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CR 151419

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IDENTIFICATION OF NATURAL FREQUENCIES AND MODAL DAMPING RATIOS OF AEROSPACE STRUCTURES FROM RESPONSE DATA

by
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(NASA-CR-151419) IDENTIFICATION OF NATURAL FREQUENCIES AND MODAL DAMPING RATIOS OF AEROSPACE STRUCTURES FROM RESPONSE DATA (Houston Univ.) 36 p HC A03/MF A01 CSCL 13M
N77-26532
Unclas
G3/39 30384

Technical Report No. NC-1

November 1976

JUN 1977
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NASA STI FACILITY
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Submitted by: C. D. Michalopoulos
Principal Investigator

Sponsor: National Aeronautics and Space Administration

Contract: NAS 9-14347, Modification 5S

UNIVERSITY OF HOUSTON
Founded 1927

University of Houston
Department of Mechanical Engineering
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ABSTRACT

A method is presented by which response data can be processed in the frequency domain to obtain natural frequencies and modal damping ratios of multi-degree-of-freedom systems. Primary attention is focused on purely mechanical systems which possess classical normal modes. Systems that do not possess such modes, but whose damping matrix [C] is such that the matrix $[\varphi]^T[C][\varphi]$, $[\varphi]$ being the modal matrix, is not diagonal but whose off-diagonal elements are small, are also considered. When the power spectral density of the excitation is unknown, the method requires it to be reasonably flat. Numerical examples are given for systems with one and two degrees of freedom.

1. INTRODUCTION

The problem of processing response data of an aerospace structure to controlled excitation or random excitation during flight in order to determine the dynamic characteristics of the structure has recently received considerable attention in the United States [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]* as well as abroad [12, 13, 14]. From an applications point of view, dynamic signature analysis is very important since it can be used to detect flaws or to predict the onset of flutter during subcritical wind-tunnel or flight tests.

In references 1, 2, 8, 10, 12 and 14 the autocorrelation function of the random response is utilized to obtain natural frequencies and modal damping ratios in the frequency domain. This method of modal identification assumes that the frequency spectrum of the random exciting forces is flat (white noise) over the frequency range of the modes of interest. The method becomes inaccurate if the excitation is not white noise and/or if the structure has closely-spaced vibrational modes. Also, errors are introduced due to statistical causes associated with signatures of finite duration.

The random decrement method, introduced by Cole [1], has some definite advantages over the autocorrelation function and

---

*Numbers in brackets designate references.

has been utilized by the inventor for the on-line failure detection and damping measurements of aerospace structures. Some of the advantages of this method are: (1) changes of intensity of the input do not affect the level of the signature, which is at constant amplitude; (2) the signature distortion is considerably less, than in the case of the autocorrelation method, if the input spectral density is not flat.

The work by Cole is primarily concerned with systems having one degree of freedom. Very recently, Chang [11] studied the dynamic characteristic of aeroelastic systems using randomdec signatures. Multi-degree-of-freedom were analyzed, but numerical experiments were carried out only for systems with two degrees of freedom (purely mechanical as well as aeroelastic).

The purpose of the current investigation is to develop a method by which response data of an aerospace structure (obtained during flight) can be processed in order to determine its natural frequencies and modal damping ratios. The analysis is to be sufficiently general to include systems other than just purely mechanical with classical normal modes, in order that it may be applied to the Shuttle Pogo Instability program.

2. ANALYSIS OF ONE- AND MULTI-DEGREE-OF-FREEDOM SYSTEMS WITH CLASSICAL DAMPING

First, systems with one degree of freedom are analyzed in detail. Then, multi-degree-of-freedom systems possessing classical normal modes are considered. Finally, in Section 3, purely mechanical, linear systems with nonclassical (non-proportional) damping are discussed.

2.1 Systems with One Degree of Freedom

Consider a one-degree-of freedom system whose motion is governed by the differential equation

$$\ddot{x} + 2\Omega\zeta\dot{x} + \Omega^2 x = \Omega^2 f(t) \tag{1}$$

where $f(t)$ is the random excitation, that is $f(t)$ is a function representing an ergotic random process, Ω is the natural frequency, and ζ the damping ratio. The power spectral density of the (random) response $x(t)$ is given by the familiar relation

$$S_x(\omega) = H^*(\omega)H(\omega)S_f(\omega), \tag{2}$$

where $S_x(\omega)$ is the power spectral density of the response,

$S_f(\omega)$ is the power spectral density of the excitation,

$$H(\omega) = \frac{1}{1 - \left(\frac{\omega}{\Omega}\right)^2 + i2\zeta\frac{\omega}{\Omega}},$$

$H^*(\omega)$ is the complex conjugate of $H(\omega)$.

Equation (2) can be rewritten in the form

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega), \tag{3}$$

where

$$|H(\omega)|^2 = \frac{1}{\left[1 - \left(\frac{\omega}{\Omega}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\Omega}\right)^2} \tag{4}$$

Note that the power spectral density $S_x(\omega)$ is the Fourier transform of the autocorrelation $R_x(\tau)$, that is

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau}d\tau, \tag{5}$$

where

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt \tag{6}$$

The autocorrelation in terms of the spectral density is given by

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)e^{i\omega\tau}d\omega \tag{7}$$

Equations (5) and (7) is Fourier transform pair. Analogous relations exist between $S_f(\omega)$ and $R_f(\tau)$.

Let X_n be the finite Fourier transform of $x(t)$, defined by

$$X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)e^{-i\omega_n\tau}d\tau, \tag{8}$$

where

$$\omega_n = n\left(\frac{2\pi}{T}\right); \quad n = 1, 2, 3, \ldots$$

and T is the time interval for which the (random) response has been measured. The function

$$S_x'(\omega_n) = \frac{1}{T}X_nX_n^* \tag{9}$$

is an estimate of $S_x(\omega)$ at the frequency $\omega = \omega_n$ (see, for example Reference 15). The spectral density $S'_x(\omega_n)$ can be obtained from a random response record x(t) from (9) by means of an FFT (fast Fourier transform) or by means of a real time analyzer.

