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RESEARCH STUDY ON STABILIZATION AND CONTROL

MODERN SAMPLED-DATA CONTROL THEORY

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PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA
BIMONTHLY REPORT

DIGITAL CONTROLLER DESIGN

subtitle:

CONTINUOUS AND DISCRETE
DESCRIBING FUNCTION ANALYSIS
OF THE IPS SYSTEM

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I. MODEL DEVELOPMENT OF THE CONTINUOUS-DATA IPS
CONTROL SYSTEMS

The objective of this chapter is to develop the dynamic equations and
the mathematical model of the continuous-data IPS control system. The IPS
model considered includes one flexible body mode and is hardmounted to the
Orbiter/Pallet. The model contains equations describing a torque feed-forward
loop (using accelerometers as inputs) which will aid in reducing the pointing
errors caused by Orbiter disturbances.

The equations of motion of the IPS system are written as

\[ MR + DR + KR + Q\dot{R}dt = F \]  \hspace{1cm} (1-1)

where

\[ R = \begin{bmatrix}
X_s \\
Z_s \\
\theta_s \\
\theta_i \\
x \\
z \\
\eta
\end{bmatrix} \]  \hspace{1cm} (1-2)

\( X_s, Z_s \): translations of the Orbiter
\( \theta_s \): attitude of the Orbiter
\( \theta_i \): attitude of the instrument
\( X, Z \): accelerometer measurements
\( M \): \( 7 \times 7 \) mass matrix
\( D \): \( 7 \times 7 \) damping matrix
\( K \): \( 7 \times 7 \) stiffness matrix
\( Q \): \( 7 \times 7 \) integral control matrix
\( F \): \( 7 \times 1 \) generalized force vector
The elements of $M$ are:

$m_{11} = m_0 + m_i = 91,200$

$m_{13} = m_i (d_{sm} + c)_z = -226$

$m_{14} = m_i (m_{bz} c \phi + r_{bx} S\phi) = 2,270$

$m_{22} = m_0 + m_i = 91,200$

$m_{23} = -m_i (d_{sm} + c)_x = 16,600$

$m_{24} = -m_i (-r_{bz} S \phi + r_{bx} C\phi) = 3,920$

$m_{31} = m_{13} = m_i (d_{sm} + c)_z = -226$

$m_{32} = m_{23} = 16,600$

$m_{33} = l_{0y} + l_{iy} + m_i ((d_{sm} + c)_x^2 + (d_{sm} + c)_z^2) = 7.06 \times 10^6$

$m_{34} = l_{iy} + m_i ((d_{sm} + c)_z (r_{bz} C \phi + r_{bx} S\phi) + (d_{sm} + c)_x (-r_{bz} S \phi + r_{bx} C\phi)) = 26,700$

$m_{41} = m_{14} = 2,270$

$m_{42} = m_{24} = 3,920$

$m_{43} = m_{34} = 26,700$

$m_{44} = l_{iy} + m_i (r_{bx}^2 + r_{bz}^2) = 10,300$

$m_{51} = -1.0$

$m_{53} = -d_{smz} = 0.929$

$m_{55} = 1.0$

$m_{57} = h_x = -0.00113$

$m_{63} = d_{smx} = -4.72$

$m_{66} = 1.0$

$m_{67} = h_z = 0.00137$

$m_{77} = 1.0$

$m_{62} = -1$

All other elements of $M$ are zero.
The elements of \( D \) are:
\[
\begin{align*}
d_{43} &= d_{44} = K_r = 19,700 \\
d_{47} &= K_r \sigma_{rg} = -76.4 \\
d_{55} &= 2c_x \omega_x = 44 \\
d_{66} &= 2c_z \omega_z = 44 \\
d_{77} &= 2c_b \omega_b = 0.115
\end{align*}
\]
All other elements of \( D \) are zero.

The elements of \( K \) are:
\[
\begin{align*}
k_{43} &= k_{44} = K_p = 70,000 \\
k_{45} &= -m_i (r_{bz} c_\phi + r_{bx} s_\phi) \omega_y^2 = -2.23 \times 10^6 \\
k_{46} &= m_i (-r_{bz} s_\phi + r_{bx} c_\phi) \omega_y^2 = -3.87 \times 10^6 \\
k_{47} &= K_p \sigma_{ss} = -272 \\
k_{55} &= \omega_x^2 = 986 \\
k_{66} &= \omega_z^2 = 986 \\
k_{77} &= \omega_b^2 = 132
\end{align*}
\]
All other elements of \( K \) are zero.

The elements of \( Q \) are:
\[
\begin{align*}
q_{43} &= q_{44} = K_i = 1.1 \times 10^5 \\
q_{47} &= K_i \sigma_{ss} = -427
\end{align*}
\]
All other elements of \( Q \) are zero.

The generalized force vector is
\[
F = [F_x \quad F_z - r_{cx} \quad F_{cz} - r_{cz} \quad 0 \quad 0 \quad h_{cz} \quad 0 \quad h_{cx} \quad h_{cz} \quad 0 \quad h_{cx} \quad h_{cz} \quad 0 \quad 0 \quad h_{cx} \quad h_{cz} \quad 0 \quad h_{cx} \quad h_{cz} \quad 0 \quad h_{cx} \quad h_{cz} \quad 0 \quad h_{cx} ]'
\]

Given the elements of the matrices \( M, D, K \) and \( Q \), the equations of motion of Eq. (1-1) are written as
\[ m_{11} \dddot{x}_s + m_{13} \dddot{\theta}_s + m_{14} \dddot{\theta}_i = F_x \]  \hfill (1-4)

\[ m_{22} \dddot{x}_s + m_{23} \dddot{\theta}_s + m_{24} \dddot{\theta}_i = F_z \]  \hfill (1-5)

\[ m_{31} \dddot{x}_s + m_{32} \dddot{z}_s + m_{33} \dddot{\theta}_s + m_{34} \dddot{\theta}_i = r_{cx} F_z - r_{cz} F_x \]  \hfill (1-6)

\[ m_{41} \dddot{x}_s + m_{42} \dddot{z}_s + m_{43} \dddot{\theta}_s + m_{44} \dddot{\theta}_i + K_r \dot{\theta}_s + K_r \dot{\theta}_i + K_r \sigma_{rg} \]  

\[ + K_p \theta_s + K_p \theta_i + k_{45} \dot{x} + k_{46} \dot{z} + k_{47} \eta + K_I \theta_s dt \]  

\[ + K_I \int \theta_i dt + K_I \sigma_{ss} \int \eta dt = 0 \]  \hfill (1-7)

\[ m_{51} \dddot{x}_s + m_{53} \dddot{\theta}_s + m_{54} \dddot{\theta}_s + m_{55} \dddot{x}_s = 0 \]  \hfill (1-8)

\[ m_{62} \dddot{z}_s + m_{63} \dddot{\theta}_s + m_{64} \dddot{\theta}_s + m_{65} \dddot{z}_s = 0 \]  \hfill (1-9)

\[ m_{77} \ddot{\eta} + d_{77} \dot{\eta} + k_{77} \eta = h_{z_c} F_z + h_{x_c} F_x \]  \hfill (1-10)

Without the nonlinear wire-cable and flex-pivot torques, the control torque is expressed as

\[ T_c = -K_r \dot{\theta}_s - K_r \dot{\theta}_i - K_p \theta_s - K_p \theta_i - K_I \theta_s dt - K_I \theta_i dt \]  

\[ - k_{45} \dot{x} - k_{46} \dot{z} \]  \hfill (1-11)

where \( K_r, K_p, \) and \( K_I \) are the rate, position, and integral constants of the controller, respectively.

The nonlinear torque due to the flex pivot and wire cable can be lumped into one operator \( N \) which operates on \( \theta_i \). Thus, the component, \( -N(\theta_i) \), should be added to Eq. (1-11); i.e.,

\[ T_c = -K_r \dot{\theta}_s - K_r \dot{\theta}_i - K_p \theta_s - K_p \theta_i - K_I \theta_s dt - K_I \theta_i dt \]  

\[ - k_{45} \dot{x} - k_{46} \dot{z} - N(\theta_i) \]  \hfill (1-12)

Solving for \( x_s \) from Eq. (1-4), we have

\[ \dddot{x}_s = - \frac{m_{13}}{m_{11}} \dddot{\theta}_s - \frac{m_{14}}{m_{11}} \dddot{\theta}_i + \frac{1}{m_{11}} F_x \]  \hfill (1-13)
Solving for $Z_s$ from Eq. (1-5), we have

$$Z_s = -\frac{m_{23}}{m_{22}} \ddot{\psi}_s - \frac{m_{24}}{m_{22}} \ddot{\psi}_i + F_z \frac{1}{m_{22}} \tag{1-14}$$

Substitute Eq. (1-13) and (1-14) into Eq. (1-6), we have

$$-\frac{m_{31} m_{13}}{m_{11}} \ddot{\psi}_s - \frac{m_{31} m_{14}}{m_{11}} \ddot{\psi}_i + \frac{m_{31}}{m_{11}} F - \frac{m_{32} m_{23}}{m_{22}} \ddot{\psi}_s - \frac{m_{32} m_{24}}{m_{22}} \ddot{\psi}_i + \frac{m_{32}}{m_{22}} F_z$$

$$+ m_{33} \ddot{\psi}_s + m_{34} \ddot{\psi}_i = r \sigma z - r \sigma x \quad \tag{1-15}$$

The last equation is rearranged to the following form:

$$\left( \begin{array}{c} m_{33} - \frac{m_{31} m_{13}}{m_{11}} - \frac{m_{32} m_{23}}{m_{22}} \\ m_{11} \end{array} \right) \ddot{\psi}_s + \left( \begin{array}{c} m_{34} - \frac{m_{31} m_{14}}{m_{11}} - \frac{m_{32} m_{24}}{m_{22}} \\ m_{22} \end{array} \right) \ddot{\psi}_i$$

$$= r \sigma z - r \sigma x - \frac{m_{31}}{m_{11}} F_x - \frac{m_{32}}{m_{22}} F_z \quad \tag{1-16}$$

