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A MATHEMATICAL STUDY OF A RANDOM PROCESS PROPOSED AS AN ATMOSPHERIC TURBULENCE MODEL

By

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16. Abstract

A random process is formed by the product of a local Gaussian process and a random amplitude process, and the sum of that product with an independent mean value process. The mathematical properties of the resulting process are developed, including the first and second order properties and the characteristic function of general order. An approximate method for the analysis of the response of linear dynamic systems to the process is developed. The transition properties of the process are also examined.

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INTRODUCTION

Product random processes are used to model atmospheric turbulence in aeronautical applications. One example is the process developed by Press and his associates at NACA (references 1 and 2). This process is formed by the product of two Gaussian processes, one representing the local Gaussian properties of the turbulence and the other introducing a random amplitude modulation. This process is the basis of the current application of random process theory to the specification of aircraft strength requirements for flight in turbulence (references 3 and 4) and has had extensive application to the analysis and measurement of aircraft response to atmospheric turbulence (references 5 through 7, for example). The mathematical properties of the Press model have been examined in reference 8. That study shows how the random process accounts for both the Gaussian property of short term turbulence experience and the non-Gaussian form of long term experience. In the present study the Press model is referred to as either a product or an amplitude modulated random process.

Another random process which is used to model atmospheric turbulence is that developed recently by Reeves (references 9 through 12). This process is a modification of the product process by the addition of an independent mean value process, which allows more versatility in modelling the properties of measured turbulence data. The original development of references 11 and 12 was primarily concerned with the simulation of the short term properties of atmospheric turbulence. Consequently, the three components of the total process (called the local, amplitude, and mean value components in the present study) were specified to have the same integral scale value.

The present study is an extension and reinterpretation of the development of references 11 and 12. The basic random process is reinterpreted by introduction of the modulation concept: the local Gaussian component is modulated by the slowly varying amplitude component and by the addition of the slowly varying mean value component. The resulting random process is referred to as either the amplitude-modulated-plus-mean (AMPM) or the total process. The first part of the study is a general development of the first (at one time value), second (at two time values), and higher order properties of the process without restriction to the modulation condition. An approximate method, based on the modulation concept, for the analysis of the response of linear dynamic systems to the AMPM process is developed. Finally, the transition properties of the process, which explicitly show the modulation effects, are examined.

SYMBOLS

A	standard deviation of either the subscripted process or (without subscript) the R process
b	standard deviation of S process
$C(\ )$	characteristic function of subscripted process
c	standard deviation of M process
$\det(\ )$	determinant of matrix
$E[\ ]$	ensemble average
$\mathrm{erf}(\ )$	error function (reference 13)
$\mathrm{erfc}(\ )$	complementary error function = $1 - \mathrm{erf}(\ )$
$h(t)$	impulsive response function of a linear system
i	unit of imaginaries, $\sqrt{-1}$
M_4	fourth order flatness factor (or kurtosis)
m, M	mean value process
$N(\ )$	expected rate of positive slope crossings of indicated level
N_{or}	expected rate of positive slope zero crossings of R process $= \frac{1}{2\pi}\frac{A_{\dot{r}}}{A_r}$
$p(\ )$	probability density function of subscripted process
$p(w \mid s)$	conditional probability density function of the w process conditional on the value of s
r, R	local random process
s, S	amplitude random process
T	time interval
t	time
w, W	total or amplitude-modulated-plus-mean (AMPM) random process
z, Z	product or amplitude modulated random process

α	ratio of the standard deviations of the Z and M processes
θ	Fourier transformation variable for characteristic function
$\rho(\tau)$	autocorrelation function of subscripted process
σ	modulus of amplitude random process
τ	time difference variable
$\Phi\ (\omega)$	power spectral density function of subscripted process
$\phi\ (\omega)$	normalized power spectral density function of subscripted process
$\Psi\ (\tau)$	autocovariance function of subscripted process
ω	frequency, Fourier transformation variable of τ

Superscripts:

ℓ	integer index

Subscripts:

c	conditional
d	derivative
i	integer index
in	input random process
j	integer index
out	output random process
q	quasi-steady
tr	transition

DEVELOPMENT OF THE AMPLITUDE-MODULATED-PLUS-MEAN (AMPM) RANDOM PROCESS

The mathematical properties of the AMPM process are developed in the present section. The development starts with the definition of the AMPM process in terms of its three component processes. The probabilistic structure of the process, that is, the first order and the higher order properties, is then developed. No restriction is placed on the relative values of the integral scales of the three component processes. The development thus supplements that of references 9 through 12.

Process Definition

The product random process is defined by the product of two component processes.

$$z(t) = r(t)\ s(t) \tag{1}$$

The total or AMPM process is defined by the sum of the product process and a third component process.

$$w(t) = z(t) + m(t) \tag{2}$$

$$w(t) = r(t)\ s(t) + m(t) \tag{3}$$

The three component processes are specified to be statistically independent. They are identified as: the local process R, the amplitude process S, and the mean value process M.* This terminology anticipates some

*The present notation is based upon that of reference 8. The R, S, and M component processes correspond to the a(t), b(t), and c(t) processes, respectively, of references 9 and 12.

conditions which are subsequently placed on the relative values of the integral scales of the components: the rapidly varying local process R is modulated by a slower amplitude process S. The random process Z will be referred to as either the product process or the amplitude modulated process. The random process W will be referred to as either the total process or the AMPM process. In both cases the latter terms will be used when the modulating properties of the amplitude component process are being emphasized. The former terms are used in the present general development, since the modulation condition is not introduced.

