properties of cepstra of signals that are "minimum phase", where we define minimum phase in analogy with 1-D, and we follow [D-100]. Specifically $s(m,n)$ is minimum phase with respect to a given quadrant (NE,NW,SW,SE) or half plane (E,W) if the signal and its inverse $\tilde{s}(m,n)$ (under convolution) are zero outside the given sector and if $s(m,n)$ and $s(m,n)$ are the impulse responses of stable filters that are recursively implemented in the direction associated with the given sector. Examining (D.22),

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[9] Since it is only the denominator of $H(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$ that matters as far as direction of recursibility and stability, one often considers applying this procedure to $1/B(z_1,z_2)$

(D.23), the factorizations of interest have the property that the impulse response for each piece (e.g. $h_{NE}(n,m) \longleftrightarrow H_{NE}(z_1,z_2)$) is minimum phase ($H_{NE}(z_1,z_2)$ has no poles or zeroes in $\{|z_1|\geq 1\} \cap \{|z_2|\geq 1\}$ and $h_{NE}(n,m)$, $\tilde{h}_{NE}(n,m)$ are NE quadrant signals).

One then obtains the desired algorithm by using the following important property [D-100]:

> A signal is minimum phase with respect to a given sector <u>if and only if</u> its 2-D cepstrum is zero outside this sector.

Using this property, we can derive the 4 piece spectral factorization of Pistor [D-42] or the 2 piece factorization of Dudgeon via the following algorithm: Given $h(m,n)$, calculate $\hat{h}(m,n)$. Consider the restrictions of $\hat{h}$ to the various (4 or 2) sectors of interest, for example

$$\hat{h}(m,n) = \hat{h}_{NE}(m,n) + \hat{h}_{NW}(m,n) + \hat{h}_{SW}(m,n) + \hat{h}_{SE}(m,n) \tag{D.24}$$

The desired spectral factors are the complex exponentials of the transforms of these restrictions. This, in principle, solves the spectral factorization problem, but unfortunately the fundamental theorem of algebra gets in the way again. Unlike the 1-D case, the factors in (D.22) and (D.23) need <u>not</u> be ratios of finite order polynomials. Hence each piece, in principle requires an infinite amount of storage (e.g. for a NE filter we must keep <u>all</u> data points to the SW). Approximations are clearly needed, and we refer the reader to [D-33, 42] for details.

An excellent treatment of the use of cepstra for spectral factorization is given in Ekstrom and Woods [D-119]. In addition to considering the 2 and 4 factor cases, they consider an 8 factor case -- 4 factors corresponding to signals that are strictly in the 4 quadrants (i.e. they are zero along the coordinate axes) plus 4 factors for the 4 pieces of the axes ($m=0$ and $n\geq 0$, $m=0$ and $n\leq 0$, $m\geq 0$ and $n=0$, $m\leq 0$ and $n=0$). These last 4 pieces correspond to the separable part of the system function, ($H(z_1,z_2)$ is separable if and only if it is of the form $H_1(z_1)H_2(z_2)$) while the other 4 pieces can be viewed as "totally non-separable." Using this factorization applied to a NE quadrant filter, Ekstrom and Woods obtain an interesting interpretation of the two conditions of Huang's [D-50] stability test (D.19), (D.20). Essentially we factor our system function as follows

$$B_{NE}(z_1,z_2) = B_1(z_1)B_2(z_2)B_{SNE}(z_1,z_2) \tag{D.25}$$

where "SNE" means "strictly NE". Then (D.19) corresponds to $b_1(m)$ being minimum phase, while (D.20) implies that $b_2(n)*b_{SNE}(z_1,z_2)$ is NE minimum phase.

Ekstrom and Woods also discuss the likelihood that the factors are not of finite order, and, in fact, if one factorization has finite order factors, this does not imply that either of the other two factorizations do. They also discuss the numerical calculation of cepstra and of the spectral factors, and they in fact propose this as an algorithm to test for stability (B satisfies (D.19), (D.20) if and only if $\hat{b}$ is a NE quadrant function). Such a procedure

in principle requires an infinite amount of computation (we must check $\hat{b}=0$ over all three other quadrants), but one can obtain a fast approximate test by looking over a restricted part of the plane.

The question of stable filter design to approximate a given frequency magnitude-response function is considered in [D-119]. They point out that to do this one needs two one quadrant filters (NE and NW, NW and SW, SW and SE, or SE and NE), but one can make the approximation with a single half-plane filter.[10] Ekstrom and Woods also consider the finite order approximation of the infinite degree rational functions that arise as factors in the spectral factorization. Intuitively, one wants to window the denominator power series to obtain a finite order series that remains stable. In [D-119] it is shown that stability is preserved if one uses an exponential window.

In closing, let us note that in Section B we saw that one could devise state space stochastic realization procedures to perform the desired spectral factorization. As one might expect, in 2-D there are some difficulties with this type of procedure, but some results do exist. We will talk about this further when we discuss 2-D state space methods.

A final stabilization procedure is based on the guaranteed stability in 1-D of least squares inverses. The least squares inverse (LSI) is obtained using exactly the methodology one brings into play in performing linear prediction of speech (see Section B). Given the denominator B and its inverse transform b, one seeks a finite extent impulse response p that approximates the

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[10] Also, as discussed in [D-49], in order to implement a zero-phase filter by means of causal, recursive filters, one needs 4 identical quadrant filters -- one for each direction -- or 2 identical half-plane filters. This is the analog of the 1-D result, in which we realize a zero-phase filter as the cascade of a given filter, followed by an identical filter going backward in time.

convolutional inverse of b. One then seeks to choose the coefficients in p to minimize the sum of the squares of the difference between b*p and the unit impulse. In 1-D, this leads to the fast algorithms described in Section B, in which one iterates on the extent of p. One also has the guarantee that p is minimum phase (i.e. that the all pole model 1/P is stable). In [D-49] Shanks, et.al., conjectured that this minimum phase property holds in 2-D. Under this assumption, they proposed the use of a double least squares inverse to stabilize and unstable denominator. That is, given b, we calculate its LSI p, and we then calculate the LSI $\tilde{b}$ of p. By conjecture this is minimum phase, and $\tilde{B}(z_1,z_2)$ hopefully is a good approximation of $B(z_1,z_2)$ (at least in magnitude on $|z_1|=|z_2|=1$). Using this design procedure, numerous 2-D filters have been designed (see, for example, [D-49]). Unfortunately, Genin and Kamp [D-144,145] have recently shown that this conjecture is false in 2-D. Not only does this make suspect the aforementioned design procedure, but it also makes more difficult the extension of linear prediction concepts to 2-D. We will have more to say about this in the next subsection. Suffice it for us to note here that unlike the 1-D case [B-26], in the 2-D case the linear prediction solution does not match the first few correlation coefficients [D-66,67,156].

Let us make a few final comments concerning 2-D design and structures questions. Again, one finds that certain 1-D concepts and techniques do extend, while others do not. One of the earliest design methods proposed was by Treitel and Shanks [D-48], in which they suggested approximating a desired impulse response h(m,n) as a sum of separable terms

$$h(m,n) = \sum_{i=1}^{N} f_i(m) g_i(n) \tag{D.26}$$

If h is of limited extent one can in principle do this exactly, by viewing h as a matrix and then finding its spectral representation. In general this leads to no efficiencies in implementation unless N is substantially less than the extent of h. Treitel and Shanks suggest a method for truncating (D.26), essentially keeping only the dominant terms, corresponding to the largest eigenvalues of h'h, and they perform an error analysis for such approximate filters. Also having a decomposition such as (D.26) suggests several interesting structures. The summation can clearly be realized via a parallel arrangement of the various separable terms, and each separable term is a cascade of two 1-D FIR filters -- one operating vertically, the other horizontally. Thus, each of these can be implemented with an FFT, or, one might approximate each 1-D filter by a recursive filter which can be implemented even more efficiently. Thus we see that the separable and sum of separable cases can be handled essentially with 1-D techniques. We will see later that such cases have special implications in the state space framework.

Motivated by a similar desire to use 1-D design methods for 2-D problems, Shanks, et.al., [D-49] considered taking a 1-D continuous time filter F(s), which can be viwed as either a horizontal or vertical 2-D filter, and rotating it by an angle β

$$\tilde{F}(s_1,s_2) = F(s_1\cos\beta+s_2\sin\beta) \tag{D.27}$$

thus obtaining a 1-D filter that processes data along lines at an angle β with the s_1 axis. One can then apply the 2-D bilinear transformation to obtain a 2-D digital filter design. Several examples of such "rotated designs" are given in [D-49]. In addition, Costa and Venetsanopoulos [D-51] have considered this design technique in more detail. They note that since F is 1-D, it factors, and thus the stability test for the final 2-D filter can be reduced to very simple tests on the factors. They find that for given directions of recursion, there are constraints on the angle β for which the resulting filter is stable. Of course, other angles are possible if one rotates the data or changes the direction of recursion. In addition, they consider the design of filters with circular symmetry, obtained by cascading identical 1-D filters that have been rotated to be spaced evenly between 0° and 360°. Such designs have the advantages of guaranteed stability, efficient computer design, and cascade implementation due to the factorizability of the 1-D prototype filter.

The use of transformations to take 1-D designs into 2-D designs is a conceptually appealing idea. In addition to the methods mentioned above and the 1-D projections of Mersereau and Dudgeon [D-55] and Manry and Aggarwal [D-56] discussed earlier, several other methods have been devised for utilizing 1-D filter designs. One of the most powerful methods of this type for designing 2-D FIR filters involves the so-called <u>McClellan transformations</u> [D-2,36,127,128,129]. The original algorithm as developed in [D-2,36] involves transforming a 1-D filter of the form

$$G(e^{j\omega}) = \sum_{n=0}^{M} b(n)\cos n\omega \tag{D.28}$$

into a 2-D linear phase filter

$$H(e^{j\omega_1}, e^{j\omega_2}) = \exp\{-j(n_1\omega_1 + n_2\omega_2)\}\hat{H}(\omega_1, \omega_2) \tag{D.29}$$

where

$$\hat{H}(e^{j\omega_1}, e^{j\omega_2}) = \sum_{k=0}^{n_1} \sum_{p=0}^{n_2} a(k,p)\cos k\omega_1 \cos p\omega_2 \tag{D.30}$$

The specification of (D.30) is obtained from (D.28) by means of the transformation

$$\cos\omega = A\cos\omega_1 + B\cos\omega_2 + C\cos\omega_1\cos\omega_2 + D \tag{D.31}$$

where choices of A,B,C,D determine the shape of contours where ω=constant. Clearly on such contours $|H|$ is constant. For example, the choice A=B=C=-D=1/2 yields nearly circular contours, and hence one can map a low pass filter G into a low pass circularly-symmetric filter H. Thus, one can use 1-D FIR techniques to design 2-D FIR filters of high order in a reasonably efficient manner. In some cases, one can in fact show that transformations of 1-D optimal fitlers (in the Chebyshev sense -- i.e. minimizing maximum deviation from a desired frequency response) are in fact the <u>optimal</u> 2-D designs [D-36]. In [D-128] an

extension of this design criterion was considered, in which (D.31) was replaced by

$$\cos\omega = \sum_{p=0}^{P} \sum_{q=0}^{Q} t(p,q)\cos p\omega_1 \cos q\omega_2 = H_p(e^{j\omega_1}, e^{j\omega_2}) \tag{D.32}$$

By careful choice of the parameters $t(p,q)$, one can obtain a variety of contour shapes in the 2-D frequency plane, and [D-128] contains details of algorithms for choosing these parameters to obtain best approximations to given desired contours. Having chosen the contours, the second part of the design procedure involves the design of the 1-D FIR filter, for which there are numerous procedures [D-2].

One of the nice features of these transformation designs is that they lead directly to efficient structures. The development of these structures and a study of their relative merits based on number of multiplies, coefficient sensitivity, and roundoff noise is given in [D-127,129]. We briefly illustrate the idea by following the development of Chan and McClellan in [D-127]. Examining (D.28), we note that

$$\cos n\omega = T_n[\cos\omega] \tag{D.33}$$

where T_n is the nth Chebyshev polynomial, which satisfies the recursion

$$\begin{aligned} T_0(x) &= 1 \quad , \qquad T_1(x) = x \\ T_n(x) &= 2x\, T_{n-1}(x) - T_{n-2}(x) \end{aligned} \tag{D.34}$$

Rewriting (D.28) as

$$G(e^{j\omega}) = \sum_{n=0}^{M} b(n) T_n[\cos\omega] \tag{D.35}$$

and replacing $\cos\omega$ by (D.32), we can directly obtain a realization of $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$ as an interconnection of M copies of $H_p(e^{j\omega_1}, e^{j\omega_2})$, where one uses the recursion (D.34) in interconnecting the copies of H_p to obtain realizations of each of the $T_n[H_p(e^{j\omega_1}, e^{j\omega_2})]$. We refer the reader to [D-127] for details.

Another 2-D design method adapted from 1-D was proposed by Shanks, et.al., [D-49], who modified the time-domain design technique of Burrus and Parks [B-114]. As in the 1-D case, given a desired impulse response $h(m,n)$, we want to find a rational transfer function $A(z_1,z_2)/B(z_1,z_2)$ that yields an impulse response "close" to h. We first solve for the denominator B by a method quite similar to that use in computing the least-squares inverse (and which evidently will have the same stability problems as those mentioned earlier). One can then solve for the numerator using the analog of the method described in [B-114] and in Section B.

One of the most widely used FIR design methods in 1-D is the optimum Chebyshev design method, where the Remez exchange algorithm leads to an extremely efficient computer design technique [D-2]. Kamp and Thiran [D-74] have extended this algorithm to 2-D, but not without a number of severe complications. Firstly, the Haar condition does not hold in the 2-D case and this can lead to degeneracies that can keep the algorithm from converging. Also, unlike the ordered 1-D case in which one can show [D-2] that errors between

the optimal design and the desired response alternate between $\pm$ the maximum error (the Chebyshev norm), in the 2-D case one has no such alternation theorem. This makes the exchange algorithm far more complex, and this plus several other factors make the algorithm extremely slow. Hence it is limited to low order impulse responses. We refer the reader to [D-74] for details.

In the 1-D case, one can use the so-called differential correction method to find optimal Chebyshev rational frequency responses [D-52] and this method has been extended by Bednar [D-13] to the 2-D case. As pointed out in [D-3], this method requires a great deal of computation time, and also the algorithm produces as its output an optimal rational magnitude-squared frequency response. Thus, to obtain the actual filter specification, one must perform a spectral factorization, which, as we have seen, leads in general to an infinite order numerator and denominator.

In addition to the design methods mentioned above, a number of other methods have been proposed. These include windowing [D-133], frequency sampling [D-134], transformations of (z_1,z_2) to obtain new designs from old [D-149], and the extension of wave digital filters [D-151] to 2-D, with all of the pseudopassivity and stability properties of their 1-D counterparts. We refer the reader to these references for details.

The issue of 2-D filter structures and of their effects on required storage, number of multiplies, coefficient sensitivity, and roundoff noise has been raised several times in this section and is clearly of great importance. The issue is complicated significantly by the fact that one cannot factor general 2-D polynomials. This immediately rules out cascade and parallel realizations

unless one is dealing with one of the special classes of filters described earlier. Mitra, et.a. [D-26] show, however that one can write down the generalizations of 1-D direct form realizations for NE recursions. They also comment as we did earlier on the dependence of storage requirements not only on the order of the filter but also on the output array dimensions. In addition, for several special classes of NE rational filters, they developed structures based on continued fraction expansions. We refer the reader to [D-26] for details.

As in the 1-D case, a critical question in the design of 2-D IIR filters is the existence of limit cycles and the effect of roundoff noise filter output. Maria and Fahny [D-28,73] have considered the limit cycle problem for first order 2-D recursive filters, both singly [D-73] and in cascade [D-28]. The results in [D-28] on the existence of horizontal, vertical, and noninteracting diagonal limit cycles parallel the results of Jackson [A-20] quite closely, and their method for bounding the magnitude of limit cycles is quite similar to the 1-D result of Sandberg and Kaiser [A-4], although the bounds become far more complex as one looks at limit cycles on rows or columns other than the first ones. Open questions involve the extension of this type of result to higher order filters. In addition, an intriguing question is whether one can extend any of the other techniques discussed in Section A. Do the passivity-Tsypskin-positive real-frequency domain results of Claasen, et.al., [A-15] and others extend to the 2-D case? What about the Lyapunov techniques of Willson [A-2]? Of course in this case one would need 2-D state space models and a 2-D Lyapunov theory.

The analysis of roundoff noise in 2-D filters can be carried out much as for 1-D filters, and we refer the reader to the references for examples of this type of analysis. Another open question concerns the extension of the Lyapunov equation-state covariance noise analysis method described in Section C for 1-D roundoff analysis. Again one would need a state space model in order to consider this question. We will come back to this question in a moment.

Finally, we note that Chan [D-107] has proposed a unified state space framework for the study of 1-D and 2-D structures. In Section C we discussed the 1-D aspects of this approach, in which all structures can be viewed as factorizations of the map that transforms the present state and next input into the next state and output. In the 2-D case, one must process inputs sequentially according to any order function that is compatible with the recursion precedence relation (D.11). Also, as we have noted before, the resulting 1-D state space realization is finite dimensional if and only if the data is defined on a domain that is bounded in one direction. Using the scan order described earlier, Chan develops a time-varying state realization. The time-variations arise for precisely the reason mentioned earlier-- we must take account of the edge effects as we finish scanning one line and begin scanning at the start of the next. Chan develops a realization using the scan order for a general NE recursive filter. He conjectures that this realization is minimal in the recursive case, but shows that it is <u>not</u> in the FIR case. On the other hand, in the FIR case, we have mentioned earlier that one can realize the 2-D filter with the scan order and a <u>time-invariant</u> 1-D filter by padding the ends of each line with zeroes (this is essentially what Mersereau and Dudgeon did in [D-55]). Chan show that he can do the same in his setting by finding a nonminimal, (caused by padded

zeroes) time-invariant realization. This leads to an interesting tradeoff--nonminimality of one realization versus the more complex control needed in order to implement the time-varying minimal one. The utility of such 1-D state space models and the additional degree of freedom one has in choosing the order relation (and hence the state space as Witsenhausen [D-93,94] pointed out) makes this an interesting area for further research.

In addition to the above 1-D state space descriptions for recursively ordered 2-D systems, some work has been done in the past few years involving the definition and analysis of 2-D state space models. Roesser [D-110] considers NE models of the form

$$\begin{aligned} v(i+1,j) &= A_1 v(i,j) + A_2 h(i,j) + B_1 x(i,j) \\ h(i,j+1) &= A_3 v(i,j) + A_4 h(i,j) + B_2 x(i,j) \\ y(i,j) &= C_1 v(i,j) + C_2 h(i,j) + D x(i,j) \end{aligned} \tag{D.36}$$

here x is the input, y is the output, and v and h together play the role of a "state" variable. Here v carries information vertically, and h conveys it horizontally. In addition, Roesser takes (D.36) to be a NE recursion $(i,j \geq 0)$.

Given this model, Roesser considers several issues. He solves (D.36), and the solution resembles the variation of constants formula for usual finite-dimensional 1-D linear systems. The one main difference is that boundary conditions $v(0,j)$, $j \geq 0$ and $h(i,0)$, $i \geq 0$ must be specified. Roesser also considers a 2-D version of the Cayley-Hamilton Theorem. Taking the 2-D transform of (D.36), we obtain

$$\frac{Y(z_1,z_2)}{X(z_1,z_2)} = [C_1,C_2]\begin{bmatrix} z_1I-A_1 & -A_2 \\ -A_3 & z_2I-A_4 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \qquad (D.37)$$

and hence in this setting the role the characteristic polynomial is played by

$$p(z_1,z_2) = \det\begin{bmatrix} z_1I-A_1 & -A_2 \\ -A_3 & z_2I-A_4 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = A^{1,0} + A^{0,1} \qquad (D.38)$$

where

$$A^{1,0} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \qquad A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix} \qquad (D.39)$$

represent the required dynamics to advance the system in the vertical and horizontal directions, respectively. We can then define the transition matrix over a number of vertical and horizontal steps

$$A^{0,0} = I$$

$$A^{i,j} = A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1} \tag{D.40}$$

Then, if we define

$$E^{i}F^{j}A \doteq F^{j}E^{i}A = A^{i,j} \tag{D.41}$$

we have the 2-D Cayley Hamilton theorem

$$p(E,F)A = 0 \tag{D.42}$$

Roesser uses this result to obtain an efficient method for computing the transition matrix. The result is also used to obtain finite rank tests as in the 1-D case for controllability and observability, which are defined in analogy with 1-D. Specifically a state (v,h) is observable if whenever it appears as the initial state at $(0,0)$, with <u>all</u> other boundary conditions zero, the resulting output $y(i,j)$, $i,j \geq 0$ is not identically zero when all zero inputs are applied. The state is controllable if there is some $(i,j) \geq (0,0)$ and set of inputs so that $(v(i,j),h(i,j)) = (v,h)$ when the boundary conditions are all zero.

Several questions and issues arise in considering Roesser's model. First of all, not all NE quadrant rational transfer functions can be realized by systems of the form (D.36), although this can be remedied by a modification of the output equations [D-164], we refer the reader to [D-164] for more on realization theory and canonical forms for these systems. Also, in obtaining his algorithm for recursively computing the $A^{i,j}$ via the Cayley-Hamilton theorem, Roesser used the notion of 2-D eigenvalues in a crucial manner, and in the usual

non-factorizable case the calculation of zeroes of $p(z_1,z_2)$ is extremely difficult. This is not only complicates his transition matrix algorithm, but it makes stability tests more difficult. One must use methods such as Siljak's [D-27] on $p(z_1,z_2)$ or the direct extension of Huang's stability test to the model (D.36) (see [D-164]). An interesting open question is the development of Lyapunov stability methods for (D.36). Furthermore, the model (D.36) is limited to quadrant-causal systems. This is perfectly reasonable for the study of quadrant-recursive filters, but its value for the analysis of other 2-D signals is unclear. For example, Roesser mentions the possibility of a 2-D filtering theory based on (D.36). In this case, one would want to model the observed signal z as

$$z(i,j) = y(i,j) + N(i,j) \tag{D.43}$$

where N is noise, and y is generated by a model as in (D.36) with x a noise process. Thus (D.36) plays the role of a "spatial shaping filter." As Ekstrom and Woods [D-119] point out, one <u>cannot</u> obtain arbitrary spectra from a NE shaping filter. Hence, one may need <u>two</u> such filters, as well as a method for modelling the spectra of the signal field. Also, the artificially imposed causality of the model (D.36) and in fact of <u>any</u> state space model may cause difficulties. For example, in an image one would not expect light intensity as a function of spatial location to have a NE causal structure. On the other hand, if a NE causal filter yields the proper shape for the intensity correlation function, there may be no difficulty in using such a model. Indeed, as Andrews and Hunt [D-81] point out, the use of such models may be

of value in leading to efficient recursive filtering methods for image processing. This remains an open area for further research, and we will have more to say about it in the next subsection.

Finally, we note that Roesser's "state" $(v(i,j),h(i,j))$ might better be termed a "local state" [D-97,138]. As we saw earlier, in recursively solving 2-D equations, the required amount of storage in general depends on the size of the arrays of interest (see Figures D.3 and D.6). Hence if the array sizes are unbounded, the required memory is infinite. Thus, v and h in Roesser's model do <u>not</u> represent the true state. Rather the model (D.36) can be viewed as arising by reducing a scalar, high order 2-D difference equation to a vector, first-order equation. In this way, we see that the dimensions of v and h correspond to the <u>order</u> of the equations of interest.

Issues of this type have been considered in more depth by Fornasini and Marchesini [D-97,138]. They consider impulse responses that lie strictly in the NE quadrant, and for such systems they define a notion of "global state" using a direct generalization of the theory of Nerode. In order to define the global state as containing all relevant information concerning "past" inputs, one needs to define "past." The definition of past inputs at the point (i,j) is all $x(k,\ell)$ where <u>either</u> $k<i$ <u>or</u> $j<\ell$ (see Figure D.8). In this way the state <u>must</u> summarize all needed boundary conditions, and Fornasini and Marchesini point out that the state is usually infinite dimensional.