Substitute Eqs. (1-13) and (1-14) into Eq. (1-7), and making use of Eq. (1-12), we have

$$m_{41} \left( \frac{\ddot{\psi}_s - \frac{m_{14}}{m_{11}} \ddot{\psi}_i + \frac{1}{m_{11}} F_x} {m_{11}} \right) + m_{42} \left( \frac{\ddot{\psi}_s - \frac{m_{24}}{m_{22}} \ddot{\psi}_i + F_z \frac{1}{m_{22}}} {m_{22}} \right)$$

$$+ m_{43} \ddot{\psi}_s + m_{44} \ddot{\psi}_i + K_r \sigma r \tilde{r} + K_{i, r} \sigma s \int \tilde{r} \, dt - T_c = 0 \quad \tag{1-17}$$

Simplifying, the last equation becomes

$$\left( m_{43} \frac{m_{13} m_{41}}{m_{11}} - \frac{m_{42} m_{23}}{m_{22}} \right) \ddot{\psi}_s + \left( m_{44} \frac{m_{14} m_{41}}{m_{11}} - \frac{m_{42} m_{24}}{m_{22}} \right) \ddot{\psi}_i$$

$$+ K_r \sigma r \tilde{r} + K_{i, r} \sigma s \int \tilde{r} \, dt - T_c = - \frac{m_{41}}{m_{11}} F_x - \frac{m_{42}}{m_{22}} F_z \quad \tag{1-18}$$

Similarly, Eq. (1-9) is written as

$$m_{62} \left( \frac{\ddot{\psi}_s - \frac{m_{24}}{m_{22}} \ddot{\psi}_i + \frac{1}{m_{22}} F_z} {m_{22}} \right) + m_{63} \ddot{\psi}_s + m_{66} \ddot{Z} + m_{67} \ddot{n} + d_{66} \ddot{Z} + k_{66} \ddot{Z} = 0 \quad \tag{1-19}$$
Equations (1-16), (1-18), (1-19), (1-8) and (1-10) are now written as

\[
\begin{align*}
M_s \ddot{\theta}_s + M_i \ddot{\theta}_i &= - \left( \frac{r_{cz} + m_{31}}{m_{11}} \right) F_x + \left( \frac{r_{cx} - m_{32}}{m_{22}} \right) F_z \\
M_k \ddot{\theta}_s + M_n \ddot{\theta}_n + K_r \dot{r} g \dot{\eta} + k_{47} \dot{\eta} + K_i \dot{r} s_s f \dot{\eta} d t - T_c &= - \frac{m_{41}}{m_{11}} F_x - \frac{m_{42}}{m_{22}} F_z
\end{align*}
\]

where

\[
\begin{align*}
M_s &= m_{33} - \frac{m_{31} m_{31}}{m_{11}} - \frac{m_{32} m_{23}}{m_{22}} \\
M_i &= m_{34} - \frac{m_{31} m_{41}}{m_{11}} - \frac{m_{32} m_{24}}{m_{22}} \\
M_k &= m_{43} - \frac{m_{13} m_{41}}{m_{11}} - \frac{m_{42} m_{23}}{m_{22}} \\
M_n &= m_{44} - \frac{m_{14} m_{41}}{m_{11}} - \frac{m_{42} m_{24}}{m_{22}} \\
M_p &= - \frac{m_{62} m_{23}}{m_{22}} + m_{63} \\
M_r &= - \frac{m_{62} m_{24}}{m_{22}} \\
M_u &= M_{53} - \frac{m_{51} m_{13}}{m_{11}} \\
M_v &= - \frac{m_{51} m_{14}}{m_{11}}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{m_{53} - \frac{m_{51} m_{13}}{m_{11}}}{} \right) \ddot{\theta}_s - \frac{m_{51} m_{14}}{m_{11}} \ddot{\theta}_i + m_{55} \dddot{\chi} + m_{57} \dddot{\eta} + m_{55} \dddot{\chi} + k_{55} \dddot{\chi} &= - \frac{m_{51}}{m_{11}} F_x \\
m_{77} \dddot{\eta} + d_{77} \dddot{\eta} + k_{77} \dddot{\eta} &= h_{zc} F_z + h_{xc} F_x
\end{align*}
\]

These last five differential equations are rearranged so that a block diagram
can be constructed.

\[ \ddot{\phi}_s = -\frac{M_i}{M_s} \ddot{\phi}_s - \frac{1}{M_s} \left( \frac{m_{31}}{m_{11}} \right) F_x + \frac{1}{M_s} \left( \frac{r_{cz}}{m_{21}} - \frac{m_{32}}{m_{22}} \right) F_z \]  

(1-33)

\[ \ddot{\phi}_i = -\frac{M_k}{M_n} \ddot{\phi}_s - \frac{K_r}{M_n} \ddot{\phi}_i - \frac{k_{47}}{M_n} \eta - \frac{1}{M_n} \int \eta dt + \frac{1}{M_n} T_c \]

\[ = -\frac{m_{41}}{m_{11}} F_x - \frac{m_{42}}{m_{11}} F_z \]  

(1-34)

\[ \ddot{z} = -\frac{M_p}{m_{66}} \ddot{\phi}_s - \frac{M_r}{m_{66}} \ddot{\phi}_i - \frac{m_{67}}{m_{66}} \eta - \frac{d_{66}}{m_{66}} \ddot{z} - \frac{k_{66}}{m_{66}} z - \frac{m_{62}}{m_{22}m_{66}} F_z \]  

(1-35)

\[ \ddot{x} = -\frac{M_u}{m_{55}} \ddot{\phi}_s - \frac{M_v}{m_{55}} \ddot{\phi}_i - \frac{m_{57}}{m_{55}} \eta - \frac{d_{55}}{m_{55}} \ddot{x} - \frac{k_{55}}{m_{55}} x - \frac{m_{51}}{m_{11}m_{55}} F_x \]  

(1-36)

\[ \ddot{\eta} = -\frac{d_{77}}{m_{77}} \ddot{\eta} + \frac{k_{77}}{m_{77}} \eta + \frac{h_{xc}}{m_{77}} F_z + \frac{h_{xc}}{m_{77}} F_x \]  

(1-37)

The control torque \( T_c \) is given by Eq. (1-12).

The block diagram which portrays the differential equations of Eqs. (1-33) through (1-37) is shown in Fig. 1-1. The nonlinear element which represents the nonlinear torque due to the flex pivot and wire cable is also included in the block diagram. The signal flow graph representation of Fig. 1-1 for the purpose of evaluating the determinant is shown in Fig. 1-2. The dynamics of \( \eta \) is eliminated since they do not enter the determinant of the signal flow graph.

Applying Mason's gain formula to Fig. 1-2, the determinant of the signal flow graph is evaluated as follows:

\[ \Delta = 1 + \frac{N_s + K_s + K_1 + K_r s^2 - \frac{M_v}{m_{55}} k_{45} G_x s^3 - \frac{M_r}{m_{66}} k_{46} G_z s^3}{\frac{M_n s^3}{m_{11}} + \frac{M_{55}}{m_{55}} k_{45} G_x s^3 - \frac{M_{66}}{m_{66}} k_{46} G_z s^3} \]
Figure 1-1. Block diagram of the IPS system with wire-cable and flex-pivot nonlinearities.
Figure 1-2. Signal flow graph of the IPS system.
\[ M_s M_s n^3 + M_n \left( Ns + K_p s + K_i + K_r s^2 - \frac{M_v}{m_{55}} k_{45} G x s^3 - \frac{M_r}{m_{66}} k_{46} G z s^3 \right) \]
\[ + M_1 \left( \frac{M_u}{m_{55}} k_{45} G x s^3 + \frac{M_p k_{46}}{m_{66}} G z s^3 - K_p s - K_i - K_r s^2 - M_k s^3 \right) = 0 \] (1-38)

where
\[ G_x = \frac{1}{s^2 + \frac{d_{55}}{m_{55}} s + \frac{k_{55}}{m_{55}}} \] (1-39)
\[ G_z = \frac{1}{s^2 + \frac{d_{66}}{m_{66}} s + \frac{k_{66}}{m_{66}}} \] (1-40)

Rearranging both sides of Eq. (1-38), we have
\[ M_s M_s n^3 + M_n \left( Ns + K_p s + K_i + K_r s^2 - \frac{M_v}{m_{55}} k_{45} G x s^3 - \frac{M_r}{m_{66}} k_{46} G z s^3 \right) \]
\[ + M_1 \left( \frac{M_u}{m_{55}} k_{45} G x s^3 + \frac{M_p k_{46}}{m_{66}} G z s^3 - K_p s - K_i - K_r s^2 - M_k s^3 \right) = 0 \] (1-41)

Dividing both sides of the last equation by the terms that do not contain \( N \), we get the equivalent linear transfer function that the nonlinear element \( N \) sees,
\[ G_{eq}(s) = \frac{M_s}{(M_s M_n - M_i k) s^3 + M_n \left( K_p s + K_i + K_r s^2 - \frac{M_v}{m_{55}} k_{45} G x s^3 - \frac{M_r}{m_{66}} k_{46} G z s^3 \right)} \]
\[ + M_1 \left( \frac{M_u}{m_{55}} k_{45} G x s^3 + \frac{M_p k_{46}}{m_{66}} G z s^3 - K_p s - K_i - K_r s^2 \right) \] (1-42)

For the system parameters given, \( G_x = G_z \); thus, Eq. (1-42) is simplified to
\[ G_{eq}(s) = \frac{M_s \left( s^2 + \frac{d_{55}}{m_{55}} s + \frac{k_{55}}{m_{55}} \right)}{\left( (M_s M_n - M_i k) s^3 + (M_n - M_i) (K_p s^2 + K_i s + K_i) \right) \left( s^2 + \frac{d_{55}}{m_{55}} s + \frac{k_{55}}{m_{55}} \right)} \]
\[ + \left( \frac{M_u}{m_{55}} k_{45} s^3 + \frac{M_p}{m_{66}} k_{46} s^3 \right) \] (1-43)
Substitution of the system parameters into Eq. (1-43) with

\[ M_s = 7056977.95 \]
\[ M_k = M_l = 25992.12 \]
\[ M_n = 10074.90 \]
\[ M_u = 0.926522 \]
\[ M_p = -4.537982 \]
\[ M_r = 0.042982 \]
\[ M_v = 0.02489 \]

we have

\[ G(s) \text{eq } = \frac{1.002077 \times 10^{-4} s(s^2 + 44s + 986)}{s^5 + 45.96676s^4 + 1107.4783s^3 + 2257.769s^2 + 7374.0808s + 10828.48765} \]

(1-44)

It is interesting to note that the system shown in Fig. 1-1 is of the 11th order. However, since the dynamics of \( \eta \) do not enter the picture on the stability of the nonlinear system, and since \( G_x = G_z \), the transfer function seen by \( N \) is only of the 5th order.
II. PREDICTION OF SELF-SUSTAINED OSCILLATIONS OF THE CONTINUOUS-DATA IPS SYSTEM WITH WIRE-CABLE AND FLEX-PIVOT NONLINEARITIES

The equivalent transfer function \( G_{eq}(s) \) in Eq. (1-44) can be plotted in the gain-phase coordinates together with the plot of \(-1/N\) of the nonlinear element for stability analysis. The describing function of the combined flex-pivot and wire-cable nonlinearity using the Dahl model has been derived in [1]. In this earlier report a simplified model of the IPS system was obtained by assuming that all but motion about two of the seven degrees of freedom of axes are negligible. Only motion about the scientific instrument axis and the mount rotation were considered.