2.1.1 Definition and Minimization of Error Function

It is possible to define an error function in the mean square sense such that minimization of this error will yield the parameters of the system Ω and ζ. Let us define the error according to

$$e(\Omega,\zeta) = \int_0^\infty [S_x(\omega) - S'_x(\omega_n)]^2 d\omega \tag{10}$$

Define a vector $\underset{\sim}{\xi}$ by

$$\underset{\sim}{\xi} \equiv \{\xi\} = \begin{Bmatrix} \Omega \\ \zeta \end{Bmatrix} \tag{11}$$

For $e(\Omega,\zeta)$ to be a minimum

$$\frac{\partial e}{\partial \xi_i} = 2\int_0^\infty [S_x(\omega) - S'_x(\omega)] \frac{\partial S_x(\omega)}{\partial \xi_i} d\omega = 0; \; i = 1, 2 \tag{12}$$

Let

$$\left\{h(\Omega,\zeta)\right\} = \frac{\partial e}{\partial \underset{\sim}{\xi}} \tag{13}$$

Expanding the vector {h} into Taylor series,

$$\left\{h\left(\underset{\sim}{\xi}^{n+1}\right)\right\} = \left\{h\left(\underset{\sim}{\xi}^{n}\right)\right\} + \frac{\partial\left\{h\left(\underset{\sim}{\xi}^{n}\right)\right\}}{\partial \underset{\sim}{\xi}} \Delta\underset{\sim}{\xi}^{n} + o\|\Delta\underset{\sim}{\xi}^{n}\|$$

$$\left(\Delta\underset{\sim}{\xi}^{n} \equiv \underset{\sim}{\xi}^{n+1} - \underset{\sim}{\xi}^{n}\right) \tag{14}$$

In order for

$$\left\{h\left(\underset{\sim}{\xi}^{n+1}\right)\right\} \cong 0, \tag{15}$$

the increment $\Delta\underset{\sim}{\xi}^{n}$ must be obtained from

$$\Delta\underset{\sim}{\xi}^{n} = -\left[\frac{\partial\left\{h\left(\underset{\sim}{\xi}^{n}\right)\right\}}{\partial\underset{\sim}{\xi}}\right]^{-1}\left\{h\left(\underset{\sim}{\xi}^{n}\right)\right\} \tag{16}$$

This equation can be rewritten in the form

$$\{\Delta\xi^{n}\} = [D^{n}]\ \{h^{n}\}, \tag{17}$$

where

$$[D^{n}] = -\left[\frac{\partial^{2}e}{\partial\xi_{i}\partial\xi_{j}}\right]^{-1};\ i,\ j = 1,\ 2 \tag{18}$$

The elements of the derivative matrix [D] can easily be determined from Eq. (10) (see Appendix I). Thus, Eqs. (12) can be solved by iterating. Starting with an initial guess $\{\xi^{1}\}$, $[D^{1}]$ and $\{h^{1}\}$ can be computed by numerical integration. Then, $\{\Delta\xi^{1}\}$ can be determined from (17) and $\{\xi^{2}\}$ from

$$\{\xi^{2}\} = \{\xi^{1}\} + \{\Delta\xi^{1}\}$$

The process is continued until $\{\Delta\xi^{n}\} \cong \{0\}$, i.e. until

$$\left|\frac{\xi_{i}^{n+1} - \xi_{i}^{n}}{\xi_{i}^{n}}\right| < \epsilon;\ i = 1,\ 2$$

2.1.2 White Noise Excitation

In the above, it has been assumed that the power spectral density of the forcing function, $S_f(\omega)$, is known. It is possible, however, to determine the parameters Ω and ζ even if $S_f(\omega)$ is not known, provided it is constant (white noise) or nearly constant. If the excitation is ideal white noise, $S_f(\omega) = S_0 = \text{const}$. Notice that for lightly damped systems for which $|H(\omega)|^2$ has a very steep peak in the neighborhood of Ω (and is practically zero away from Ω), the behavior of $S_x(\omega) = |H(\omega)|^2 S_f(\omega)$ depends almost entirely on $|H(\omega)|^2$ (and only slightly on $S_f(\omega)$), provided that $S_f(\omega)$ is flat, that is, it has no peaks in the neighborhood of Ω. See Figure 1.

Assuming that $S_f(\omega)$ is reasonably flat, it can be set equal to a constant, S_0, without appreciable loss of accuracy in determining Ω and ζ (for lightly-damped systems). This follows from the discussion of the preceeding paragraph. Equation (9) for the error function can be rewritten in the form

$$e(\Omega,\zeta) = \int_0^\infty \left[S_0 \, |H(\omega)|^2 - S_x'(\omega) \right]^2 d\omega \tag{19}$$

The actual value of S_0 is not important in the minimization of e and it can be set equal to unity. This is shown in Appendix II.

2.2. Systems with N Degrees of Freedom

The differential equations governing small oscillations of viscously-damped systems with N degrees of freedom are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \tag{20}$$

where all symbols have their usual meaning. Let $[\varphi]$ be the modal matrix assumed normalized with respect to the inertia matrix, $[M]$. Then

$$[\varphi]^T[M][\varphi] = [I]$$

and (21)

$$[\varphi]^T[K][\varphi] = [\omega^2]$$

2.2.1. Systems with Classical Normal Modes

Assume that the system, governed by (20), possesses classical normal modes, that is, assume that the damping matrix is such that

$$[\varphi]^T[C][\varphi] = [R] = [2\zeta\omega] \tag{22}$$

The identification problem for the N-degree-of-freedom system, analogous to the problem for the simple oscillator discussed in Section 2.2.1, is to determine ω_i, ζ_i $(i = 1, 2, \ldots, N)$ from one or more random responses, $x_i(t)$, due to the random forcing function $\{F(t)\}$.

Uncoupling the differential equations of motion (by means of the normal-coordinate transformation) and proceeding as in the case of the single-degree-of-freedom system, it is easy to show (see, for example, Reference 16) that

$$\overline{x_i(t)x_j(t+\tau)} = \frac{1}{2\pi}\sum_{r=1}^{N}\sum_{s=1}^{N}\frac{\varphi_{ir}\varphi_{js}}{\omega_r^2\omega_s^2}\int_{-\infty}^{\infty}H_r^*(\omega)H_s(\omega)S_{f_rf_s}(\omega)e^{i\omega\tau}\,d\omega \tag{23}$$

where

$R_{x_i x_j}(t) \equiv \overline{x_i(t) x_j(t+\tau)}$ is the cross-correlation of the response

$S_{f_i f_j}(\omega)$ = cross-spectral density of the forcing function $\{f\} = [\varphi]^T\{F\}$

$$H_r(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_r}\right)^2 + i\left(2\zeta_r \frac{\omega}{\omega_r}\right)} \tag{24}$$

$H_r^*(\omega)$ = the complex conjugate of $H_r(\omega)$ and all other symbols have been previously defined.