Attention in [D-97] then shifts to local NE state space descriptions of the form

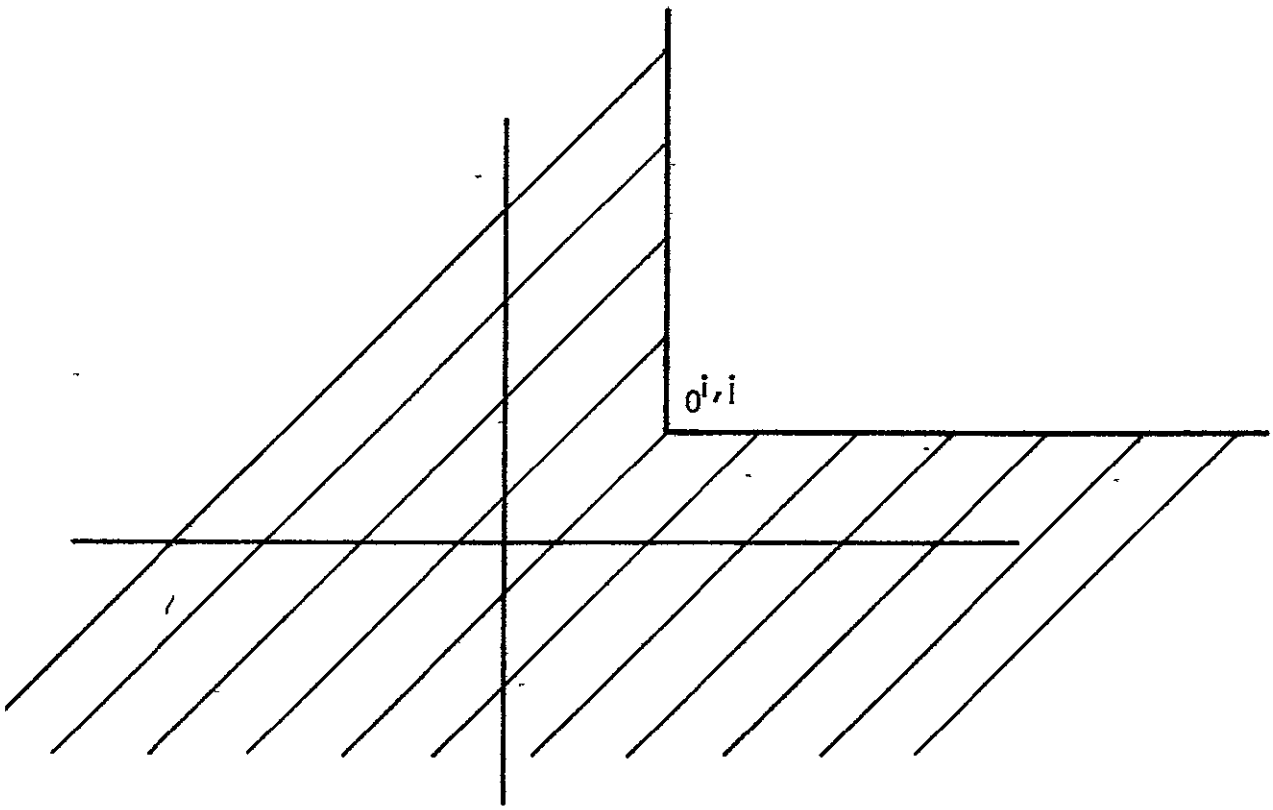

Figure D.8: Illustrating the"Past" in the Definition of Marchesini and Fornasini.

$$x(m+1,n+1) = A_0x(m,n) + A_1x(m+1,n) + A_2x(m,n+1) + Bu(m,n) \qquad (D.44)$$

$$y(m,n) = Cx(m,n)$$

Note here that vertical and horizontal information is conveyed by a single state vector. Having this model, it is then shown that a NE IIR filter can be realized as in (D.44) if and only if the transform of the impulse response is rational. The "if" part of this result involves a procedure for constructing a realization in a form that is some type of generalization of the 1-D "standard controllable form".

Having such realizations, attention naturally focusses on minimality--obtaining a local state space model (D.44) with as small a state space as possible. This leads directly to the notions of controllability and observability, with finite rank conditions for these properties being developed in a manner analogous to that of Roesser. In fact, a simple proof of the 2-D Cayley-Hamilton result is given in [D-97] for systems as in (D.44). The main minimality result of Marchesini and Fornasini is that minimality implies local controllability and observability (an algorithm for reducing the dimension of uncontrollable and/or unobservable realizations is given) but that local controllability and observability <u>do not</u> imply minimality. This is done by means of a counterexample that we will discuss shortly.

It should be noted that the work in [D-97] is phrased in terms of the algebraic notion of <u>formal power series</u> (essentially (D.3) with no convergence properties attached to it). The most thorough treatments of the uses of this theory to study topics in formal language theory, automata theory, nonlinear

systems analysis, and 2-D processes are the works of Fliess [D-98,139,140]. Fliess studies the properties of rational power series[11] in detail using in part a generalization of the Hankel matrix, and he shows that the rank of this matrix equals the dimension of the minimal global state space. This is infinite dimensional, in general, but Fliess notes [D-98] that the global state space is finite dimensional if and only if the formal power series is "recognizable", which simply means that it has a separable denominator. As we have seen one can do a great deal of analysis for separable 2-D systems, since many 1-D concepts and results directly extend in this case

Attasi [D-6,35,96] has studied such systems in great detail. His basic model is a special case of (D.44)

$$x(m+1,n+1) = F_1x(m,n+1) + F_2x(m+1,n) - F_1F_2x(m,n) + Gu(m,n) \tag{D.45}$$

$$y(m,n) = Hx(m,n)$$

where it is assumed that

$$F_1F_2 = F_2F_1 \tag{D.46}$$

With these assumptions, one finds that the impulse response is strictly NE, and it and its transform are given by

$$h(i,j) = HF_1^{i-1}F_2^{j-1}G, \qquad i,j>0 \tag{D.47}$$

$$H(z_1,z_2) = H(z_1I-F_1)^{-1}(z_2I-F_2)^{-1}G \tag{D.48}$$

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[11] In general the indeterminates in this theory are taken to be noncommuting. However in the 2-D case, the two shifts z_1 and z_2 do commute.

Clearly any FIR filter can be realized as in (D.45), and thus any stable impulse response can be approximated arbitrarily closely by a system of this form. This, of course, is neither startling nor necessarily very useful, since the dimension of the resulting state-space system may be extremely large.

Having this framework, Attasi defines dual notions of local controllability and observability and derives conditions somewhat simpler than in [D-97,110] because of the special nature of (D.45). Attasi also considers minimal realizations of the form of (D.45), obtains a state space decomposition result and minimal realization algorithm much like those in 1-D (here the 2-D Hankel matrix plays a crucial role), and shows that minimality implies controllability and observability. He <u>also</u> proves the converse of this last result, but this is only true if one looks for the minimal realization in the class of models given by (D.45). Consider the example constructed by Fornasini and Marchesini [D-97]

$$\begin{aligned} H(z_1,z_2) &= \frac{z_1^{-1}z_2^{-1}(1+z_1^{-1}+z_2^{-1})}{1+z_1^{-1}+z_2^{-1}+z_1^{-1}z_2^{-1}} \\ &= \frac{z_1^{-1}z_2^{-1}(1+z_1^{-1}+z_2^{-1})}{(1+z_1^{-1})(1+z_2^{-1})} \end{aligned} \tag{D.49}$$

The minimal realization of the form of (D.45) is of dimension ≥ 3, but one can find a realization of the form (D.44) of dimension 2. This clearly points out another of the many complications that arises in going from 1-D to 2-D.

Undoubtedly the major contribution of Attasi's work is that he <u>did</u> something with his models[12]. He <u>was</u> able to develop a 2-D Lyapunov equation. More

[12] That may very well be because this is the one case in which one can readily see what to do.

specifically, to show northern and eastern asymptotic stability, we simply need to check the 1-D systems along vertical or horizontal lines. This leads to 1-D Lyapunov equations and nothing new. However, Attasi did obtain an "invariance principle" type of result (see Section A): if F_1 and F_2 are stable, then (D.45) is controllable if and only if the equation

$$P - F_1 P F_1' - F_2 P F_2' + F_1 F_2 P F_1' F_2' = GG^1 \tag{D.50}$$

has a unique positive definite solution P. The exact implication of this result for 2-D stability theory and its potential utility in such areas as limit cycle analysis are at present unclear and remain intriguing questions for further work.

Attasi also considers systems as in (D.45) which are driven by white noise. Again he obtains a 2-D Lyapunov equation for the state covariance, and this result may be of some value in performing roundoff noise analysis for 2-D filters (see the analogous 1-D discussion in Section C). Also, Attasi shows that any 2-D stationary covariance function can be approximated arbitrarily closely by a system of this type, and he develops a stochastic realization theory that exactly parallels the 1-D case with one rather surprising exception. In the 1-D case, there are in general a whole family of stochastic realizations, each of which essentially factors the spectral density S(z) of the output process y. In the 2-D case, assuming that one <u>can</u> factor the spectrum $S(z_1,z_2)$ of y, the stochastic realization is essentially unique. This is due primarily to the additional constraints on S imposed by the fact that we use a single quadrant shaping filter (D.45). Specifically, in addition

to the constraints imposed by NE and SW correlations, an additional constraint arises in considering NW and SE correlations. This constraint leads to the uniqueness result.

We note that this stochastic realization- spectral factorization result suffers from all of the numerical problems mentioned in Section B and from the difficulties of 2-D factorization. The one novel feature of Attasi's development is the use and in fact the necessity for using non-square factors -- i.e. to perform the required factorization

$$S(z_1,z_2) = H(z_1,z_2)H'(z_1^{-1},z_2^{-1}) \tag{D.51}$$

where H is NE causal and of the form (D.48), one must consider rectangular factors.[13] For example, if y is a scalar process, then H in general must be 1xm, and, in fact, the aforementioned uniqueness result fixes the value of m. We refer the reader to [D-6,35,96].

We remark that the primary motivation for Attasi's work was to develop a 2-D stochastic framework in which to study 2-D Kalman filtering and its application to image processing. Several other authors have consider such problems, and we will consider them in the next subsection.

Recently, Morf, et.al., [D-162,163] have made several noteworthy contributions to 2-D state space theory. In [D-162] they consider the properties of polynomial and rational matrices in two variables. The motivation for this study, which leads naturally to multi-input, multi-output 2-D systems, is the generalization of the scalar 2-D polynomial results of Bose [D-147,166] and the matrix 1-D polynomial results of Rosenbrock [D-168] and Wolovich [D-169].

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[13] Rectangular factors are considered in the general 1-D stochastic realization theory described in Section B, but they are not necessary in order to factor 1-D scalar spectra.

Morf, et.al., generalize the scalar notion of primitive factorization to the matrix polynomial case, and they provide an existence and uniqueness proof for such a factorization. By regarding a 2-D polynomial $p(z_1,z_2)$ as a 1-D polynomial (say in z_2) with coefficients that are rational functions in the other variable and by introducing several notions from algebraic geometry, they are able to use many 1-D techniques to obtain 2-D generalizations of the Euclidean algorithm, Hermite and Smith forms, tests for relative coprimeness of polynomial matrices, matrix fraction descriptions of rational matrices, and the extraction of greatest common right divisors. In 1-D Rosenbrock and Wolovich utilize many of these properties to study multi-input, multi-output state space models. In [D-163] the results of [D-162] are used to study 2-D state space models. The models of Roesser, Fornasini-Marchesini, and Attasi are reviewed, and Morf, et.al., argue in favor of Roesser's model. Their reasoning is that (D.36) is a true first order system, and hence v and h together comprise a valid local state. The model (D.44), on the other hand is not first order, and hence x is not a local state -- i.e. the order of the system (D.44) may be larger than the dimension of x. The importance of this is not totally clear, since, as we've seen, the required storage depends on more than the order of the system.

The concepts of local controllability and observability for the Roesser model are explored in [D-163], and the authors point out that these conditions neither imply <u>or</u> are implied by the minimality of the realization (this is done with several instructive examples). This difficulty can be partially overcome by redefining local controllability and observability for (D.36) by requiring these properties to hold <u>separately</u> in the horizontal and vertical directions (but not necessarily jointly). With this definition, minimality

implies but is not implied by local controllability and observability.

To obtain notions of controllability and observability that are equivalent to minimality, Morf, et.al., generalize the approach of Rosenbrock in which coprimeness of polynomial matrices plays a crucial role. This leads to the notions of modal controllability and observability and a related concept of minimality and also allows one to use the algebraic and geometric concepts developed in [D-162] in order to study the 2-D realization problem. In this setting the existence of minimal realizations becomes a difficult problem, and one may not even exist if we restrict ourselves to systems with real parameters (see [D-163] for an example). In related work, Sontag [D-143,154,E-29] has also found realizations of lower dimension than those proposed by Fornasini and Marchesini, and he has shown that minimal realizations need not be unique up to a change of basis. All of these facts indicate that the 2-D state space model is an extremely complex one and offers some extremely difficult mathematical and conceptual problems. As with all other topics concerning 2-D systems, there are many possible ways to generalize 1-D concepts. It remains to be seen whether any of these state models and realization theories can provide a useful framework for solving 2-D analysis and synthesis problems.

A number of authors have considered state space and other dynamic models defined with very general independent variables. Motivated to a large degree by the partially-ordered feedback structures of Ho and Chu [D-87,88] and Witsenhausen [D-93,94], Mullans and Elliott [D-95] and Wyman [D-143,160,161] have considered the development of an algebraic state space theory on partially ordered sets. In addition, Seviora and Sablatash [D-114-116] have placed

algebraic (specifically, abelian group) structures on the independent variable in order to consider a generalized transform and digital filter theory with the aid of tools from the theory of abstract harmonic analysis. Their framework is quite abstract and general, and it includes such possible time sets as the integers, the usual 2-D plane of integer pairs, and a variety of "cylindrical time sets." We will have occasion to use such a "time set" in the next section.

Finally, we have noted at several points that the issues arising in the analysis of 2-D discrete time systems have many similarities with results in other areas. For example, Ansell [D-64] and Youla [D-77] studied continuous-time transfer functions in two variables that arise in the consideration of networks containing lumped and distributed elements. Along similar lines, Kamen [E-30] has developed an algebraic theory for considering continuous-time systems that contain time delays. In addition, as mentioned earlier, Sontag [D-143,154,E-29] has considered a general algebraic framework of this type and has tied together some of the time delay and 2-D results.

Other classes of systems have also been analyzed in a similar manner. Kamen [D-142] has developed a theory for time-varying 1-D systems that bears some resemblance to the 2-D theory. Also, Fliess [D-98,139,140], Fornasini and Marchesini [D-138,E-36], and Bush [D-155] have noted and have taken advantage of some of the rather striking relationships among certain nonlinear and 2-D system results. To illustrate the basic idea, consider the following three systems:

Volterra (single input)

$$y(m) = \sum_{k,\ell} h(m-k,m-\ell)x(k)x(\ell) \tag{D.52}$$

Bilinear (two inputs)

$$y(m) = \sum_{k,\ell} h(m-k,m-\ell)x_1(k)x_2(\ell) \tag{D.53}$$

Two Dimensional (single input)

$$y(m,n) = \sum_{k,\ell} h(m-k,n-\ell)x(k,\ell) \tag{D.54}$$

One immediately sees the striking relationship among these three classes of systems, and it is not surprising that similar methods of analysis can be used on all of them. Indeed, Fliess' formal power series formulation leads directly to a methodology for analyzing algebraic properties of each kind of system. Also, Fornasini and Marchesini were led to the study of 2-D systems by their earlier results on bilinear systems. Finally, we note that in his work on bilinear systems Bush considered the 2-D transform $H(z_1,z_2)$ of the weighting function h that appears in (D.53). He showed that if one could write

$$H(z_1,z_2) = \frac{p(z_1^{-1},z_2^{-1})}{q_1(z_1^{-1})q_2(z_2^{-1})q_3(z_1^{-1}z_2^{-1})} \tag{D.55}$$

where p is a two-variable polynomial and the q_i are polynomials in a single variable, then the system could be realized by three finite dimensional linear systems and a single multiplier. Again the fundamental theorem of algebra makes it difficult to find representations as in (D.55) (a condition slightly weaker than separability). We refer the reader to the references for details of these ideas.

In this subsection we have surveyed a large number of issues involving systems over a 2-D parameter space. We have seen that a number of 1-D concepts can be extended to the 2-D case (e.g. 2-D FIR implementation schemes using the FFT), while others cannot (e.g., cascade structures). In many cases there are several possible extensions from 1-D to 2-D (as with the several notions of causality and the variety of directions of recursion), and in most situations the 2-D counterparts of 1-D results are far more complex (as with the 2-D stability tests). We have mentioned several of the reasons for difficulties in 2-D -- difficulties in defining notions of causality, recursibility, and "state" (local or global) in 2-D, the absence of a 2-D factorization theorem, and the absence of the Haar condition. Also we have speculated on a wide range of open problems in such areas as filter design, filter structures, the accompanying issues of storage, sensitivity, and roundoff effects, and the development of useful state space models and tools such as the 2-D Lyapunov equation. In the next subsection we will open up several additional issues involving 2-D random processes.

D.2 Image Processing, Random Fields, and Space-Time Systems

Digital processing of images for data compression, noise removal, or enhancement is one of the major areas of applications of 2-D digital signal processing techniques. In addition, image processing has spurred a great deal of work in the analysis of spatially-distributed stochastic variables -- random fields. In this subsection we will discuss some of the work concerning

image processing and random fields and will point out what we consider to be several particularly intriguing areas for further work. The reader who is interested in obtaining a detailed understanding of image formation and processing and of the response of the human visual system should consult the references. In particular, we refer the reader to the survey paper of Hunt [D-4], the book written by Andrews and Hunt [D-81], and the paper of Stockham [D-82]. We will refer to these references often as we sketch some of the issues involved in image processing.

Let $g(x,y)$ denote the image radiant energy as a function of two spatial variables, where, for the time being, we will assume that the system is free of noise. The image results from an image formation process that transforms the original object radiant energy $f(x,y)$ into the observed image. A general model that is often used for the image formation process is

$$g(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x,y,x_1,y_1,f(x_1,y_1))dx_1dy_1 \tag{D.56}$$

Although in some cases the formation process may be nonlinear (see [D-81] for examples), in many cases it is valid to assume a linear model

$$g(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x,y,x_1,y_1)f(x_1y_1)dx_1dy_1 \tag{D.57}$$

Here $h(x,y,x_1,y_1)$ is called the <u>point-spread function</u> (PSF), as it represents the image that results from a point source located at (x_1,y_1) (i.e. $f(x,y) = \delta(x-x_1)\delta(y-y_1)$).

This function models the smoothing and blur that take place in the image formation process. Sources of such blur abound. See [D-4,19,24,65,81] for detailed discussions of some of these. Examples include blur due to motion, defocused systems, and the effects of atmospheric turbulence.

The model (D.57) represents a spatially-varying 2-D linear system. In some cases, one can take advantage of simplifying assumptions, such as

<u>shift-invariance</u>

$$h(x,y,x_1,y_1) = h(x-x_1,y-y_1) \tag{D.58}$$

<u>separability</u>

$$h(x,y,x_1,y_1) = h_1(x,x_1)h_2(y,y_1) \tag{D.59}$$

or both

$$h(x,y,x_1,y_1) = h_1(x-x_1)h_2(y-y_1) \tag{D.60}$$

As one might expect, these simplifications lead to gains in analytical tractability and computational efficiency.

It is clear that the continuous-space model of (D.57) is inappropriate for digital storage or processing of images, and one usually obtains a discrete model by sampling the left-hand side of (D.57) and by approximating the right-hand side using some type of quadrature formula (see [D-4,23,81] for discussions of the errors involved in this approximation). One then ends up with a model of the form

$$g(i,j) = \sum_{k,\ell} h(i,j,k,\ell) f(k,\ell) \tag{D.61}$$

where the $g(i,j)$ form the 2-D image array, the $f(k,\ell)$ form the object array, and the $h(i,j,k,\ell)$ form the discrete point-spread function. Note that the simplifications (D.58)-(D.60) can also be imposed in the discrete domain. For example, shift invariance yields the 2-D convolution

$$g(i,j) = \sum_{k,\ell} h(i-k,j-\ell) f(k,\ell) \tag{D.62}$$

Most digital image processing schemes involve the analysis of equations (D.61) or (D.62), and we will spend most of our time with them.[14] As all images of interest are of finite extent, we assume that the range of i,j,k, and ℓ in (D.61) and (D.62) is $1,\ldots,N$.[15]

In addition to the image formation process, one must take into account the process of image recording and storing. As discussed in [D-4,81,82], two well-developed and related image models for photographic images are the <u>intensity</u> and <u>density</u> images, which are related in an essentially logarithmic manner. Let $g_1(x,y)$ be the intensity of light reflected from a photographic film on which is stored the image represented by the intensity function $g(x,y)$.

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[14] We refer the reader to [D-81] in which a mixed continuous-discrete digital scheme is discussed. The image g is sampled, but the continuous form of the right-hand side of (D.57) is left intact. Spline approximations are used to estimate the image between samples.

[15] There is no loss of generality in assuming a square picture, as we can always pad a rectangular image array with zeroes in order to make it square.

Then (see [D-4]) the <u>intensity image model</u> is

$$g_i(x,y) = N_i(x,y)[g(x,y)]^{\gamma} \tag{D.63}$$

where γ is known for the given type of film (it essentially controls contrast), and $N(x,y)$ is <u>film grain noise</u> due to random fluctuations of silver density on the film. On the other hand, the <u>density image model</u> is essentially the logarithm of (D.63)

$$g_d(x,y) = \gamma \log[g(x,y)] + n_d(x,y) \tag{D.64}$$

As described in [D-4,81], the complexities of these models have been avoided in most cases. Equation (D.63) has been replaced by an additive model

$$g_i(x,y) = g(x,y) + n_i(x,y) \tag{D.65}$$

while the <u>low contrast assumption</u> [D-4,81] has been used to justify replacing (D.64) with

$$g_d(x,y) = \gamma g(x,y) + n_d(x,y) \tag{D.66}$$

It is not our purpose here to justify these models and assumptions, and we refer the reader to the references for more details of the modelling of imaging systems.

Given the above discussion, we now have the following mathematical model: a discretized object $f(i,j)$ and "noise-free" image $g(i,j)$, where $i,j=1,\ldots,N$, and f and g are related by (D.61) or (D.62); an observed image

$$q(i,j) = g(i,j) + v(i,j) \tag{D.67}$$

where v is and additive noise process.[16] We now turn our attention to the analysis of this model. We will return to consider the nonlinear models (D.63),(D.64) somewhat later.

At various points in this development, it will be more convenient to view f,g,q, and v as vectors by performing a scan (lexicographic) ordering. For example

$$f = \begin{bmatrix} f(1,1) \\ f(1,2) \\ \vdots \\ f(1,N) \\ f(2,1) \\ \vdots \\ f(2,N) \\ \vdots \\ f(N,N) \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \tag{D.68}$$

where $f_i' = (f(i,1),\ldots,f(i,N))$. In this case the relevant equation is

$$q = Hf + v \tag{D.69}$$

where H is an $N^2 x N^2$ matrix formed from the PSF. Examination of (D.61) and (D.68) yields the following form for H

[16] This noise may include more than film grade noise. Specifically, the effects of light from sources other than the object can be included in v.

$$H = \begin{bmatrix} H_{11} & H_{12} & \cdots\cdots & H_{1N} \\ H_{21} & H_{22} & \cdots\cdots & H_{2N} \\ \vdots & \vdots & & \vdots \\ H_{N1} & H_{N2} & \cdots\cdots & H_{NN} \end{bmatrix} \tag{D.70}$$

where H_{ij} is NxN and its (m,n) element is $h(i,m,j,n)$. If the imaging system is shift-invariant -- i.e. if (D.62) holds, it is readily seen that H is block Toeplitz -- i.e.

$$H_{ij} = H_{i-j} \tag{D.71}$$

and, in fact, each of the blocks is itself a Toeplitz matrix.[17] This fact will be extremely important when we discuss the computational aspects of certain processing algorithms. Note also that if H is separable, than

$$H = A_1 \otimes A_2 \tag{D.72}$$

where $\otimes$ denotes the tensor or Kronecker product, and A_1 and A_2 are NxN matrices given by

[17] Note that all that is needed for (D.71) is "horizontal stationarity"--i.e. $h(i,m,j,n) = h(i-j,m,n)$. Vertical stationarity in turn implies that each block is Toeplitz.

$$A_1 = \begin{bmatrix} h_1(1,1) & h_1(1,2) & .. & h_1(1,N) \\ h_1(2,1) & h_1(2,2) & .. & h_1(2,N) \\ \vdots & \vdots & & \vdots \\ h_1(N,1) & h_1(N,2) & .. & h_1(N,N) \end{bmatrix} \tag{D.73}$$

where

$$h(i,j,m,n) = h_1(i,m)h_2(j,n) \tag{D.74}$$

Note that horizontal stationarity implies that A_1 is Toeplitz, while vertical stationarity implies that A_2 is Toeplitz.

It is evident from the preceding development that probabilistic and statistical methods must play some role in image processing. In this context, f,g,v, and perhaps h are <u>random fields</u>. Such a random field $s(i,j)$ is characterized by some type of statistical description -- the joint density of the values of the field at different points or perhaps a statistical model

such as a 2-D ARMA model. We will consider some of these more complex descriptions at a later point, but for now all we will use is the mean and covariance

$$\bar{s}(i,j) = E[s(i,j)] \tag{D.75}$$

$$r(i,j,m,n) = E\{[s(i,j)-\bar{s}(i,j)][s(m,n)-\bar{s}(m,n)]\} \tag{D.76}$$

The field will be called (wide-sense) <u>stationary</u> if[18]

$$r(i,j,m,n) = r(i-m,j-n) \tag{D.78}$$

Note that if s and $\bar{s}$ are ordered lexicographically, then

$$E[(s-\bar{s})(s-\bar{s})'] = R \tag{D.79}$$

where R is the $N^2 \times N^2$ matrix obtained from r in the same manner that H in (D.70) is obtained from the PSF h. We also observe that R is block Toeplitz if s is stationary in the horizontal direction, and each block is itself Toeplitz if we have vertical (and hence full) stationarity. In addition, if the covariance is separable

$$r(i,j,m,n) = r_1(i,m)r_2(j,n) \tag{D.80}$$

we can obtain a representation for R much as the one for H in (D.72). Note that in some sense (D.80) says that correlations in the data have horizontal and vertical as "preferred directions". While this may be reasonable in some cases (perhaps for cases in which one variable is space and the other is time) and may be acceptable in others (because it leads to mathematical tractability and good results), in many cases the assumption of (D.80) may be totally inappropriate. We will comment on this further later in this section.