For the purpose of comparison, the equivalent transfer function that the nonlinearity sees for the simplified IPS in [1] is repeated as follows:

\[
G_{eq}(s) = \frac{0.0013946s(s^2 + 0.0012528s + 0.0036846)}{s^5 + 16.7s^4 + 222.6s^3 + 279.806s^2 + 5.857s + 5.13855} \] (2-1)

when the integral control constant \( K_I \) is \( 10^6 \). The zeros of this transfer function are \( s = -0.0006264 + j0.06069764 \) and \( s = -0.0006264 - j0.06067674 \). The five poles are at \( s = -1.38135, s = -0.0031956 + j0.135895, s = -0.0031956 - j0.135895, s = -7.65618 + j11.946, \) and \( s = -7.65618 - j11.946 \). It was concluded in [1] from the gain-phase plot of \( G_{eq}(s) \) versus the \(-1/N(A)\) plot that the simplified IPS may have a sustained oscillation that is characterized by the frequency of \( \omega = 0.16 \) rad/sec. Depending on the values of the parameters of the wire-cable and flex-pivot nonlinearities, \( K_{WT} \) and \( h_{WT} \), the amplitude of oscillation of \( \epsilon \), which is comparable to \( \Theta_i \) in this report, lies between \( 3 \times 10^{-8} \) to \( 3 \times 10^{-6} \) for \( K_I = 10^6 \).

The equivalent transfer function of Eq. (1-44) is plotted as shown in Fig. 2-1. It is noted that the IPS model used in the present work does not
Figure 2-1. Frequency response plot and describing function plot of IPS system with flex-pivot and wire-cable nonlinearities (Dahl model).
have the same system parameters as the simplified IPS system studied in [1].
Thus, the natural frequencies of the two systems are different. From Fig.
2-1 it is observed that the $G_{eq}(s)$ and $-1/N$ loci would intersect only at
very low frequencies and when the amplitude of oscillation is extremely
small. For instance, at $\omega = 10^{-3}$ rad/sec, the amplitude of oscillation
is approximately $10^{-10}$, and for all practical purposes this can be regarded
as zero. Thus, the IPS model considered here would not exhibit sustained
oscillations due to the wire-cable and flex-pivot nonlinearities.

It is of interest to investigate the poles and zeros of the transfer
function of Eq. (1-44). The poles and zeros of $G_{eq}(s)$ of Eq. (1-44) are
tabulated as follows:

zeros:

$s = 0, \ -22 + j22.4053565, \ -22 - j22.4053565$

poles:

$s = -1.66963, \ -0.123519 + j2.5232$

$s = -0.123519 - j2.5232$

$s = -22.025 + j23.0467$

$s = -22.025 - j23.0467$

The natural frequency of the dominant poles of the $G_{eq}(s)$ of Eq. (1-44)
is found to be at 2.5232 rad/sec. This is much higher that that of the
transfer function of Eq. (2-1). Furthermore, the two zeros listed above
are very close to two of the poles.
III. DIGITAL COMPUTER SIMULATION OF THE CONTINUOUS-DATA NONLINEAR IPS CONTROL SYSTEM WITH DAHL MODEL AND ONE FLEXIBLE BODY MODE

The IPS control system with one flexible body mode and the nonlinear flex pivot torque modelled by the Dahl solid friction model \((i = 2)\) is simulated on the digital computer. A block diagram of the IPS system is shown in Fig. 1-1. For the Dahl model with \(i = 2\), the following parameters are used:

\[
\gamma = 9.2444 \times 10^4 \\
T_{FP0} = 2.25 \times 10^{-3} \text{ N-M}
\]

The objective of the computer simulation of the system is to verify and/or clarify the analysis results obtained by the describing function method in Chapter II. Although the describing function method analysis in Chapter II does not yield precise results for possible amplitudes and frequencies of sustained oscillations, it does indicate that for the sets of system parameters used if sustained oscillations were to exist the amplitude of oscillation would be approximately less than \(10^{-3}\) radians and the frequencies would be less than \(10^{-3}\) radians/sec.

Since the significant time constant of the IPS system is very long, it would require a long computer simulation time to verify the existence or nonexistence of a limit cycle. Normally, the possibility of sustained oscillations could be checked by using two computer simulations, one with initial conditions that correspond to above and the other with initial conditions that correspond to below the value of the predicted amplitude of oscillations. If a sustained oscillation exists then the simulation with initial conditions below the predicted amplitude will result in a response that oscillates near the predicted frequency and increases in
amplitude. Also, the simulation with initial conditions that correspond to an initial amplitude above the predicted amplitude will result in a response that decreases in amplitude and oscillates near the predicted frequency. However, since the analysis of Chapter II does not predict any specific sustained oscillations, the method described above must be changed.

Since the analysis of Chapter II provides no specific candidates for amplitudes and frequencies of limit cycles (or the amplitude and frequencies are all extremely small), it may be necessary to check for the absence of limit cycles. The absence of limit cycles can be verified by showing that the responses of the system due to any initial conditions actually decay to zero as time increases indefinitely.

Again, since the time constant of the IPS system is very long, the simulation of the system response by a digital computer may involve excessive computer time. Since if sustained oscillations were to exist, the possible frequencies of oscillations would be less than $10^{-3}$ rad/sec, the simulation time required would be in excess of $2\pi \times 10^3$ sec. Also, a double-precision program with 30 digits of accuracy would be required, since the amplitude of oscillation must be less than $10^{-9}$ radians. Instead, the results from the simulation will be used to verify that the oscillations of the system are damped out at a mode that is practically independent of the nonlinearity.

As a means of exciting the system, an initial perturbation of $\Theta_i(0) = 10^{-9}$ radians was applied to the system. Two simulations of the system were completed for two extreme sets of wire cable parameters ($K_{WT}$ and $H_{WT}$). The responses of $\Theta_i$ for both of these cases are shown in Figs. 3-1 and 3-2. These responses indicate that the influence of the wire cable parameters on the system response is very insignificant. In addition, the frequency of
Figure 3-1. Time response of $G_i(t)$ with $K_{MR} = 0.1$, $G_i(0) = 10$.
oscillation is approximately $2.5 \text{ rad/sec}$, which is approximately $10^9$ times greater than the maximum possible frequency of oscillation. Also, the envelope of the response is decaying with a mode of $\sigma = -0.1239$, where $1/\sigma$ is considered as the time constant. Hence, the system over the 50 sec interval of simulation time has a response due to a pair of poles at $s = -0.1239 + j2.5$ and $s = -0.1239 - j2.5$, which is approximately one of the pairs of poles of $G_{eq}(s)$. Thus, the system oscillates at the dominant mode of $G_{eq}(s)$ as expected, if the system had no limit cycles and the nonlinearity were replaced by a unity gain element. This implies that the nonlinearity has very little influence on the the system response under the simulated conditions, and that no limit cycles exist.

Since the size of the feasible amplitudes predicted is so small, at least 15 digits are needed to carry out the 50 seconds of computer simulation.

The listing of the computer simulation program is given in Table 3-1.
Figure 3-1. Computer simulation program of continuos-data IPS system.

```fortran
PROGRAM IPSSIM (INPUT,IPSD2,TAPE6=IPSD2,OUTPUT=IPSD2)
IMPLICIT REAL (6)
COMMON SS,AKs,AKP,AK45,AK66,SIG66,D77,M77,AK77,HZC,HXC
MM,IR,AR,MF,MT,MR,AK66,AK77,HZC,HXC
C MAIN CALL RK'S (R(K,T,Y,DERY,i4DTM,HLF,1'CTY,OUT,F11X)
STOP
E1'1 D
```

The program snippet includes declarations and initialization of variables, followed by the main call to a subroutine (presumably RK's).
SUBROUTINE OUTP (TIME, Y, DERY, IHLF, NDTIM, PRMT)
DIMENSION Y(NDIM), DERY(NDIM), PRMT(7)
DELTIME = PRMT(7)
IF (ABS (PRMT(1) - DEL) .LE. 10) GO TO 1
IF (ABS (PRMT(5) - DEL) .LE. 10) GO TO 1
RETURN
1
PRMT(7) = TIME
WRITE (15, IHLF, TIME, Y(3), Y(8))
RETURN
STOP
END

SUBROUTINE FCY (TIME, STATE, STDOT, KERP, NDTIM)
IMPLICIT REAL (8)
COMMON STS(1), AK(4), AKP, AKS, AK4, AK1, AK45, AK46, SIGR, D77, M77, AK77, HZC
IF (STATE(3) .LT. 2) GOTO 7
IF (STATE(3) .GT. 2) GOTO 8
IF (STATE(3) .LT. 2) GOTO 9
IF (STATE(3) .GT. 2) GOTO 10
RETURN
7
8
9
10
RETURN
STOP
END

Table 3-1 (continued).
Table 3-1 (continued),

SUBROUTINE RKGS (PRMT,Y,DER,Y,NDIM,ILFL,FCT,OUTP,AUX)

DIMENSION Y(NDIM), DER(NDIM), AUX(8,NDIM), A(4), B(4)

C
C

DO 1 I=1,NDIM
1 AUX(I,1)=.75566667*DERY(I)

X=PRMT(I)
XEND=PRMT(I+1)
X=PRMT(I+1)+.2.
PRMT(I+2)=0.
KEEP=0.
CALL FCT(X,Y,DER,KEEP,NDIM)

ERROR TEST IF(H*(X+XEND-H))/39, 37, 2

PREPARATIONS FOR RUNGE-KUTTA METHOD

A(1) = .5.
A(2) = 2.92932.
A(3) = 1.666657.
A(4) = 2.3.
B(1) = 1.
B(2) = 1.
B(3) = 1.
B(4) = 1.
C(1) = 2.
C(2) = 2.92932.
C(3) = 1.707107.
C(4) = 5.