Since

$$R_{x_i x_j}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{x_i x_j}(\omega) e^{i\omega\tau} d\omega, \tag{25}$$

where $S_{x_i x_j}(\omega)$ is the cross-spectral density of the response, it follows that

$$S_{x_i x_j}(\omega) = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{\varphi_{ir}\varphi_{js}}{\omega_r^2 \omega_s^2} H_r^*(\omega) H_s(\omega) S_{f_r f_s}(\omega) \tag{26}$$

or

$$S_{x_i x_j}(\omega) = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{\varphi_{ir}\varphi_{js} S_{f_r f_s}(\omega)}{\left[\omega_r^2 - \omega^2 - i(2\zeta_r \omega\omega_r)\right]\left[\omega_s^2 - \omega^2 + i(2\zeta_s \omega\omega_s)\right]}$$

Assume now that the cross-spectral density of the excitation is zero, i.e.

$$S_{f_r f_s}(\omega) = 0, \qquad r \neq s$$

and let

$$S_{f_r f_r}(\omega) = S_r(\omega)$$

Then,

$$S_{x_i x_j}(\omega) = \sum_{r=1}^{N} \frac{\varphi_{ir}\varphi_{jr}S_r(\omega)}{(\omega_r^2-\omega^2)^2+4\zeta_r^2\omega_r^2\omega^2} \tag{27}$$

The finite Fourier transform of $x_j(t)$ is

$$X_{jn} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_j(\tau)e^{-i\Omega_n\tau}\,d\tau, \tag{28}$$

where

$$\Omega_n = n\left(\frac{2\pi}{T}\right); \quad n = 1, 2, 3, \ldots$$

Notice that the function

$$S'_{x_j x_j}(\omega) = \frac{1}{T}X_{jn}X_{jn}^* \tag{29}$$

is an approximation of $S_{x_j x_j}(\omega)$ at $\omega = \Omega_n$.

2.2.2 Definition and Minimization of Error Function

Define an error function e_j according to

$$e_j(\omega_1, \omega_2, \ldots, \omega_N, \zeta_1, \zeta_2, \ldots, \zeta_N) = \int_0^\infty [S_{x_j x_j}(\omega)-S'_{x_j x_j}(\omega)]^2 d\omega \tag{30}$$

Suppose that p natural frequencies lying in the frequency band (Ω_L, Ω_u) and the associated modal damping ratios are to be identified by processing m records $x_i(t)$, $i = 1, 2, \ldots, m$. Following a procedure similar to that for a system with one degree of freedom, let

$$\{h\}_j = \frac{\partial e_j}{\partial \xi_r} \quad (j = 1, 2, \ldots, m;\ r = 1, 2, \ldots, 2p), \tag{31}$$

where the vector $\{\xi\}$ is defined by

$$\{\xi\} = (\omega_1, \omega_2, \ldots, \omega_p, \zeta_1, \zeta_2, \ldots, \zeta_p)^T \tag{32}$$

The increments $\{\Delta\xi\}_j$ at each iteration are computed from

$$\{\Delta\xi\}_j = [D]_j \{h\}_j, \tag{33}$$

where

$$[D]_j = - \left[\frac{\partial^2 e_j}{\partial\xi_r \partial\xi_s}\right]^{-1} \tag{34}$$

The elements of the derivative matrix in (34) can easily be derived by differentiation, as in the case of the simple oscillator (App. I).

Note that the m random records, $x_j(t)$, $j = 1, 2, \ldots, m$, will give rise to m iterative matrix equations like Eq. (33). Each system of equations yields (after convergence) values for $\omega_1, \omega_2, \ldots, \omega_p, \zeta_1, \zeta_2, \ldots, \zeta_p$. The results are then averaged.

It should be pointed out that for lightly-damped systems the products $|H_r(\omega)||H_s(\omega)|$ are small for $r \neq s$ in comparison to the same products for $r = s$. This implies that the double sum of Eq. (26) reduces to a single sum for lightly-damped systems even though the power cross-spectral densities of the excitation, $S_{f_r f_s}(\omega)$, may not be zero, provided that they are reasonably flat.

3. SYSTEMS WITH NONCLASSICAL NORMAL MODES

This section deals with nonclassically damped linear systems. Using a matrix perturbation technique, general expressions are derived for the correlation and spectral density matrices. Special forms of these matrices considering only a first-order perturbation are also given. Returning to Eq. (22), let us assume that [R] is not diagonal, but the damping matric [C] is such that

$$[R] = \lceil D \rfloor + \epsilon [\mathcal{D}], \tag{35}$$

where ϵ is a small quantity. It is shown in Appendix III that the cross correlation matrix $[R_x(\tau)]$ is given by

$$[R_x(\tau)] = \frac{1}{2\pi}[\varphi]\int_{-\infty}^{\infty}[\Psi^*(\omega)][S_f(\omega)][\Psi(\omega)]^T e^{i\omega\tau}d\omega[\varphi]^T, \tag{36}$$

where

$$[S_f(\omega)] = [\varphi]^T[S_F(\omega)][\varphi] \tag{37}$$

The power spectral density matrix $[S_x(\omega)]$ is given by

$$[S_x(\omega)] = [\varphi][\Psi^*(\omega)][S_f(\omega)][\Psi(\omega)]^T[\varphi]^T \tag{38}$$

In Eqs. (36) and (37), the matrix $[\Psi(\omega)]$ is defined by

$$[\Psi(\omega)] = \lceil H(\omega) \rfloor \left[[I] + i\epsilon\omega[\mathcal{D}]\lceil H(\omega) \rfloor \right]^{-1}, \tag{39}$$

where

$$\lceil H(\omega) \rfloor = \left[-\omega^2[I] + i\omega\lceil D \rfloor + \lceil \omega^2 \rfloor \right]^{-1} \tag{40}$$

3.1 Results for a First-Order Perturbation

The matrix $[\Psi(\omega)]$ for a first-order perturbation is

$$[\Psi(\omega)]_1 = [H(\omega)] - i\epsilon\omega[H(\omega)][\mathcal{D}][H(\omega)] \tag{41}$$

Substituting this expression in (36) and (38), yields

$$\begin{aligned}[R_x(\tau)]_1 &= \frac{1}{2\pi}[\varphi]\int_{-\infty}^{\infty}[H^*(\omega)][S_f(\omega)][H(\omega)]e^{i\omega\tau}d\omega[\varphi]^T \\ &+ \frac{i\epsilon}{2\pi}[\varphi]\int_{-\infty}^{\infty}\omega H^*[\mathcal{D}HS_f - S_fH\mathcal{D}]He^{i\omega\tau}d\omega[\varphi]^T\end{aligned} \tag{42}$$

and

$$\begin{aligned}[S_x(\omega)]_1 &= [\varphi][H^*(\omega)][S_f(\omega)][H(\omega)][\varphi]^T \\ &+ i\epsilon\omega[\varphi][H^*\mathcal{D}HS_f - S_fH^*\mathcal{D}H][\varphi]^T\end{aligned} \tag{43}$$

Notice that the leading terms of (42) and (43) are identical to the cross-correlation and power spectral density matrices, respectively, of classically damped systems (for convenience, the brackets have been ommitted in the second terms of (42) and (43)).