First Order Properties

The probabilistic structure of the total process is completely defined by the structures of the three component processes. The development starts with the formulation of the first order properties, that is, the random process is considered at one value of time only. The formulation is developed in terms of the characteristic functions, which are the Fourier transformation of the probability density functions (reference 14).

$$C_w(\theta) = E[e^{iw\theta}] \tag{4}$$

The characteristic function of the product process is developed from its two components, equation (1), by using the relation for the product of two independent random processes (references 8 and 14).

$$C_z(\theta) = \int_{-\infty}^{\infty} C_r(\theta s)\, p(s)\, ds \tag{5}$$

The characteristic function of the total process follows from the defining relation, equation (2), which states that the process is the sum of two independent random processes.

$$C_w(\theta) = C_z(\theta)\, C_m(\theta) \tag{6}$$

Equations (5) and (6) express the characteristic function of the total process in terms of the characteristic and probability density functions of the three component processes.

The probability density function of the total process similarly can be expressed in terms of the functions of the three component processes. The relationship is formulated by writing the joint probability density of the total process and two of its components. This is developed by using the coordinate transformation of equation (3) and the independence of the component processes (reference 14).

$$p(w,s,m) = \frac{1}{|s|}\, p_r\left(\frac{w - m}{s}\right) p(s)\; p(m) \tag{7}$$

The probability density function of the total process is obtained from the joint function by integrating over the range of the two component processes, that is, as a marginal function of the joint density function.

$$p(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(w,s,m)\; ds\; dm \tag{8}$$

Equations (7) and (8) are equivalent to equations (5) and (6). This equivalence can be shown by using the appropriate Fourier transformation relations (reference 15).

The total process can also be developed in terms of a conditional process, which is related to the local R component. The conditional process is the value of the total process for given values of the amplitude and mean value components. The associated conditional probability density function follows from the joint density function, equation (7), and the definition of conditional probability.

$$p_c(w|s,m) = \frac{1}{|s|} p_r\left(\frac{w - m}{s}\right) \tag{9}$$

The preceding development has considered the characteristic and probability density functions of the component processes in general terms; no specific form has been assumed for these processes. Following the developments of Press (references 1 and 2) and Reeves (references 9 through 12), the three independent component processes are specified to be stationary and Gaussian with zero mean values. The first order properties of the component processes are completely determined by their variances, which are

$$E[r^2] = A^2$$

$$E[s^2] = b^2$$

$$E[m^2] = c^2 \tag{10}$$

This notation is guided by that used for the Press model of atmospheric turbulence in the aeronautical literature.* The probability density and

*The notation of A, b, and c for the standard deviations of the component processes in the present study corresponds to that of σ_a, σ_b, and σ_c respectively in reference 9.

characteristic functions of the three component processes have the Gaussian functional form. For the R component process the functions are

$$p_r(r) = \frac{1}{\sqrt{2\pi}\, A} e^{-r^2/2A^2} \tag{11}$$

$$C_r(\theta) = e^{-\frac{1}{2}\theta^2 A^2} \tag{12}$$

The probability density and characteristic functions of the other components have similar functional forms; the notation for the associated variances is given in equation (10).

The statistical moments of the total process are developed from the defining relation, equation (3), and the moments of the three independent components. All odd order moments are zero. The relations for the variance and some of the higher order moments are

$$\begin{aligned} E[w^2] &= A^2b^2 + c^2 \\ E[w^4] &= 9A^4b^4 + 6A^2b^2c^2 + 3c^4 \\ E[w^6] &= 225A^6b^6 + 135A^4b^4c^2 \\ &\quad + 45A^2b^2c^4 + 15c^6 \end{aligned} \tag{13}$$

The relations for the moments can be expressed in an alternate notation. Since the total process is the sum of the independent product and mean value processes, the ratio of the standard deviations of these two processes is a fundamental parameter of the total process.

$$\alpha = Ab/c \tag{14}$$

If the standard deviation of the product process approaches zero, then the value of α becomes zero. In this case the total process is dominated by the mean value process. If the standard deviation of the mean value process approaches zero, then the value of α becomes infinite. In this case the total process is dominated by the product process.

The statistical moments can be expressed in terms of the α parameter by using equations (13) and (14).

$$\begin{aligned} E[w^2] &= c^2(\alpha^2 + 1) \\ E[w^4] &= 3c^4(3\alpha^4 + 2\alpha^2 + 1) \\ E[w^6] &= 15c^6(15\alpha^6 + 9\alpha^4 + 3\alpha^2 + 1) \end{aligned} \tag{15}$$

The associated fourth order flatness factor (or kurtosis) of the total process is

$$M_4(w) = \frac{E[w^4]}{E^2[w^2]} = \frac{3(3\alpha^4 + 2\alpha^2 + 1)}{(\alpha^2 + 1)^2} \tag{16}$$

This factor shows the probabilistic structure of the total process in a concise manner. In the limit of small values of the α parameter, the total process becomes the Gaussian mean value process; the flatness factor has the Gaussian value of three. In the limit of large values of the α parameter, the total process becomes the product process, which is the product of two independent Gaussian processes. The resulting flatness factor is equal to nine, which is the square of the Gaussian value.