PREPARATIONS OF FIRST RUNGE-KUTTA STEP

DO 3 I=1,NDIM
AUX(I,1)=Y(I)
AUX(I,2)=DERY(I)
AUX(I,3)=AUX(I)
3 H=H+H.
ILFL=-1.
ISTEP=0.
END=0.

START OF A RUNGE-KUTTA STEP IF((X+H-XEND)*H)/7, 6, 5

5 H=XEND-X.
6 END=1.

RECORDING OF INITIAL VALUES OF THIS STEP CALL OUTP(X,Y,DER,IPFC,NDIM,PRMT)

8 ISTEP=0.
9 ISTEP=ISTEP+1.

START OF INNERMOST RUNGE-KUTTA LOOP

J=1.
10 AJ=A(J)
CJ=C(J).
DO 11 I=1,NDIM
11 R=H*DERY(I)
A2=AJ+(1-JO)*AUX(6,I))
Y(I)=Y(I)+.33
R=32+332.
12 AUX(5,I)=AUX(6,I)+32-CJ*R1
13 IF(J-4)=12, 15, 15.
14 J=J+1.
IF(J-3)=13, 14, 13.
15 X=X+.5.
16 CALL FCT(X,Y,DER,KEEP,NDIM)
GOTO 10.

END OF INNERMOST RUNGE-KUTTA LOOP

REPRODUCIBILITY OF 

ORIGINAL PROGRAM
Table 3-1 (continued).

<table>
<thead>
<tr>
<th>Line</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>IF (ITEST) 15, 16, 29</td>
</tr>
<tr>
<td>15</td>
<td>IF (ITEST) 15, 16, 29</td>
</tr>
<tr>
<td>16</td>
<td>IN CASE ITEST = 0 THERE IS NO POSSIBILITY FOR TESTING OF ACCURACY</td>
</tr>
<tr>
<td>17</td>
<td>DO 17 I = 1, NDIM</td>
</tr>
<tr>
<td>18</td>
<td>AUX (4, I) = Y (I)</td>
</tr>
<tr>
<td>19</td>
<td>ISTEP = ISTEP + ISTEP - 2</td>
</tr>
<tr>
<td>20</td>
<td>IHLP = IHLP + 1</td>
</tr>
<tr>
<td>21</td>
<td>H = H + 5</td>
</tr>
<tr>
<td>22</td>
<td>DO 19 I = 1, NDIM</td>
</tr>
<tr>
<td>23</td>
<td>Y (I) = AUX (1, I)</td>
</tr>
<tr>
<td>24</td>
<td>DFY (I) = AUX (2, I)</td>
</tr>
<tr>
<td>25</td>
<td>AUX (5, I) = Y (I)</td>
</tr>
<tr>
<td>26</td>
<td>GOTO 9</td>
</tr>
<tr>
<td>27</td>
<td>IN CASE ITEST = 1 TESTING OF ACCURACY IS POSSIBLE</td>
</tr>
<tr>
<td>28</td>
<td>IMOD = ISTEP / 2</td>
</tr>
<tr>
<td>29</td>
<td>IF (ISPEP = IMOD - IMOD) 21, 23, 21</td>
</tr>
<tr>
<td>30</td>
<td>CALL FCT (X, Y, DFY, KEEP, NDIM)</td>
</tr>
<tr>
<td>31</td>
<td>DO 22 I = 1, NDIM</td>
</tr>
<tr>
<td>32</td>
<td>AUX (5, I) = Y (I)</td>
</tr>
<tr>
<td>33</td>
<td>GOTO 9</td>
</tr>
<tr>
<td>34</td>
<td>COMPUTATION OF TEST VALUE DELT</td>
</tr>
<tr>
<td>35</td>
<td>DELT = 0.</td>
</tr>
<tr>
<td>36</td>
<td>DO 24 I = 1, NDIM</td>
</tr>
<tr>
<td>37</td>
<td>DELT = DELT + AUX (8, I) * ABS (AUX (4, I) - Y (I))</td>
</tr>
<tr>
<td>38</td>
<td>IF (DELT = PRMT (1 + 3)) 28, 29, 25</td>
</tr>
<tr>
<td>39</td>
<td>ERROR IS TOO GREAT</td>
</tr>
<tr>
<td>40</td>
<td>IF (IHLP = 1) 26, 35, 36</td>
</tr>
<tr>
<td>41</td>
<td>DO 27 I = 1, NDIM</td>
</tr>
<tr>
<td>42</td>
<td>AUX (4, I) = AUX (5, I)</td>
</tr>
<tr>
<td>43</td>
<td>ISTEP = ISTEP + ISTEP - 4</td>
</tr>
<tr>
<td>44</td>
<td>X = H</td>
</tr>
<tr>
<td>45</td>
<td>END = 0</td>
</tr>
<tr>
<td>46</td>
<td>GOTO 13</td>
</tr>
<tr>
<td>47</td>
<td>RESULT VALUES ARE GOOD</td>
</tr>
<tr>
<td>48</td>
<td>KEEP = 0</td>
</tr>
<tr>
<td>49</td>
<td>CALL FCT (X, Y, DFY, KEEP, NDIM)</td>
</tr>
<tr>
<td>50</td>
<td>DO 29 I = 1, NDIM</td>
</tr>
<tr>
<td>51</td>
<td>AUX (1, I) = Y (I)</td>
</tr>
<tr>
<td>52</td>
<td>AUX (2, I) = DFY (I)</td>
</tr>
<tr>
<td>53</td>
<td>AUX (3, I) = AUX (6, I)</td>
</tr>
<tr>
<td>54</td>
<td>Y (I) = AUX (5, I)</td>
</tr>
<tr>
<td>55</td>
<td>DFY (I) = AUX (7, I)</td>
</tr>
<tr>
<td>56</td>
<td>CALL FCT (X, Y, DFY, IHLP, NDIM, PRMT)</td>
</tr>
<tr>
<td>57</td>
<td>IF (PRMT (1 + 4)) 40, 36, 45</td>
</tr>
<tr>
<td>58</td>
<td>DO 31 I = 1, NDIM</td>
</tr>
<tr>
<td>59</td>
<td>Y (I) = AUX (1, I)</td>
</tr>
<tr>
<td>60</td>
<td>DFY (I) = AUX (2, I)</td>
</tr>
<tr>
<td>61</td>
<td>KEEP = IHLP</td>
</tr>
<tr>
<td>62</td>
<td>IF (KEEP = 2) GOTO 39</td>
</tr>
<tr>
<td>63</td>
<td>IF (END) 32, 32, 39</td>
</tr>
<tr>
<td>64</td>
<td>INCREMENT SETS DOUBLED</td>
</tr>
<tr>
<td>65</td>
<td>IHLP = IHLP - 1</td>
</tr>
<tr>
<td>66</td>
<td>ISTEP = ISTEP / 2</td>
</tr>
<tr>
<td>67</td>
<td>H = H + H</td>
</tr>
<tr>
<td>68</td>
<td>IF (IHLP) 4, 33, 33</td>
</tr>
<tr>
<td>69</td>
<td>IMOD = ISTEP / 2</td>
</tr>
<tr>
<td>70</td>
<td>IF (ISTEP - IMOD - IMOD) 4, 34, 4</td>
</tr>
<tr>
<td>71</td>
<td>IF (DELT = PRMT (1 + 3)) 35, 35, 4</td>
</tr>
<tr>
<td>72</td>
<td>IHLP = IHLP - 1</td>
</tr>
<tr>
<td>73</td>
<td>ISTEP = ISTEP / 2</td>
</tr>
<tr>
<td>74</td>
<td>H = H + H</td>
</tr>
<tr>
<td>75</td>
<td>GOTO 4</td>
</tr>
</tbody>
</table>
Table 3-1 (continued),

36 LHP=11
CALL TCF (X,Y,DERY,KEEP,NMAX)
GOTO 39
37 LHP=12
GOTO 39
38 LHP=13
CALL JUP (X,Y,DERY,LHP,NMAX,PRNT)
40 RETURN
END

PROGRAM PLOT (INPUT,TAPE5=INPUT,GPH2,TAPP1=GPH2,IPS02,
TAPE6=IPS02)
CALL GRAP (1,"THET",0.,100.,0.1,2,"THDOSH",5,"DUMP",11
1."FIN",132)
END

 SUBROUTINE GRAP (NUNCOL,HEAD,INIT,FINAL,NUMPT,NUMP1,HEAD1,
* NUM2,HEAD2,NUMSP,WAIT,NUMTY)
 DIMENSCIT DATA (100),STOTP (61),DATA1 (100),
* DATA2 (1000),CARD (36),4115

THE GRAPHING PARAMETERS ARE THE FOLLOWING:
NUNCOL= THE NO. OF THE COL. OF THE DATA FILE TO BE PLOTTED
AND PRINTED.
HEAD = HEADING FOR THE PLOTTED VARIABLE (IN QUOTE MARKS)
INIT = THE INITIAL PLOTTING TIME
FINA = THE FINAL PLOTTING TIME
NUMPT = THE NO. OF OTHER VARIABLES TO BE PRINTED ALONG SIDE
THE GRAPH (0,1,2)
NUMP1 = NO. OF THE COL. OF THE DATA FILE TO BE THE FIRST
PRINTED VARIABLE
HEAD1 = HEADING FOR THE FIRST PRINTED VARIABLE
NUM2 = NO. OF THE COL. OF THE DATA FILE TO BE THE SECOND
HEAD2 = HEADING FOR THE SECOND PRINTED VARIABLE
NUMSP = NO. OF LIMITS OF DATA FILE TO BE SKIPPED BETWEEN
SUCCESSIVE PLTS (THIS ALLOWS THE GRAPH TO BE COMPRESSED)
WAIT = "TH CAUSE THE POG. TO WAIT FOR A CARTRIDGE RETURN INPUT
BEFORE IT STARTS TO PRINT. THIS ALLOWS TIME SO THE PAPER
CAN BE POSTIONED TO THE TOP OF A NEW PAGE.
NUMTY = THE NO. OF CHARACTERS PER LINE OF TTY OUTPUT. (90,132)
IF 39 IS USED THEN NO ADDITIONAL COLS. ARE PRINTED.