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[18] This is not quite standard, since one usually also requires $\bar{s}(i,j)$=constant. Clearly any process which is stationary in our sense can be transformed into one in this stronger sense by subtracting out the mean.

One important problem in image processing is the efficient representation of images for storage or transmission [D-4,31,37,76,241]. For such applications, one wishes to represent the image with as few pieces of information as possible but with a reasonable level of accuracy. Intuitively, one then wants the redundancy in the pieces of information kept to a minimum. Suppose we are given an image s[19] with covariance R. The off-diagonal elements of R tell us how much correlation there is among the various pixels ("picture elements" -- i.e., components of s), and this correlation can be interpreted as a measure of the redundancy in the picture. One method for obtaining a less redundant representation is to transform s

$$\sigma = Ts \tag{D.81}$$

(where $T^{-1}=T'$) so that the covariance of σ

$$\Sigma = TRT' \tag{D.82}$$

is diagonal -- i.e. T is the matrix of eigenvectors of R and the components of σ are uncorrelated. This transformation is called the Karhunen-Loeve transform, and its use in efficient coding can be seen as follows (see, for example, [D-4]). Let us order the eigenvalues of R in order of decreasing magnitude. Then we store or transmit only those components of σ corresponding

[19] Either an NxN array or an N^2 vector. We shall use these two forms interchangeably and without comment unless there is a chance of confusion.

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to the $M<N^2$ largest eigenvalues. We are guaranteed to have retained those "coordinates" of the image that contain the most information, and we can obtain an approximate image by inverting the transform:

$$\hat{s} = T'\hat{\sigma} \tag{D.83}$$

where $\hat{\sigma}$ is formed by setting to zero those components of σ that were discarded. We can in fact decide how many terms to keep on the basis of the size of the reconstruction error

$$e = s-\hat{s} \tag{D.84}$$

As discussed in [D-4,37], the Karhunen-Loeve transform leads to a very efficient coding scheme. However, in general, this transform involves exorbitant amounts of computation. We must find the eigenvectors and eigenvalues of R (usually just once off-line for a class of images with the same covariance), and then we must perform the transform coding (D.81) or decoding (D.83). This can involve a great deal of on-line computation (see [D-4,37] for estimates), since there is no "fast" method for performing this transform, in general. There are, however, several special cases in which this transform can be calculated efficiently. One of these [D-241] involves the use of a more detailed model of the image as a random field, and we will defer discussion of it until we begin our treatment of more detailed models for fields and images. Another case, motivated by similar analysis performed by Hunt [D-4,46] and Andrews and Hunt [D-81], is quite instructive, and, as we will use this idea on several occasions, we will develop it here in detail.

Suppose that s is stationary. Then R is a block Toeplitz matrix with Toeplitz blocks. Following [D-81], suppose further that a particular pixel is correlated with a number of surrounding pixels, but is uncorrelated with ones some distance d away (Andrews and Hunt cite d=20-30 pixels as a typical number). Then the block Toeplitz covariance matrix takes the form

$$R = \begin{bmatrix} R_0 & \cdots & R_{d-1} & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ R_{1-d} & & \ddots & & & 0 \\ 0 & \ddots & & \ddots & & R_{d-1} \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & R_{1-d} & \cdots & R_0 \end{bmatrix} \tag{D.85}$$

where each R_1 is an NxN Toeplitz matrix

$$R_1 = \begin{bmatrix} R_0^1 & \cdots & R_{d-1}^1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ R_{1-d}^1 & & \ddots & & & 0 \\ 0 & \ddots & & \ddots & & R_{d-1}^1 \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & R_{1-d}^1 & \cdots & R_0^1 \end{bmatrix} \tag{D.86}$$

We now modify R and the R_i to make R block circulant and R_i circulant. A block circulant matrix is block Toeplitz with each row a cyclic shift to the right of the preceding one, where the last block on the right of one row becomes the first block on the left in the next row. Examining (D.85), (D.86), we see that this merely means replacing some of the zeroes with nonzero entries.

The reasons for doing this and its interpretation can be found in the following observations:

1. Let R_c denote the circulant approximation to R, and let T_c be the matrix of eigenvectors of R_c. Then the product $T_c s$ can be computed efficiently using the fast Fourier transform. This is shown in Appendix 2 and is the reason for using this approximation.

2. For N large compared to d, $||R-R_c||$ is small, where $||\cdot||$ is any matrix norm. In addition, this error can be made arbitrarily small by choosing N large enough (see [D-81]).

3. Let us see what the circulant approximation means. For R_c to be block circulant, we must have that

$$r(i,j,m,n) = r[(i,j)\text{mod}N,m,n] \tag{D.87}$$

Intuitively, instead of thinking of the image as a flat array, think of it as a cylinder, so that horizontal distance matters only modulo N. Furthermore,

if we also have that each block is itself circulant, we should think of the image as the surface of a torus (connect the two ends of the cylinder).[20] See Figure D.9 for an illustration of this.

As discussed in [D-37] the Karhunen-Loeve expansion can also be performed quickly if the covariance is separable. In this case, we perform the expansion separately in the horizontal and vertical directions -- essentially 1-D transforms on data records of length N. Hence in the stationary or separable cases, there appear to be relatively efficient methods to perform the transform. However, motivated by the complexity of the general Karhunen-Loeve expansion, researchers have applied other, more efficient transform techniques such as the FFT and the Hadamard transform to the problem of image compression and coding (see [D-4,37,81] for a discussion of several of these). Many of these work nearly as well as Karhunen-Loeve [D-4]. This is not surprising given the preceding discussion concerning circulant approximations.

As discussed in Section B, one of the most widely used coding or compression schemes for 1-D time series, such as speech, is linear prediction, in which we design a one-step predictor or inverse whitening filter (depending upon your point of view) for the time series. This method has several appealing features in 1-D -- it is efficient (if one uses the Levinson algorithm), it leads to recursive coding and decoding algorithms, and it yields excellent performance. In 2-D the situation is not nearly as clear. What direction

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[20] Seviora [D-114] and Seviora and Sablatash [D-115] dealt with a general framework that included transforms on cylindrical and toroidal spaces for the purpose of digital signal processing.

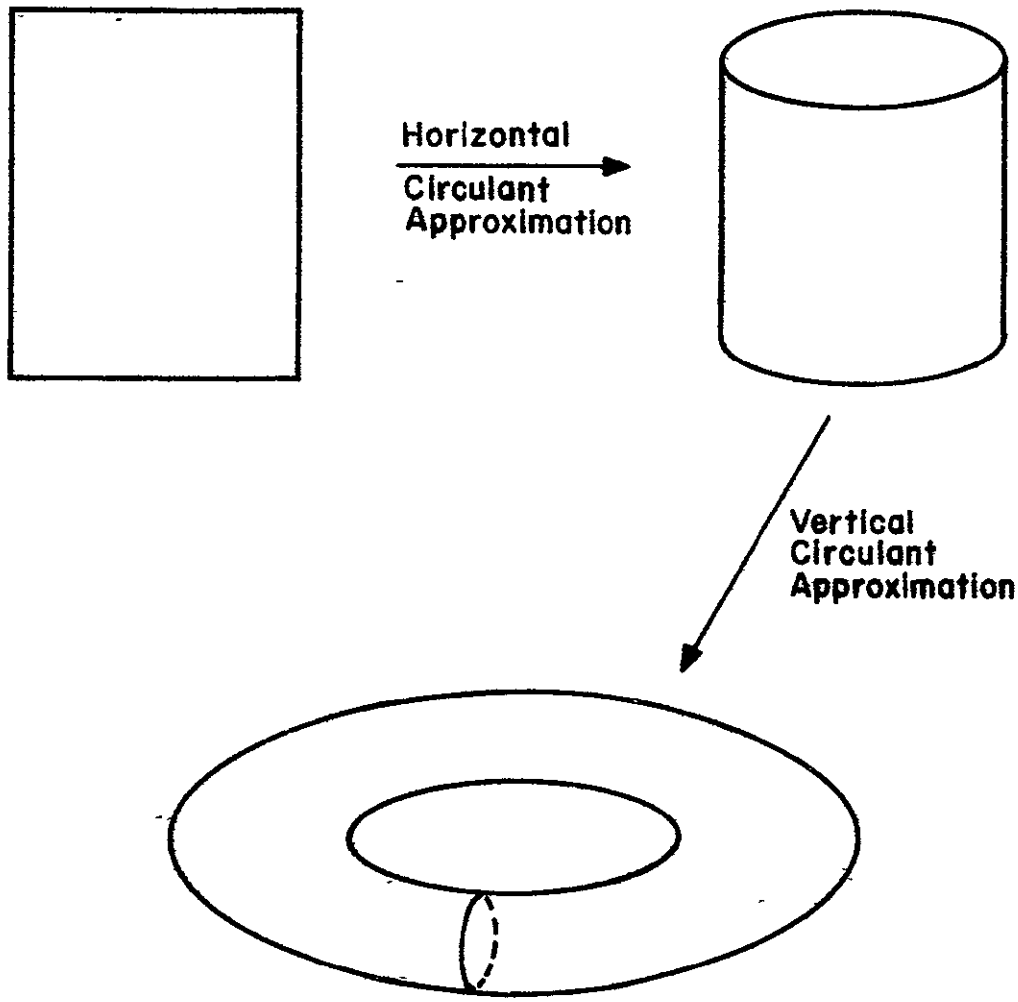

Figure D.9: Illustrating the Circulant Approximation

do we predict in and what old data do we use to do the prediction? Genin and Kamp [D-144,145] have shown that 2-D least-squares inverse filters need not be stable; can this problem be overcome? Are there efficient 2-D algorithms along the lines of Levinson's method? We will address some of these questions later as we develop more detailed stochastic models. Let us point out here, however, that for a particular ordering of the points in a 2-D array, Habibi [D-76] and Habibi and Robinson [D-37] have obtained encouraging results using a predictive encoder. In comparison with transform methods, they found the predictive coding scheme to be superior as far as system complexity, time delay due to the coding operation, and coding performance at high bit rates, but the transform methods were more robust to errors in the knowledge of the image covariance and required lower bit rates. In addition, Habibi and Robinson [D-37] suggest a hybrid scheme in which we transform the data horizontally line by line and then perform 1-D linear prediction on each column. They report that the performance of this system is excellent. These promising results and the questions mentioned earlier concerning the direction of prediction are sufficient to warrant further investigation of such methods.

We now turn our attention to the problem of restoring blurred and noise-corrupted images. Initially we will concentrate on the linear model (D.61), (D.62), (D.67) or, equivalently (D.69). For details concerning these methods we refer the reader to the references and in particular to the survey papers [D-4,19,38] and the text [D-81].

One of the first methods proposed for image restoration is aimed solely at the removal of the effects of blur and essentially ignores the presence of additive noise. This is the <u>inverse filter</u>

$$\hat{f} = H^{-1}q \tag{D.88}$$

In the space-invariant case, (D.62), we can take transforms

$$\hat{F}(z_1,z_2) = \frac{Q(z_1,z_2)}{H(z_1,z_2)} \tag{D.89}$$

In addition, in this case H is block Toeplitz with Toeplitz blocks, and hence we can make the circulant approximation (assuming that the extent of the PSF is much smaller than the size of the picture -- see [D-81]) and hence can take the DFT of (D.62), yielding

$$\hat{F}(m,n) = \frac{Q(m,n)}{H(m,n)} \tag{D.90}$$

where, for example,

$$H(m,n) = \sum_{k,\ell=0}^{N-1} h(k,\ell)W_N^{-km-\ell n} \tag{D.91}$$

Note that as an alternative to making the circulant approximation, we can use the 2-D version of a standard 1-D idea -- we embed the 2-D acyclic convolution (D.62) in a larger 2-D cyclic convolution by padding each row and column with a sufficient number of zeroes. Equivalently, we intersperse

zeroes in the appropriate places in the lexicographically ordered vectors q, $\hat{f}$, etc., and in the block matrix H [D-46]. The resulting matrix H is block circulant with circulant blocks (see Appendix 2 for the correspondence between circulant matrices and cyclic convolution). Thus, we can directly apply (D.90) with no approximation to this padded image.

Let us make several comments concerning the inverse filter. First of all, the image formation process (D.61), (D.62) may not be invertible, and thus, we cannot even perform the calculation indicated by (D.88). One might consider using a pseudo-inverse, and we will discuss this in the context of another restoration methodology. In addition, examining the transformed versions (D.89), (D.90), we see the possibility of two further problems. The frequency response H usually falls off at high frequencies. Thus, assuming that high frequency noise is present, we may observe extreme noise amplifications. In addition, the inverse filter transfer function flows up at the zeroes of H, and this can cause severe difficulties. Looking at these equations in the space domain, Sondhi [D-19], Hunt [D-4], and Andrews and Hunt [D-81] argue that the difficulty arises due to the severe problems encountered in a attempting to invert integral equations such as (D.57). In the discrete domain, this implies the ill-conditioning of the matrix H, and thus, even if its inverse exists, the solution suggested by (D.88):

$$\hat{f} = f + H^{-1}v \tag{D.92}$$

may be dominated by the noise.

In order to overcome difficulties such as these, one must explicitly take the presence of noise into account. This leads to the discrete Wiener filter formulation [D-4,17,19,38,40,81]. Consider (D.69) with

$$E(ff')=P\ , \qquad E(vv')=R, \qquad E(fv')=0 \tag{D.93}$$

and suppose we wish to choose our estimate $\hat{f}$ as the minimum mean square error (MMSE) estimate

$$\min_{\hat{f}} E[(f-\hat{f})'(f-\hat{f})] \tag{D.94}$$

If we limit ourselves to linear transformations on the data or if we assume Gaussian statistics,[21] we obtain the optimal estimate

$$\hat{f} = PH'(HPH'+R)^{-1}q \tag{D.95}$$

Again let us note that in the space-invariant, zero-mean,[22] stationary case,

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[21] The Gaussian assumption can clearly only be made for convenience, since we know a priori that all components of f must be ≥ 0. We note that although this eliminates the Gaussian assumption in theory, in practice one often makes it anyway, since it leads to tractable problem formulations and acceptable system performance (see, for example, [B-104], where the same type of positivity assumption was encountered).

[22] The zero mean assumption is included to guarantee the block-Toeplitz structure of P and R. If we have nonzero means for f and v, we can subtract out their effects from (D.69) and proceed with the analysis. In this case, the estimate produced by (D.96) is the estimate of the deviation of f from its a priori mean.

we can perform (D.95) in the frequency domain, obtaining an expression analogous to (D.89). In addition, in this case all of the matrices are block Toeplitz, and we can use the same block circulant approximation to obtain an expression analogous to (D.90):

$$\hat{F}(m,n) = \frac{H^*(m,n)Q(m,n)}{|H(m,n)|^2 + \frac{\Phi_v(m,n)}{\Phi_f(m,n)}} \tag{D.96}$$

where "*" denotes complex conjugate, Φ_v is the 2-D DFT of the noise covariance, and Φ_f is the DFT of the image covariance.

Note from (D.95), (D.96) that the problem observed with the inverse filter has been removed -- i.e. the inverse in (D.95) and the denominator in (D.96) won't blow up, since we have explicity included the effects of noise. The Wiener filter does, however, have some difficulties and limitations as an image processing system. To a great extent this is due to the fact that the MMSE criterion is not particularly well-suited to the way in which the human visual system works (see Stockham's paper [D-82] for a discussion of the visual system). In particular, the Wiener filter is overly concerned with noise suppression. In addition, in order to make the filter computationally feasible, one often assumes stationarity. This in turn leads to a filter that is insensitive to abrupt changes -- i.e. it tends to smooth edges and reduce contrast. On the other hand, in high contrast regions, the human visual system will readily accept more noise in order to obtain greater resolution. Thus,

the Wiener filter sacrifices too much in resolution in favor of noise suppression. We will return to this key image processing tradeoff later in this section.

Another difficulty with the Wiener filter is the amount of a priori information that is required. For the inverse filter all we need is the PSF,[23] while for the Wiener filter we need the PSF and the second order statistics of the original image and the noise. This is a great deal of information to assume to be known, and a serious question here concerns the robustness of the Wiener filter to errors in this a priori knowledge.

Several schemes have been proposed that are aimed at trading-off between the potentially high-resolution, poor noise performance of the inverse filter and the lower-resolution, good noise performance of the Wiener filter. One of these is the constrained least squares filter, suggested by Sondhi [D-19] and developed and discussed by Hunt [D-46] and Andrews and Hunt [D-81]. In this formulation, we wish to choose $\hat{f}$ to minimize

$$J(\hat{f}) = \hat{f}'C'C\hat{f} \tag{D.97}$$

subject to the constraint

$$(H\hat{f}-q)'(Hf-q)=e \tag{D.98}$$

[23] As discussed in [D-81], the PSF is usually assumed to be known, and for certain types of blur, this is a reasonable assumption. However, in many cases, either the entire PSF or several of its parameters are not known a priori and must be estimated. We will discuss this shortly.

The solution is

$$\hat{f} = (H'H + \gamma C'C)^{-1} H'q \tag{D.99}$$

where γ is a Lagrange multiplier found by iteration in order to satisfy (D.98). Again one can obtain transform versions of (D.99) in the shift-invariant case.

Several comments are in order concerning this approach, which has been shown in several experiments to perform at a level superior to that of the Wiener and inverse filters [D-4,81]. Note first of all from (D.99) that we have eliminated the need for covariance information for f and v. In addition, by adjusting the size of e in (D.98) (or equivalently of γ in (D.99)), we can effectively control the amount of noise suppression. Also, we have some freedom in the choice of C, and several possibilities and their interpretations are discussed in [D-81]. For example, choosing C=I, essentially leads to a "pseudo-inverse" filter -- i.e. this filter resembles the inverse filter but avoids the illconditioning by adding γI to H'H before inverting. In addition, one can choose C as a "finite difference matrix," which leads to our minimizing some measure of the rate of fluctuation in the estimated image. One can also choose C in order to match the characteristics of the human visual system [D-4], and the choice,

$$C = P^{-1/2} R^{1/2} \tag{D.100}$$

leads to a "parametric Wiener filter," closely resembling (D.95) in structure.

Another approach, proposed by Stockham, et.al. [D-4,81,E-4], leads to a filter that is the geometric mean of the inverse and Wiener filters (hence it directly trades off between the properties of these systems):

$$\hat{F}(m,n) = \left[\frac{1}{|H(m,n)|^2 + \frac{\Phi_v(m,n)}{\Phi_f(m,n)}} \right]^{1/2} Q(m,n) \qquad \text{(D.101)}$$

This filter, obtained by designing a system so that the output power spectral density equals that of the original image, has worked extremely well in several experiments [D-4,81,E-4]. We note that (D.101) is not precisely correct, as it does not include the phase effect of the restoring filter. Since phase is extremely important in image processing and viewing, one must take it into account. This has been done for several specific types of PSF's, and we refer the reader to [E-4] and the references therein. In addition, in examining (D.101) it appears that we again require a great deal of a priori information; however, this particular filter is particularly well-suited to the use of on-line estimates of quantities such as the PSF. We will discuss this in more detail later in this section, and we refer the reader to [E-4] for details.

In addition to these techniques, a number of other approaches along these lines have been developed, and we refer the reader to the references for details. At this point we want to make several observations concerning these processing systems. Note first of all that they are nonrecursive and in principle require the block processing of the entire image or substantial

sections of the image [D-47]. Hence the computational burden of these schemes can be quite high. In the shift-invariant, stationary case this problem can be somewhat alleviated with the aid of FFT techniques, but the required amount of calculation is still substantial. The situation is even more complicated if the PSF is shift-varying. Examples of such imaging systems are given by Sawchuk [D-65] and Robbins and Huang [D-20]. In his paper, Sawchuk suggests breaking the PSF into shift-invariant pieces, followed by the use of some of the techniques we have discussed. Sawchuk and Robbins and Huang also discuss the possibility of inverting nonlinear distortions in the imaging system, followed by the use of shift-invariant methods. Clearly the PSF must be of a special form for this to be possible.

The use of the FFT or the inversion of nonlinear distortions notwithstanding, it is clear that the processing methods described so far require a great deal of on-line calculation. In 1-D, one finds that recursive methods are often preferable to nonrecursive ones because of their computational advantages. Although the situation is not as clear in 2-D (as we saw in subsection D.1), it certainly seems worthwhile to investigate recursive 2-D image processing methods. As discussed in [D-81] the 1-D Kalman filter offers great computational savings over nonrecursive methods, and an appealing question is the extension of such filters to 2-D. Anyone familiar with 1-D Kalman filtering theory realizes that the design of the filter relies heavily on a dynamic -- i.e. recursive -- representation of the received signal. Hence,

to develop such techniques in 2-D, we need more complex models of images than that provided by the mean and covariance. The need for the use of such models is an obvious drawback to this approach, but the potential gains in computational efficiency represent a distinct advantage. We now will describe several of the approaches taken in the application of recursive estimation techniques to 2-D processing. This research topic is still in its early stages of development, and many open questions remain.

One approach to recursive processing of images involves the 1-D processing of the scan-ordered image (see Section D.1). This work has been developed by Nahi, Silverman, and their colleagues [D-8,18,21,24,58,174]. Suppose we have an image $f(m,n)$ (assumed to be zero mean for convenience) with stationary covariance

$$r(k,\ell) = E[f(m,n)f(m+k,n+\ell)] \tag{D.102}$$

Suppose we observe

$$q(m,n) = f(m,n) + v(m,n) \tag{D.103}$$

where the additive noise v is, for simplicity, assumed to be zero mean and white, with

$$E[v(m,n)v(k,\ell)] = R\delta_{m,k}\delta_{n\ell} \tag{D.104}$$

We now take the scan ordering of the NxN grid on which q, f, and v are defined. Let us use the same symbols to denote the resulting 1-D processes.

We then have

$$q(k) = f(k) + v(k) \tag{D.105}$$

$$E[f(k)f(\ell)] = S(k,\ell) \tag{D.106}$$

$$E[v(k)v(\ell)] = R\delta_{k\ell} \tag{D.107}$$

where $S(k,\ell)$ can be calculated from knowledge of $r(m,n)$.

Note that the scanned image $f(k)$ is <u>not</u> stationary, just as in Section D.1 we found that scanned 2-D systems did not become time-invariant 1-D systems. The problem is clearly due to the abrupt change that occurs when the scanner reaches the end of one line and begins the next. For example, it is clear that one will have

$$S(i,i+1) = S(i+1,i+2) = r(0,1)$$

if and only if i, $i+1$, and $i+2$ come from the same line of the image. On the other hand, it is clear that 2-D stationarity plus the periodicity of the scanner should yield some structure for S, and, in fact, it is easily seen that

$$S(k,\ell) = S(k+N,\ell+N) \qquad \forall\, k,\ell \tag{D.108}$$

A process with this property is called <u>cyclostationary</u>, and many of its properties have been analyzed in detail [D-43,80,83].

Given the model (D.105)-(D.107), one wishes to use Kalman filtering techniques in order to suppress the noise. In order to do this, we need a state space model for f. That is, we have a stochastic realization problem:

find a finite-dimensional linear system driven by white noise that yields an output with correlation function given by (D.106). Unfortunately, as pointed out in [D-21], $S(k,\ell)$ does not have the required separability that is needed in order for such a realization to exist. Hence, some sort of approximation is needed, and several have been developed. The simplest of these involves finding a stationary approximation to (D.106), much as Manry and Aggarwal found shift-invariant approximations to the shift-varying scanned filters they studied in [D-56]. The basic idea here, due to Franks [D-43], is to use to stationary covariance

$$R(k) = \frac{1}{N} \sum_{m=1}^{N} S(m,m+k) \tag{D.109}$$

This is equivalent to randomizing the variable m over the scan of one line in the computation of $E[f(m)f(m+k)]$.

Having $R(k)$, one can then use some realization procedure to find a Markov model

$$x(k+1) = Ax(k) + \omega(k) \tag{D.110}$$

$$f(k) = c'x(k) \tag{D.111}$$

$$E[\omega(k)\omega(j)] = Q\delta_{kj} \tag{D.112}$$

that realizes or approximates the given correlation function. We refer the reader to [D-18] for a method used by Nahi and Assefi.

We can now obtain an image restoration scheme by direct application of Kalman filtering to the model (D.105), (D.107), (D.110)-(D.112). Several comments are in order. We first note that the filter has an artificial causality -- only the points above and to the left on the same line affect the estimate of a given pixel. This can be partially removed by smoothing the data -- i.e. by estimating each f(k) based on all the data. With the model we have developed, this can be done efficiently with two Kalman filters, scanning in opposite directions and starting at opposite ends of the image. The resulting estimate still has difficulties because of the randomizing used to obtain (D.110)-(D.112). This causes problems much like those caused by Manry-Aggarwal's shift-invariant approximation. In this case, one can remove some of these difficulties by transposing the image and performing the same type of processing again (2 more Kalman filters scanning in a direction orthogonal to the other 2 filters). This appears to be reminiscent of Pistor's four quadrant decomposition [D-42] -- we have NE, NW, SE, and SW Kalman filters.