The probability density and characteristic functions of the total process are obtained from the functions of the three component processes. The characteristic function of the product process is obtained by using equation (5) (references 8 and 11).

$$C_z(\theta) = (A^2b^2\theta^2 + 1)^{-1/2} \tag{17}$$

The characteristic function of the total process is obtained by using equation (6).

$$C_w(\theta) = (A^2b^2\theta^2 + 1)^{-\frac{1}{2}} e^{-\frac{1}{2}\theta^2c^2} \tag{18}$$

The development of the corresponding probability density function involves an inverse Fourier transformation which appears to be intractable. The formulation can be extended to include the first order characteristic functions of a vector random process. However these functions are easily obtained from the general form of the higher order function, which is developed subsequently.

In the original development (references 1 and 2) the product process was formulated in terms of the associated conditional process, equation (9), rather than in terms of the local R component process. The probability density function of the conditional process associated with the total process is determined by equation (9) and by the specific form of the density function of the R component, equation (11).

$$p_c(w|s,m) = \frac{1}{\sqrt{2\pi}\, A|s|} \, e^{-\frac{(w-m)^2}{2A^2 s^2}} \tag{19}$$

Thus the conditional process is Gaussian with given values of the standard deviation and mean. In this approach the total process is formed from the conditional process by introducing the random variations of the standard deviation and mean value. In the present formulation the amplitude component is not the standard deviation of the conditional process, but can be related to it by giving the standard deviation factor A a unit value and by introducing a new random process which is restricted to nonnegative values.

$$\sigma = |s| \tag{20}$$

The probability density function of the new process follows from that of the amplitude process by using the indicated transformation. Since only the modulus of the amplitude random variable appears in the probability density function of the conditional process, equation (19), and in the joint probability density function, equation (7), the formulation of the first order properties can be developed in terms of either of these two processes.

Second and Higher Order Properties

The formulation is extended to the higher order properties, that is, the relations for the total process considered at more than one value of time. The defining relation of the total process, equation (3), is valid at all time alues. It defines the development in time of the total process in terms of the development of the three component processes. The higher order properties of the total process are thus defined by those of the three component processes.

The autocovariance function is the joint second moment of the random process (with zero mean value) at two time values. Since the component processes are specified to be stationary, the autocovariance of the total process is a function of the difference between the two time values only.

$$E[w(t_1)\ w(t_2)] = \Psi_w(\tau) \tag{21}$$

where $\tau = |t_2 - t_1|$

Using the defining relation, equation (3), and the independence of the component processes, the autocovariance function of the total process is determined by the functions of the three components.

$$\Psi_w(\tau) = \Psi_z(\tau) + \Psi_m(\tau)$$

where (22)

$$\Psi_z(\tau) = \Psi_r(\tau)\ \Psi_s(\tau)$$

The power spectral density function is the Fourier transformation of the autocovariance function. Using equation (22), the power spectral density function of the total process is expressed in terms of the functions of the component processes (reference 16).

$$\Phi_w(\omega) = \Phi_z(\omega) + \Phi_m(\omega) \tag{23}$$

where

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} \Phi_r(\omega - \tilde{\omega})\Phi_s(\tilde{\omega})\ d\tilde{\omega}$$

The indicated integral is the convolution of the functions of the local and the amplitude component process. The convolution operation follows from the Fourier transformation of the product of the two autocovariance functions in equation (22).

The derivative of the total process is expressed in terms of the derivatives of the component processes by using the defining relation, equation (3).

$$\dot{v}(t) = \dot{r}(t)s(t) + r(t)\dot{s}(t) + \dot{m}(t) \tag{24}$$

(This assumes the existence of the indicated derivatives.) The derivative is thus formed by the sum of two product processes and one mean value process. The autocovariance function of the derivative can be expressed in terms of the autocovariance functions of the components either from equation (24) or from the autocovariance relation, equation (22).

The higher order probability density and characteristic functions of the total process are determined from the corresponding functions of the three component processes. The characteristic functions are generally easier to develop. The second order characteristic function of the product process is

$$C_z(\theta_1,\theta_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_r(\theta_1 s_1,\theta_2 s_2)\; p(s_1,s_2)\; ds_1\; ds_2 \tag{25}$$

where $$s_1 = s(t_1)$$

$$s_2 = s(t_2)$$

Since the total process is the sum of the independent product and mean value processes, equation (2), the characteristic function is the product of the functions of those two processes.

$$C_w(\theta_1,\theta_2) = C_z(\theta_1,\theta_2)\ C_m(\theta_1,\theta_2) \tag{26}$$

The formulation can also be developed in terms of an associated conditional process. The relation for the second order probability density function of the conditional process is similar to that for the first order function, equation (9).

$$p_c(w_1,w_2|s_1,m_1;\ s_2,m_2) = \frac{1}{|s_1 s_2|}\, p_r\left(\frac{w_1 - m_1}{s_1}\ ,\ \frac{w_2 - m_2}{s_2}\right) \tag{27}$$