An error function similar to that defined by Eq. (30) can be set up using Eq. (38) or (43). Minimization of this function would yield the natural frequencies of the system, the elements of $[D]$ and the elements of $[\mathcal{D}]$. For systems with nonclassical linear damping, however, it may be more efficient to determine the complex eigenvalues of the reduced system of 2N equations rather than using the approach discussed herein.

4. AN ALTERNATE METHOD OF MATCHING POWER SPECTRAL DENSITIES

4.1 Systems with One Degree of Freedom

A trial-and-error method can be employed to minimize the error function defined by either Eq. (10) or (30). The procedure, which is straight-forward, is described below.

As in the iterative scheme discussed in Section 2, an initial estimate of Ω and ζ is required. The difference between the two methods (Newton-Raphson and trial-and-error) is that the latter requires initial specification only of the ranges in which Ω and ζ lie. Given these ranges, the damping ratio ζ is held fixed and the natural frequency Ω is allowed to vary within the assumed range until a value of Ω is found which minimizes the error. The frequency is then fixed (at this value) and the damping ratio is varied through its range until the error is minimized further. The procedure is repeated until the error becomes smaller than a desired value.

4.2 Multi-Degree-of-Freedom Systems

For a system with N degrees of freedom, initial estimates of ω_i, ζ_i ($i = 1, 2, \ldots, N$) or ranges within which these parameters lie are required. All ω_i's and ζ_i's except one, say ω_k, are held fixed and ω_k is varied within its assumed range until the error given by Eq. (30) is minimum. The procedure is repeated for all ω_k and ζ_k. After each sweep, a new set of ranges in ω_i and ζ_i

(around the values determined by the previous set of iterations) is used and the procedure is repeated. The ranges used each time are made progressively smaller.

It is clear that the trial-and-error procedure can be used initially and, once sufficiently good estimates of ω_i and ζ_i are established, the Newton-Raphson iterative technique can be employed efficiently to yield accurate results in a few iterations.

5. PRELIMINARY NUMERICAL RESULTS

Some preliminary numerical results are given in Tables 1, 2 and 3. Table 1 shows that convergence is reached after six iterations for a one-degree-of-freedom system with parameters $\Omega = 1.0$, $\zeta = 0.04$. The Newton-Raphson scheme discussed in Section 2 was used. Two sets of iterations are shown, one with initial guesses $\Omega = 0.8$, $\zeta = 0.06$ and the other with $\Omega = 1.3$, $\zeta = 0.06$. White-noise excitation with power spectral density 1 was assumed.

In Tables 2 and 3, numerical results are shown for systems with two degrees of freedom. The parameters of the systems are: $\omega_1 = 1.0$, $\zeta_1 = 0.04$, $\omega_2 = 1.5$, $\zeta_2 = 0.04$ (Table 2); $\omega_1 = 1.0$, $\zeta_1 = 0.04$, $\omega_2 = 1.1$, $\zeta_2 = 0.04$ (Table 3). The power spectral densities $S_{x_1 x_1}(\omega)$ for these systems, assuming white-noise excitation with power spectral density 1, are shown in Figs. 2 and 3. The trial-and-error method discussed in Section 4.2 was employed to generate these tables. It is seen that six complete sweeps are sufficient for convergence. Within each sweep, each of the parameters was incremented in ten equal steps (within its range) until the error was minimum.

APPENDIX I

Elements of Derivative Matrix of Equation 17

Consider the function $|H(\omega)|^2$ in the form

$$|H(\omega)|^2 = \frac{\Omega^4}{(\Omega^2 - \omega^2)^2 + 4\zeta^2\Omega^2\omega^2}$$

and let

$$A = (\Omega^2 - \omega^2)^2 + 4\zeta^2\Omega^2\omega^2,$$

$$B = \frac{\partial A}{\partial \Omega} = 4(\Omega^3 - \Omega\omega^2) + 8\Omega\omega^2\zeta^2,$$

$$C = \frac{\partial B}{\partial \Omega} = 4(3\Omega^2 - \omega^2) + 8\omega^2\zeta^2,$$

$$D = \frac{\partial A}{\partial \zeta} = 8\Omega^2\omega^2\zeta,$$

$$E = \frac{\partial D}{\partial \Omega} = 16\Omega\omega^2\zeta.$$

The derivatives of $|H(\omega)|^2$ are

$$\frac{\partial}{\partial \Omega}|H(\omega)|^2 = \frac{\partial}{\partial \Omega}\left(\frac{\Omega^4}{A}\right) = \frac{4\Omega^3}{A} - \frac{B}{A^2}\Omega^4$$

$$\frac{\partial^2}{\partial \Omega^2}|H(\omega)|^2 = \frac{12}{A}\Omega^2 - \frac{8B}{A^2}\Omega^3 - \left(\frac{C}{A^2} - \frac{2B^2}{A^3}\right)\Omega^2$$

$$\frac{\partial}{\partial \zeta}|H(\omega)|^2 = -\frac{D}{A^2}\Omega^4$$

$$\frac{\partial^2}{\partial \zeta^2}|H(\omega)|^2 = \left(\frac{2D^2}{A^3} - \frac{D/\zeta}{A^2}\right)\Omega^4$$

$$\frac{\partial^2}{\partial \Omega \partial \zeta}|H(\omega)|^2 = -\frac{4D}{A^2}\Omega^3 + \left(\frac{2DB}{A^3} - \frac{E}{A^2}\right)\Omega^4$$

Thus, the elements of the derivative matrix of Eq. (17) can be determined (by inversion) from the following derivatives of the error function:

$$\frac{\partial^2 e}{\partial \xi_j \partial \xi_i} = 2S_0 \int_0^\infty \Big[S_0 \frac{\partial}{\partial \xi_j} |H(\omega)|^2 \frac{\partial}{\partial \xi_i} |H(\omega)|^2$$

$$+ \left(S_0 |H(\omega)|^2 - S_x'(\omega) \right) \cdot \frac{\partial^2}{\partial \xi_i \partial \xi_i} |H(\omega)|^2 \Big] d\omega$$

$$\frac{\partial^2 e}{\partial \xi_1^2} = 2S_0 \int_0^\infty \Big\{ S_0 \left(\frac{4\Omega^3}{A} - \frac{B}{A^2}\Omega^4 \right)^2$$

$$+ \left(\frac{S_0 \Omega^4}{A} - S_x'(\omega) \right) \cdot \left(\frac{12\Omega^2}{A} - \frac{8B\Omega^3}{A^2} - \left[\frac{C}{A^2} - \frac{2B^2}{A^3} \right] \right) \Big\} d\omega$$