A number of other comments can be made concerning this approach to image processing. First of all, like the Wiener filter, the Kalman filter is based on a MMSE criterion, and hence we can expect it to sacrifice resolution for noise suppression. In addition, this method relies heavily on a priori knowledge of the image covariance, and the robustness of the approach in the presence of modeling errors remains an open question. We have already commented on the problems inherent in the stationary approximation of the cyclostationary covariance of the scanned image. In [D-21] Nahi suggests that one

use a piecewise stationary approximation over various sections of each scanned line. This leads to a time-varying, piecewise-constant state variable description for the scanned process.

Several alternative methods exist for reducing the affect of the stationary approximation. Nahi and Franco [D-58] suggest the simultaneous scanning of a number of lines ("vector scanning"). One can then model correlations both along the scan and along the components of the vector of the scan. If one scans all lines simultaneously, we can take all of these correlations into account. Note that in this case we have turned a 2-D, scalar signal into a 1-D, multivariable signal, much as we discussed in the preceding subsection. Of course, this leads to problems with the dimensionality of the resulting processor. Thus, Nahi and Franco suggest a "section-scan", scheme which is in fact far more efficient than the scalar system described previously. This sectioning approach is much like that of Manry and Aggarwal, in which a number of lines are processed together, and different sections are processed independently. An interesting point here is that Manry and Aggarwal discussed the use overlapping sections to avoid problems at the edges. A similar approach might work well in the framework developed by Nahi and Franco. We note, however, that the vector modelling in [D-58] requires the separability of the image covariance. In fact, Nahi and Franco [D-58] and Franks [D-43] argue that a good model to be used is the exponential model

$$r(m,n) = \rho_1^{|m|}\rho_2^{|n|} \tag{D.113}$$

The necessity for using separable covariances is clearly a limitation, but it does allow one to obtain detailed results. In addition to the work mentioned above, Powell and Silverman [D-8] used the separability assumption on $r(m,n)$ to develop exact dynamic models for each line of the scalar and vector scan processes. These models involve time-delays in the output equation (due to the nonseparability of $S(k,\ell)$), and the dimension of the models increases in proportion to the width of the scan. This last fact is not surprising, since we saw in subsection D.1 that the dimension of the global state of a 2-D system grows in proportion to the extent of the plane on which the system is defined.

The recursive methods discussed so far have assumed that there is no blurring due to a nontrivial PSF. If there is such blurring, essentially we must develop a 1-D dynamical model for the effect of the blur along the scan. The simplest example of this -- motion blur along the direction of the scan -- was considered by Aboutalib and Silverman [D-24]. In the absence of noise, they design the line-by-line inverse system to remove the blur both in the space-invariant and space-variant cases. The inverse they propose is a recursive one, and hence can be implemented with relatively small computational demands. If noise is present, one augments the scalar or vector scan dynamic models of Nahi, Assefi, and Franco with the dynamic model of the blur, and uses the Kalman filter line by line (or section by section) to remove the blur and to suppress the noise. Again this system offers computational advantages over nonrecursive schemes, but the inverse system may be very sensitive to

errors in the knowledge of the PSF. The robustness properties of the Kalman filter in this case are not yet known.

All of the recursive scan techniques have basically been one-dimensional, in that no 2-D model for the image (beyond the usual covariance description) has been used. Recently, however, a number of researchers [D-6,22,34,35,71,96, 148,173,174,229,236] have considered 2-D recursive models for images. The first work along this line was that of Habibi [D-22] who considered the separable covariance function given in (D-113). Habibi noted that this covariance could be obtained from a 2-D, recursive, auto-regressive shaping filter

$$x(k+1,\ell+1) = \rho_2 x(k+1,\ell) + \rho_1 x(k,\ell+1) - \rho_1\rho_2 x(k,\ell) + \sqrt{(1-\rho_1^2)(1-\rho_2^2)}\; w(k,\ell) \tag{D.114}$$

where $w(k,\ell)$ is a white, zero mean process with

$$E[w(k,\ell)w(m,n)] = \delta_{km}\delta_{\ell n} \tag{D.115}$$

Assuming measurements of the form

$$y(k,\ell) = x(k,\ell) + v(k,\ell) \tag{D.116}$$

Habibi then developed an estimator to estimate $x(k+1,\ell+1)$ based on $\{y(m,n) \mid m\leq k,\ n\leq \ell\}$ -- i.e. this estimator is a one-step NE predictor. Habibi

chose an estimator structure of the form

$$\hat{x}(k+1,\ell+1) = \rho_2\hat{x}(k+1,\ell) + \rho_1\hat{x}(k,\ell+1) - \rho_1\rho_2\hat{x}(k,\ell) + F(k,\ell)[y(k,\ell)-\hat{x}(k,\ell)] \tag{D.117}$$

and determined a value for the gain $F(k,\ell)$. Unfortunately, this estimator is suboptimal, as pointed out by Strintzis [D-165]. The problem is that in 1-D Kalman filtering, it is well-known that in order to obtain the optimal estimate recursively, one must estimate the entire state of the process. However, as discussed in the preceding section, the global state has dimension proportional to the extent of the 2-D domain under consideration. Hence $x(k,\ell)$ is not the global state, and we cannot expect its estimate alone to suffice for recursive optimal estimation. In fact, as Morf, et.al. [D-162, 163] point out $x(k,\ell)$ is not the complete local state, and this makes the meaning of (D.117) even more questionable. Still, as Strintzis mentions, the structure of this estimator is so simple and intuitively appealing, it would be worthwhile to determine just how suboptimal it is.

The most complete study of optimal 2-D Kalman filtering has been performed by Woods and Radewan [D-173,229,236]. We assume that we have a one-sided causal dynamic model (see Fig. D.5, D.6, Equation (D.12)) for the random field

$$x(m,n) = \sum_{k=1}^{M} \sum_{\ell=-M}^{+M} b(k,\ell)x(m-k,n-\ell)$$

$$+ \sum_{\ell=1}^{M} b(0,\ell)x(m,n-\ell) + w(m,n) \tag{D.118}$$

This model can be assumed to be given or can be obtained from the image power spectral density by means of 2-D spectral factorization [D-119]. This latter method in general leads to infinite order factors which must be truncated. A third method for obtaining the model (D.118) is by direct parameter estimation using a method such as 2-D linear prediction. We will comment on methods such as these later in this section.

Woods and Radewan consider the observation equation

$$q(m,n) = x(m,n) + v(m,n) \tag{D.119}$$

where v is zero mean and white with variance R. Suppose we want to estimate $x(m,n)$ given all values of q in the past, where past is defined relative to the direction of recursion in (D.118) -- i.e. $\{q(i,j) | i \leq m-1, \text{ all } j\} \cup \{q(m,j) | j \leq n\}$. Woods and Radewan point out that this can be done optimally with an extremely high dimensional Kalman filter to estimate the global state of the system, which in this case has dimension on the order of MN (M=order of the filter, N=width of the image). In fact, a valid global state is (see Figure D.10)

$$s(m,n)' = [x(m,n), x(m,n-1), \ldots, x(1,m); x(N,n-1), \ldots, x(1,n-1);$$
$$\ldots; x(N,n-M), \ldots, x(m-M,n-M)] \tag{D.120}$$

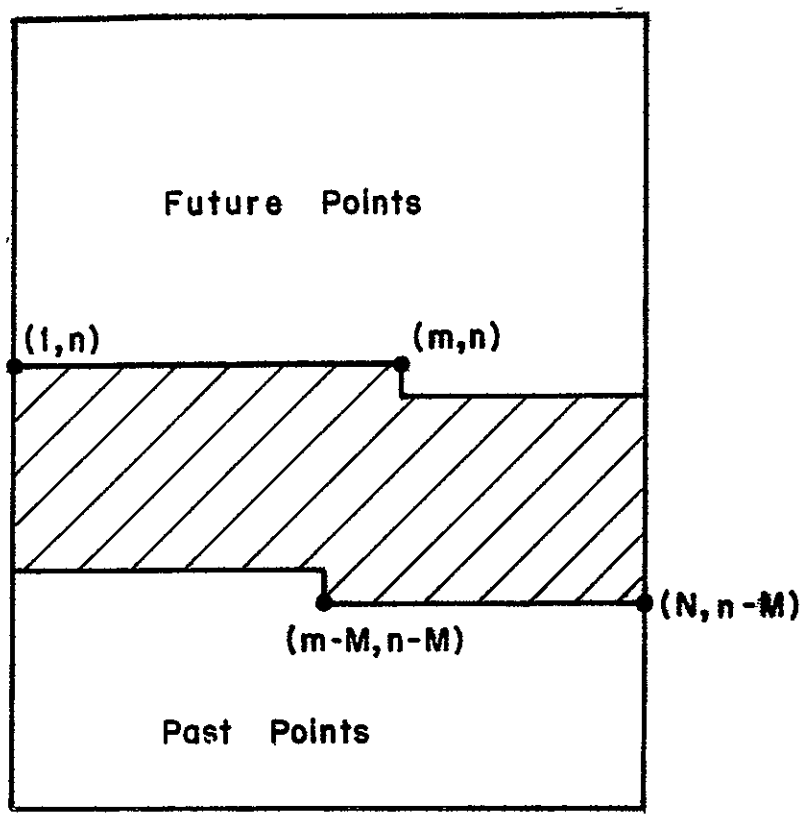


Figure D.10: Illustrating the Global State of Woods-Radewan.

By inspection of the state and the recursion (D.118), it is clear that we can write a 1-D equation for the scan-ordered process. Note that this model will be time-varying, since we must take into account the initiation of a new line. We can also write a relation between q and s

$$q(m,n) = Hs(m,n) + v(m,n) \tag{D.121}$$

where H merely picks off the first element of s. Given this development, a rather enormous Kalman filter can be written down. In addition, one can obtain a more efficient optimal estimator by processing one line of data at a time (see [D-229]).

As developed in [D-173,229,236], this filter does not correct for image blur. However, it does appear that one can modify the development so that it can. Suppose our observation is

$$t(m,n) = \sum_{j=-P}^{P} \sum_{k=-P}^{P} h(j,k)x(m-j,n-k) + \xi(n,m) \tag{D.122}$$

where ξ is additive white noise. Note that in terms of the ordering implied by the recursion (D.118), t(m,n) involves values of x that occur in the future. This can be corrected by a time delay of the observations

$$q(m,n) = t(m-P,n-P) \tag{D.123}$$

In this case, assuming $2P<M$, we may write a relation of the form of (D.121), where in this case v is a shifted version of ξ, and H is such that we obtain the proper blurring. If $2P>M$, we must increase the dimension of s -- keep

more data from the past -- in order to make sure that s contains all the components of x that affect q(m,n). We illustrate these ideas in Figure D.11. From the figure it is clear that H in (D.121) will not be constant, since we must take end-of-line effects into account, when portions of the diagonally shaded region in Figure D.11 lie outside the range of the image. This clearly can be done, and, as before, we obtain a giant Kalman filter. Another method for optimal Kalman filtering in the presence of blurring has been developed by Hart, et.al. [D-71]. They also use a global state for the filter, but they assume that the different pixels are all independent -- i.e. that all of the b(i,j) are zero in (D.118).

Optimal line-by-line Kalman filtering for images has also been considered by Attasi and his colleagues [D-6,34,35,96] using a stochastic version of the model discussed in subsection D.1. Specifically, consider noisy observations of an image f(i,j)

$$q(i,j) = f(i,j) + v(i,j) \tag{D.124}$$

where the image is assumed to be generated by a separable vector analog of the model used by Habibi [D-22]

$$\begin{aligned} x(i,j) &= F_1 x(i-1,j) + F_2 x(i,j-1) - F_1 F_2 x(i-1,j-1) + w(i-1,j-1) \\ f(i,j) &= Hx(i,j) \end{aligned} \tag{D.125}$$

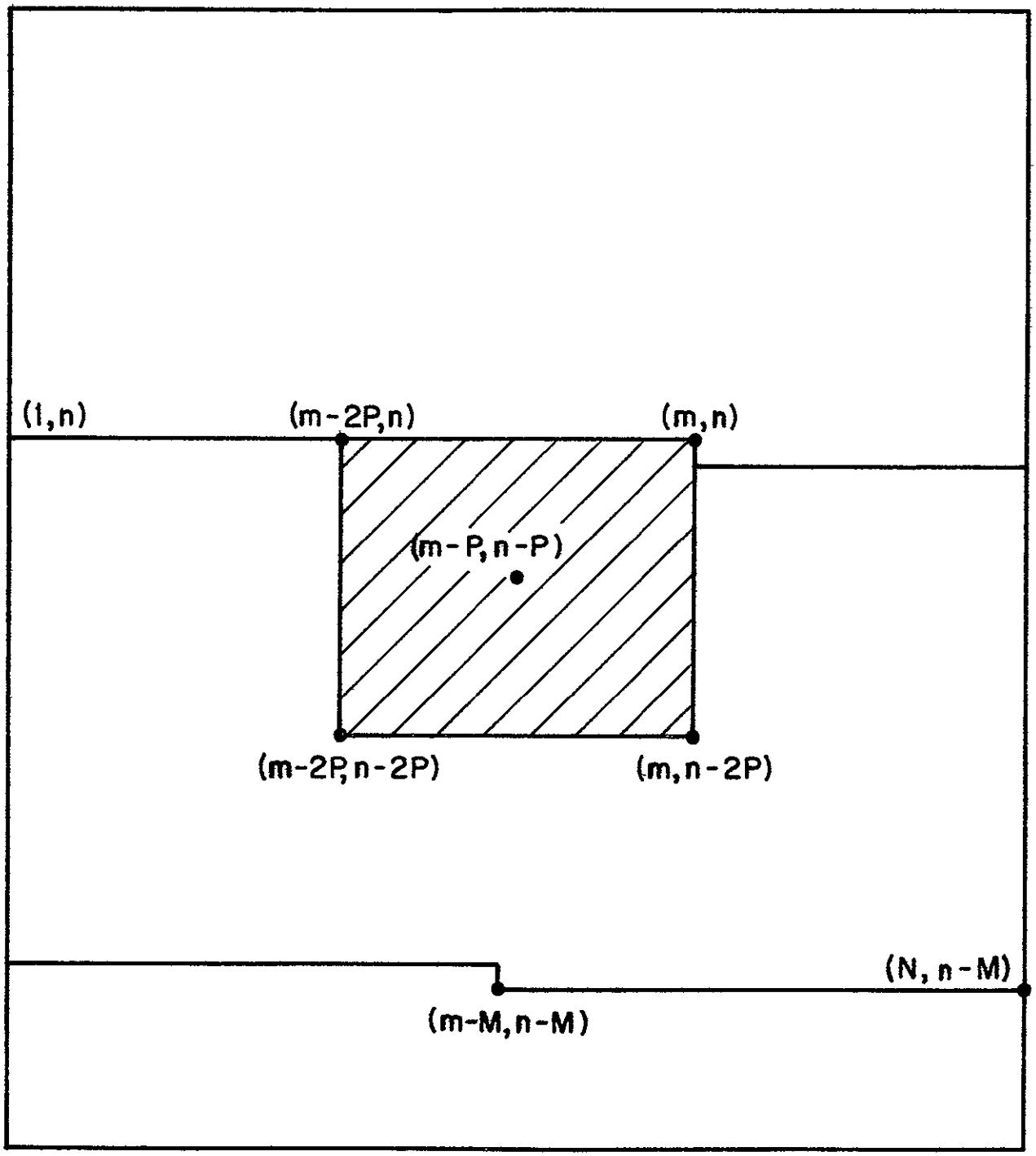


Figure D.11: Illustrating the Adaptation of the Woods-Radewan Model to Allow Blurring.

We wish to obtain the optimal estimate $\hat{x}(m,n)$ of $x(m,n)$ given $q(i,j)$ for $i \leq m$ and all j. The optimal estimate in this case consists essentially of two 1-D operations. Suppose we have $\hat{x}(m-1,n)$ for all n. We first predict ahead one line to obtain

$$\bar{x}(m,n) = F\hat{x}(m-1,n) \qquad \forall n \tag{D.126}$$

Note that each of these estimates is calculated independently. We now observe the new line of measurements $q(m,n)$ for all n, and we create the error process and the error measurement

$$e(m,n) = x(m,n) - \bar{x}(m,n) \tag{D.127}$$

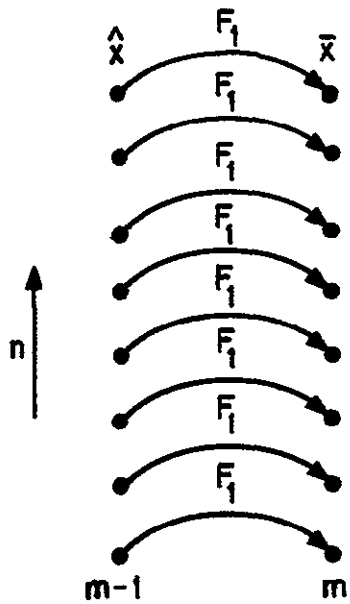
$$y(m,n) = q(m,n) - H\bar{x}(m,n) = He(m,n) + v(m,n) \tag{D.128}$$

Thus we have a 1-D estimation problem -- estimate $e(m,n)$ for all n, given $y(m,n)$ for all n. Attasi shows that one can obtain a finite dimensional 1-D realization for $e(m,n)$ as a function of n. Hence, this estimation problem reduces to the usual 1-D smoothing problem. The solution consists of two 1-D Kalman filters starting at opposite ends of the line. The estimates produced by these filters are then combined to produce $\hat{e}(m,n)$ and

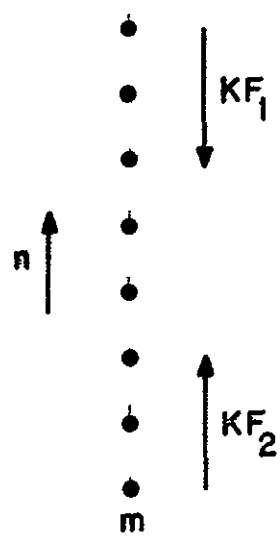
$$\hat{x}(m,n) = \bar{x}(m,n) + \hat{e}(m,n) \tag{D.129}$$

For details, we refer the reader to the references. The "geometry" of the estimator is illustrated in Figure D.12.

Let us make several comments concerning this estimator. First of all, we see that the decoupled structure of the estimator yields a far more



(a) Predicting Ahead One Column



(b) Processing the New Column of Data with Two Kalman Filters

Figure D.12: Illustrating the Structure of Attasi's Estimator

efficient estimator than that of Woods and Radewan. This is apparently due to the separability of the underlying model (D.125). Also, for this model it is not clear if we can perform the same modifications in order to incorporate blurring. This and the separbility restriction are obvious drawbacks, but the appealing structure of the filter is reason enough for further investigation, especially given the compatability of this algorithm with parallel processing techniques. Furthermore, we note that the optimal smoother can again be implemented with two filters of the type devised by Attasi -- one sweeping the columns in order of increasing m, and the other in order of decreasing m. Again, this is reminiscent of the decomposition of zero phase filters into two half-plane filters [D-42,119].

The method of proof used by Attasi involves the taking of z-transforms along the n direction and the treatment of m as a time variable. Essentially we are regarding the 2-D system as a high-dimensional (infinite if the domain of n is unbounded) 1-D system, where we can use a spatial transform "along" the 1-D state vector in order to simplify the calculations. The key step in Attasi's development is a derivation of a set of Riccati equations, parametrized by the transform variable z, for the power spectral density $S_m(z)$ of e(m,n) considered as a function of n. One can then factor these spectra to obtain the 1-D realizations of the e's. As Attasi points out, the dimension of the realization for e(m,n) is on the order of m times the dimension of x -- i.e. it grows linearly with m. One can avoid this difficulty by using reduced order estimators. For example we may choose to use the steady-state filter,

in which case we can obtain a finite dimensional system whose spectrum approximates $S_\infty(z)$.

Let us note that Attasi's work brings out several crucial issues. Specifically we have seen the effective use of the equivalent representations of signals as multivariable 1-D and scalar 2-D. We have also seen that transforms along one of these variables can be useful in obtaining solutions. Later in this section we will discuss the relation between 2-D processing and distributed and decentralized control. The issues just mentioned will be of great importance then as well.

As we have seen, optimal 2-D Kalman filtering algorithms require large amounts of storage and computation. Thus, a number of researchers [D-34,148, 173,174,229,236] have developed suboptimal estimators that require less computation. We will briefly describe several of these and refer the reader to the references for more on this subject. Let us begin with the technique of Woods and Radewan [D-173,229,236]. They consider two types of suboptimal filters. The first involves breaking the picture up into strips of width $W<N$. One then processes across and up these strips individually, much as with the Manry-Aggarwal section-scan. This reduces the dimension of the global state, as we replace N in (D.120) with W. Woods and Radewan also suggest overlapping the strips in order to avoid the edge effects caused by incorrect boundary conditions between strips.

The other suboptimal filter developed in [D-229] is the reduced update Kalman filter. Examining the optimal filter of Woods and Radewan, we see that

the predict cycle is computationally straightforward -- one simply uses the recursion (D.118) assuming no noise and using preceding estimates. The measurement update part of the optimal filter, on the other hand, involves updating the estimates of all of the components of the state. Assuming N>>M, we expect that a given pixel is most correlated only with a small percentage of the elements of the state vector. Therefore, it seems reasonable only to update the estimates of those components of the state that are within a certain distance of the point being processed. This should greatly simplify the filter with minimal effect on performance. In other words, we are essentially designing a constrained Kalman filter in which we constrain many of the gain elements to be zero and essentially allow only "near neighbor updates." We remark that a similar idea was proposed by Pratt [D-17] for the Wiener filter and by Murphy and Silverman [D-174] in the Kalman filtering context. In addition, it is interesting to note that similar ideas have been proposed for large-scale systems in which measurements on a particular subsystem are used to update only those subsystems that are "near" to it as determined by some measure of dynamic interaction (see, for example, [D-201,205,208] for related results for problems of freeway traffic control and estimation). We will have more to say about this later.

Motivated by the simplicity of the filter proposed by Habibi [D-22] and by the recursive local state-space model proposed by Roesser [D-110], Barry, et.al. [D-148] have developed a class of constrained filters. Specifically, they consider a noisy version of Roesser's model (D.36)

$$\begin{bmatrix} v(i+1,j) \\ h(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} v(i,j) \\ h(i,j) \end{bmatrix} + w(i,j) \tag{D.130}$$

$$y(i,j) = C_1 v(i,j) + C_2 h(i,j) + \nu(i,j) \tag{D.131}$$

where w and ν are white noise processes. Their suboptimal estimator is then taken to be the optimum estimator of the form

$$\begin{bmatrix} \hat{v}(i+1,j) \\ \hat{h}(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \hat{v}(i,j) \\ \hat{h}(i,j) \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \left(y(i,j) - [C_1, C_2] \begin{bmatrix} \hat{v}(i,j) \\ \hat{h}(i,j) \end{bmatrix} \right) \tag{D.132}$$

All of the recursive estimators that we have examined up to this point have had two things in common -- they have involved discrete 2-D space and have used recursive random field models. Recently Wong [D-172,187] reported on some work on 2-D continuous-space estimation. This theory involves the development of a stochastic calculus in 2-D, and this in turn has led to a number of interesting theoretical results. We defer the discussion of this topic until later in this subsection.

At this time it is worth mentioning that there has been work performed on recursive processing of fields that come from nonrecursive models. Specifically, Jain and Angel [D-32] have considered fields described by a nearest neighbor, interpolative equation

$$x(m,n) = \alpha_1 [x(m,n+1) + x(m,n-1)]$$
$$+ \alpha_2 [x(m+1,n) + x(m-1,n)] + w(m,n) \tag{D.133}$$

Fields of this type have been studied by several authors and were proposed by Woods [D-9] as the prototype of discrete, 2-D Markov fields. We will have more to say about the properties and other uses of these fields in a short while. For now, we concentrate on the estimation problem when we observe

$$y(m,n) = x(m,n) + v(m,n) \tag{D.134}$$

Following [D-32], let us consider the vector scan process -- i.e. we process an entire line of data at a time. Define the resulting 1-D vector processes x_m, y_m, w_m, and v_m. For example

$$x_m = \begin{bmatrix} x(m,1) \\ \vdots \\ x(m,N) \end{bmatrix} \tag{D.135}$$

Then, one can write (D.134), (D.135) as

$$x_{m+1} = Qx_m - x_{m-1} + w_m \tag{D.136}$$

$$y_m = x_m + v_m \tag{D.137}$$

where Q is a symmetric, tridiagonal, Toeplitz matrix

$$Q = \frac{1}{\alpha_2}\begin{bmatrix} 1-\alpha_1 & 0 & \cdots & \cdots & 0 \\ -\alpha_1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -\alpha_1 \\ 0 & \cdots & \cdots & 0 \; -\alpha_1 & 1 \end{bmatrix} \tag{D.138}$$

Examining the structure of (D.138), one is tempted to utilize the same type of circulant approximation as that used by Andrews and Hunt [D-81] in order to diagonalize the system efficiently with the aid of the FFT. However, as Jain and Angel point out, the diagonalization of Q

$$M'QM = \mathrm{diag}(\lambda_1,\ldots,\lambda_N) \tag{D.139}$$

can be performed with the aid of the FFT without any approximation. Thus, if we define the transformed quantities $\bar{x}_m$, $\bar{y}_m$, etc., where, for example,

$$\bar{x}_m = M'x_m \tag{D.140}$$

we obtain a set of N <u>decoupled</u> estimation problems, indexed by j (which indexes the components of the transformed vectors):

$$\hat{\bar{x}}_{m+1,j} = \lambda_j \bar{x}_{m,j} - \bar{x}_{m-1,j} + \bar{w}_{m,j} \tag{D.141}$$

$$\hat{\bar{y}}_{m,j} = \bar{x}_{m,j} + \bar{v}_{m,j} \tag{D.142}$$

Each of these problems can be solved using a second-order Kalman filter (see [D-32] for an alternative method of derivation), and we obtain the efficient implementation illustrated in Figure D.13. Again, if one wishes to utilize all of the data to estimate each pixel, we can implement the smoother by including a second bank of filters which sweeps the lines in the opposite direction (m runs from N to 1). One can also implement a one step smoother -- which estimates x_m based on data through line m+1. This requires only one back of filters, as in Figure D.13. We refer the reader to [D-32] for details.