The associated second order characteristic function is obtained by taking the Fourier transformation of the probability density function. Using the appropriate identities or the Fourier transformation (reference 15), the relation is

$$C_c(\theta_1,\theta_2|s_1,m_1;\ s_2,m_2) = C_r(\theta_1 s_1,\theta_2 s_2)\ e^{i\theta_1 m_1 + i\theta_2 m_2} \tag{28}$$

Since the three independent components are specified to be Gaussian random processes, specific relations can be developed for the higher order characteristic functions of the total process. The properties of the Gaussian component processes are defined by their autocovariance functions, which are conveniently expressed in terms of the standard deviations and autocorrelation functions.

$$\Psi_z(\tau) = A^2b^2\rho_z(\tau)$$

$$\Psi_m(\tau) = c^2\rho_m(\tau) \tag{29}$$

where

$$\rho_z(\tau) = \rho_r(\tau)\rho_s(\tau)$$

(The term autocorrelation function is used herein to identify the normalized autocovariance function.) In a similar manner the power spectral density functions of the three component processes are defined in normalized form.

$$\Phi_z(\omega) = A^2b^2\phi_z(\omega)$$

$$\Phi_m(\omega) = c^2\phi_m(\omega) \tag{30}$$

The normalized power spectral density functions are the Fourier transformations of the corresponding autocorrelation functions. The power spectral density functions are normalized in the sense that the integrals of the functions over the total frequency range have unit values. This property corresponds to that of the autocorrelation functions having a unit value when the time difference variable is set to zero.

The autocorrelation and normalized power spectral density functions of the total process are related to the corresponding functions of the product and mean value processes. The relations are developed by combining equations (22) and (23), which relate the autocovariance and spectral functions of the various random processes, with equations (29) and (30), which relate the total and normalized functions of the component processes.

$$\rho_w(\tau) = \frac{\alpha^2}{\alpha^2 + 1} \rho_z(\tau) + \frac{1}{\alpha^2 + 1} \rho_m(\tau) \tag{31a}$$

$$\phi_w(\omega) = \frac{\alpha^2}{\alpha^2 + 1} \phi_z(\omega) + \frac{1}{\alpha^2 + 1} \phi_m(\omega) \tag{31b}$$

These two relations show the contributions of the product and mean value processes to the functions of the total process. In the limit of small values of the α parameter, the functions of the total process become those of the mean value process. In the limit of large values of the α parameter, the functions of the total process become those of the product process.

The expression for the second order characteristic function of the total process is developed from the characteristic and probability density functions of the three component processes. For example, the second order characteristic function of the R component process is

$$C_r(\theta_1,\theta_2) = \exp\{-\tfrac{1}{2} A^2[\theta_1^2 + 2\rho_r(\tau)\theta_1\theta_2 + \theta_2^2]\} \tag{32}$$

The function is dependent upon the difference between the two values of time, which appears in the autocorrelation function. The second order characteristic functions of the other component processes have the same form, with the appropriate notation for the corresponding moments indicated in equation (29).

The second order characteristic function of the total process is developed from equations (25) and (26) by using the appropriate functions for the components and performing the required integrations.

$$C_w(\theta_1,\theta_2) = [(1 - \rho_r^2)(1 - \rho_s^2)\beta_1^2\beta_2^2 + \beta_1^2 + 2\rho_r\rho_s\beta_1\beta_2$$

$$+ \beta_2^2 + 1]^{-1/2} \cdot \exp[- \frac{1}{2} c^2(\theta_1^2 + 2\rho_m\theta_1\theta_2 + \theta_2^2)] \tag{33}$$

where

$$\beta_i = Ab\theta_i \quad i = 1,2$$

The associated second order probability density function is obtained by the inverse Fourier transformation of the characteristic function in both variables. The resulting inversions appear to be intractable.

The second order characteristic function can be used to develop other quantities of interest. An example is the development of the characteristic function of the first derivative of the total process, which is obtained by using the approach of reference 17.

$$C_{\dot{w}}(\theta_d) = \lim_{\tau \to o} C_w(- \frac{\theta_d}{\tau}, \frac{\theta_d}{\tau})$$

$$= [1 - A^2b^2\rho_r''(0)\theta_d^2]^{-1/2}[1 - A^2b^2\rho_s''(0)\theta_d^2]^{-1/2} \tag{34}$$

$$\exp[\frac{1}{2} c^2\rho_m''(0)\theta_d^2]$$

The characteristic function of the derivative is the product of three separate characteristic functions. The first two of these are the characteristic functions of the product of two independent Gaussian processes, equation (17). The third is the characteristic function of a Gaussian process. Thus the derivative of the total process is the sum of three independent processes of the form indicated. This corresponds to the form of the derivative of the total process indicated by equation (24).