INPUT FORMAT REQUIRED:
FORMAT (6X,1P12.4,1P1021.3)
6X= THIS 6 FIELD WIDTH CAN BE USED TO STORE THLF OR SOME OTHER PARM.
1P121.4= THIS FIELD IS USED TO STORE THE INDEPENDENT VARIABLE (TIME)
1P1021.3= THESE FIELD WIDTHS CONERN UPTO 10 COLS. OF DATA
FOR 10 DIFFERENT VARIABLES

IF (WAIT NE "A") GO TO 1001
PRINT 10)
1002 FORMAT ("IS TTY SET FOR 132 OR 80. CHARACTERS ?",/,
* "CARRY CARTRIDGE RETURN TO BEGIN OUTPUT")
READ (5,1001) (CARD (L),L=1,9C)
1001 FORMAT (98M)
1002 FORMAT (98M)
TTYYM M=79,L=132,79
1003 IF (NUMTY .L .132) GO TO 10
TTYYMM=35,79
1004 NUMTY=0
1005 NAME="TIME"
MAXYMLE=9
MAXCOL=NUNCOL
MOD (NUM1,CT,MAXCOL) MAXCOL=NUM1
MOD (NUM2,CT,MAXCOL) MAXCOL=NUM2
20 IF (DUMP ) TIME (X(L),L=1,MAXCOL)
MOD (POSI(5),NE,0) GO TO 106 (X(L),L=1,MAXCOL)
30 FORMAT (6X,1G10.4,1G11.3)
Table 3-1 (continued).

IF (1. D2-EQ. TIME) .LT. AINIT) GO TO 20
ISkip = ISkip + 1
IF (ISkip .EQ. NUMSKP) GO TO 20
ISkip = 0
IF (CUM = 1, C3 = 5) GT FINAL) GO TO 100
MAXNUM = MAXNUM + 1
TTimes (MAXNUM, 1) = TME
DATA (MAXNUM) = X (NUMCOL)
DATA1 (MAXNUM) = X (NUMP1)
DATA2 (MAXNUM) = X (NUMP2)
GO TO 20
130 IF (NUMINT, EQ. C) WRITE (' ', 1), NAME, HEAD
FORM1 (' ', X10, 3X, A10)
110 IF (NUMINT, EQ. 1) WRITE (' ', 112) NAME, HEAD, HEAD1
FORM1 (' ', X10, 3X, A10, 3X, A10)
120 IF (NUMINT, EQ. 2) WRITE (' ', 1120) NAME, HEAD, HEAD1, HEAD2
FORM1 (' ', X10, 3X, A10, 3X, A10, 3X, A10)
DMax = DATA
DMin = DATA
DO 50 K = 1, MAXNUM
IF (DATA, LT. DMAX) GO TO 40
DMax = DATA
50 IF (DATA, GT. DMIN) GO TO 50
CONTINUE
WIDTH = DMAX - DMIN
DO 90 M = 1, MAXNUM
K = 1 + (DATA (M) - DMIN) / WIDTH * TTYNUM
IF (K, EQ. 1) GO TO 75
12001 = K - 1
DO 70 J = 1, 12001
CSTORE (M) = ""
CSTORE (K) = ""
12002 = K - 1
DO 77 J = 12002
CSTORE (J) = ""
90 IF (NUMINT, EQ. 0) WRITE (1, 96) TIMES (M), DATA (M), (CSTORE (L), L = 1, K)
FORM1 ("", X1, 1E10.4, 1X, 1PE11.3, 1X, 80A1, 1X, 1PE11.3)
91 IF (NUMINT, EQ. 1) WRITE (1, 96) TIMES (M), DATA (M), (CSTORE (L), L = 1, 80)
* DATA1 (M)
92 IF (NUMINT, EQ. 2) WRITE (1, 96) TIMES (M), DATA (M), (CSTORE (L), L = 1, 80)
* DATA2 (M)
CONTINUE
93 RETURN
END
IV. DISCRETE DESCRIBING FUNCTION OF THE COMBINED WIRE-CABLE TORQUE AND FLEX-PIVOT NONLINEARITY

In the last three chapters the continuous-data IPS system with the combined nonlinearity of the wire-cable torque and flex pivot is studied. The describing function analysis is applied to the continuous-data system.

In the present chapter the digital IPS system with the combined nonlinearity is considered. The block diagram of the system is shown in Fig. 4-1. The combined nonlinearity is represented by the block N. The digital system is characterized by having the sample-and-hold units which have the notation S/H. In order to apply the discrete describing function technique, a sample-and-hold unit is inserted in the path of the nonlinearity N.

The objective of the investigation is to study whether self-sustained oscillations will occur in the digital IPS system. If self-sustained oscillations do occur, as they most likely will in a digital nonlinear system, what are the amplitudes and frequencies of these oscillations?

4.1 Discrete Describing Function of the Combined Nonlinearity

The discrete model of the combined nonlinearity of the wire-cable torque and the flex-pivot characteristics is shown in Fig. 4-2. The zero-order hold at the input of the combined nonlinearity is deleted, since there is already a zero-order hold at the output of N.

The wire-cable torque $T_{wc}$ is modelled as shown in Fig. 4-2 and is functionally represented by

$$T_{wc} = H_{WT} \text{SGN}(\dot{\theta}_i) + K_{WT} \theta_i$$

or

$$T_{wc}^+(\theta_i) = H_{WT} + K_{WT} \theta_i, \quad \theta_i > 0$$

(4-1)
Figure 4-1. Block diagram of the digital IPS system with flex-pivot and wire-cable nonlinearity.
Figure 4-2. Discrete model of the combined nonlinearity of the wire-cable and flex-pivot torques.

\[ T_{wc}(\theta_i) = -H_{WT} + K_{WT}\theta_i \quad \dot{\theta}_i \leq 0 \quad (4-3) \]

where \( H_{WT} \) is in N-m, \( K_{WT} \) in N-m/rad, \( \theta_i \) is in rad, and \( T_{wc}(\theta_i) \) in N-m.

It has been established that the Dahl solid rolling friction characteristics can be approximated by the nonlinear relation,

\[ \frac{dT_{FP}(\theta_i)}{d\theta_i} = \gamma(T_{FP1} - T_{FP0})^i \quad (4-4) \]

where

- \( i \) = positive number
- \( \gamma \) = positive constant
- \( T_{FP1} = T_{FP} \text{SGN}(\dot{\theta}_i) \)
- \( T_{FP0} = \text{saturation level of } T_{FP} \)

For \( i = 2 \), Eq. (4-4) is integrated to give

\[ \theta_i + C_1 = \frac{-1}{\gamma(T_{FP} + T_{FP0})} \quad \dot{\theta}_i \leq 0 \quad (4-5) \]

\[ \theta_i + C_2 = \frac{-1}{\gamma(T_{FP} - T_{FP0})} \quad \dot{\theta}_i \geq 0 \quad (4-6) \]
where $C_1$ and $C_2$ are constants of integration, and

$$T_{FP}^+ = T_{FP} \quad \dot{\theta}_i \geq 0 \quad (4-7)$$

$$T_{FP}^- = T_{FP} \quad \dot{\theta}_i \leq 0 \quad (4-8)$$

The constants of integration are determined at the initial point where

$$\theta_{ii} = \text{initial value of } \theta_i$$

$$T_{FPi} = \text{initial value of } T_{FP}$$

Then,

$$C_1 = -\theta_{ii} - \frac{1}{Y(T_{FPi}^- - T_{FP0})} \quad \dot{\theta}_i \geq 0 \quad (4-9)$$

$$C_2 = -\theta_{ii} - \frac{1}{Y(T_{FPi}^+ + T_{FP0})} \quad \dot{\theta}_i \leq 0 \quad (4-10)$$

The describing function analysis depends on the assumption that the input to the nonlinearity is a sine wave. Let $\theta_i(t)$ be described by the cosinusoidal function,

$$\theta_i(t) = A \cos \omega t \quad (4-11)$$

Then,

$$\dot{\theta}_i(t) = -A \omega \sin \omega t \quad (4-12)$$

Thus,

$$\theta_{ii} = -A \quad \dot{\theta}_i \geq 0 \quad (4-13)$$

$$\theta_{ii} = A \quad \dot{\theta}_i \leq 0 \quad (4-14)$$

The constant of integration in Eqs. (4-9) and (4-10) become

$$C_1 = A - \frac{1}{Y(T_{FPi}^- - T_{FP0})} \quad (4-15)$$

$$C_2 = -A - \frac{1}{Y(T_{FPi}^+ + T_{FP0})} \quad (4-16)$$

Substitution of Eqs. (4-11) and (4-15) in Eq. (4-5) and simplifying, the solution of $T_{FP}^+$ is written as
\[ T_{FP}^+ (t) = T_{FP0} \frac{R}{R - 1} + \frac{a}{2} \frac{(1 - \cos \omega t)}{1 - \cos \omega t} + \frac{1}{R - 1} \] (4-17)

which is valid for \( \omega_i > 0 \) or \((2k + 1) \pi < \omega t \leq 2(k + 1) \pi\), \(k = 0, 1, 2, \ldots\),

where

\[ a = 2 \gamma A T_{FP0} \] (4-18)

\[ R = - \frac{1}{a} + \sqrt{\frac{a^2 + 1}{a^2}} = \frac{T_{FPi}}{T_{FP0}} \] (4-19)

Similarly, for \( \omega_i < 0 \), using Eqs. (4-11) and (4-16) in Eq. (4-6), we have

\[ T_{FP}^- = T_{FP0} \frac{R}{R + 1} - \frac{a}{2} \frac{(1 - \cos \omega t)}{1 - \cos \omega t} + \frac{1}{R + 1} \] (4-20)

which is valid for \( 2k \pi < \omega t \leq (2k + 1) \pi\), \(k = 0, 1, 2, \ldots\).