The approach in [D-32] deserves some comment. Again as in Attasi's work, we have seen that transforming variables in one dimension and processing in the other can lead to extremely efficient processing schemes.[24] Just as with the block circulant approach of Andrews and Hunt, the spatial stationarity of the 1-D equation (D.136) is such that the FFT can be used to great advantage. This observation leads one to seek other formulations that possess structure that can be exploited in this manner. Jain and Angel mention several other random field models that lead to symmetric, tridiagonal, Toeplitz evolution equations when scanned line by line, and in [D-30] Jain uses similar analysis for the efficient recursive filtering of one of these models, the so-called semicausal model:

$$\begin{aligned} x(m,n) = {} & \alpha_1 [x(m-1,n) + x(m+1,n)] \\ & -\rho\alpha_1 [x(m+1,n-1) + x(m-1,n-1)] \\ & + \rho x(m,n-1) + w(m,n) \end{aligned} \tag{D.143}$$

[24] Recall that the use of a transform in one direction followed by linear prediction in the other was proposed as an image coding scheme by Habibi and Robinson [D-37].

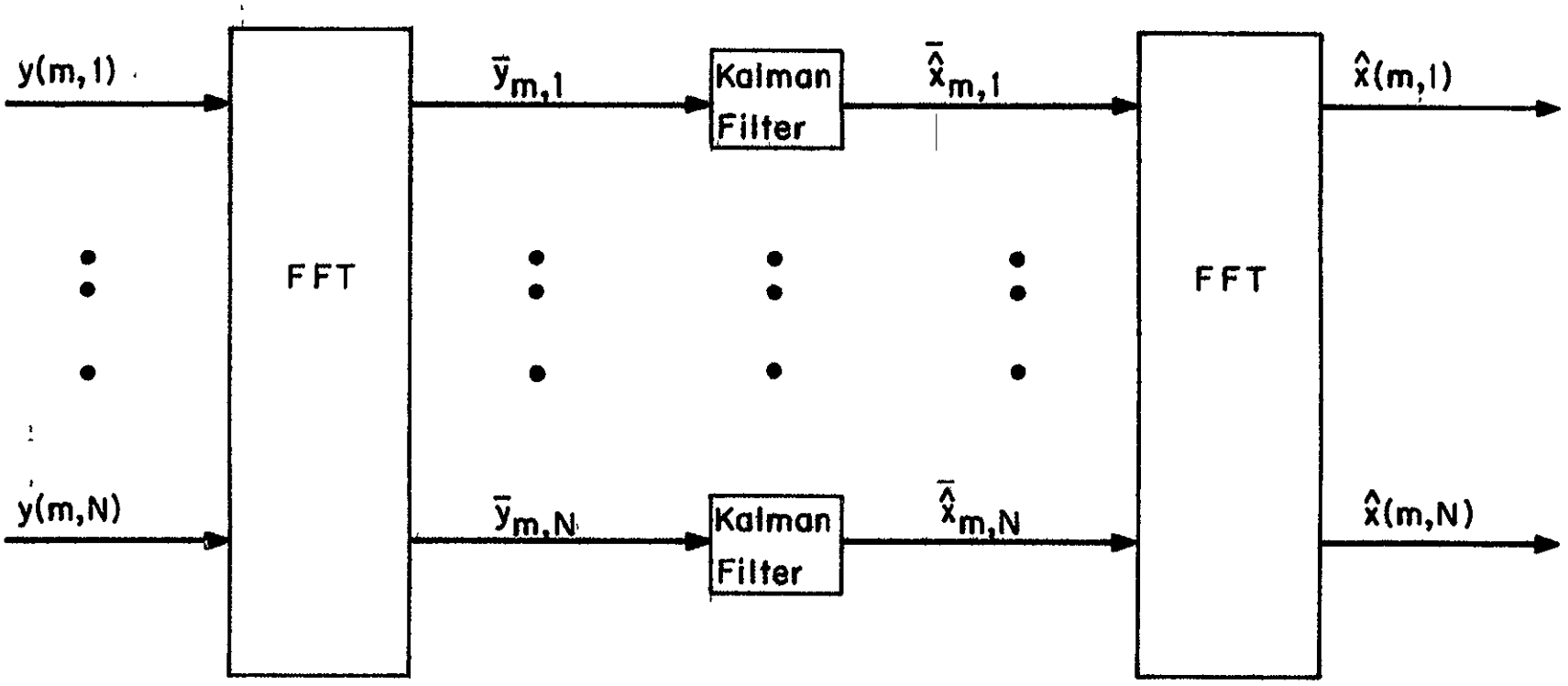


Figure D.13: Illustrating the Optimal Filter of Jain and Angel

The model was given this name since $x(m,n)$ depends only on $x(i,j)$ with $j \leq n$ (note, however, that (D.143) is not recursive). We note that throughout the development in [D-30,32] it is assumed that no blurring occurs. It is not clear if the approach adopted in these references can be extended to include the effect of a PSF, but the efficiency of the algorithms developed by Jain and Angel indicates that it is certainly worth trying to find such an extension. As we shall see, the use of structure in this manner can be applied in a number of different settings.

We have now surveyed a number of nonrecursive and recursive estimation methods, and the techniques discussed to this point deserve some comment. The recursive techniques come with many of the same criticisms that were made concerning nonrecursive filters. They require detailed models of the image statistics and image formation process, and they are essentially based on the MMSE criterion. Hence, they in general, will sacrifice resolution in favor of noise suppression. In addition, these recursive techniques necessarily affect the image because of the assumed model structure. The effect of this in some cases (such as in the Kalman filter based on a stationary approximation to the scanned image) may require additional processing (of the transposed image, for example), while in other cases, such as the 2-D causal models of Woods-Radewan and Attasi or the noncausal models of Jain and Angel, the effects may not be so noticeable. We have seen that some of the recursive techniques allow the inclusion of image blur, while in other cases the extensions to include blur have yet to be developed. Also, we have seen that in some cases optimal Kalman filtering is extremely complex, and suboptimal,

but intuitively appealing, recursive filter structures must be used. In other cases -- specifically in the work of Attasi and Angel and Jain -- we have observed that the use of the structure of the assumed model can lead to extremely efficient optimal estimation algorithms. In addition, although work in this area has been limited in extent [D-24,174], the recursive techniques are directly amenable to the analysis of space-varying and nonstationary models. Thus, in spite of the many qualifications, we find enough positive attributes to warrant continued study of recursive techniques for image restoration.

Let us now comment and speculate on several aspects of image processing that we have only mentioned in passing previously. First of all, we have the problem of nonlinearities in image sensing. Consider first the multiplicative noise model (D.63). As discussed in [D-82,E-2] and in Section E, one can often filter signals corrupted by multiplicative noise by first taking the logarithm, then filtering with a linear system, and then exponentiating. This process -- an example of homomorphic filtering -- is described in Section E. We note here that this technique has been applied with great success [D-82,E-2], and in [D-82] it is argued that this type of processing is extremely compatible with the response characteristics of the human visual system.

Equation (D.64) illustrates another kind of measurement nonlinearity, in which the noise is additive but the signal is distorted in a nonlinear fashion. Hunt [D-4,81] has studied such image processing problems in the context of nonrecursive restoration techniques. Specifically, he has devised

an iterative scheme for computing the maximum a posteriori image estimate given the observations. In the case of linear measurements, this reduces to the Wiener filter. The analog of this technique for the recursive methods is the extended Kalman filter (EKF), which essentially involves a continual relinearization about the present best estimate. This method can readily be derived for all of the recursive methods discussed. The interested reader is referred to [D-21] for a discussion of this method in the context of the Nahi-Assefi scalar-scan recursive technique. There are, of course, many other nonlinear 1-D recursive estimation techniques besides the EKF, and most of these can be applied in this framework. For an example of one other such technique (again applied to the Nahi-Assefi method), we refer the reader to [D-75].

Another issue that we have mentioned on several occasions is the incorporation of constraints, such as the positivity of the image estimate, into the estimation procedure. As mentioned earlier (see footnote 21) in many cases we needn't worry about this constraint explicitly.[25] However, it is worth understanding the implications of such constraints. Andrews and Hunt [D-81] consider the constrained least squares formulation together with the additional positivity constraint. In this case there is no closed-form solution, and iterative nonlinear programming methods must be used. Mascarenhas and Pratt [D-23] also consider the incorporation of upper bounds on pixel intensities in order to improve the conditioning of the restoration problem, and similar types of bounds on the pixels and on the values of the

[25] And for homomorphic techniques we have no reason to worry at all, since exponentiation at the end guarantees positivity.

PSF (assumed unknown in this case) were considered by MacAdam [D-39]. In the case of recursive techniques, one can also include positivity constraints. In [D-239] Jain discusses a recursive, iterative method for incorporating this constraint into the Nahi-Assefi model. Thus, we see that constraints such as these can be incorporated into the methods discussed previously. The cost is a great increase in computational complexity, and it is not clear that it is worth the trouble.

A third problem area with many of the restoration techniques is in the reliance on a priori information. As mentioned earlier, one often can assume knowledge of the PSF or can determine it by observing known test scenes through the imaging system. In other cases, we may not have such information and must estimate the PSF as well as the image. Based on the assumption that the extent of the PSF is far less than that of the image, Stockham, et.al. [E-4] suggest a "blind homomorphic deconvolution" procedure, in which one breaks the received image into pieces, takes 2-D transforms and the logarithm of the transforms, and then averages over the various pieces. This, combined with the specification of a prototype transform (corresponding to the average of the logarithm of the transform of the original image) allows one to estimate the PSF and the other parameters needed for the geometric mean filter described earlier. We refer the reader to [E-4] for details.

The question of parameter uncertainty is clearly of major importance for the various recursive techniques, all of which require a great deal of a priori information. Thus one important question concerns the robustness of

these techniques in the face of modelling errors. As mentioned in Section C, techniques do exist for the sensitivity analysis of 1-D state-space models and 1-D Kalman filters (see [A-65,C-23]). Can we extend these methods to the 2-D case, and how well do the 2-D algorithms perform? Is there any way to make them more robust? In addition, methods abound in 1-D for on-line parameter identification and adaptive estimation in the presence of unknown parameters (see the various techniques described in Section B). Can we apply these methods with any success to the 2-D problem? The successes of such methods in 1-D and the several appealing features of 2-D recursive estimation techniques make these worthwhile questions for future research.

A final area of concern is the resolution-noise suppression tradeoff. As mentioned earlier, the human visual system is willing to accept more noise in certain regions, such as edges, in order to improve resolution. Thus, in relatively slowly varying regions of the image, we would like to remove noise, while where there are abrupt scene changes or other high frequency fluctuations of interest, we would prefer to forego noise suppression in favor of resolution. Backus and Gilbert [D-78] (see also [D-19]) have devised a nonrecursive technique for taking this tradeoff into account. They define a quantitative measure of the blur induced in the image by filtering. Then for any given value of this measure, one can determine the restoration scheme that minimizes the effects of noise subject to this constraint. We refer the reader to [D-19,78] for details (see also [D-79]). Anderson and Netravali [D-99] have developed another nonrecursive approach involving a performance index that provides a tradeoff between blur introduced by the filter and the level of

noise suppression. Their criterion utilizes the results of certain psychovisual experiments that were designed to measure the relative importance of a unit of noise in high and low contrast conditions, but the evidence is still inconclusive as to whether or not a standard measure can be obtained for a large class of images. In addition to these methods, we refer the reader to [D-38,44,225] for discussions of several other nonrecursive image enhancement techniques.

In the context of simultaneous image enhancement and noise suppression, an important problem involves the detection of edges or boundaries between different regions in an image. Within each of these regions one may be able to utilize one of the restoration techniques developed earlier, and in this manner we can suppress noise while preserving the resolution of the boundaries. We also note that in many applications the determination of the boundaries themselves may be the key issue [D-175]. In recent years a variety of techniques have been developed for detecting and recognizing various types of boundaries in 2-D data. Many of these methods are based on pattern recognition techniques [D-243], and we will not discuss them here. We simply refer the reader to several references on this subject, [D-15,210,240].

In 1-D, a variety of recursive techniques have been developed for estimation and detection of abrupt changes in signals (see [B-103] for a survey of many of these). These techniques have been successfully applied in a wide variety of applications, including automatic detection of cardiac arrhythmias [B-104] and the detection of sensor and actuator failures [B-103]. An important question then is the extension of methods such as these to the detection of

boundaries in images. To a large extent this remains an open problem, but there has been some work along these lines. Specifically, Nahi and Habibi [D-25] considered the problem of the detection of an object superimposed on a background scheme. Their approach involved the modification of the methods of Nahi [D-18,21,58] and of Habibi [D-22] to incorporate a binary variable that indicates whether a particular pixel is in the object or in the background. The scheme devised in [D-25] involves the recursive calculation of likelihood ratios for the existence of boundaries, and it also incorporates the use of a bank of two filters (based on object and background statistics, respectively) for the suppression of noise once the boundaries have been determined. In [D-175] Nahi and Lopez-Mora were primarily concerned with the estimation of the boundary. Here, the 1-D Markov scan model of [D-18,21,58] is augmented to include several states used to model the boundary. As the resulting model is nonlinear, a nonlinear estimation scheme is employed, and some promising results are presented in [D-175]. These results notwithstanding, a great deal of work remains to be done in the development of recursive methods for the detection of boundaries in images. It is our feeling that this may prove to be one of the most important uses of 2-D recursive estimation techniques

We now turn our attention to the detailed analysis of statistical and probabilistic models for random fields. Applications for such techniques extend far beyond image processing into fields such as seismic signal processing [D-68,70,199,209,216,227,245], gravity mapping [D-1,211,212,224], meteorology

and atmospheric modelling, [D-69,214,231], biomedical imagery and image reconstruction [D-11,29,213,223], modelling of scattering fields [D-222,232-235], modelling of the distribution of earth resources [D-15,61,246], analysis and modelling of turbulence [D-217], and the modelling and analysis of random transport and wave propagation phenomena [D-170,189,193,194,215,220,226]. With such a wide variety of potential applications, there clearly is a need for a general methodology for the analysis of random fields. Much has been done in this direction, but, as with all multidimensional topics, much remains to be done. We will describe some of the work that has been done, will touch on several of the applications mentioned above, and will speculate on some open questions.

Motivated to a great extent by their utility in 1-D, many researchers have investigated the extension of the concept of a Markov process to several dimensions. Perhaps the first of these was developed by Levy for continuous parameter spaces [D-12,197,198,230]. The situation in two dimensions is depicted in Figure D.14. Suppose we have a 2-D random field $f(x,y)$. Then f is called Markov of degree p if it essentially has the following property:[26] let ∂G be any smooth closed curve encircling the origin and separating the plane into the "past" (G_P), the "present" (∂G), and the "future" (G_F); then, given f and its first $p-1$ derivatives at the present, the values of f in the future are independent of the values of f in the past. The field f is called Markov if it is Markov of degree 1. This definition is quite intuitive, and one can imagine fields in a variety of physical situations that have this type

[26] We say "essentially" here since f may not be differentiable. For the technically precise definition, we refer the reader to the references.

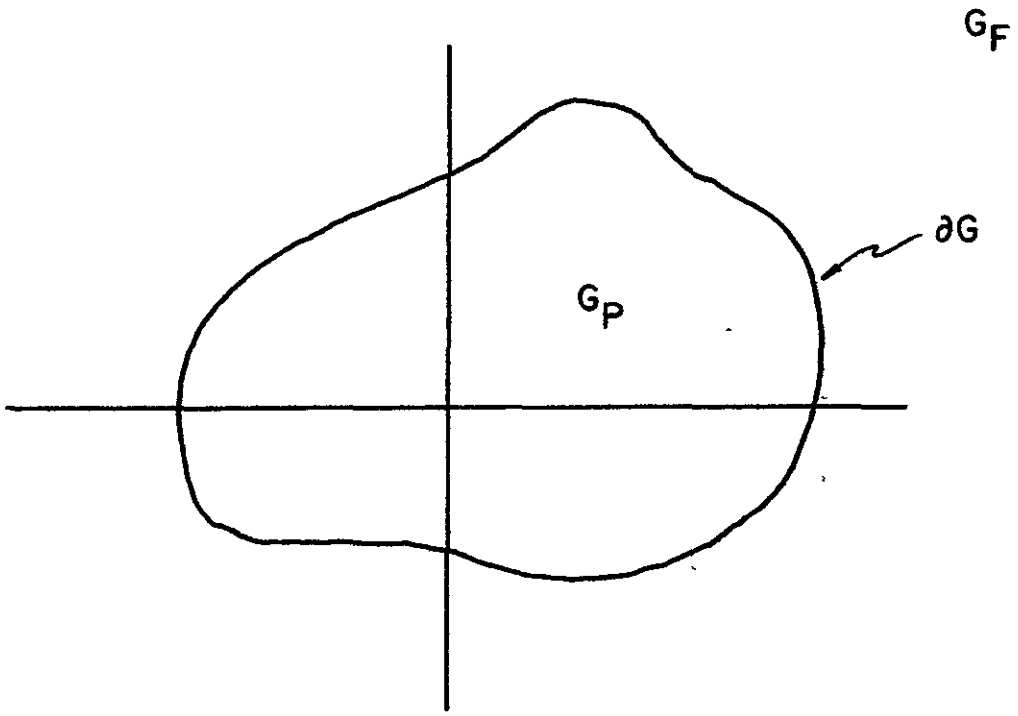


Figure D.14: Illustrating Levy's 2-D Markov Property

of radial causality.

Levy also defined a multidimensional Brownian motion process $x(t)$, $t = (t_1,\ldots,t_d)$, which is a Gaussian process with statistics

$$E[x(a)] = 0 \tag{D.144}$$

$$E[x(a)x(b)] = \frac{1}{2}\left(|a| + |b| - |b-a|\right) \tag{D.145}$$

(here $|\cdot|$ is the usual Euclidean distance). McKean [D-198] showed that for d odd, $x(t)$ is Markovian of degree $(d+1)/2$, and for d even, $x(t)$ has <u>no</u> Markovian property. Since Brownian motion and its Markovian properties proved to be so useful in developing 1-D tools of stochastic analysis, the above result is disappointing. This disappointment is in fact compounded by the analysis of Wong [D-12] who showed that there are essentially no continuous Gaussian random fields in two or more dimensions that are simultaneously stationary, isotropic (the covariance function is invariant if we rotate the coordinates of the parameter space), and Markov (of degree one). Thus it is evident that this setting will not lead to a useful multidimensional stochastic calculus for the study of random fields. To do this, we must turn to a recursive formulation, and we shall do this shortly.

It is interesting to note that the analog of Levy's notion for discrete space systems, as developed by Woods in [D-9], leads to far more useful results. Stationary Gaussian fields of this type can be generated by interpolative filters of the form[27]

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[27] Such models have been considered by several authors including Whittle [D-61] and Larimore and Beavers [D-1].

$$x(n,m) = \sum_{D_p} h(k,\ell)x(n-k,m-\ell) + u(n,m) \tag{D.146}$$

where $u(n,m)$ is stationary, and

$$D_p = \{(k,\ell)\,|\,k^2+\ell^2 \leq p^2,\ k,\ell \text{ not both } 0\} \tag{D.147}$$

$$E[x(n,m)u(k,\ell)] = c\delta_{nk}\delta_{m\ell} \tag{D.148}$$

$$E[u(n,m)u(0,0)] = \begin{cases} c & m=n=0 \\ -h(m,n)c & (m,n)\varepsilon D_p \\ 0 & \text{otherwise} \end{cases} \tag{D.149}$$

Thus, we see that the driving noise in this case is not white but is finitely correlated.

We have already seen in the work of Jain and Angel [D-32] that interpolative models can be used for efficient recursive estimation of random fields. Such models also have several other uses. One possibility is in the area of spectral estimation. In [D-237] Woods proposes the fitting of observed correlation data to an interpolative Markov model. In this case one again obtains a set of normal equations for the coefficients of the model that yields the minimal interpolation error in a least squares sense. Unfortunately, as Woods points out, these equations cannot be inverted efficiently as in the 1-D linear prediction case, and Woods [D-237] proposes a complex algorithm for obtaining the desired spectral estimate.

Fortunately, as we have seen in [D-32], nearest neighbor models have a great deal of structure that can be exploited to obtain efficient computational schemes. In [D-31] Jain proposes a nearest neighbor interpolative filter for imaging coding. Basically, Jain assumes a separable, stationary, isotropic model for the image

$$E[x(n,m)x(0,0)] = \rho^{|n|+|m|} \tag{D.150}$$

and in this case he finds the optimum first order (p=1 in (D.147)) interpolative error filter. In this simple case, one can solve the normal equations by inspection. Having this filter, one can consider a coding scheme in which we transmit only the interpolation error. Thus, the decoder must essentially solve the interpolative, and hence nonrecursive, equation. In general, this is a difficult task in its own right. However, one can use techniques analogous to those in [D-32] to perform the reconstruction efficiently. That is, we can consider reconstructing the image line by line, and the resulting vector equations display the same type of tridiagonal structure that was exploited earlier in the development of an efficient restoration scheme. Similarly in this case we can also use FFT algorithms for efficient reconstruction. In addition, as discussed in [D-241,242], the use of interpolative models leads to efficient Karhunen-Loeve transform coding using the FFT.

Thus, we have seen that interpolative models have a number of appealing properties. They also have their drawbacks, such as in efficient spectral

estimation,[28] and one is naturally led to seek other models and statistical methods for fitting 2-D data to parametric forms for such models. One immediate generalization from 1-D that we have mentioned before is the development of 2-D linear prediction techniques -- i.e. the identification of 2-D, causal, autoregressive models by means of least squares predictive error filter design. Immediately we see that one problem that arises is the choice of the direction of recursion for the AR model -- i.e. which elements of the field will be used to predict which other elements. Another problem is the stability of the resulting filter, which is guaranteed in 1-D but not in 2-D, as Genin and Kamp have pointed out [D-145] (see also the work of Marzetta [D-66,67]). In addition, even if stability is not a problem, one faces the question of finding efficient algorithms for the solution of the normal equations that specify the filter parameters -- i.e., is there a fast 2-D Levinson algorithm?[29] In [D-145] Genin and Kamp develop sets of recurrence relations for 2-D orthogonal polynomials. Can these relations be used to devise fast algorithms as they can in 1-D (see Section B)?

The above questions remain open in general, but recently Marzetta [D-67] developed a fast algorithm for 2-D linear prediction that involves the use of 1-D techniques and the same 2-D, scalar/1-D, vector interplay that we have seen before. Consider the situation depicted in Figure D.15a. We have a stationary 2-D field $x(k,\ell)$, and we wish to predict $x(m,n)$ based on the array of $x(i,j)$ to the SW that are indicated in the figure. We do this in two steps. First,

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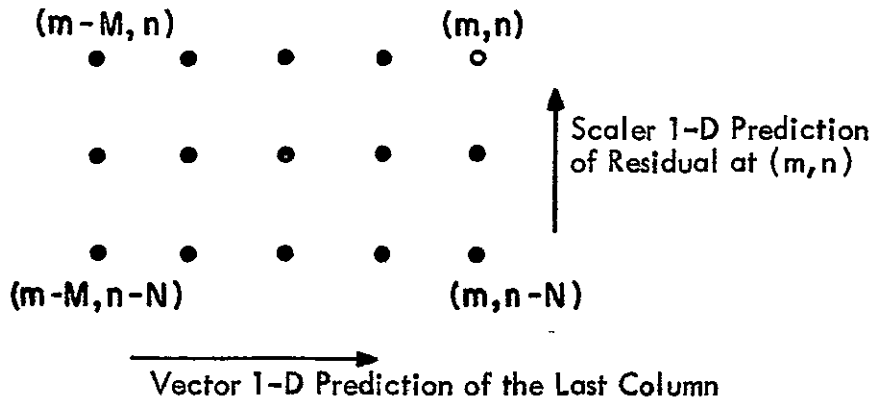
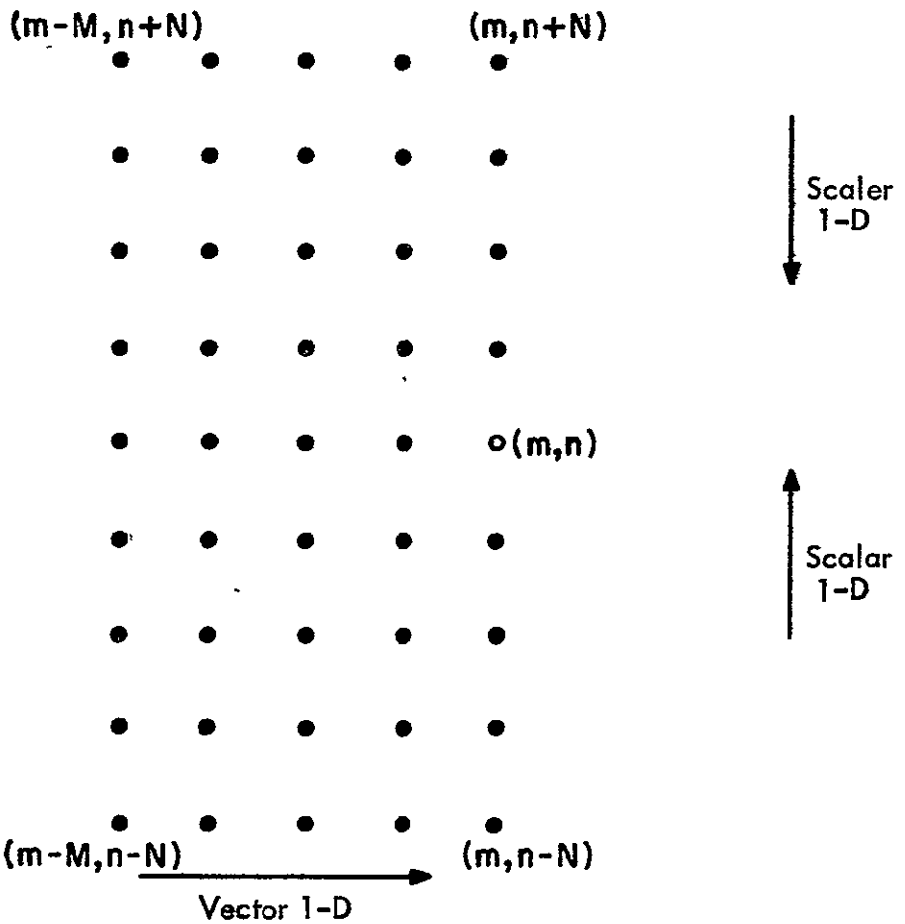
[28] See also [D-62] for another difficulty that arises with such discrete-time, nonrecursive, 2-D Markov models.