$$\frac{\partial^2 e}{\partial \xi_2^2} = 2S_0 \int_0^\infty \Big\{ S_0 \Omega^8 \frac{D^2}{A^4}$$

$$+ \left(\frac{S_0 \Omega^4}{A} - S_x'(\omega) \right) \cdot \left(\frac{2D^2}{A^3} - \frac{D/\zeta}{A^2} \right) \cdot \Omega^4 \Big\} d\omega$$

$$\frac{\partial^2 e}{\partial \xi_1 \partial \xi_2} = 2S_0 \int_0^\infty \Big\{ S_0 \left(\frac{4\Omega^3}{A} - \frac{B}{A^2}\Omega^4 \right) \left(- \frac{D}{A^2}\Omega^4 \right)$$

$$+ \left(\frac{S_0 \Omega^4}{A} - S_x'(\omega) \right) \cdot \left(- \frac{4D\Omega^3}{A^2} + \left[\frac{2DB}{A^3} - \frac{E}{A^2} \right] \Omega^4 \right) \Big\} d\omega$$

Note white noise excitation of power spectral density S_0 has been assumed, and that $\xi_1 = \Omega$, $\xi_2 = \zeta$.

APPENDIX II

Proof that for Ideal White Noise the Constant Value of the Power Spectral Density Need not be Known

Consider a linear system with one degree of freedom excited by two different forcing functions $f_1(t)$ and $f_2(t)$, representing ergodic random processes. Let $x_1(t)$ and $x_2(t)$ be the responses to f_1 and f_2, respectively. If the excitations are related by

$$f_2 = \alpha f_1 ,$$

where α is a constant, since the system is linear, the responses will be related by

$$x_2 = \alpha x_1$$

From the definition of power spectral density,

$$S_{x_2}(\omega) = \alpha^2 S_{x_1}(\omega)$$

and

$$S_{f_2}(\omega) = \alpha^2 S_{f_1}(\omega)$$

Noting that

$$X_{n_2} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2(\tau) e^{-i\omega\tau} d\tau$$

$$= \alpha X_{n_1} ,$$

the estimates of the power spectral densities $S'_{x_2}(\omega)$ and $S'_{x_1}(\omega)$ of the responses are related by

$$S'_{x_2}(\omega) = \alpha^2 S_{x_1}(\omega)$$

The error functions e_{f_1} and e_{f_2},

$$e_{f_1} = \int_0^\infty \left[S_{x_1}(\omega) - S'_{x_1}(\omega) \right]^2 d\omega,$$

$$e_{f_2} = \int_0^\infty \left[S_{x_2}(\omega) - S'_{x_2}(\omega) \right]^2 d\omega,$$

are obviously related by $e_{f_2} = \alpha^4 e_{f_1}$. Therefore, from

$$\frac{\partial e_{f_2}}{\partial \xi_i} = \alpha^4 \frac{\partial e_{f_1}}{\partial \xi_i} = 0, \qquad \left(\{\xi\} = \begin{Bmatrix} \Omega \\ \zeta \end{Bmatrix} \right)$$

it is clear that the value of α does not enter in the determination of Ω and ζ (through the minimization of an error function defined according to equation 10), since it cancels out.

APPENDIX III

Response of Linear Nonclassically Damped Systems to Random Excitation

The differential equations governing small oscillations of a viscously-damped, linear system with N degrees of freedom, in matrix form, are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}, \tag{1}$$

where [M], [C] and [K] are the inertia, damping and stiffness matrices, respectively, $\{x\}$ is the response vector, and $\{F(t)\}$ is a vector of applied random forces. Let $[\varphi]$ be the modal matrix of the undamped system. If $[\varphi]$ is normalized with respect to [M],

$$\begin{aligned} [\varphi]^T[M][\varphi] &= [I] \\ [\varphi]^T[M][\varphi] &= [\omega^2] \end{aligned} \tag{2}$$

Let

$$[\varphi]^T[C][\varphi] = [R] \tag{3}$$

The system is said to possess classical normal modes if [R] is diagonal. Conditions for the existence of such modes have been given by Caughey [16]. It is assumed herein that [R] is not diagonal but that its off-diagonal elements are small. That is, we assume that [C] is such that [R] can be written as

$$[R] = [D] + \epsilon[\Delta], \tag{4}$$

where ϵ is small. Introducing the normal coordinate transformation

$$\{x\} = [\varphi]\{p\} \tag{5}$$

in (1) and pre-multiplying by $[\varphi]^T$, yields

$$[I]\{\ddot{p}\} + [R]\{\dot{p}\} + \lceil\omega^2\rfloor\{p\} = [\varphi]^T\{F(t)\} \tag{6}$$

Equation (6) is now solved using a matrix perturbation technique. Assume the following expansion for the vector of normal coordinates:

$$\{p\} = \sum_{k=0}^{n} \epsilon^k\{p_k\} \tag{7}$$

substitution of (4) and (7) in (6) leads to

$$\begin{aligned}\sum_{k=0}^{n}\epsilon^k\{\ddot{p}_k\} + \Big[\lceil D\rfloor + \epsilon[\Delta]\Big]\sum_{k=0}^{n}\epsilon^k\{\dot{p}_k\} \\ + \lceil\omega^2\rfloor\sum_{k=0}^{n}\epsilon^k\{p_k\} = \{f(t)\},\end{aligned} \tag{8}$$

where $\{f(t)\} = [\varphi]^T\{F(t)\}$. Equation (8) can be rewritten in the form

$$\begin{aligned}\sum_{k=0}^{n}\Big(\{\ddot{p}_k\} + \lceil D\rfloor\{\dot{p}_k\} + \lceil\omega^2\rfloor\{p_k\}\epsilon^k \\ + \sum_{k=0}^{n}\epsilon^{k+1}[\Delta]\{\dot{p}_k\} = \{f\}\end{aligned} \tag{9}$$

or

$$\begin{aligned}&[I]\{\ddot{p}_0\} + \lceil D\rfloor\{\dot{p}_0\} + \lceil\omega^2\rfloor\{p_0\} \\ &+ \sum_{k=1}^{n}\epsilon^k\Big(\{\ddot{p}_k\} + \lceil D\rfloor\{\dot{p}_k\} + \lceil\omega^2\rfloor\{p_k\} + [\Delta]\{\dot{p}_{k-1}\}\Big) = \{f\}\end{aligned}$$