The relations for \( T_{FP}^+ \) and \( T_{FP}^- \) in Eqs. (4-17) and (4-20) together with those of \( T_{WC}(\omega_i) \) in Eqs. (4-2) and (4-3) are used for the derivation of the discrete describing function of the combined nonlinearity. As shown in Fig. 4-2, the total torque disturbance due to the combined nonlinearity is given by

\[ T_n^+ = T_{WC}^+ + 3T_{FP}^+ \]

\[ = H_{WT} + K_{WT} A \cos \omega t + 3T_{FP0} \frac{R}{R - 1} + \frac{a}{2} \frac{(1 - \cos \omega t)}{1 - \cos \omega t} + \frac{1}{R - 1} \quad \omega_i > 0 \] (4-21)

\[ T_n^- = T_{WC}^- + 3T_{FP}^- \]

\[ = -H_{WT} + K_{WT} A \cos \omega t + 3T_{FP0} \frac{R}{R + 1} - \frac{a}{2} \frac{(1 - \cos \omega t)}{1 - \cos \omega t} + \frac{1}{R + 1} \quad \omega_i < 0 \] (4-22)

For the discrete model, we let

\[ \omega_i (t) = A \cos (\omega t + \phi) \] (4-23)

where \( \phi \) denotes the phase in radians. The z-transform of \( \omega_i (t) \) is
\[ \Theta_i(z) = \sum_{k=0}^{\infty} A_\cos \left( \frac{2\pi k}{N} + \phi \right) z^{-k} \]  \hspace{1cm} (4-24)

When \( N = 2, 3, \ldots \), or in closed form,

\[ \Theta_i(z) = \frac{Az \left[ (z - \cos 2\pi/N) \cos \phi - \sin 2\pi/N \sin \phi \right]}{z^2 - 2z \cos 2\pi/N + 1} \]  \hspace{1cm} (4-25)

The \( z \)-transform of \( T_n(t) \) is denoted by \( T_n(z) \). Then, the discrete describing function of the combined nonlinearity is defined as

\[ N(z) = \frac{T_n(z)}{\Theta_i(z)} \]  \hspace{1cm} (4-26)

The discrete describing function (DDF) for \( N = 2 \) is derived separately in the following section.

4.2 The DDF of the Combined Nonlinearity For \( N = 2 \)

Let \( T_n(kT) \) denote the value of \( T_n^r(t) \) at \( t = kT \). For \( N = 2 \), the signal \( T_n^r(t) \) is a periodic function with a period of \( 2T \). The \( z \)-transform of \( T_n^r(t) \) is written as

\[ T_n(z) = T_n(0)(1 + z^{-2} + z^{-4} + \ldots) + T_n(T)(z^{-1} + z^{-3} + \ldots) \]

\[ = \frac{T_n(0)z^2 + T_n(T)z}{z^2 - 1} \]  \hspace{1cm} (4-27)

For the cosinusoidal input of Eq. (4-23), the corresponding expression for \( T_{FP}^+(t) \) and \( T_{FP}^-(t) \) are

\[ T_{FP}^+(t) = T_{FP0} \frac{R - 1}{R} + \frac{a}{2} \left[ 1 - \cos(\omega t + \phi) \right] \]  \hspace{1cm} \( \phi_i > 0 \) \hspace{1cm} (4-28)

\[ T_{FP}^-(t) = T_{FP0} \frac{1}{R + 1} - \frac{a}{2} \left[ 1 - \cos(\omega t + \phi) \right] \]  \hspace{1cm} \( \phi_i \leq 0 \) \hspace{1cm} (4-29)

respectively. For \( t = kT \), the last two equations become
\[ T^+_{FP}(kT) = T_{FP0} \frac{R}{R - 1} + \frac{a}{2}(1 - \cos(\frac{2\pi k + \phi}{N})) \quad \Theta_i \geq 0 \quad (4-30) \]

\[ T^-_{FP}(kT) = T_{FP0} \frac{1}{R + 1} + \frac{a}{2}(1 - \cos(\frac{2\pi k + \phi}{N})) \quad \Theta_i \leq 0 \quad (4-31) \]

For \( N = 2 \), Eq. (4-25) is simplified to

\[ \Theta_i(z) = \frac{Az cos \phi}{z + 1} \quad (4-32) \]

Substitution of Eqs. (4-32) and (4-27) into Eq. (4-26), we have

\[ N(z) = \frac{T_n(z)}{\Theta_i(z)} = \frac{T_n(0)z + T_n(T)}{A(z - 1) cos \phi} \quad (4-33) \]

Also, for \( N = 2 \), \( z = -1 \); the last equation is simplified to

\[ N(z) = \frac{T_n(0) - T_n(T)}{2A cos \phi} \quad (4-34) \]

For \( N = 2 \),

\[ T_n(0) = T_n^-(0) \quad 0 \leq \phi < \pi (\Theta_i \geq 0) \]

\[ = T_n^+(0) \quad \pi < \phi < 2\pi (\Theta_i \leq 0) \]

\[ T_n(T) = T_n^+(T) \quad 0 \leq \phi < \pi (\Theta_i \geq 0) \]

\[ = T_n^-(T) \quad \pi < \phi < 2\pi (\Theta_i \leq 0) \]

Then, Eq. (4-34) becomes

\[ N(z) = \frac{T_n^-(0) - T_n^+(T)}{2A cos \phi} \quad 0 \leq \phi < \pi \quad (4-35) \]

\[ N(z) = \frac{T_n^+(0) - T_n^-(T)}{2A cos \phi} \quad \pi < \phi < 2\pi \quad (4-36) \]

For stability analysis, it is convenient to define
Using Eqs. (4-2), (4-3), (4-30) and (4-31), we have

\[ T_0^+ = H_{WT} + K_{WT} \cos \phi + 3T_{FP0} \left( \frac{R}{R - 1} + \frac{a}{2} \frac{1 - \cos \phi}{1 + \cos \phi} \right) \]

\[ T_0^- = -H_{WT} + K_{WT} \cos \phi + 3T_{FP0} \left( \frac{R}{R + 1} - \frac{a}{2} \frac{1 - \cos \phi}{1 - \cos \phi} \right) \]

\[ T^+(T) = H_{WT} + K_{WT} \cos (\phi + \pi) + 3T_{FP0} \left( \frac{R}{R - 1} + \frac{a}{2} \frac{1 - \cos (\pi + \phi)}{1 + \cos (\pi + \phi)} \right) \]

\[ T^-(T) = -H_{WT} - K_{WT} \cos \phi + 3T_{FP0} \left( \frac{R}{R + 1} - \frac{a}{2} \frac{1 + \cos \phi}{1 + \cos \phi} \right) \]

Thus, for \( 0 \leq \phi < \pi \),

\[ T_c^-(0) - T_c^+(T) = 2K_{WT} \cos \phi - 2H_{WT} + 3T_{FP0} \left( \frac{R}{R + 1} + \frac{a}{2} \frac{1 - \cos \phi}{1 - \cos \phi} \right) \]

\[ - \left( \frac{R}{R - 1} + \frac{a}{2} \frac{1 + \cos \phi}{1 + \cos \phi} \right) \]

For \( \pi \leq \phi < 2\pi \),

\[ T_c^+(0) - T_c^-(T) = 2K_{WT} \cos \phi + 2H_{WT} + 3T_{FP0} \left( \frac{R}{R - 1} + \frac{a}{2} \frac{1 - \cos \phi}{1 - \cos \phi} \right) \]

\[ - \left( \frac{R}{R + 1} - \frac{a}{2} \frac{1 + \cos \phi}{1 + \cos \phi} \right) \]
4.3 Properties of $F(z) = -1/N(z)$ for $N = 2$ as $A \to 0$ and $A \to \infty$

The properties of $F(z)$ for $N = 2$ as $A \to 0$ and $A \to \infty$ are now investigated. These properties will be useful in the determination of the critical regions of $F(z)$ for stability studies.

**Theorem 4-1.**

For $N = 2$,

$$\lim_{A \to \infty} F(z) = \lim_{A \to \infty} -1/N(z) = -1/K_{WT} \text{ for all } \phi \quad (4-44)$$

**Proof:** From Eq. (4-19),

$$\frac{R}{R - 1} = -1 + \sqrt{a^2 + 1}$$

$$\frac{R}{R + 1} = -1 + \sqrt{a^2 + 1} \quad (4-45)$$

$$\frac{R}{R - 1} = -1 - \sqrt{a^2 + 1} \quad (4-46)$$

$$\frac{R}{R + 1} = -1 + \sqrt{a^2 + 1} \quad (4-47)$$

$$\frac{1}{R - 1} = -1 - \sqrt{a^2 + 1} \quad (4-48)$$

and $a = 2\gamma AT_{FPO}$.

Let $F^+(z) = F(z)$ for $0 \leq \phi < \pi$. Substituting Eq. (4-35) into Eq. (4-37), we have

$$F^+(z) = \frac{-2A\cos\phi}{T_n(0) - T^*_n(T)} \quad (4-49)$$

Then,

$$\lim_{A \to \infty} F^+(z) = \lim_{A \to \infty} \frac{-2A\cos\phi}{T_n(0) - T^*_n(T)} \quad (4-50)$$

Substituting Eqs. (4-39) and (4-40) into the last equation and using Eqs. (4-45) through (4-48), we get
\[
\lim_{A \to \infty} F^+(z) = \lim_{A \to \infty} \frac{-2A\cos \phi}{3T_{FP0} \left[ \frac{1}{2} \left( 1 - \cos \phi \right) - a + \frac{a}{2} \right] + 2K_{WT} \cos \phi - 2H_{WT}}
\]
\[
= \lim_{A \to \infty} \frac{-2A\cos \phi}{3T_{FP0} \left[ \frac{1}{2} \left( 1 - \cos \phi \right) - a + \frac{a}{2} \right] + 2K_{WT} \cos \phi - 2H_{WT}}
\]
\[
= \lim_{A \to \infty} \frac{-2A\cos \phi}{-6T_{FP0} + 2K_{WT} \cos \phi - 2H_{WT}} = \lim_{A \to \infty} \frac{-A\cos \phi}{K_{WT} \cos \phi} = -\frac{1}{K_{WT}} \text{ Q.E.D.}
\]

Similarly, for \( \pi \leq \phi < 2\pi \), \( F^-(z) = F(z) \),

\[
\lim_{A \to \infty} F^-(z) = \lim_{A \to \infty} \frac{-2A\cos \phi}{T_{n}(0) - T_{n}(-1)}
\]
\[
= \lim_{A \to \infty} \frac{-2A\cos \phi}{3T_{FP0} \left[ \frac{1}{2} \left( 1 - \cos \phi \right) - a + \frac{a}{2} \right] + 2K_{WT} \cos \phi - 2H_{WT}}
\]
\[
= \lim_{A \to \infty} \frac{-2A\cos \phi}{-6T_{FP0} + 2K_{WT} \cos \phi - 2H_{WT}} = \lim_{A \to \infty} \frac{-A\cos \phi}{K_{WT} \cos \phi} = -\frac{1}{K_{WT}} \text{ Q.E.D.}
\]
Theorem 4-2.