[29] A related question, given the perspective of Section B, is the existence of fast algorithms for the calculation of the gains of recursive 2-D Kalman filters.



(a) Illustrating Marzetta's Fast Algorithm



(b) Does a Fast Algotirhm Like This Exist?

Figure D.15: Known and Conjectured Fast Algorithms for 2-D Linear Prediction and Interpolation.

regarding each column as a 1-D vector, we use the preceding columns to predict the mth column. This can be done with standard fast algorithms for vector 1-D linear prediction. In fact, in this case we can effectively use the faster scalar 1-D algorithm since the block Toeplitz matrix to be inverted is in fact Toeplitz because of vertical stationarity. Having completed this step, we compute the prediction errors in the last column and use these to predict the error at (m,n) by performing a scalar 1-D prediction to the North. This algorithm bears a striking resemblance in style to that of Attasi (see Figure D.12). The only difference is that Attasi uses two 1-D Kalman filters -- one North, one South -- to perform a smoothing along the last column. This observation leads us to speculate on the existence of a fast algorithm for linear interpolation for the semicausal [D-30] structure illustrated in Figure D.15b.

Identification of parametric 2-D models has attracted the attention of several statisticians over the years [D-1,59,61], and several of their results are definitely worth noting. Whittle [D-61,63] was one of the first researchers to consider the properties of 2-D stationary processes. One of the topics he considered was the "unilateral" representation of a 2-D process, which is simply a half-plane recursive representation of a given field. Using a method exactly along the lines developed by Dudgeon [D-33,102], Ekstrom and Woods [D-119], and Marzetta [D-66], Whittle obtained an in general infinite order representation of this type by factoring the 2-D power spectral density of the process. In addition, in [D-61] Whittle also relates various recursive and nonrecursive autoregressive discrete-space models to analogous stochastic

partial differential equations. Such equations were examined by Heine [D-60], who examined the properties of linear stochastic equations of the parabolic, elliptic, and hyperbolic forms. Whittle noted that the nearest neighbor model corresponds to an elliptic equation for which Heine showed that the correlation function takes the form of a modified Bessel function of the second kind. Whittle then uses this fact to argue that in the discrete space-case, such correlation function forms are preferable to decaying exponentials.

In addition to considering these issues, Whittle also discussed the maximum likelihood and least squares estimation of the parameters of a 2-D autoregressive model. This subject is also considered in far greater detail by Larimore and Beavers [D-1], and the results bring to light a rather important point. In 1-D, assuming Gaussian statistics, finding the maximum likelihood parameter estimates is equivalent to finding the parameters of an inverse filter that yields the least squares prediction error -- i.e., the log-likelihood ratio is up to an additive constant, proportional to the negative of the sum of squared estimation errors. In the 2-D problem, <u>this is not the case</u> if the field model is not causal. This is due to the fact that in this case the Jacobean of the transformation from prediction errors to the field is not unity and is, in general, a rather complicated function of the parameters. This greatly complicates parameter and spectral estimation, as we already noted in discussing the work of Woods [D-237]. We refer the reader to [D-1] for details of the problem of 2-D parametric model identification and for the consideration of other problems, such as the design of a 1-D shaping

filter for the part of a 2-D field observed by a point tracing a path in the plane. This problem is of great practical value in problems such as accurate inertial navigation and gravity field estimation [D-1,193,194][30]

As discussed earlier, in 1-D the use of stochastic calculus greatly facilitates the analysis of continuous-time random processes. It seems natural, then to attempt to extend concepts such as the Markov property, Brownian motion, and stochastic calculus to 2-D. We have seen, however, that the intuitively appealing approach of Levy does not provide a useful framework, and the reason for this is the lack of causality in this framework. Specifically, in 1-D the basic tools of analysis of Brownian motion, Poisson processes, stochastic differential equations, etc., essentially are based on the principles of martingale theory (see, for example, [D-247]). Simply put,[31] a martingale $M(t)$ is a 1-D random process such that for $t>s$ the best estimate $M(t)$ given $M(\tau)$, $\tau \leq s$ is $M(s)$:

$$E[M(t) \mid M(\tau),\ \tau \leq s] = M(s) \tag{D.151}$$

To extend these notions to 2-D, we immediately run into a problem: what does $t>s$ mean? That is, we must be able to specify at least a partial order

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[30] The problem of modelling random perturbations in gravitational fields has been considered by a number of authors [D-211,212,224]. A common approach to this problem is the use of the spatial transform most appropriate for such problems--spherical harmonics. Wong [D-12,230] has considered such transform methods in the general setting of isotropic random fields on spaces with constant curvature. The use of geometric concepts such as spherical harmonics greatly facilitates the analysis of random fields. We also refer the reader to the work of Swerling [D-10], in which many of the statistical properties of random contours are discussed at some length.

[31] The following discussion is greatly oversimplified, and we refer the reader to the references for the full story.

on the plane, and, as discussed in Subsection D.1, this can be done if we impose a causal structure on the processes considered.

In recent years 2-D martingales (and higher dimensional generalizations) with a NE causal structure have been investigated by a number of authors [D-172, 176,179-187,191,192,196]. Basically, we consider processes $M(z_1,z_2)$ defined on the NE quadrant, on which we place the partial order

$$(z_1,z_2) \succ (\xi_1,\xi_2) \iff z_i \geq \xi_i, \; i=1,2, \tag{D.152}$$

Then $M(z_1,z_2)$ is a (NE) martingale if whenever $(z_1,z_2) \succ (\xi_1,\xi_2)$

$$E[M(z_1,z_2) \,|\, M(s_1,s_2),\; (\xi_1,\xi_2) \succ (s_1,s_2)] = M(\xi_1,\xi_2) \tag{D.153}$$

The relevant geometry is depicted in Figure D.16. Here $\mathscr{F}(\xi_1,\xi_2)$ denotes the set of all $M(s_1,s_2)$ with $(s_1,s_2) \succ (\xi_1,\xi_2)$.

Having this framework one can then begin to develop all of the tools for a usable 2-D stochastic calculus. The results obtained indicate that such a calculus can be developed, but it is not without its surprises and limitations.[32] One of the major surprises is that given a NE Martingale, the lack of a total order leads directly to the construction of a second martingale, and, in fact this second martingale, which in some sense involves products of the original martingale at unordered points, is essential to the

[32] The same comment can, of course, be made with regard to just about any topic in 2-D system analysis.

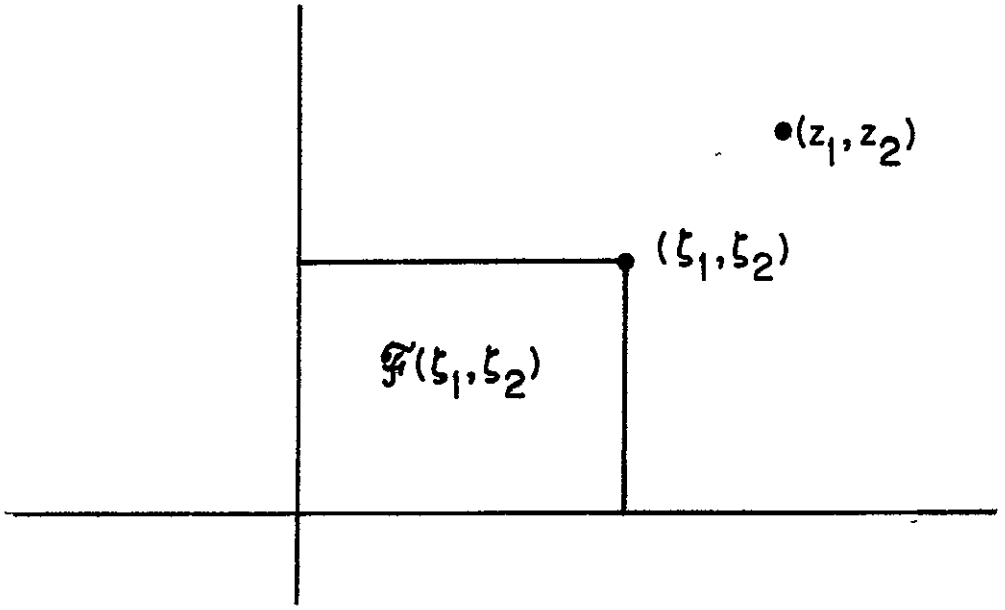

Figure D.16: Illustrating the Structure of a NE Martingale.

development of a full set of stochastic differentiation rules.[33] In addition, one of the major limitations of this approach appears to be the restriction to quadrant causality. But is this really a restriction? In 1-D one of the most important dynamic models involves the representation of a random process as the output of a causal stochastic differential equation driven by a martingale. Perhaps in 2-D we must break the process into two parts, one driven by a NE martingale and one by a SE martingale. Recalling Ekstrom and Woods assertion [D-119] that any power spectral density can be created by using white noise to drive one half-plane or two quadrant filters, this idea may not be that far-fetched.

In any event, there certainly appear to be enough reasons to pursue the utility of such a continuous parameter 2-D stochastic calculus. In 1-D one often finds that the continuous-time solution is far simpler computationally and conceptually than the corresponding discrete-time solution and, in fact, for digital systems one often solves the continuous problem and discretizes rather than discretizing the problem at the start. Examination of the recursive 2-D optimal estimation and detection results derived by Wong in the continuous case [D-172,176,179,187] and comparison of them to the analogous discrete-time results discussed earlier in this section, we see that the same may be true here. In addition to applications such as these, it appears that a 2-D stochastic calculus may be of use in the analysis of processes that evolve in both space and time, which is the next topic of discussion. It is our feeling that the preceding remarks and the following development provide

[33] In [D-184] it is argued that this second martingale arises naturally from the deterministic rules involving Stieltjes differentials on the plane.

ample motivation for the continued study of 2-D stochastic calculus.

Throughout this subsection we have seen numerous examples of 2-D signal processing problems in which good use is made of the transformation of the signals obtained by considering them to be 1-D vector time signals, in which the other independent spatial variable is used to index components of the vectors. We now will briefly examine several problems in which the processes are truly of this from -- i.e. they are space-time processes -- or at least in which one can benefit by viewing multivariable 1-D systems as systems with two independent variables.

One of the best examples of space-time processes arises in the consideration of seismic signal processing (see [D-68,70,199,209,215,216,227,245]), in which we observe the response of the earth to excitation through an array of sensors. In such a system the sensors receive signals due to reflections from different layers in the earth. In addition, there is often coherent noise, resulting from various types of waves, and there also is incoherent noise. Hence, we obtain a 2-D signal $y(j,t)$, where t is time and j denotes the jth sensor (here j can be thought of as a measure of distance from the sensor to the location of the original excitation). If $S(t)$ denotes the response of the earth to the excitation, we can model $y(j,t)$ as follows [D-68]:

$$y(j,t) = S(t-\tau_j) + N(t-\delta_j) + w(j,t) \tag{D.154}$$

where τ_j and δ_j are the time delays incurred by the earth response and the coherent noise, respectively, in travelling to the jth sensor. Also $w(j,t)$

is the incoherent noise. Given this 2-D signal, we want to estimate S(t) and the time delay τ_j (called "moveouts").

A number of solutions have been developed for this problem. In the context of 2-D signal processing, if we assume constant but different speeds of propagation for S and N -- i.e.

$$\tau_j = \frac{d_j}{v_s} , \qquad \delta_j = \frac{d_j}{v_N} \tag{D.155}$$

where d_j is the distance to the jth sensor, we can use "fan" filters to discriminate between these signals. Basically, if we consider the 2-D Fourier transform of these space time signals (let us assume for simplicity that we have a continuum of sensors)

$$Y(\omega_1,\omega_2) = \iint y(x,t)e^{-j\omega_1 x-\omega_2 t}dxdt \tag{D.156}$$

then the point (ω_1,ω_2) corresponds to a plane wave traveling with velocity ω_2/ω_1 = slope of the line connecting this point to the origin. Hence all of the velocities within a given range are obtained by points in a sector in (ω_1,ω_2)-space, and thus if we design a filter to pass only the frequencies in the appropriate sector, we can achieve the desired velocity discrimination (see Figure D.17).

In addition to this type of approach, one can consider the design of optimal filters for the estimation of S and the τ_j. In [D-68] Sengbush and

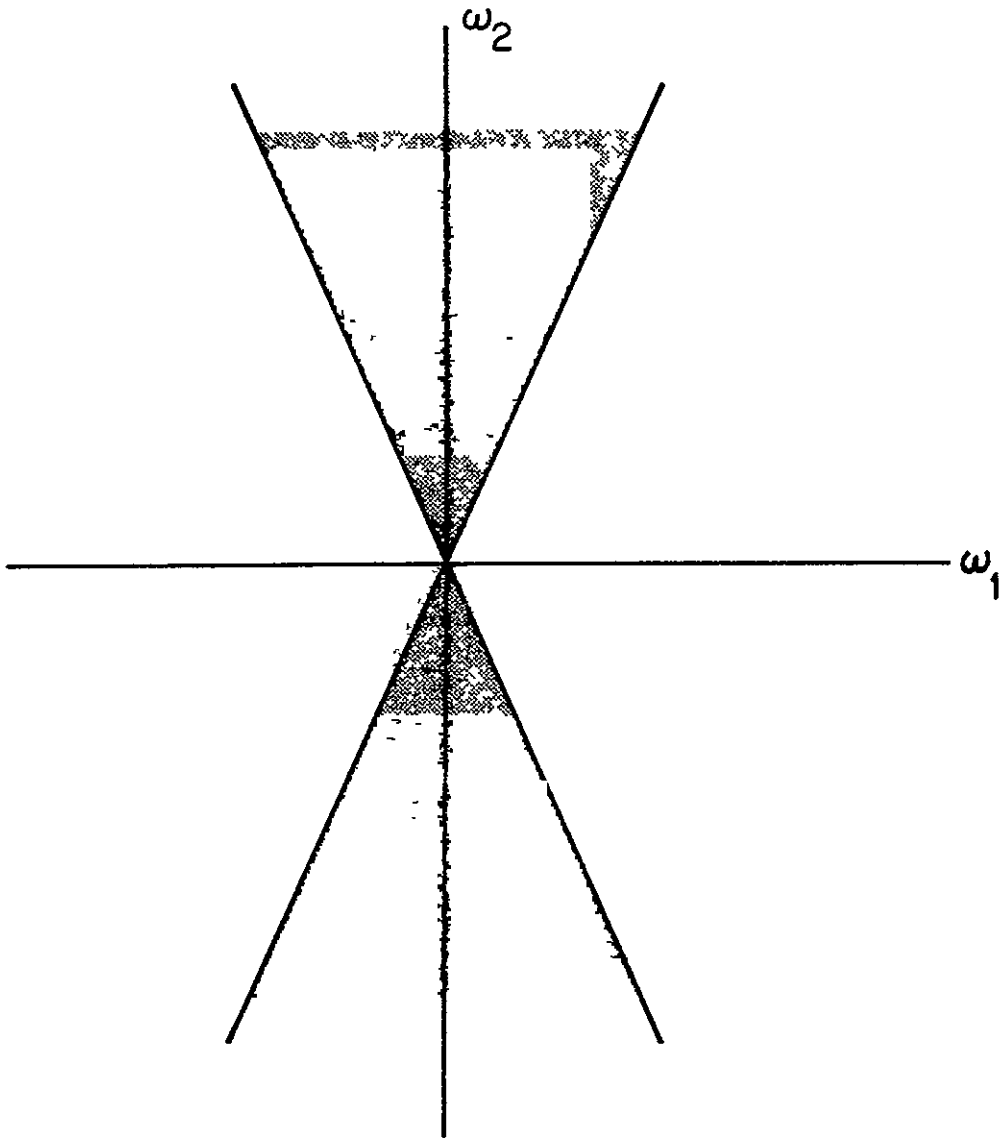

Figure D.17: An Ideal Fan Filter (Passband is Shaded).

Foster derive the optimal nonrecursive Wiener filter for this problem and analyze its properties as a 2-D filtering system. We refer the reader to [D-68] for the details of this development and for a discussion of other 2-D nonrecursive techniques.

An interesting question involves the development of recursive estimation techniques for problems such as these. Such algorithms may be particularly useful given the apparent need for using space-varying models [D-68]. We will discuss the problem of recursive techniques very shortly.

Another class of space-time problems is essentially 3-D. This involves the observation of a sequence of 2-D images in order to determine motion or scene changes. Such problems arise in meteorological problems such as the tracking of cloud motion [D-69,231]. In addition, if one is performing image processing on a sequence of images, one might expect that the use of temporal as well as spatial correlations would improve overall processor performance. The development of systematic recursive or nonrecursive approaches to problems such as these is an appealing area for future work.

A final area in which one finds space-time processes is in the consideration of random vector (transport) or force fields which affect the motion of particles or waves. Applications for models such as these abound. How does the statistical description of a random gravitational field affect the motion of a satellite [D-224], and by observing the motion of the satellite, how can we obtain better estimates of the gravitational field? Given a statistical description of wind currents, predict the space-time distribution of pollutants coming from some source, and determine the optimal locations for the placement of pollution sensors.

Techniques exist for all of the problems mentioned above, but at this time there is no systematic theory for the probabilistic analysis and recursive estimation of general space-time stochastic processes, although major steps have been taken in this direction for space-time point processes [D-11,223], and some work has been done towards developing a calculus for isotropic random vector fields [D-226]. In addition, motivated by several of these applications, Kam and Willsky [D-193-195] and Washburn [D-196] have attempted to utilize the tools of 1-D and 2-D stochastic calculus in order to develop recursive techniques for space-time processes. We briefly describe several of these results.

The results in [D-193-195] are basically separable in nature, in that 1-D stochastic models are developed separately for the spatial and temporal variations. Motivated by time delay problems such as those that arise in seismic signal processing, we have considered the following problem: a source at spatial location s=0 transmits a random signal $\phi(t)$, $t \geq 0$. This signal is modeled as the output of a possibly time-varying linear shaping filter

$$\dot{x}(t) = A(t)x(t) + w(t) \tag{D.157}$$

$$\phi(t) = C(t)x(t) \tag{D.158}$$

The signal is then propagated in the positive s direction by a random velocity field v(s) with given statistics. At points $s_1,\ldots,s_n$ we have sensors which measure delayed versions of the signal

$$y(i,t) = \phi(t-\tau_i) + \nu(i,t) \tag{D.159}$$

$$\tau_i = \int_0^{s_i} \frac{ds}{v(s)} \tag{D.160}$$

Given this problem formulation we consider the problem of recursive optimal estimation of ϕ and of the τ_i[34]. This is an extremely difficult problem, and implementable solutions have been found only in certain special cases. However, the work in [D-193, 194] represents a useful first step in the development of such techniques, and the results obtained can be used to devise suboptimal recursive schemes. Work continues along these lines.

Another problem considered in [D-193,195] has certain aspects in common with problems considered in [D-1,24]. Specifically, we have a random field and a point sensor that traces a 1-D track along the field. As considered in [D-1], suppose we can model the spatial variations along this 1-D track by a spatial shaping filter

$$\dot{x}(s) = Ax(s) + w(s) \tag{D.161}$$

$$f(s) = Cx(s) \tag{D.162}$$

Let $v(t)$ and $s(t)$ denote the velocity and position of the point sensor as a function of time. The time history of the observations of the point sensor may then be modeled by

$$y(t) = f(s(t)) + \nu(t) \tag{D.163}$$

---

[34] We also allow the possibility of delayed versions of ϕ being transmitted from other locations. This can be used to model multiple reflections.

or, if we include the possibility of blurring

$$y(t) = \int_0^t h(t-\tau)f(s(\tau))d\tau + \nu(t) \tag{D.164}$$

Although only the case of (D.163) was considered in [D-193,195], the analysis can be readily extended to the case of (D.164). This extension is presently being developed.

Given this formulation, one can ask several questions. For example one might wish to estimate the field f given these measurements. If the velocity history is known, this is not difficult, and this problem resembles that of [D-24] at least in spirit. If the velocity is unknown -- i.e. we have random motion blur -- the problem is more complex. Methods are developed in [D-193,195] for the suboptimal solution of this problem. Note that in this case we have one more difficulty -- the mapping problem. At any point in time we don't know which point s(t) we're looking at. Note also that intuitively in all of these problems the velocity v(t) must affect the accuracy of our observations -- the faster we move, the less we observe. Thus, one can consider the problem of <u>controlling</u> the speed of the sensor in order to achieve certain performance specifications. An optimal control problem along these lines is considered in [D-193,195].

A third class of separable space-time problems, motivated by the random force field problem, is presently being studied. We have a 1-D random acceleration field a(s) which has a spatial shaping filter representation

$$\dot{x}(s) = Ax(s) + w(s) \tag{D.165}$$

$$a(s) = Cx(s) \tag{D.166}$$

Suppose a particle is subject to this field. The equations governing its motion are

$$\dot{s}(t) = v(t) \tag{D.167}$$

$$\dot{v}(t) = a(s(t)) \tag{D.168}$$

We wish to estimate the shape of the random field from noisy observations of the position of the particle

$$y(t) = s(t) + \nu(t) \tag{D.169}$$

Results for problems of this type will be forthcoming.

Clearly all of these problems represent vast simplifications of real problems, but they also represent a start. One must now consider the extension of these ideas to several spatial dimensions and the use of non-separable space-time stochastic models. The use of a multidimensional stochastic calculus such as that described earlier is clerly essential. As an indication that at least in some cases the NE causal structure of this calculus may not be a problem and in fact may be natural, we mention an observation of Washburn [D-196]. Suppose we consider a space-time system with one spatial dimension, and suppose that because of fundamental limitations (due, for example, to the finite speed of light) events at any given spatial point can affect those at another only with a certain time delay. This leads to the usual "light cone" description of the future and past of

a given space-time point. Assuming we scale the axes appropriately, this cone can be assumed to have an angle of 90°, as indicated in Figure D.18. Hence, by rotating the coordinates by 45°, we obtain a NE causal structure. The utility of this observation when combined with 2-D stochastic calculus and a variety of space-time analysis problems will be reported in the future [D-196].