The formulation can be extended to the development of higher order properties. The characteristic functions of higher order can be developed from the higher order functions of the Gaussian component processes. The following notation is used for the autocorrelation functions of the R component process.

$$E[r(t_i)r(t_j)] = A^2\rho_{r,ij} \tag{35}$$

Similar notation is used for the amplitude and mean value components. Using the independence of the product and the mean value processes, the characteristic function of general order is the product of the functions of these two processes.

$$C_{\bar{w}}(\bar{\theta}) = C_{\bar{z}}(\bar{\theta})C_{\bar{m}}(\bar{\theta}) \tag{36}$$

The characteristic function of the mean value component is that of a Gaussian process. The characteristic function of the product process is given in reference 8.

$$C_{\bar{z}}(\bar{\theta}) = [\det(q_{ij} + \delta_{ij})]^{-1/2}$$

where

$$q_{ij} = A^2b^2\theta_i \sum_{\ell=1}^{n} \rho_{r,i\ell}\theta_\ell\rho_{s,\ell j} \tag{37}$$

$$\delta_{ij} = \text{Kronecker delta function}$$

This relation is developed as the general order characteristic function of a scalar process, that is, for a scalar process at a general number (equal to the dimension n) of time points. The relation is also the joint characteristic function of a vector process at a single time point, with appropriate interpretation of the correlation coefficients of the component processes. Similarly the relation is the characteristic function of a vector process at several time points, with appropriate interpretation of the correlation coefficients. Thus equations (36) and (37) are the general development of the characteristic functions of the total random process.

QUASI-STEADY APPROXIMATION

An approximate method for the analysis of the response of linear dynamic systems to the AMPM process is developed by using the modulation concept. This method, termed the quasi-steady approximation in reference 8, was inherent in the original development of the amplitude modulated process (references 1 and 2). The resulting analytical technique has had extensive application in the analysis of aircraft dynamic response to atmospheric turbulence (references 5 and 6, for example). The concept of the quasi-steady approximation is applied to the AMPM process in the present section.

The modulation concept introduces the idea of random processes with significantly different time scales. The amplitude component is assumed to be sufficiently slowly varying relative to both the local R process and the system dynamic properties that it does not significantly influence the dynamic aspects of the system response. The system dynamic response to the amplitude modulated process is thus due to the rapidly varying R component process; the amplitude component influences the response only in a static manner. Since the dynamic response is entirely due to the local R process, which is Gaussian, the probabilistic structure of the response of the linear system can be completely determined. The addition of the mean value component introduces no fundamental difficulty in the analysis of linear system response, since that component is Gaussian and independent of the amplitude modulated process.

The development of the quasi-steady approximation is shown by the analysis of the response of a linear dynamic system to the AMPM process. For linear dynamic systems the response or output is the convolution of the impulsive response of the system and the input function.

$$w_{out}(t) = \int_0^\infty h(\tau) w_{in}(t - \tau) d\tau \tag{38}$$

Using the defining relation, equation (3), the input process is expressed in terms of its three independent components.

$$\begin{aligned} w_{out}(t) &= \int_0^\infty h(\tau) r_{in}(t - \tau) s_{in}(t - \tau) d\tau \\ &\quad + \int_0^\infty h(\tau) m_{in}(t - \tau) d\tau \end{aligned} \tag{39}$$

The system output process is formed by the two terms of equation (39), which correspond to the system response to the amplitude modulated and the mean value terms of the input process, equation (2). The response of the linear system to the mean value component process can be analyzed exactly.

$$m_{out}(t) = \int_{o}^{\infty} h(\tau)\, m_{in}(t - \tau)d\tau \tag{40}$$

Since the input mean value process is Gaussian, then so is the output mean value process. This is a general relation for linear systems; there is no restriction on the value of the integral scale of the mean value process.

The quasi-steady approximation is applied to the analysis of the response of linear dynamic systems to the amplitude modulated process, which is the first term of equation (39). Two basic assumptions are made. First, it is assumed that the impulsive response function of the system is essentially zero after some time period T, so that the function can be set to zero with negligible change in the computed system response. The upper limit on the time integral of the amplitude modulated term in equation (39) can accordingly be set to T. Second, the amplitude process is assumed to be sufficiently slowly varying, so that the amplitude is essentially constant over the system response period T. Since the amplitude is then constant during the integration period, it can be removed from the integration. Using these two assumptions, the system response, equation (39), becomes

$$\begin{aligned} w_{out}(t) \approx\ & s_{in}(t) \int_{o}^{T} h(\tau) r_{in}(t - \tau)d\tau \\ & + m_{out}(t) \end{aligned} \tag{41}$$

The integral, which now involves only the local R component process, is the same basic relation as equation (40).

$$r_{out}(t) = \int_0^T h(\tau)\, r_{in}(t - \tau)\, d\tau \tag{42}$$

The output R component process is Gaussian since it is the response of a linear system to a Gaussian process.

The expression for the response of the linear system to the AMPM process is obtained by combining the previous relations.

$$w_{out}(t) \approx r_{out}(t)\, s_{in}(t) + m_{out}(t) \tag{43}$$

Both the system input and output processes have the same probabilistic structure since they have the same defining relation, equations (3) and (43), between the three independent and stationary Gaussian component processes. Thus the input and the system output are both AMPM random processes; they differ only in the forms of the autocovariance functions of the associated R and mean value component processes. The quasi-steady approximation is a method for developing the response of linear dynamic systems to the AMPM process. The restrictions of the method are that the local R component process is Gaussian and that the amplitude component process is much slower than both the R component process and all significant aspects of the system dynamic response. The mean value component process must be Gaussian*. In the present development all three component processes are stationary and Gaussian.