For \( N = 2 \),

\[
\lim_{A \to 0} F(z) = \lim_{A \to 0} \frac{1}{N(z)} = \frac{0/180^\circ + \tan^{-1} \left( \frac{\cos\phi}{\cos\phi} \right)}{\cos\phi} \quad \text{for all } \phi \quad (4-51)
\]

Proof: For \( 0 \leq \phi < \pi \),

\[
\lim_{A \to 0} F^+(z) = \frac{-2A \cos\phi}{3T_{FP0} \left( \frac{-1 + \sqrt{a^2 + 1}}{1 + a + \sqrt{a^2 + 1}} + \frac{a(1 - \cos\phi)}{2(1 - \cos\phi)} \right)} - \frac{2A \cos\phi}{\cos\phi} \quad (4-52)
\]

\[
= \lim_{A \to 0} \frac{a}{1 + a} \left( \frac{-1 + \sqrt{a^2 + 1}}{1 + a + \sqrt{a^2 + 1}} + \frac{a(1 - \cos\phi)}{2(1 - \cos\phi)} \right) \quad (4-53)
\]

\[
= \lim_{A \to 0} \frac{-2A \cos\phi}{\cos\phi} = \frac{0/180^\circ}{\cos\phi} \quad \text{for } 0 \leq \phi < \pi/2
\]

\[
= \frac{0/180^\circ}{\cos\phi} \quad \text{for } \pi/2 < \phi < \pi
\]

Similarly, for \( \pi \leq \phi < 2\pi \),

\[
\lim_{A \to 0} F^-(z) = \frac{-2A \cos\phi}{3T_{FP0} \left( \frac{-1 + \sqrt{a^2 + 1}}{1 + a + \sqrt{a^2 + 1}} + \frac{a(1 - \cos\phi)}{2(1 - \cos\phi)} \right)} - \frac{2A \cos\phi}{\cos\phi} \quad (4-54)
\]

\[
= \lim_{A \to 0} \frac{a}{1 + a} \left( \frac{-1 + \sqrt{a^2 + 1}}{1 + a + \sqrt{a^2 + 1}} + \frac{a(1 - \cos\phi)}{2(1 - \cos\phi)} \right) \quad (4-55)
\]

\[
= \lim_{A \to 0} \frac{-2A \cos\phi}{\cos\phi} = \frac{0/180^\circ}{\cos\phi} \quad \text{for } 0 \leq \phi < \pi/2
\]

\[
= \frac{0/180^\circ}{\cos\phi} \quad \text{for } \pi/2 < \phi < \pi
\]
\[ \lim_{A \to 0} F(z) = \lim_{A \to 0} \frac{-2A \cos \phi}{2K_{WT} \cos \phi + 2H_{WT}} \]

\[ = \lim_{A \to 0} \frac{-2A \cos \phi}{2K_{WT} \cos \phi + 2H_{WT}} \]

\[ = \frac{0/180^\circ + \tan^{-1} \frac{\cos \phi}{|\cos \phi|}}{3\pi/2 < \phi < 2\pi} \]

\[ = 0/0^\circ \quad \pi \leq \phi < 3\pi/2 \quad \text{Q.E.D.} \]

4.4 The DDF of the Combined Nonlinearity For \( N \geq 3 \)

A general relation can be derived for the combined nonlinearity for \( N \geq 3 \).

In general, the \( z \)-transform of the output of the combined nonlinearity may be written as

\[ T_\phi(z) = \sum_{m=0}^{\infty} \sum_{k=0}^{N-1} T_n(kT)z^{-k-mN} \]

\[ = \sum_{k=0}^{N-1} T_n(kT)z^{N-k} \]

The discrete describing function then becomes

\[ \frac{T_n(z)}{N(z)} = \frac{\sum_{k=0}^{N-1} T_n(kT)z^{N-k}}{(z^N - 1) \sum_{k=0}^{\infty} (A \cos \frac{2\pi k}{N} + \phi)z^{-k}} \]

The last equation is simplified to

\[ N(z) = \frac{\sum_{k=0}^{N-1} T_n(kT)z^{N-k-1}}{A \sum_{k=0}^{N-1} (\cos \frac{2\pi k}{N} + \phi)z^{N-k-1}} \quad (N \geq 3) \]

Or, alternately,
\[
N(z) = \frac{\sum_{k=0}^{N-1} T_n(kT)z^{N-k}(z^2 - 2z\cos\frac{2\pi}{N} + 1)}{\sum_{k=0}^{N-1} (z - e^{j2\pi k/N})A(z\cos\frac{2\pi}{N}\cos\phi - \sin\frac{2\pi}{N}\sin\phi)}
\]

Or,
\[
N(z) = \frac{\sum_{k=0}^{N-1} T_n(kT)z^{N-k-1}}{A(z - 1)\sum_{k=2}^{N-2} (z - e^{j2\pi k/N})(z - \cos\frac{2\pi}{N}\cos\phi - \sin\frac{2\pi}{N}\sin\phi)}
\]

For \(N = 3\), \(z = e^{j2\pi/3}\),
\[
N(z) = \frac{T_n(0)z^2 + T_n(T)z + T_n(2T)}{A(z - 1)((z + 0.5)\cos\phi - 0.866\sin\phi)}
\]

Similar expressions can be obtained for \(N = 4, 5, \ldots\) with \(z\) identified with \(e^{j2\pi/N}\).

In general,
\[
T_n(kT) = T_n^-(kT) \quad 0 \leq \frac{2\pi k}{N} + \phi < \pi
\]

\[
= T_n^+(kT) \quad \pi < \frac{2\pi k}{N} + \phi < 2\pi
\]

4.5 Asymptotic Properties of \(-1/N(z)\) for \(N \geq 3\) as \(A\) Approaches Infinity

The following properties of \(-1/N(z)\) are found for \(N \geq 3\). The proofs of these properties can be obtained the same way as those for \(N = 2\) by replacing \(\phi\) by \(\phi + 2\pi k/N\).

(a) \(\lim_{A \to \infty} (-1/N(z)) = -\frac{1}{K_{WT}}\) for all \(\phi\) \hspace{1cm} (4-59)

(b) \(\lim_{A \to \infty} \left|\frac{-1/N(z)}{A} = \frac{1}{K_{WT}}\right|\) for all \(\phi\) \hspace{1cm} (4-60)

(c) \(\lim_{A \to \infty} (\text{Arg}(-1/N(z))) = 180^\circ\) for all \(\phi\) \hspace{1cm} (4-61)

4.6 Asymptotic Properties of \(-1/N(z)\) for \(N \geq 3\) as \(A\) Approaches Zero

The following properties of \(-1/N(z)\) for \(N \geq 3\) are obtained as \(A\)
approaches zero.

(a) For \( N \geq 3 \), and for all \( \phi \),

\[
\lim_{A \to 0} |1/N(z)| = 0 \quad (4-62)
\]

(b) For \( N \geq 3 \), \( (N = \text{odd integers}) \)

\[
\lim_{A \to 0} \text{Arg}(1/N(z)) = -\left(\frac{1 + 3N + 2k}{2N}\right)\pi + \phi \quad (4-63)
\]

for \( k\pi/N \leq \phi < (k + 1)\pi/N \), \( k = 0, 1, 2, \ldots, (2N - 1) \).

(c) For \( N \geq 4 \), \( (N = \text{even integers}) \)

\[
\lim_{A \to 0} \text{Arg}(-1/N(z)) = -\left(\frac{2 + 3N + 4k}{2N}\right)\pi + \phi \quad (4-64)
\]

for \( 2k\pi/N \leq \phi < 2(k + 1)\pi/N \).

In view of the properties of \(-1/N(z)\) listed above, the following theorems are generated.

**Theorem 4-3.**

For even integral \( N \geq 3 \), the magnitude and phase of \(-1/N(z)\) repeat for every \( \phi = 2\pi/N \) radians.

**Theorem 4-4.**

For odd \( N \) \( (N \geq 3) \), the magnitude and phase of \(-1/N(z)\) repeat for every \( \phi = \pi/N \) radians.

### 4.7 Discrete Describing Function Plots of the Combined Wire-Cable and Flex-Pivot Nonlinearity - The Critical Regions

The discrete describing function, \( N(z) \), for the combined wire-cable and flex-pivot nonlinearity is derived in the preceding sections. The plots of \( F(z) = -1/N(z) \) together with the plot of \( G_{\text{eq}}(z) \), which is the linear transfer function that \( N(z) \) sees, in the frequency domain allow the study of the condition of self-sustained oscillations of the digital IPS system.
Computer programs for the evaluation of the \(-1/N(z)\) for \(N = 2\) and \(N \geq 3\) have been prepared. The listings of these programs are given in Tables 4-1 and 4-2, respectively.

For \(N = 2\), the expression for \(F(z) = -1/N(z)\) is given in Eqs. (4-35), (4-36) and (4-37). Figure 4-3 shows the \(F(z)\) plot for \(N = 2\) in the gain-phase coordinates with \(0 \leq A < \infty\) and all values of \(\phi\).

The following set of parameters are used for the nonlinear elements:

\[
\begin{align*}
T_{FP0} & = 0.00225 \\
\gamma & = 9.2444 \times 10^4 \\
K_{WT} & = 100 \\
H_{WT} & = 1
\end{align*}
\]

In view of Theorem 4-1, Eq. (4-44), the most important parameter among those listed above is \(K_{WT}\), since when \(A\) approaches infinity the magnitude of \(F(z)\) approaches \(1/K_{WT}\). However, as shown in Fig. 4-3, the \(F(z)\) plot stays on the \(-180^0\) and \(-360^0\) axes for \(N = 2\).