Since $\{f\}$ does not depend on ε,

$$\{\ddot{p}_0\} + [D]\{\dot{p}_0\} + [\omega^2]\{p_0\} = \{f\} \tag{10}$$

and

$$\sum_{k=1}^{n} \varepsilon^k\left(\{\ddot{p}_k\} + [D]\{\dot{p}_k\} + [\omega^2]\{p_k\} + [\Delta]\{\dot{p}_{k-1}\}\right) = \{0\}$$

Hence,

$$\{\ddot{p}_k\} + [D]\{\dot{p}_k\} + [\omega^2]\{p_k\} = -[\Delta]\{\dot{p}_{k-1}\} \tag{11}$$

Taking the Fourier transform of (10)

$$\left[-\omega^2[I] + i\omega[D] + [\omega^2]\right]\{\overline{p}_0(\omega)\} = \{\overline{f}(\omega)\}$$

or

$$\{\overline{p}_0(\omega)\} = [H(\omega)]\{f(\omega)\}, \tag{12}$$

where

$$[H(\omega)] = \left[-\omega^2[I] + i\omega[D] + [\omega^2]\right]^{-1}$$

Since

$$\begin{aligned}\{\overline{\dot{p}}_0\} &= i\omega\{\overline{p}_0\} \\ &= i\omega[H]\{\overline{f}\},\end{aligned}$$

from (11) with K=1, we find

$$\begin{aligned}\{\overline{\dot{p}}_1(\omega)\} &= -[H][\Delta]\{\overline{\dot{p}}_0(\omega)\} \\ &= -i\omega[H][\Delta][H]\{\overline{f}\}\end{aligned}$$

Similarly,

$$\begin{aligned}\{\overline{p}_2(\omega)\} &= -[H][\Delta]\{\overline{\dot{p}}_1(\omega)\} \\ &= (-i\omega)^2[H][\Delta][H][E][H]\{\overline{f}(\omega)\} \\ &= (-i\omega)^2[H]\left[[\Delta][H]\right]^2\{\overline{f}(\omega)\}\end{aligned}$$

and

$$\{\bar{p}_k(\omega)\} = (-i\omega)^k [H]\left[[\mathcal{D}][H]\right]^k \{\bar{f}(\omega)\} \tag{13}$$

Now,

$$\{\bar{p}(\omega)\} = \sum_{k=0}^{n} \epsilon^k \{\bar{p}_k(\omega)\}$$

$$= \sum_{k=1}^{n} (-i\omega\epsilon)^k [H]\left[[\mathcal{D}][H]\right]^k \{\bar{f}(\omega)\}$$

or

$$\{\bar{p}(\omega)\} = [H] \sum_{k=1} (-i\omega\epsilon)^k \left[[\mathcal{D}][H]\right]^k \{\bar{f}(\omega)\} \tag{14}$$

Considering an infinite-order perturbation ($K=\infty$), Eq. (14) can be rewritten in the form

$$\{\bar{p}\} = [H]\left[[I] + i\epsilon\omega[\mathcal{D}][H]\right]^{-1} \{\bar{f}\} \tag{15}$$

Substituting in (5)

$$\{\bar{X}(\omega)\} = [\varphi][H(\omega)][B(\omega)]^{-1}\{\bar{f}(\omega)\} \tag{16}$$

where

$$[B(\omega)] = [I] + i\epsilon\omega[\mathcal{D}][H(\omega)] \tag{17}$$

Let

$$[\Psi(\omega)] = [H(\omega)][B(\omega)]^{-1} \tag{18}$$

Equation (15) can be rewritten as

$$\{\bar{p}(\omega)\} = [\Psi(\omega)]\{\bar{f}(\omega)\} \tag{19}$$

Now consider the matrix

$$\{p(t)\}\{p(t+\tau)\}^T$$

Using the convolution theorem the inverse transform of $\{p(t)\}$ is

$$\{p(t)\} = \int_{-\infty}^{\infty} [\Psi(\lambda)]\{F(t - \lambda)\}d\lambda \tag{20}$$

Similarly,

$$\{p(t + \tau)\}^T = \int_{-\infty}^{\infty} \{F(t + \tau - \xi)\}^T [\Psi(\xi)]^T d\xi \tag{21}$$

The cross-correlation matrix of the response $[R_p(\tau)]$ is defined by

$$[R_p(\tau)] = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \{p(t)\}\{p(t + \tau)\}^T dt$$

or, using (20) and (21),

$$[R_p(\tau)] = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\int_{-\infty}^{\infty} [\Psi(\lambda)]\{F(t-\lambda)\}d\lambda\right]\left[\int_{-\infty}^{\infty} \{F(t+\tau-\xi)\}^T [\Psi(\xi)]^T d\xi\right]dt \tag{22}$$

From (22), it can be shown that

$$[R_p(\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\Psi^*(\omega)][S_f(\omega)][\Psi(\omega)]^T e^{i\omega\tau} d\omega \tag{23}$$

where

$$[S_f(\omega)] = [\varphi]^T [S_F(\omega)][\varphi]$$

Since

$$\{x(t)\} = [\varphi]\{p(t)\}, \qquad \{x\}^T = \{p\}^T[\varphi]^T,$$

$$[R_x(\tau)] = \frac{1}{2\pi} [\varphi]\int_{-\infty}^{\infty} [\Psi^*(\omega)][S_f(\omega)][\Psi(\omega)]^T e^{i\omega\tau} d\omega [\varphi]^T$$

REFERENCES

[1] Cole, Henry A., Jr., "On-The-Line Analysis of Random Vibrations," AIAA/ASME 9th Structures, Structural Dynamics and Materials Conference, Palm Springs, California, April 1-3, 1968, Paper No. 68-288.

[2] Wilcox, Philip R. and Crawford, William L.: A Least Squares Method for Reduction of Free Oscillation Data. NASA TN D-4503, 1968.

[3] Cole, Henry A., Jr., "Failure Detection of a Space Shuttle Wing Flutter Model by Random Decrement," NASA TM X-62041, May 1971.

[4] Baird, Eugene F., and W. B. Clark, "Recent Developments in Flight Flutter Testing in the United States," Presented at the 34th Meeting of the AGARD Structures and Materials Panel, Lyngby, Denmark, 9-14 April 1972.

[5] Cole, Henry A., Jr., "Method and Apparatus for Measuring the Damping Characteristics of a Structure." United States Patent No. 3,620,069, Nov. 16, 1971.

[6] Cole, H. A., Jr., "On-Line Failure Detection and Damping Measurement of Aerospace Structures by Random Decrement Signatures," NASA CR-2205, March 1973.

[7] Rosenbaum, Robert, "Survey of Aircraft Subcritical Flight Flutter Testing Methods," NASA CR-132 479, August 1974.

[8] Coupry, Gabriel, "Random Techniques for Flutter Testing in Wind-Tunnel and in Flight," Paper Presented at the Fifteenth Israël Annual Conference on Aviation and Astronautics, Tel-Aviv, Israël, March 15 and 16, 1973.