As mentioned earlier, in addition to systems which truly have a space-time character, one can view any multivariable 1-D system as a 2-D system by considering the "space" variable to be the index of the elements of the various vector functions of time. While this may not be particularly natural in general, this philosophy appears to have some merit for large scale systems which consist of a number of interconnected subsystems. In this case we let the spatial variable index subsystem variables which may be vector quantities themselves. A general linear model for such a system is

$$x(k+1,i) = \sum_j A_{ij} x(k,j) + \sum_j B_{ij} u(k,j) + w(k,j) \tag{D.170}$$

$$y(k,i) = \sum_j C_{ij} x(k,j) + v(k,j) \tag{D.171}$$

Clearly this is a recursive 2-D model. Examples of large-scale systems of this type abound in practice. Examples include power systems, communication

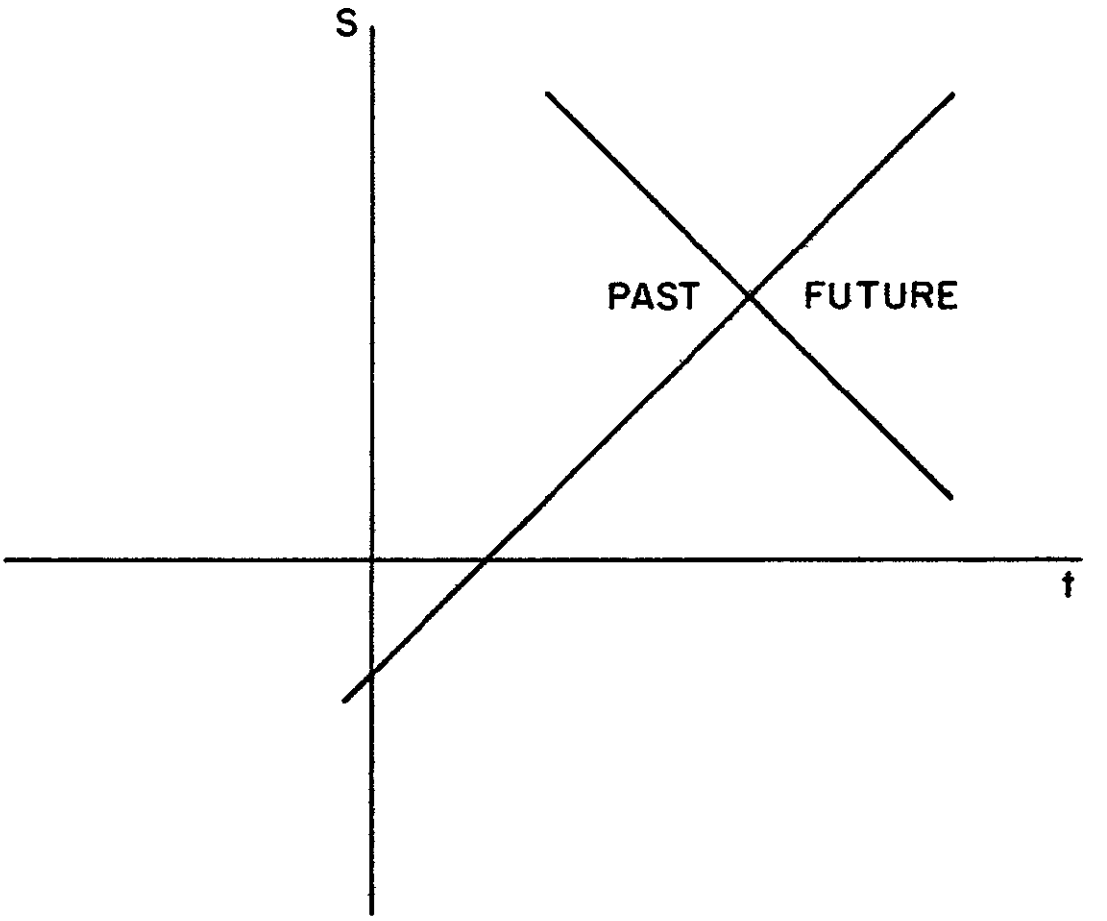

Figure D.18: The 2-D Causality Structure for Space-Time Processes

networks, and freeway traffic systems.[35] We refer the reader to [D-84-89, 91-94,190,201,203-208] for other examples and for some insight into the problems associated with such systems.

The problems with these systems are of two types: (1) the analysis of these systems using tools such as the Lyapunov equation and the determination of optimal filter and controller designs is far too complex to be carried out using standard methods because of the high dimensionality of the overall system; and (2) the implementation of standard controllers and estimators is out of the question, since these systems require totally centralized processing of all subsystem data in order to determine each subsystem control; what is needed is a decentralized scheme.

We have seen similar questions in our study of recursive image processing techniques. The full-state optimal Kalman filter of Woods and Radewan [D-173,229,236] was of enormous dimension, and one would never dream of attempting to solve the Riccati equation in this case. In addition, the on-line Kalman filter update is far too complex, and, as we discussed, Woods and Radewan suggested a "nearest neighbor" constrained Kalman filter, in which only those pixels near the one presently being processed are themselves updated. This clearly is a decentralization of sorts, as are the techniques proposed by Murphy and Silverman [D-174] and Pratt [D-17]. What these methods have in common is the following: we specify some constraints on information transfer

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[35] In this last case, the subsystem index does represent a spatial variable, as each subsystem describes the aggregate behavior of traffic on a link of a freeway (see [D-208]). In this case, the choice of the size of each link is a type of sampling problem, and the issues of spatial sampling, such as those raised by Mascarenhas and Pratt [D-23] and Hunt [D-4] in the context of image processing, are clearly relevant here.

-- i.e. we limit the extent of the update portion of the filter -- and then we optimize the filter gains subject to these constraints. This same philosophy is precisely what is used in many decentralized control and estimation problems [D-86,201,205]. That is, we specify some constraints on the information pattern[36] -- i.e. which data are available for each subsystem -- and then we optimize the estimator and controller gains subject to these constraints.

Thus, we've seen that large-scale systems can be viewed as 2-D systems and that constrained optimization for efficient or decentralized processing is common in both settings. Is there any other insight can be gained or new results that can be obtained by examination of large scale systems as 2-D systems? The answer is perhaps, and we will relate some preliminary observations that make us feel that the answer will ultimately be yes

First of all, suppose that the model (D.170), (D.171) falls into the class considered by Attasi [D-6,35,96]. Then the optimal centralized Kalman filter is nothing more than Attasi's line by line optimal processor. In this context, let us re-examine the structure of this processor as pictured in Figure D.12. One may argue that this processor may not be a good image restoration system, but it certainly is an extremely efficient centralized Kalman filter! The predict cycles for each subsystem are carried out in a totally decoupled fashion, and in the update stage, each subsystem need only communicate with its nearest neighbors (we have two streams of information flowing, corresponding to the two Kalman filters). Whether optimal centralized controllers also have this structure remains an open question.

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[36] The problem of choosing a good information pattern in the first place is an extremely important and complex one, but it is beyond the scope of our present discussion. We refer the reader to the references and in particular to [D-85].

As a second example, consider the case in which (D.170), (D.171) are spatially invariant[37]

$$x(k+1,i) = \sum_j A_{i-j} x(k,j) + \sum_j B_{i-j} u(k,j) + w(k,j) \tag{D.172}$$

$$y(k,i) = \sum_j C_{i-j} x(k,j) + v(k,j) \tag{D.173}$$

In the case of an infinite string of subsystems, no noise, and a spatially-invariant quadratic cost function, Melzer and Kuo [D-203] determined an efficient method for determining the optimal centralized controller, and Chu [D-205] used the same method to determine the optimal, constrained decentralized controller. The basic idea is identical to that used by Hunt [D-46], Andrews and Hunt [D-81], Jain and Angel [D-32], and Attasi [D-6,35,96] -- we take the z-transforms of (D.172), (D.173) in the subsystem variable to obtain a system of decoupled optimal control problems (parametrized by z) of dimension equal to that of each x_i.

To make these ideas more clear, let us consider the case in which we have a finite string of subsystems [D-190], $i=0,\ldots,N-1$. In this case, if we rewrite (D.172), (D.173) in terms of one giant state, input, and output vector, we find that the resulting A, B, and C matrices are block Toeplitz. As Andrews and Hunt [D-81] discuss, we then make the block circulant approximation to obtain[38]

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[37] Such models arise, for example, in the longitudinal control of a string of vehicles [D-204] such as one finds in personal rapid transit systems.

[38] Approximations such as these often arise in the discretization of partial differential equations such as the wave equation (see, for example, [E-31]).

$$x(k+1,i) = \sum_{j=0}^{N-1} A_j x(k,i-j) + \sum_{j=0}^{N-1} B_j u(k,i-j) + w(k,i) \tag{D.174}$$

$$y(k,i) = \sum_{j=0}^{N-1} C_j x(k,i-j) + v(k,i) \tag{D.175}$$

with

$$E[w(k,i)w'(j,\ell)] = S_{i-j}\delta_{k\ell} \tag{D.176}$$

$$E[v(k,i)v'(j,\ell)] = \Theta_{i-j}\delta_{k\ell} \tag{D.177}$$

where all subsystem indices are to be interpreted modulo N. Suppose we wish to design a controller to minimize the criterion

$$J = E\left\{\sum_{k=0}^{\infty} \sum_{i,j=0}^{N-1} [x'(k,i)Q_{i-j}x(k,j) + u'(k,i)R_{i-j}u(k,j)]\right\} \tag{D.178}$$

As discussed in Appendix 2, we take subsystem transforms. For example

$$\bar{x}(k,\ell) = \sum_{i=0}^{N-1} x(k,i)W_N^{-i\ell} \tag{D.179}$$

We then obtain a set of decoupled problems, indexed by ℓ

$$\bar{x}(k+1,\ell) = \bar{A}(\ell)\bar{x}(k,\ell) + \bar{B}(\ell)\bar{u}(k,\ell) + \bar{w}(k,\ell) \tag{D.180}$$

$$\bar{y}(k,\ell) = \bar{C}(\ell)\bar{x}(k,\ell) + \bar{v}(k,\ell) \tag{D.181}$$

$$E[\bar{w}(k,\ell)\bar{w}^*(j,m)] = \bar{S}(\ell)\delta_{kj}\delta_{\ell m} \tag{D.182}$$

$$E[\bar{v}(k,\ell)\bar{v}^*(j,m)] = \bar{\Theta}(\ell)\delta_{kj}\delta_{\ell m} \tag{D.183}$$

$$J_\ell = \frac{1}{N} E\sum_{k=0}^{\infty} [\bar{x}^*(k,\ell)'\bar{Q}(\ell)\bar{x}(k,\ell) + \bar{u}^*(k,\ell)'\bar{R}(\ell)\bar{u}(k,\ell)] \tag{D.184}$$

Here * = complex conjugate. Note also that since all original variables are real, we have $A(-\ell) = A^*(\ell)$, etc. Thus, we need only solve approximately one-half of these problems to obtain the optimal centralized controller which is efficiently implemented in Figure D.19. The reader is urged to compare this figure with Jain and Angle's optimal image restoration scheme as depicted in Figure D.13. The similarity here is rather striking as is the similarity in method and philosophy underlyi - both systems. Work involving the system (D.174), (D.175) is continuing. We are examining such issues as the effects of the block circulant approximation, the use of this method for fast algorithms for Lyapunov equations, Riccati equations, pole placement, etc., and the design of decentralized controllers. Note that one possible decentralization can be obtained by spatially windowing the optimal centralized filter and control gains. As the properties of various windows are well-known (see, for example, [C-1]), it may be possible to obtain detailed performance evaluations for such schemes.

Thus, we have seen that there are points of contact between 2-D processing concepts and large-scale 1-D system analysis. Whether these points will lead to major new results or exciting concepts remains to be seen, but there certainly appear to be some intriguing possibilities.

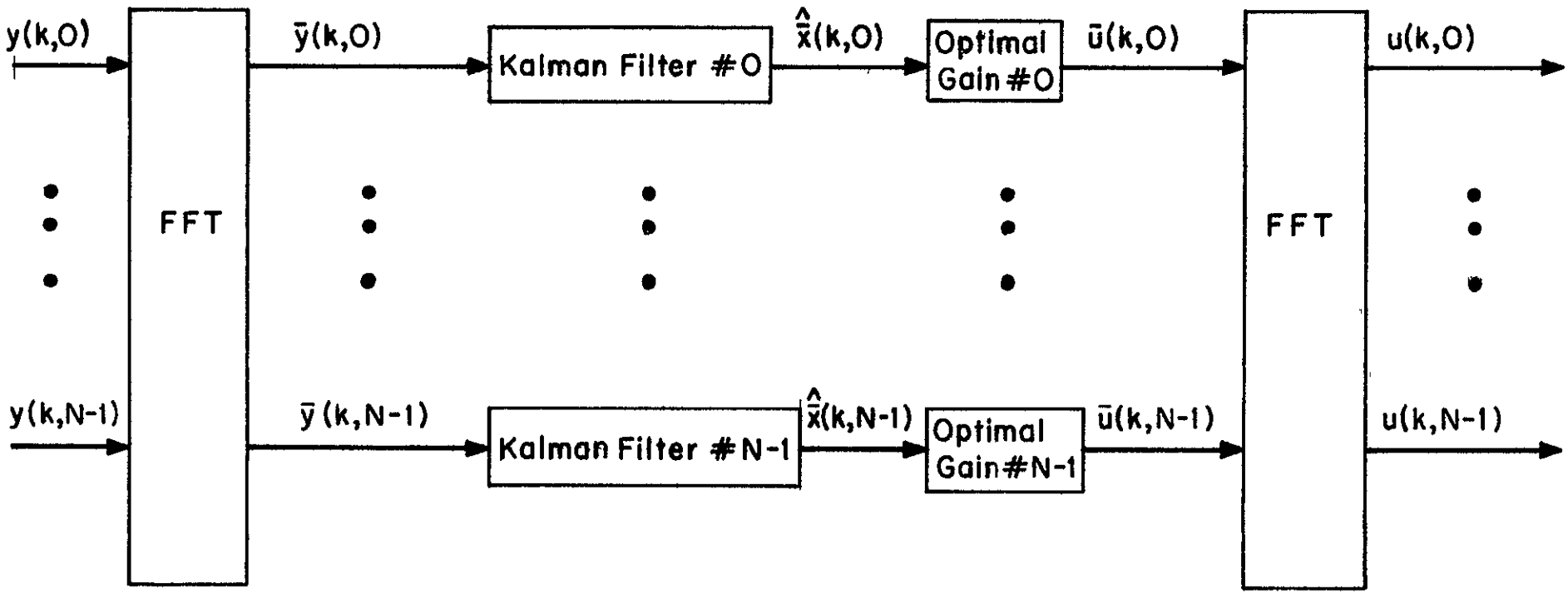

Figure D.19: Illustrating the Optimal Circulant Feedback Systems.

E. Some Issues in Nonlinear System Analysis: Homomorphic Filtering, Bilinear Systems, and Algebraic System Theory

Most of the discussion to this point has dealt with the analysis and synthesis of linear systems, perhaps distorted by nonlinear effects such as quantization. However, there has been much work on the analysis and design of systems which are fundamentally nonlinear in both digital signal processing and in control and estimation theory. It is beyond the scope of this paper to consider the research in this area at any depth, and we refer the reader to the references and to the literature in the two disciplines for the full story. In this section we limit ourselves to a brief look at two particular directions of research that have a common thread involving the use of algebraic concepts to study nonlinear systems possessing particular types of structure. The philosophy underlying these results is that many of the concepts and techniques from linear system theory can be carried over to the analysis of certain nonlinear systems. Not only is this of use in allowing one to solve certain nonlinear problems, but it is also of value in providing insight into the properties of linear systems -- i.e. one gets a clearer picture of which system properties carry over to nonlinear systems with particular structure and which properties are fundamentally tied to linearity.

In digital signal processing, Oppenheim [C-1,E-1,2] abstracted the key concept in linear system analysis -- superposition -- and developed what he termed homomorphic signal processing. Following [C-1], the basic idea is as follows. Let X and Y be spaces with two operations

defined on each -- a binary operation

$$\begin{array}{ll} x_1, x_2 \varepsilon X & x_1 * x_2 \varepsilon X \\ y_1, y_2 \varepsilon Y & y_1 \circ y_2 \varepsilon Y \end{array} \qquad (E.1)$$

and an operation of scalar action

$$\begin{array}{ll} c \varepsilon R \text{ or } C,\ x \varepsilon X & c \cdot x \varepsilon X \\ c \varepsilon R \text{ or } C,\ y \varepsilon Y & c \cdot y \varepsilon Y \end{array} \qquad (E.2)$$

A homomorphism is then a map H from X to Y which preserves these operations -- i.e. it satisfies a "generalized superposition principle"

$$\begin{aligned} H(x_1 * x_2) &= H(x_1) \circ H(x_2) \\ H(c \cdot x_1) &= c \cdot H(x_1) \end{aligned} \qquad (E.3)$$

If the operations (E.1), (E.2) satisfy the axioms of a vector space (e.g. this means that all the operations are commutative), then (E.3) looks very much like a linear system. In fact, one can show in this case [E-1] that any such system can be represented as the cascade of three homomorphic systems

$$H = D_y^{-1} \circ L \circ D_x \qquad (E.4)$$

where L is a standard linear system, and D_x and D_y are called <u>characteristic systems</u>. They translate the operations in X and Y into usual vector addition

and scalar multiplication.

Let us take a look at an example of this. Let X be the space of input sequences in which each input is strictly positive. We make X into a vector space with the operations

$$[(\alpha \cdot x_1) * (\beta \cdot x_2)](n) = x_1(n)^{\alpha} x_2(n)^{\beta} \tag{E.5}$$

and the system D_x is clearly seen to be the map

$$x(n) \longrightarrow \log[x(n)] \tag{E.6}$$

with the inverse

$$\xi(n) \longrightarrow e^{\xi(n)} \tag{E.7}$$

One can similarly define vector space operations in which X consists of all nonzero complex numbers or all of those of modulus one [C-1,E-1], although there are some difficulties due to the nonuniqueness of the complex logarithm. We will not go into these here and refer the reader to [E-1,C-1].

Having this framework, one can consider the filtering of signals corrupted by multiplicative effects. That is, suppose we observe

$$z(n) = x(n)u(n) \tag{E.8}$$

(all quantities assumed to be >0), and we wish to recover x from z. If we take the logarithm of both sides

$$\xi(n) = \log z(n) = \log x(n) + \log u(n) \tag{E.9}$$

we can use linear techniques to filter $\xi(n)$, yielding the output $\eta(n)$, and we then obtain the desired filtered version as

$$\hat{x}(n) = e^{\eta(n)} \tag{E.10}$$

For applications of multiplicative homomorphic processing, we refer the reader to [C-1,E-2]. In the case in which log (x) and log (u) are Gaussian random variables -- i.e. when x and u are lognormal variables [E-9,12-16,20] -- the filtering of $\xi(n)$ is simply a Kalman filter (this result is developed thoroughly in [E-20]). The continuous-time version of this multiplicative noise model has been studied in [E-2], and its stochastic analog was developed in [E-9]. Let us examine this case at some length. Let w(t) be a two-dimensional Gauss-Markov process satisfying the equation

$$\dot{w}(t) = Aw(t) + v(t) \tag{E.11}$$

where v(t) is a two-dimensional white noise process

$$E(v(t))=0,\ E(v(t)v(\tau)) = Q\delta(t-\tau) \tag{E.12}$$

Suppose we transmit the "frequency modulated signal"[1]

$$x(t) = \exp\left[\int_0^t [w_1(s)+jw_2(s)])ds\right] \tag{E.13}$$

Due to some effect (e.g. atmospheric turbulence [E-21]), the received signal is corrupted by multiplicative noise

[1] Here we are allowing both the usual type of modulation on the phase and a "homomorphic" modulation on the amplitude.

$$r(t) = \mu(t)x(t) \tag{E.14}$$

where

$$\mu(t) = \exp(\eta_1(t)+j\eta_2(t)) \tag{E.15}$$

and η is a two dimensional Brownian motion process

$$E(\eta(t))=0, \quad E(\dot{\eta}(t)\dot{\eta}(\tau)) = R\delta(t-\tau) \tag{E.16}$$

Because of the continuity of $r(t)$, there is no difficulty in taking the complex logarithm (see[E-9] ; essentially, continuous monitoring of phase allows one to unravel it and determine the number of revolutions, as well as the value of the phase modulo 2π). In this case $r(t)$ is equivalent to the observations

$$\begin{aligned} d\xi_1(t) &= w_1(t)dt + d\eta_1(t) \\ d\xi_2(t) &= w_2(t)dt + d\eta_2(t) \end{aligned} \tag{E.17}$$

Using standard Kalman filtering techniques, we can obtain the least squares estimates $\hat{w}_1(t)$ and $\hat{w}_2(t)$. However, the best estimate of x is <u>not</u>

$$\exp \int_0^t [\hat{w}_1(s)+j\hat{w}_2(s)]ds$$

essentially because the integral of a best estimate is <u>not</u> the best estimate

of the integral. However, in this case we can obtain the desired estimate as follows:

Let

$$\rho_1(t) = \int_0^t w_1(s)ds \ , \qquad \rho_2(t) = \int_0^t w_2(s)ds \tag{E.18}$$

Then by adjoining these integrals to w_1 and w_2 to form a four-dimensional "state", we can again design a Kalman filter (with measurements (E.17)) and obtain the best estimates $\hat{w}_1(t)$, $\hat{w}_2(t)$, $\hat{\rho}_1(t)$, and $\hat{\rho}_2(t)$. Then the desired estimate is

$$\hat{x}(t) = \exp(\hat{\rho}_1(t) + j\hat{\rho}_2(t)) \tag{E.19}$$

The details of this development are given in [E-9,12]. Also, in these references it is shown that the solution of the discrete-time version -- i.e. when we observe only $r(k\Delta)$, where r is as in (E.14) -- is much more difficult, essentially due to the ambiguity in the complex logarithm which <u>cannot</u> be resolved in this case.

In digital signal processing, multiplicative homomorphic systems represent only one half of the picture. As discussed in [C-1,E-1,2] one can study systems in which vector addition is the operation of <u>convolution</u> and multiplication by an integer n corresponds to convolution of a signal with itself n times (multiplication by a non-integer is a generalization of this [E-1,2,22]). The key to the development of homomorphic filtering techniques for convolutional noise is the z-transform of signals. Let X be a vector space of signals under the operations of convolution as vector addition and scalar multiplication as

defined above. Then we have that following transform relations

$$(x_1 * x_2)(n) \longleftrightarrow X_1(z)X_2(z)$$

$$(\alpha \cdot x_1)(n) \longleftrightarrow X_1(z)^{\alpha}$$

and we see that homomorphic convolution systems look like multiplicative homomorphic systems in the frequency domain. This allows one to develop a rather complete theory of convolution-homomorphic filtering, and we refer the reader to [C-1,E-2,22] for details. Techniques such as "homomorphic deconvolution" have found application in speech analysis [C-1,E-3], dereverberation of signals such as those arising in seismic applications [C-1,E-2,22,23], and in several other disciplines (see [C-1,E-2]).

A recent direction of research in control and estimation theory has been study of <u>bilinear systems</u> [C-26,E-5-16], and the multiplicative homomorphic system (E.11)-(E.16) represents one of the simplest examples. Consider (E.13). We can easily obtain a stochastic differential equation for x:

$$\dot{x}(t) = (w_1(t)+jw_2(t))x(t) \tag{E.21}$$

If we regard w_1, w_2 as inputs -- controls and/or noises -- we see that the right-hand side of (E.21) consists of a product of inputs and the state -- i.e. it is a bilinear function of the two. Generalizing this, we obtain the class of bilinear systems

$$x(t) = \left[A_0 + \sum_{i=1}^{N} A_i u_i(t)\right] x(t) \tag{E.22}$$

where the A_i are known nxn, possibly complex-valued matrices, the u_i are scalar inputs, and x is either an n-vector or an nxn matrix.

The question of the control, estimation, and stability of bilinear systems such as (E.22) has received a great deal of attention in the recent past and has applications in a wide range of disciplines (see [(C-26,E-5,12,14-16,24]). We will not examine the control or stability issues here and refer the reader to the references. Rather, we content ourselves with a brief look at the estimation problem in order to uncover some of the main issues in "bilinear signal processing". Note that in the scalar case (E.21), one can readily obtain a representation for x(t) of the form (E.13). However, in the vector case this is not true in general. In fact, the solution of (E.22) has the representation

$$x(t) = \exp\left\{A_0 t + \sum_{i=1}^{N} A_i \int_0^t u_i(s)\,ds\right\} x(0) \tag{E.23}$$

<u>if and only if</u> all of the matrices $A_0, A_1, \ldots, A_N$ commute (a very restrictive condition). In fact, the commutativity or noncommutativity properties of these matrices plays a central role in the analysis of bilinear systems, and the introduction of concepts from the theory of Lie algebras and Lie groups allows one to study these systems in great detail [E-5,6,8-17,25,26].

Let us see what this noncommutativity can do by examining a problem that is motivated by (E.11)-(E.16). If we examine those equations and consider

only the phse effects -- i.e. w_2 and η_2 -- we see that this problem is the estimation of a phase given noisy measurements of that phase. By performing a transformation on the measurement, we obtain a noisy measurement of the angular frequency. From we can apply standard Kalman filtering techniques to estimate the angular frequency <u>and</u> its integral, and then the desired phase estimate is just the complex exponential of the estimate of the integral. A natural extension of this problem is the consideration of rotation in three dimensions. We follow [E-5,12,14-16]. Suppose we have a satellite, equipped with an inertial platform. The orientation of the satellite with respect to an inertial frame can be specified by coordinatizing a body-fixed orthonormal basis in inertial coordinates. The resulting set of three 3-vectors is called the <u>direction cosine matrix</u> X(t) and it has the property

$$X'(t)X(t)=I, \qquad \det X(t)=1 \tag{E.24}$$

Let w(t) be the angular velocity of the body with respect to inertial space, coordinatized in the body frame. Then, it is known that the evolution of the direction cosine matrix is described by the bilinear equation

$$\dot{X}(t) = \left[\sum_{i=1}^{3} R_i w_i(t) \right] X(t) \tag{E.25}$$

where

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} ; \quad R_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \; R_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{E.26}$$

Suppose that our only observation of satellite attitude is from the inertial platform -- i.e. we observe the direction cosine matrix M(t) of the body with respect to the platform, which is supposed to remain fixed in inertial space (in which case M=X). However, because of various errors (e.g. gyro drift), the platform drifts, and our actual observation is

$$M(t) = X(t)V(t) \tag{E.27}$$

where the "platform misalignment term", V(t) is the direction cosine matrix of inertial space with respect to the platform. As described in (E-14), this can be modeled by a bilinear equation of the form[2']

$$\dot{V}(t) = V(t)\left[\sum_{i=1}^{3} R_i v_i(t)\right] \tag{E.28}$$

where the v_i represent gyro drift and for simplicity are taken to be white.