*In the subsequent development the mean value component is also assumed to be slowly varying in the same sense as the amplitude component, although this property of the mean value component is not necessary for the validity of equation (43).

Using the quasi-steady approximation, the problem of the analysis of the response of linear dynamic systems is reduced to the determination of the variances (or the covariance functions) of the system response. The following notation is used for the variances of the components of the input random process.

$$E[r_{in}^2] = 1$$
$$E[s_{in}^2] = b^2 \tag{44}$$
$$E[m_{in}^2] = c^2$$

The input R component process is defined to have unit variance. This condition establishes an arbitrary factor between the R and amplitude component processes, which is inherent in the defining product relation, equation (1). The following notation is used for the variances of the components of the output process.

$$E[r_{out}^2] = A_r^2$$
$$E[m_{out}^2] = A_m^2 c^2 \tag{45}$$

The relations for the variances of the system input and output process follow from the defining relation for the AMPM process, equation (3), and from equations (43) through (45).

$$E[w_{in}^2] = b^2 + c^2$$

$$E[w_{out}^2] \approx A_r^2 b^2 + A_m^2 c^2 \tag{46}$$

These relations express the variances of the two AMPM processes directly in terms of those of the component processes. Alternately, the variances can be expressed in terms of the variance of the mean value component process and the α parameter, which is the ratio of the standard deviations of the amplitude modulated and the mean value processes, equation (14).

$$E[w_{in}^2] = c^2(\alpha_{in}^2 + 1)$$

$$E[w_{out}^2] \approx A_m^2 c^2(\alpha_{out}^2 + 1) \tag{47}$$

The relation between the α parameters of the input and output processes is obtained by combining the previous relations.

$$\alpha_{out} = \frac{A_r b}{A_m c} = \frac{A_r}{A_m} \alpha_{in} \tag{48}$$

This relation shows the effects of the system dynamics upon the probabilistic structure of the system output process. If the system is static, then the variances of the local R and mean value component processes are changed by the same factor; the α parameters for the input and the output processes are equal. The system input and output processes then have the same probability density functions, except for the difference in the values of the standard deviation of the total process. In the general case the dynamic properties of the system will change the value of the α parameter. For a dynamic system which acts as a low-pass filter, the slower mean value component is increased relative to the R component process, resulting in a decreased value of the α parameter. For a system which acts as a high-pass filter, the R component is increased relative to the mean value component process, resulting in an increased value of the α parameter. Thus the system dynamic properties can change the relative contributions of the amplitude modulated and the mean value processes to the total random process, resulting in different functional forms of the probability density functions of the input and output processes.

The relations for the quasi-steady form of the AMPM process are obtained from those for the exact process by setting the dynamic properties of the amplitude and mean value component processes to zero. The dynamic properties of the quasi-steady process are thus due to the R component process only. An example of this procedure is the development of the relations for the derivative of the quasi-steady process. The derivative of the exact process is the sum of three independent processes.

$$\dot{w}(t) = \dot{r}(t)\ s(t) + r(t)\ \dot{s}(t) + \dot{m}(t) \tag{24}$$

The characteristic function of the derivative of the exact process, equation (34), is accordingly the product of three characteristic functions. The characteristic function of the quasi-steady process is obtained from that for the exact process by setting derivatives of the autocorrelation functions of the amplitude and mean value component processes to zero.

$$c_{\dot{w}_q}(\theta_d) = [1 - A^2 b^2 \rho_r''(0)\theta_d^2]^{-1/2} \tag{49}$$

Since this is the characteristic function for product of two independent Gaussian processes, the derivative of the quasi-steady process is

$$\dot{w}_q(t) = \dot{r}(t)\ s(t) \tag{50}$$

The quasi-steady approximation has the effect of removing the mean value variation from the derivative of the AMPM process. The derivative operation acts as a high-pass filter which completely removes the mean value component, at least to the accuracy of the quasi-steady approximation. For the quasi-steady derivative the value of the α parameter is infinite. The fourth order flatness factor of the derivative has a value of nine as opposed to the values between three and nine for the original process, as indicated by equation (16).

One important application of the quasi-steady approximation is the development of the associated exceedance expression, which is the expected frequency of positive slope crossings of a given level of the random process. The exceedance expression is developed from the joint probability density function of the random process and its first derivative (reference 18). Using the quasi-steady approximation, this joint probability density function is developed from that for two uncorrelated random variables at one time point. This in effect omits the local variation of the amplitude and mean value component processes. The resulting relation for the exceedance expression is

$$N(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(w|s,m)\ p(s)\ p(m)\ ds\ dm \tag{51}$$

The exceedance expression of the conditional process is equal to the product of one-half of the first absolute moment of the derivative and the Gaussian conditional probability density function (references 18 and 19).

$$N(w|s,m) = \frac{\dot{A}_r}{\sqrt{2\pi}} p_r\left(\frac{w-m}{s}\right) \tag{52}$$