The discrete describing function \(N(z)\) for \(N \geq 3\) is given by Eq. (4-54) or Eq. (4-56). Figure 4-4 shows the gain-phase plot of \(F(z) = -1/N(z)\) for \(N = 3\). The curves for several values of \(\phi\) between \(0^0\) and \(60^0\) are plotted to illustrate the effect of varying the phase of the input signal to the non-linearity. It should be noted that for \(N = 3\), Theorem 4-3 states that the values of \(F(z)\) repeat every 60 degrees starting from \(\phi = 0^0\). As the magnitude of the input signal, \(A\), approaches infinity, the magnitude of \(F(z)\) becomes \(1/K_{WT}\) which is \(-40\) db in this case, since \(K_{WT}\) is 100. On the other hand, as \(A\) approaches zero, the magnitude of \(F(z)\) becomes zero or \(-\infty\) db, and the phase of \(F(z)\) is bounded by \(-300^0\) and \(-240^0\) for all values of \(\phi\). When the value of \(K_{WT}\) is varied, the curves of \(F(z)\) will shift up or down according to
Table 4-1. Computer program for the computation of the discrete describing function of the combined flex-pivot and wire-cable nonlinearity of the IPS. N = 2.

```fortran
C CALCULATION FOR -1/N(Z) FOR COMBINED NONLINEARITY, N=2
        ! COMPLEX GN1, GN2
        REAL*8 PI, RAD, T0, GAMMA, ASTART, AA, AR, CSS, CSP, CSC
        REAL*8 AA2, R1, R2, R3, R4, T0F, T0M, T0M, TTF, T1, T2
        PI=3.14159D0
        RAD=180.0D0/PI
        CSP=1.0D0
        T0=0.00225D0
        KWT=100.0D0
        HWT=1.0D0
        GAMMA=9.2444D4
        ASTART=1.00-10
        NP=5
        ND=15
        WRITE(5,100)

100 FORMAT(5,101)
        DO 1 J=1, ND
        DO 1 I=1, NP
        CSS=1.0D0-CSP
        CSC=1.0D0+CSP
        A=ASTART*DFLOAT(I)*(10.0D0**(J-1))
        AA=2.0D0**GAMMA**A**T0
        R=(-1.0D0/AA)+DSQRT((AA**AA+1.0D0)/(AA**AA))
        AA2=AA/2.0D0
        R1=1.0D0/(R-1.0D0)
        R2=R*R1
        R3=1.0D0/(R+1.0D0)
        R4=R*R3
        TOF=(R2+AA2*CSP)/(R1+AA2*CSP)
        T0M=(R4-AA2*CSP)/(R3+AA2*CSP)
        T0M=(R2+AA2*CSP)/(R1+AA2*CSP)
        T1=3.0D0*T0F*(T0F-T0M)+2.0D0*(HWT*KWT*AA*CSP)
        T2=3.0D0*T0F*(T0M-TTF)+2.0D0*(HWT*KWT*AA*CSP)
        G11=2.0D0*AA*CSP/T1
        G12=2.0D0*AA*CSP/T2
        G11=REAL(GN1)
        G12=AIMAG(GN1)
        G21=REAL(GN2)
        G22=AIMAG(GN2)
        GMAG1=CABS(GN1)
        GMAG2=CABS(GN2)
        GDB1=20.0D0*LOG10(GMAG1)
        GDB2=20.0D0*LOG10(GMAG2)
        GPH1=RAD*ATAN2(G12, G11)
        GPH2=RAD*ATAN2(G22, G21)
        IF (GPH2 .GE. 0.0) GPH2=GPH2-360.
        IF (GPH1 .GE. 0.0) GPH1=GPH1-360.

1 CONTINUE
100 FORMAT(' DISCRETE DESCRIBING FUNCTION FOR IPS ')
101 FORMAT(7x,8X,'\$\alpha\$', '7X', '\$\theta\$', '7X', '\$\alpha\$', '7X', '\$\theta\$', '7X', 'PHASE')
102 FORMAT (1F4E14.5)
        STOP
        END
```
Table 4-2. Computer program for the computation of the discrete describing function of the combined flex-pivot and wire-cable nonlinearity of the IPS. N GE. 3.

```
C DISCRETE DESCRIBING FUNCTION FOR COMBINED NONLINEARITY, N GE. 3
REAL PHI,P(15),PI,RAD,GAMMA,ASTART,A,AA,R,HWT,KWT,TO
REAL PP,RR,AA,R1,R2,R3,R4,AA2,TC,PHIK,PHID,PIK,TWN,TWP
COMPLEX GV,TTSUM
COMPLEX TSUMM,ZSUMNZSUM,TZ,TSUM,Z,THETA,GNNZNM
PI=3.14159
RAD=180./PI
HWT=1.0
KWT=100.
GAMMA=9.2444E4
ASTART=1.E-8
ND=9
NP=1
TO=0.00225
RR=0.
NI=50
AN=FLOAT(NI)
N1=NI-1
N2=NI-2
P(1)=1.0
NPHI=3600
PP=2.*PI/FLOAT(NI)
THETA=CMPLX(RR,PP)
Z=CEXP(THETA)
ZSUMM=CMPLX(1.0,0.0)
TSUMM=CMPLX(0.0,0.0)
WRITE(5,100)
WRITE(5,102) HN
WRITE(5,110) GAMMA
WRITE(5,111) HWT
WRITE(5,112) KWT
WRITE(5,113) TO
WRITE(5,101)
DO 8 I=0,144536
PHI=(2.0*PI+FLOHT(I))/(FLOHT(MPHI))
PHID=PHI*RAD
WRITE(5,103)PHID
DO 1 J=1,ND
DO 9 L=1,NP
ZSUMM=CMPLX(1.0,0.0)
TSUMM=CMPLX(0.0,0.0)
AA=ASTART+FLOHT(L)*(10.**(J-1))
AA2=2.0*GAMMA*AA+TO
8 CONTINUE
9 CONTINUE
```

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Table 4-2. (Continued)

R = (-1.0/AA) + SORT((AA+AA+1.0)/(AA+AA))
R1 = 1.0/(R-1.0)
R2 = R*R1
R3 = 1.0/(R+1.0)
R4 = R*R3
DO 2 K=0,N1
PIK= (2.0+PI*FLOAT(K))/(FLOAT(N1))
PHIK=PHI+PIK
IF(PHIK.GT.(2.0+PI)) PHIK=PHIK-2.0*PI
TWN=-HWT+KWT*A*COS(PHIK)
TFP=HWT+KWT*A*COS(PHIK)
IF(PHIK.LT.PI) GO TO 6
TC=3.0*TO+(R2-AA2-AA2*COS(PHIK))/(R1+AA2-AA2*COS(PHIK)) + TWP
GO TO 7
6 TC=3.0*TO+(R4-AA2-AA2*COS(PHIK))/(R3+AA2-AA2*COS(PHIK)) + TWN
7 TSUM=TC*Z**((N1-K)
TSMN=TSUMN+TSUM
2 CONTINUE
IF(N1.LE.3) GO TO 10
DO 3 M=2,N2
ZSUM=Z-Z**M
ZSUMN=ZSUMN+ZSUM
3 CONTINUE
10 CONTINUE
TZ=(Z-COS(PP)) *COS(PHI)-SIN(PP)*SIN(PHI)
G6=TSUMN/(R+TZ**(Z-1.)*ZSUMN)
GNN=-1./G6
6V=GHN
GI=REAL(GV)
G2=AIMAG(GV)
GMAG=ABS(GV)
6DB=20.*ALOG10(GMAG)
6PHASE=RAD*ATAN2(G2,G1)
IF(6PHASE.GE.0.) 6PHASE=6PHASE-360.
WRITE(5,104) A,6PHASE,6DB,GMAG
105 FORMAT(1PE24.15)
9 CONTINUE
1 CONTINUE
8 CONTINUE
100 FORMAT(7X,'DESCRIBING FUNCTION OF COMBINED NONLINEARITY')
101 FORMAT(7X,'A',10X,'PHASE',10X,'DB',10X,'MAGNITUDE')
102 FORMAT(5X,'N=',F4.1)
110 FORMAT(5X,'GAMMA=',1PE12.4)
111 FORMAT(5X,'HWT=',F8.2)
112 FORMAT(5X,'KWT=',F8.2)
113 FORMAT(5X,'TO=',1PE12.2)
103 FORMAT(5X,'PHI=',F5.1)
104 FORMAT(1PE14.5)
STOP
END
Figure 4-3. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 2.
Figure 4-4. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. $N = 3$. 

$N(z) = 1$
$1/K_{WT}$ in db; when the values of $T_{FPO}$, $\gamma$, and $H_{WT}$ are varied, the shape of the curved portions of the plots in Fig. 4-4 will be changed. However, in general, the impact of the variation of $K_{WT}$ will be the greatest.

The $F(z)$ plots for $N = 4, 5, 6, 8, 10, 20,$ and $50$ are shown in Figs. 4-5 through 4-11, respectively. For $N = 4$, the $F(z)$ plot extends from $-315^\circ$ to $-225^\circ$ as $\phi$ varies. For $N = 5$, the span of the plot is $-288^\circ$ to $-252^\circ$.

For stability analysis, it is sufficient to consider only the bounds of the $F(z)$ plot for a fixed $N$. Self-sustained oscillations characterized by the period $T_c = NT$, where $T$ is the sampling period, may occur if $G_{eq}(z)$ intersects with any part of the $F(z)$ plot. The region bounded by all the $F(z)$ curves for a given $N$ is defined as the critical region. The critical regions for the combined nonlinearity for $N > 2$ are the regions that are bounded by the $F(z)$ curves for $\phi = 0^\circ$ and $\phi = 2\pi/N$ for $N$ = even and $\phi = \pi/N$ for $N$ = odd.

As $N$ approaches infinity, the discrete describing function $N(z)$ approaches the describing function $N$ of the continuous-data nonlinearity, as shown in Fig. 4-11. It is observed that as $N$ increases the width of the critical region becomes narrower. As $N$ approaches infinity, the critical region of $F(z) = -1/N(z)$ approaches the $-1/N$ plot shown in Fig. 2-1.
Figure 4-5. Discrete Describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 4.
Figure 4-6. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 5.
Figure 4-7. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. $N = 6$. 
Figure 4-8. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 8.
Figure 4-9. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. $N = 10$. 
Figure 4-10. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 20.
Figure 4-11. Discrete describing function plots of combined flex-pivot and wire-cable nonlinearity for the IPS. N = 50.
REFERENCES