The reader should now compare (E.25)-(E.28) with (E.13)-(E.15) (using (E.21) and an analogous equation for μ). We see that we have a direct analog of the phase -- i.e. one-dimensional rotation -- problem, including a multiplicative noise model (E.27) (see [E-9] to see that if only one w_i and the corresponding v_i are nonzero, then this problem <u>precisely</u> reduces to the phase estimation problem). Suppose we now assume that w obeys an equation such as (E.11). Then, essentially by the matrix equivalent of the complex logarithm (again we have no "mod 2π" difficulties because of our continuous observation),

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[2'] Technically, one must include a "correction term" into (E.28) if one interprets it as an Ito stochastic equation. This is not difficult but it does obscure our point with technicalities (which certainly are very important). The reader is referred to [E-16,27] for the details. Note that (E.28) can be interpreted rigorously if one uses Stratonovich calculus [E-8,14].

we can essentially differentiate M(t) to obtain noisy measurements of w corrupted by the gyro drifts v_1, v_2, v_3. The problem here is somewhat more complex than the earlier one in that one must take care in using stochastic calculus (see footnote 2) and, more importantly, because rotations in three-dimensions do not commute (see [E-16,27]). However, as derived in [E-16], one can carry the analysis through to obtain a measurement equation of the form

$$z(t) = w(t) + M(t)v(t) \tag{E.28}$$

where $v' = (v_1, v_2, v_3)$. Note that the effect of the gyro drifts on our measurement of angular velocity depends upon our attitude (this effect can be removed in the one-dimensional rotation case).

Using (E.28), we can design a Kalman filter to estimate w. However, we run into a problem in estimating X. Recall that in the one-dimensional problem, we augmented the state of our Kalman filter with the estimate of the integral of w, but in the three dimensional case the integrals of components of w are <u>not</u> simply related to X, again because of the noncommutativity of rotations in three dimensions. In fact, in this case the problem of optimal estimation of X is infinite-dimensional [E-14]. Thus, in the one-dimensional case we obtain a decomposition much like (E.4). We can convert our multiplicative process into a linear one and can operate on it with optimal linear techniques. However, the re-injection of the resulting filtered process becomes extremely complex. One must use approximate methods (see [E-12,14]) except in special cases. The case when all of the A_i commute is much like the scalar case and involves looking at the integrals of certain quantities [E-9,12]. In addition, if the A_i

obey certain (somewhat less restrictive) noncommutativity relations one can obtain a finite dimensional optimal procedure by considering several types of iterated integrals (see [E-14,15,17] for details).

Let us say a few more words about the relationship between homomorphic filtering (HF) and bilinear signal processing (BSP). Recall that HF is based on the existence of certain algebraic properties between input functions and output functions -- i.e. the validity of a superposition rule. Also, in HF, one designs a filter consisting of three parts -- a "projection" system, which "unravels" the signals so that one can use a linear filter as the second part, followed by an "injection" of the resulting process to yield the desired output. In the scalar example of BSP, as described in (E.11)-(E.19), we obtain a system of exactly this form -- i.e. a HF (logarithm-linear (Kalman) filter-exponential) --, and as we mentioned earlier, we obtain essentially the same results for the model (E.22),(E.27) if the A_i commute. However, in the general case we cannot obtain the entire picture. Specifically, we can "unravel" the signal and can perform linear (and perhaps nonlinear [E-14,15,17]) processing, but the re-injection process is much more difficult. Perhaps one of the keys to the difference between HF and BSP is the difference in the starting point of the two theories. In homomorphic filtering the fundamental assumption involves the algebraic structure of the relation between input trajectories and output trajectories (superposition). For bilinear systems analysis, the starting point is (E.22), which can be seen to impose an algebraic (multiplicative) restriction on the time rate of change of the state or output -- i.e., in some sense, (E.22) represents an "incrementally homomorphic" model, in which the fundamental assumption involves algebraically compatible <u>dynamics</u> (as opposed to input-

output relation). In the case when the A_i commute (E.22) also yields a multiplicative I/O relationship, and in the other special cases considered in [E-14,15,17], the restrictions on the A_i yield other tractable I/O relations, but in these cases the optimal filters are not homomorphic (since following the unraveling of the received signal, we perform a nonlinear filtering operation). We should note, however, that in the general case, the algebraic structure of (E.22) still allows one to perform a great deal of analysis, and we refer the reader to the references for details (see, for example, [E-5,10]).

We note that the use of algebraic and geometric concepts and techniques to study systems with algebraically compatible dynamics or input-output relations has increased greatly over the past few years as new theories and applications have been uncovered [E-5 through 19,24 through 36]. Recently, certain nonlinear systems having Volterra series representations have been studied with great success [E-6,10,11,14,17,33,34] using techniques and ideas that have grown out of the study of bilinear systems [E-5,7]. In addition, motivated by many of the same issues that motivated Oppenheim's study [E-1] of generalized superposition, several researchers [E-18,19,28-34] have examined systems whose state dynamics possess some, but not all, of the algebraic structure of linear systems. Also, several researchers [E-35,36] have studied controllability, realizability and related properties for systems which possess particularly nice input/output descriptions, much along the lines of Oppenheim's generalized superposition. By performing such analyses, new insights have been shed on the properties of linear systems, and many of the powerful tools of linear system analysis are being extended to other dynamical systems, establishing the foundations for a synthesis and analysis theory for special classes of nonlinear systems. It is

this key idea -- the use of algebraic tools to synthesize and analyze nonlinear systems with structure -- that is the major common theme of the nonlinear systems research in the two disciplines.

Concluding Remarks

In this report we have examined a number of broad research areas that have attracted workers in two disciplines -- digital signal processing and control and estimation theory. Our goal has been to explore these areas in order to gain perspective on relationships among the questions asked, methods used, and general philosophies adopted by researchers in these disciplines. Upon undertaking this study it was our feeling that such perspective would be extremely valuable in promoting collaboration and interaction among researchers in the two fields. Upon concluding this study, we think that our initial feelings have been thoroughly substantiated. Not only are there numerous examples of questions in one discipline that can benefit from the point of view of the other, but also we have found a number of new issues that naturally arose from combining the two points of view.

Each of the disciplines has its own distinct character, and clearly these will and should be maintained. On the other hand, each discipline can gain from understanding the other. State space methods have their limitations, such as in specifying useful digital algorithms and structures. On the other hand, state space methods provide extremely powerful computer-aided algorithms for noise analysis, optimal design specification, etc. State space ideas also allow one to consider multivariable and time-varying systems. All of these aspects of state space theory may prove of value to people involved in digital

signal processing. On the other side, researchers in digital filtering have answered many crucial questions related to turning design specifications into implementable designs. The deep understanding that workers in digital signal processing have concerning the problems of digital implementation is something that researchers in control and estimation would do well to gain. Thus it seems clear that a mutual understanding will prove beneficial to all concerned.

We have raised numerous questions and have speculated on various possibilities throughout this report, and it would be an impossible task to summarize these questions and speculations here. Rather, we will mention only one or two questions from each area. These may not prove to be the most exciting or promising problems, but we feel that they are representative and do summarize the tone of this report.

A. Stability Analysis -- What is the effect on overall stability of the finite arithmetic constraints of a digitally implemented feedback controller?

B. Parameter Identification, Linear Prediction, Least Squares, and Kalman Filtering -- Can state space and recursive filtering methods be applied to model and identify time-varying models of speech? Do stochastic realization and recursive maximum likelihood methods offer useful tools for pole-zero modelling of speech?

C. Synthesis, Realization, and Implementation -- Can state space realization and filter structure concepts be combined to obtain useful realizations for multivariable or time-varying digital filters? Can state space noise analysis methods aid in roundoff analysis of digital filters? Can we develop design technique (e.g. for feedback controller design) that directly take the constraints (storage, speed, word length) of digital implementation into account?

D. Multiparameter Systems, Distributed Processes, and Random Fields -- What role do state space methods (if they exist) play in the analysis and synthesis of 2-D filters? Can Lyapunov theory (if it exists) aid in understanding the effects of finite arithmetic in 2-D systems? What role should 2-D recursive estimation and detection techniques have in image processing? Can 2-D concepts provide any insight and/or results for distributed parameter, space-time, or decentralized control problems?

E. Some Issues in Nonlinear System Analysis: Homomorphic Filtering, Bilinear Systems, and Algebraic System Theory -- Is this algebraic point of view a useful approach to the analysis and synthesis of nonlinear systems and filters? Homomorphic filtering has found widespread application; can

the same be said for other algebraic concepts?

Whether any of these issues or any of the others raised in this report has useful answers is a question for the future. It is our feeling that many of them do, and it is our hope that others will think so as well.

Appendix 1: A Lyapunov Function Argument for the Limit Cycle Problem in a Second-Order Filter

Consider the second-order filter in Figure Ap.1. The ideal (undriven) dynamics of this filter are

$$x(k+1) = Ax(k) \tag{Ap.1}$$

where

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} , \quad A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \tag{Ap.2}$$

Suppose we implement the filter using a single magnitude truncation quantizer [A-3] following the summation. In this case, the actual dynamics are

$$x(k+1) = F(Ax(k)), \; F\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} Q(\xi_1) \\ \xi_2 \end{pmatrix} \tag{Ap.3}$$

Let us look for a quadratic Lyapunov function

$$V(x) = x'Bx, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \tag{Ap.4}$$

In fact, let us assume that B proves the asymptotic stability of (Ap.1) -- i.e. (see Section A)

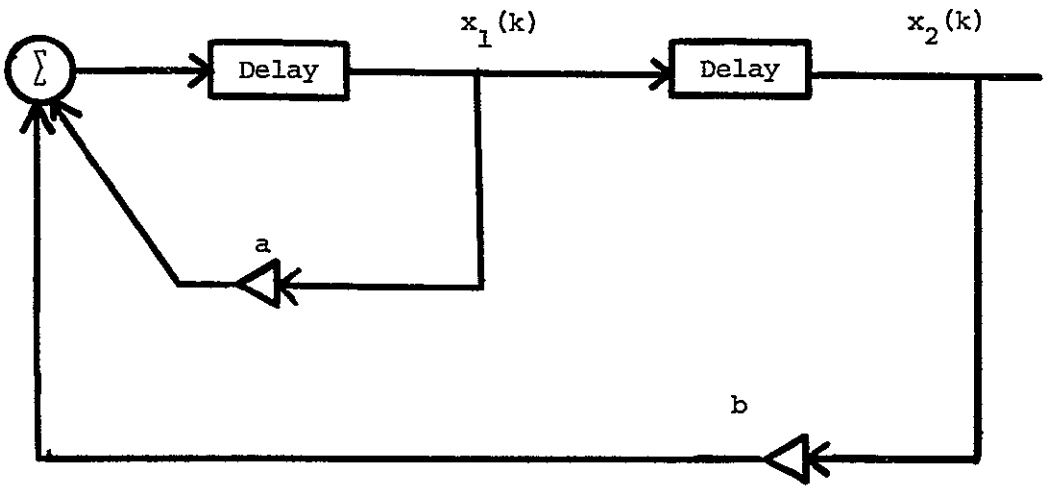

Figure Ap.1: A Second Order Filter

$$B>0, \quad B-A'BA>0 \tag{Ap.5}$$

We compute

$$\Delta V(z) = F'(Az)BF(Az) - z'Bz \tag{Ap.6a}$$

$$\begin{aligned} &= (F'(Az)BF(Az) - z'A'BAz) \\ &\quad + (z'A'BAz - z'Bz) \end{aligned} \tag{Ap.6b}$$

From this, it is clear that we will have asymptotic stability if [1]

$$F'(Az)BF(Az) - z'A'BAz \leq 0 \qquad \forall z \tag{Ap.7}$$

or if the somewhat stronger condition[2]

$$F'(\xi)BF(\xi) - \xi'B\xi \leq 0 \qquad \forall \xi \tag{Ap.8}$$

Equation (Ap.8) is equivalent to

$$b_{11}\left[Q^2(\xi_1)-\xi_1^2\right] + 2b_{12}\xi_2\left[Q(\xi_1)-\xi_1\right] \leq 0 \qquad \xi_1,\xi_2 \tag{Ap.9}$$

Using the fact that $|Q(\xi_1)| \leq |\xi_1|$, we can see that (Ap.9) holds if and only if $b_{12}=0$. Thus, we must find conditions on A such that there exists a <u>diagonal</u> B, satisfying (Ap.5) -- i.e..

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[1] This is the criterion used by Willson [A-2] for the overflow problem.

[2] This is <u>not</u> stronger if A is invertible, which is true if and only if $b \neq 0$.

$$b_{11}>0,\ b_{22}>0 \qquad B-A'BA>0 \tag{Ap.10}$$

Equation (Ap.10) can be further reduced to the following equations (after we normalize $b_{11}=1$, which we can do simply by scaling B):

$$\begin{aligned} &0<b_{22}<(1-a^2) \\ &-b^2+(b^2+1-a^2)b_{22}-b_{22}^2>0 \end{aligned} \tag{Ap.10}$$

We can rewrite the second inequality as

$$(a^2-b^2-1)b_{22} < -b_{22}^2-b^2 \tag{Ap.11}$$

and the possibilities are given in Figure Ap.2. If $(a^2-b^2-1)>0$, either we have no region in which (Ap.11) holds (b) or the region is for negative values of b_{22} (d), which biolates the first inequality in (Ap.11). Thus, we must have

$$a^2-b^2-1<0 \tag{Ap.12}$$

and in fact, we must have case (c), which means that there must be two real solutions to (Ap.11) when the inequality is made into an equality. Some algebraic manipulations yield the inequalities

$$0<b_{22}<(1-a^2) \tag{Ap.13a}$$

$$\sigma(a,b) \triangleq a^2-b^2-1<0 \tag{Ap.13b}$$

$$\rho(a,b) \triangleq (1-a^2+b^2)^2 - 4b^2>0 \tag{Ap.13c}$$

$$\frac{-\sigma(a,b)-\sqrt{\rho(a,b)}}{2} < b_{22} < \frac{-\sigma(a,b)+\sqrt{\rho(a,b)}}{2} \tag{Ap.13d}$$

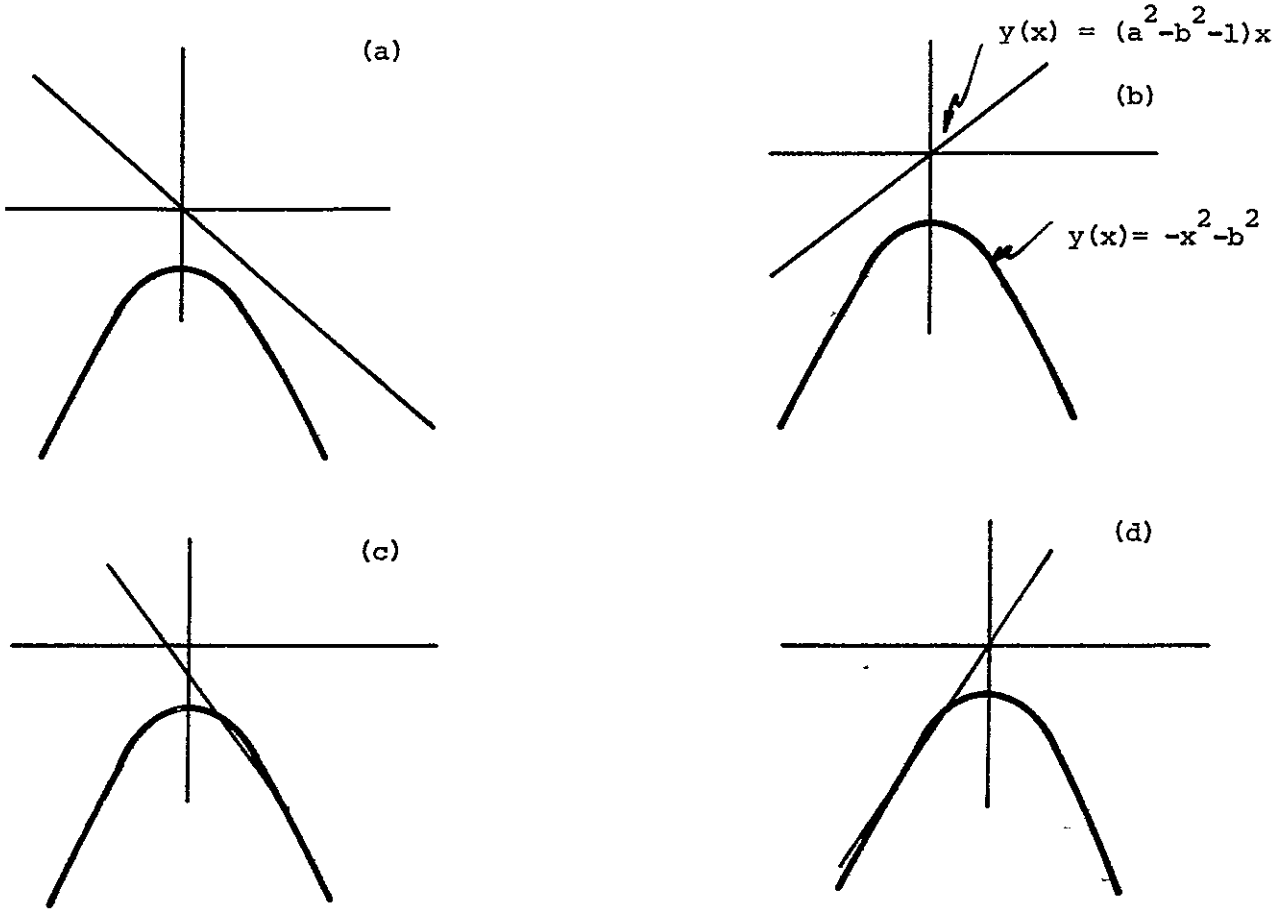

Figure Ap.2: Illustrating Inequality (Ap.11)

Using (Ap.13b), (Ap.13c) is equivalent to

$$1-a^2+b^2 > 2|b|$$

or (Ap.14)

$$|a|<|1-|b||$$

and under this condition, both (Ap.13b) and (Ap.13c) hold. Then, we will have that we can find a value of b_{22} if and only if the inequalities (Ap.13a) and (Ap.13d) overlap. Combining these, we find that the region of (a,b) -space for which we can use this technique to prove stability is

$$|a|< \ 1-|b| \ , \qquad |b|<1 \qquad \text{(Ap.15)}$$

which is illustrated in Figure Ap.3 The triangle is the region in which the linear system (Ap.1) is asymptotically stable and the cross-hatched area is (Ap.15). In the remaining part of the triangle, one must use a non-diagonal B, and this technique will not work. This is not to say that Lyapunov functions can't be found that will prove stability in these regions of (a,b) space in the one magnitude truncator case, but rather that one will have to work harder to find them if they exist (either by working directly with (A.6a) or by looking for nonquadratic Lyapunov functions). This derivation hopefully illustrates the type of argument that one can make using Lyapunov functions and also the difficulties and the limitations of the technique.

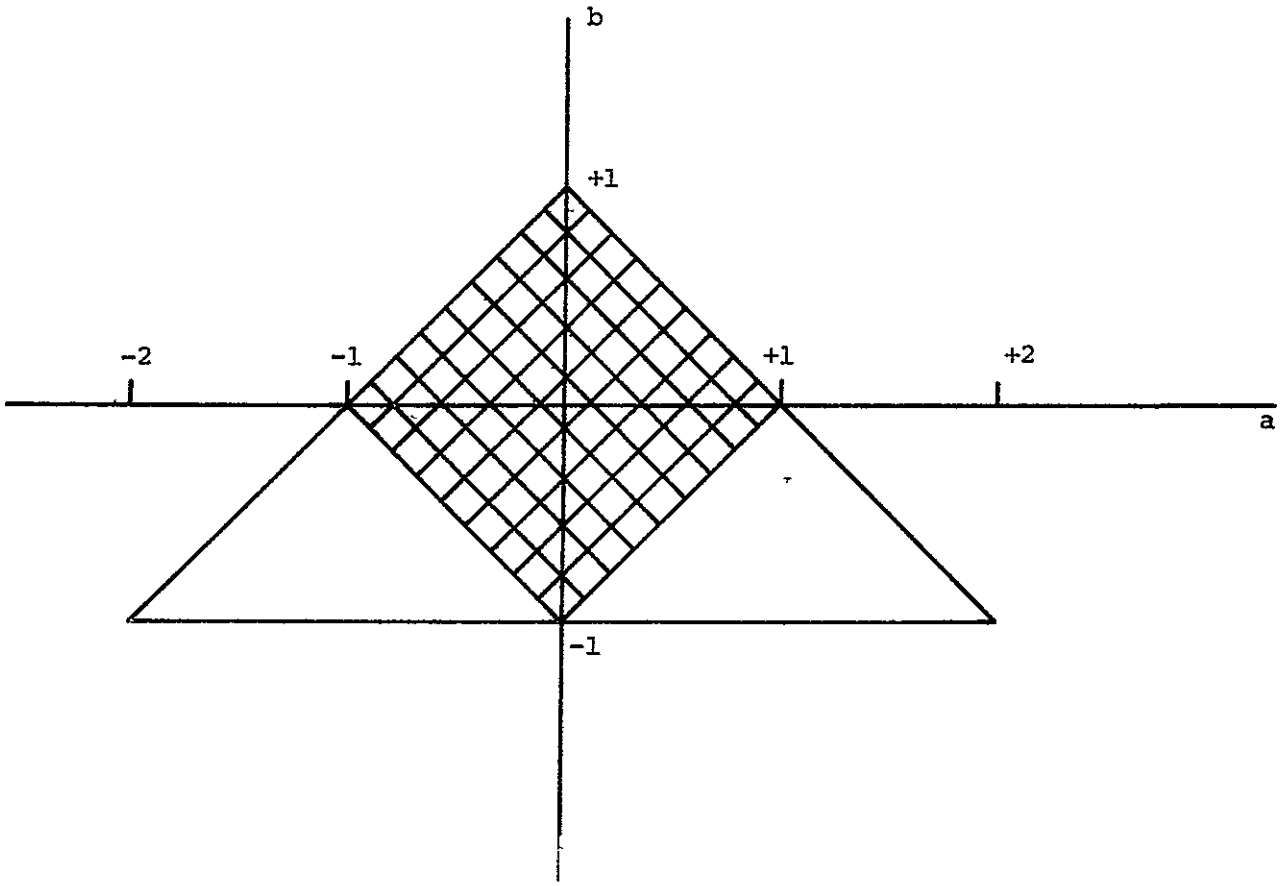

Figure Ap.3: Illustrating the Stability Result

Appendix 2: The Discrete Fourier Transform and Circulant Matrices

Circulant matrices appear in several places in Section D. In this appendix we indicate some of their properties. Suppose we have a block circulant matrix A

$$A = \begin{bmatrix} A_0 & A_{N-1} & \cdots\cdots & A_1 \\ A_1 & A_0 & \cdots\cdots & A_2 \\ \vdots & \vdots & & \vdots \\ A_{N-1} & A_{N-2} & \cdots\cdots & A_0 \end{bmatrix} \tag{Ap.16}$$

where each A_i is PxQ. Consider the equation

$$y = Ax \tag{Ap.17}$$

where y is an NP vector, partitioned into P-vectors

$$y' = (y_0', \ldots, y_{N-1}') \tag{Ap.18}$$

and x is an NQ-vector, partitioned into Q-vectors

$$x' = (x_0', \ldots, x_{N-1}') \tag{Ap.19}$$

Combining (Ap.16)-(Ap.19) we obtain

$$y_i = \sum_{j=0}^{N-1} A_j x_{i-j} \tag{Ap.20}$$

where all subscripts are to be interpreted modulo N. Hence, the right-hand side of (Ap.17) is nothing more than a cyclic convolution. Let us take the DFT of the sequences $\{y_i\}$, $\{x_i\}$, $\{A_i\}$, where, for example

$$y(\ell) = \sum_{i=0}^{N-1} y_i W_N^{-i\ell} \qquad , \ \ell=0,\ldots,N-1 \tag{Ap.21}$$

where

$$W_N = e^{j2\pi/N} \tag{Ap.22}$$

In the transformed domain, we now have N decoupled sets of equations

$$y(\ell) = A(\ell)x(\ell) \qquad , \ \ell=0,\ldots,N-1 \tag{Ap.23}$$

and we have effectively block diagonalized the block circulant matrix A. If we also have that P=Q and that each of the A_i is circulant, then each of the $A(\ell)$ is circulant, and we can diagonalize each of them by iterating the above development. Thus we can use the FFT to diagonalize A. In addition, if we write

$$\overline{y}' = (y(0)', \ldots, y(N-1)') \triangleq Ty \tag{Ap.24}$$

$$\overline{x}' = (x(0)', \ldots, x(N-1)') \triangleq Sx \tag{Ap.25}$$

(here S=T if P=Q), we observe that

$$TAS' = \mathrm{diag}\ (A(0), \ldots, A(N-1)) \tag{Ap.26}$$

Therefore, in this case, the calculation of Ms, where M is the matrix of eigenvectors of A, can be performed using the FFT.

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