Using the probability density functions of the three component processes and evaluating the integrals in equation (51), the exceedance expression of the quasi-steady process is

$$\frac{N(w)}{N_{or}} = \frac{1}{2} e^{c^2/2A^2b^2} \left\{ e^{-\frac{w}{Ab}} \left[1 + \mathrm{erf}\left(\frac{w}{\sqrt{2}c} - \frac{c}{\sqrt{2}Ab}\right)\right] + e^{\frac{w}{Ab}} \mathrm{erfc}\left(\frac{w}{\sqrt{2}c} + \frac{c}{\sqrt{2}Ab}\right)\right\} \tag{53}$$

where

$$w \geq 0$$

$$N(-w) = N(w)$$

This is expressed in terms of the standard deviations of the three component processes. An alternate form for the exceedance expression is

$$\frac{N(w)}{N_{or}} = \frac{1}{2} e^{\frac{1}{2\alpha^2}} \left\{ e^{\frac{-w}{\alpha c}} \left[1 + \mathrm{erf}\left(\frac{w}{\sqrt{2}c} - \frac{1}{\sqrt{2}\alpha}\right)\right] + e^{\frac{w}{\alpha c}} \mathrm{erfc}\left(\frac{w}{\sqrt{2}c} + \frac{1}{\sqrt{2}\alpha}\right)\right\} \tag{54}$$

where

$$w \geq 0$$

$$N(-w) = N(w)$$

This is expressed in terms of the standard deviation of the mean value component and the α parameter. The exceedance expression contains both the exponential dependence of the amplitude modulated process and the Gaussian dependence, through the error functions, of the mean value component process. The general case is a combination of these two functional forms. The expected number of zero crossings of the quasi-steady process is

$$N(w = 0) = N_{or} e^{\frac{1}{2\alpha^2}} \operatorname{erfc}\left(\frac{1}{\sqrt{2}\alpha}\right) \leq N_{or} \tag{55}$$

It is thus important to distinguish between the expected number of zero crossings of the AMPM process and those of the R component process, since these two quantities are generally not equal. The difference between the two is due to the quasi-steady effect of the mean value component.

The expected number of zero crossings, equation (55), has two limiting cases. In the limit of a vanishing contribution from the mean value component, α becomes infinite; the expected number of zero crossings is equal those of the R component process. In the limit of a vanishing contribution from the amplitude modulated process, α becomes zero; the expected number of zero crossings approaches zero.

$$\lim_{\alpha \to 0} N(w = 0) = N_{or} \sqrt{\frac{2}{\pi}} \quad \alpha \to 0 \tag{56}$$

In this case the mean value component becomes dominant. However the expected number of zero crossings of the AMPM process does not approach that of the mean value process, since the dynamic properties of that component have been eliminated by the quasi-steady approximation in the development of the exceedance expression, equations (53) and (54).

Several limiting cases of the quasi-steady exceedance expression are of interest. In the limiting case of vanishing contribution from the mean value component, α becomes infinite; the exceedance expression reduces to the exponential form of the amplitude modulated process.

$$\lim_{\alpha\to\infty} N(w) = N_{or}\, e^{-|w|/Ab} \tag{57}$$

This expression was the basis of the original development of the amplitude modulated process in reference 1. The limit of vanishing contribution from the amplitude modulated process (α becomes zero) is more complicated since the dynamics of the dominant mean value component have been eliminated by the quasi-steady approximation. However the general case of non-zero α and finite w can be considered. For small values of both the α parameter and the level of the AMPM random process, the exceedance expression approaches the Gaussian form of the mean value component process.

$$N(|w|<<c;\ \alpha<<1) \approx \sqrt{\frac{2}{\pi}}\, N_{or}\ \alpha\ e^{-\frac{w^2}{2c^2}} \tag{58}$$

For extreme values of the AMPM process, the exceedance expression approaches the exponential form of the amplitude modulated process regardless of the (non-zero) value of the α parameter.

$$N(|w| >> c\alpha) \approx N_{or}\, e^{\frac{1}{2\alpha^2}}\, e^{-|w|/\alpha c} \tag{59}$$

Thus if there is any contribution from the amplitude modulated process to the AMPM process, that is, if the α parameter is not exactly zero, the contribution from the amplitude modulated process will dominate the exceedance expression at sufficiently large values of the AMPM random process.

The preceding development of the exceedance expression is based upon the quasi-steady approximation, which eliminates the dynamic properties of the amplitude and mean value component processes. The validity of the resulting quasi-steady exceedance expression is questionable in the case of small values of both the α parameter and the level of the AMPM process. In this case it is necessary to develop the exact form of the exceedance expression of the AMPM process, following the approach of references 11 and 12. The resulting exceedance expression apparently can not be expressed in analytical form, but requires numerical integration procedures.

TRANSITION PROPERTIES

The transition properties associated with a random process show the nonstationary development of the process from an initial value of time. In the case of the AMPM process, the transition properties explicitly show the effects of the slowly varying amplitude and mean value components upon the total process. The transition properties show the development of the random process from the initial Gaussian form of the local R component process with a given amplitude and mean value to the non-Gaussian form of the fully developed AMPM process.