[9] Houbolt, J. C., "Subcritical Flutter Testing and System Identification," NASA CR-132480, August 1974.

[10] Bhat, Waman, V. and Wilby, John F.: An Evaluation of Random Analysis Methods for the Determination of Panel Damping. NASA CR-114423, Feb. 1972.

[11] Chang, C. S., "Study of Dynamic Characteristics of Aero-elastic Systems Utilizing Randondec Signatures," NASA CR-132563.

[12] Baldock, J. C. A. and C. W. Skingle, "Flutter Technology in the United Kingdom - A Survey," AIAA Paper No. 73-330, Presented at the AIAA Dynamics Specialists Conference, Williamsburg, Virginia, March 19 and 20, 1973.

[13] Dat, Rolland, "The Theoretical and Experimental Methods Used in France for Flutter Prediction," AIAA Paper No. 73-329, Presented at the AIAA Dynamics Specialists Conference, Williamsburg, Virginia, March 19 and 20, 1973.

[14] GauKroger, D. R., Heron, K. H. and C. W. Skingle, "The Processing of Response Data to Obtain Modal Frequencies and Damping Ratios," Journal of Sound and Vibration, Vol. 35, No. 4 (1974), pp. 559-571.

[15] Bendat, J. S. and A. G. Piersol, Random Data: Analysis and Measurement Procedures, Wiley-Interscience, New York, 1971.

[16] Meirovitch, L., Analytical Methods in Vibrations, MacMillan, New York, 1967.

TABLE 1 Initial Guesses and Results at Each Iteration for a System with One Degree of Freedom. $\Omega = 1.0$, $\zeta = 0.04$

Initial Guess	0.8	0.06	1.3	0.6
Iteration No.	Ω	ζ	Ω	ζ
1	.8778	.05400	.9880	.0456
2	.7855	.05911	.9880	.0437
3	.9932	.04452	1.005	.0406
4	1.003	.03692	.9980	.0403
5	1.000	.03995	1.001	.0399
6	1.000	.0400	.9996	.0400

TABLE 2 Initial Guesses and Results at Each Iteration.
$\omega_1 = 1.0$, $\omega_2 = 1.5$, $\zeta_1 = \zeta_2 = 0.04$

Initial Guess	0.8	2.0	0.06	0.06
Iteration No.	ω_1	ω_2	ζ_1	ζ_2
1	0.99200	1.5200	0.04560	0.04560
2	0.99200	1.5200	0.04128	0.04416
3	0.99968	1.5008	0.04013	0.04186
4	0.99968	1.5008	0.04013	0.04093
5	0.99968	1.5008	0.04004	0.04057
6	1.0002	1.4996	0.03999	0.04042

Assumed Ranges: $0.56 \le \omega_1 \le 1.04$

$1.12 \le \omega_2 \le 2.60$

$0.42 \le \zeta_1, \zeta_2 \le 0.78$

TABLE 3 Initial Guesses and Results at Each Iteration.
$\omega_1 = 1.0$, $\omega_2 = 1.1$, $\zeta_1 = \zeta_2 = 0.04$

Initial Guess	0.8	1.5	0.06	0.06
Iteration No.	ω_1	ω_2	ζ_1	ζ_2
1	1.0400	1.1400	.04560	.04920
2	1.0016	1.1040	.03980	.04344
3	1.0016	1.1040	.03926	.04113
4	1.0016	1.0982	.04186	.04045
5	1.0004	1.1005	.03991	.04008
6	.99988	1.0996	.04005	.04000

Assumed Ranges: $0.056 \leq \omega_1 \leq 1.04$

$1.05 \leq \omega_2 \leq 1.95$

$0.042 \leq \zeta_1, \zeta_2 \leq 0.078$

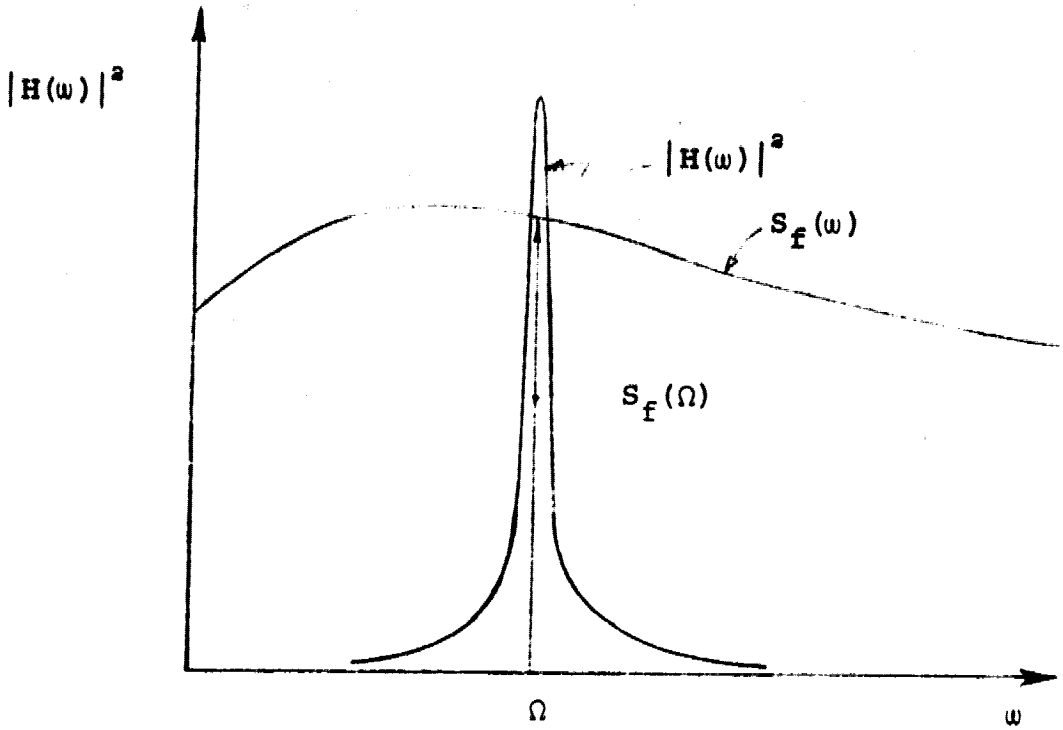

Fig. 1. Sketch of transfer function (squared) for a lightly-damped system with one degree of freedom.

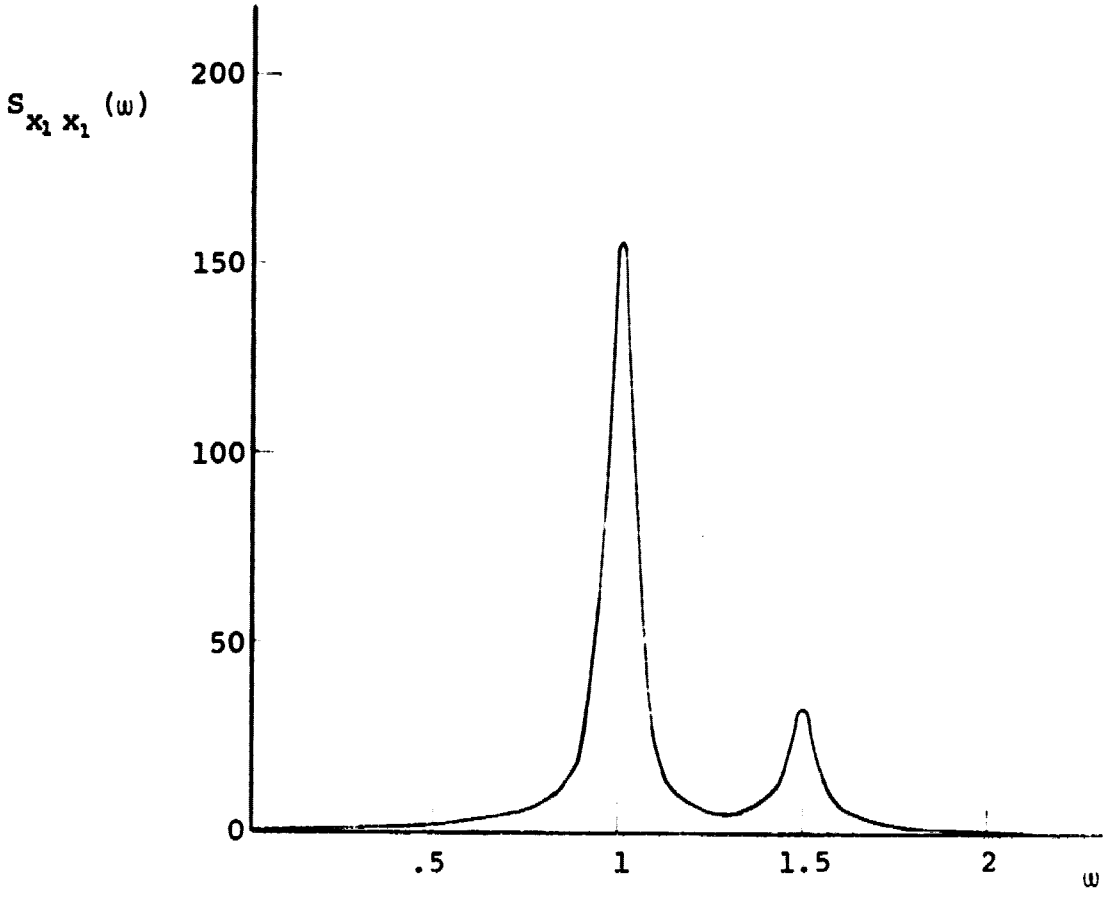

Fig. 2. Power spectral density for a two-degree-of-freedom system ($\omega_1 = 1.0$, $\omega_2 = 1.5$, $\zeta_1 = \zeta_2 = 0.04$).

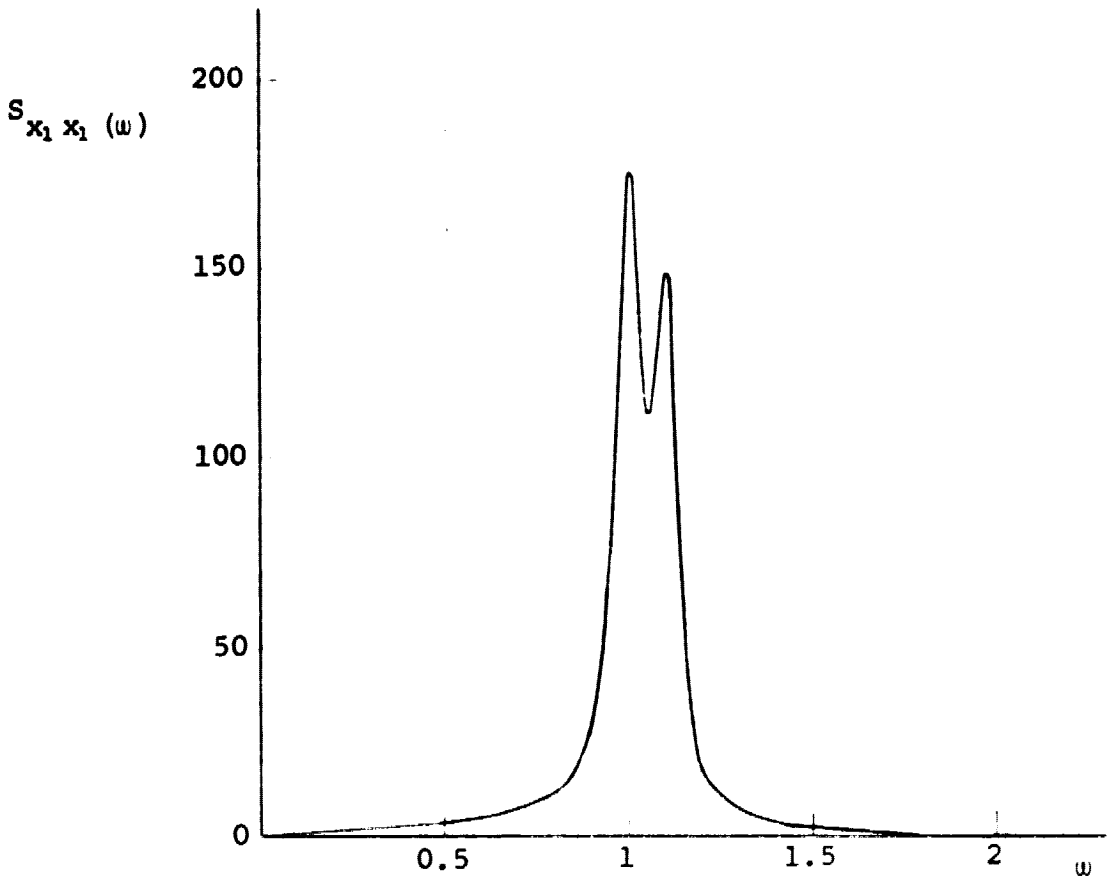

Fig. 3. Power spectral density for a two-degree-of-freedom system ($\omega_1 = 1.0$, $\omega_2 = 1.1$, $\zeta_1 = \zeta_2 = 0.04$).