The primary quantity of interest is the transition probability density function of the AMPM process at one time value, conditional on the value of the process at an earlier time value (references 19 and 20). For the AMPM process the transition probability density function is conditional on the earlier value of all three component processes.

$$p[w(t_2)|r(t_1),\ s(t_1),\ m(t_1)]$$

$$= p[w_2|r_1,\ s_1,\ m_1] \qquad t_1 \leq t_2 \tag{60}$$

The transition of the AMPM process between the two values of time occurs in two stages, since the component processes are classified as rapidly varying (the R component) and as slowly varying (the S and M components). In the first transition stage the density function becomes independent of the initial value of the local R process. This transition is not important in most applications and is omitted in the subsequent discussion. In the second transition stage the probability density function becomes independent of the initial values of the amplitude and mean value processes, thus explicitly showing the effects of the development of these two components.

The transition probability density function is obtained from the conditional function of the original process and the transition functions of the amplitude and mean value components.

$$p(w_2|s_1,m_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(w_2,\ s_2,\ m_2|s_1,\ m_1)\ ds_2\ dm_2 \tag{61a}$$

$$p(w_2|s_1,m_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(w_2|s_1,\ m_1;\ s_2,\ m_2)\ p(s_2|s_1)$$

$$p(m_2|m_1)\ ds_2\ dm_2 \tag{61b}$$

The first relation is the development of the marginal function from the joint transition probability density function. The second relation is the combination of the first with the definition of the conditional probability density function.

Equation (61b) is simplified by using a functional property of the conditional probability density of the original process.

$$p(w_2|s_1, m_1; s_2, m_2) = p(w_2|s_2, m_2) \tag{62}$$

Thus the conditional form of the AMPM process, that is, conditional on the values of the amplitude and mean value components at the same time value, is independent of the values of those two components at an earlier time value. This relation follows from the functional properties of the associated characteristic function.

$$C(\theta_2|s_1, m_1;s_2, m_2) = C(\theta_1,\theta_2|s_1,m_1;s_2,m_2)|_{\theta_1 = 0} \tag{63a}$$

$$= C_r(\theta_1 s_1,\theta_2 s_2)e^{i\theta_1 m_1 + i\theta_2 m_2}|_{\theta_1 = 0} \tag{63b}$$

The first equation expresses the characteristic function as a marginal function of the second order characteristic function. The second equation is the combination of the first with the functional relation, equation (28), for the joint conditional characteristic function of the AMPM process. This equation shows that the conditional characteristic function and thus the conditional probability density function is independent of s_1 and m_1, thereby establishing equation (62).

The relation for the transition probability density function is obtained by combining equations (61b) and (62).

$$p(w_2|s_1,m_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(w_2|s_2,m_2)\ p(s_2|s_1)\ p(m_2|m_1)\ ds_2\ dm_2 \tag{64}$$

The transition probability density thus can be developed from the first order conditional density function of the original process and the transition functions of the amplitude and mean value components. These functions are known from the basic formulation of the AMPM process. Since the component processes are stationary and Gaussian, their transition distributions are known from the associated mean and variance. For the amplitude component for example, the transition probability density function is

$$p(s_2|s_1) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho_s^2(\tau)}\ b}\ \exp\left\{\frac{-[s_2-\rho_s(\tau)\ s_1]^2}{2b^2[1-\rho_s^2(\tau)]}\right\} \tag{65}$$

Since the time dependence of the transition functions of the amplitude and mean value components appears only in the associated autocorrelation functions, the development of the transition probability density function of the AMPM process, equation (64), is dependent solely upon the autocorrelation functions of those two components.

The previous relations can be used to develop the limiting cases of the transition probability density function. The limiting functions of the amplitude component process are

$$\lim_{\tau \to 0} p(s_2|s_1) = \delta(s_2 - s_1) \tag{66a}$$

$$\lim_{\tau \to \infty} p(s_2|s_1) = p(s_2) \tag{66b}$$

where $\delta(s)$ = Dirac delta function

Thus in the limit of small time differences the amplitude component is equal to its initial value s_1 with probability one. In the limit of large time differences the amplitude component process reaches its fully developed form with the probability density function being independent of the initial value. Similar relations hold for the mean value component. The corresponding limiting cases of the transition probability density function of the total process are developed from equations (64), (66a), and (66b).

$$\lim_{\tau \to 0} p(w_2|s_1,m_1) = p(w_1|s_1,m_1) = \frac{1}{|s_1|} p_r\left(\frac{w_1 - m_1}{s_1}\right) \tag{67a}$$

$$\lim_{\tau \to \infty} p(w_2|s_1,m_1) = p(w_2) \tag{67b}$$

Equation (9) has also been used in the first relation. In the limit of small time differences, the transition function has a Gaussian form with given amplitude and mean value. In the limit of large time values, the transition function approaches the form of the fully developed process, which is independent of the initial values of the amplitude and mean value components.

The transition properties of the AMPM process can also be developed in terms of an associated transition random process. This nonstationary process corresponds to the transition probability density function, equation (64). The transition process is formed from the fully developed R process and the transitional forms of the amplitude and mean value components.

$$w_{tr} = r\, s_{tr} + m_{tr} \tag{68}$$

The definition of the transition process in terms of the associated three component processes follows from both equation (61b) and the form of the relations for the transition process, which correspond to the appropriate relations for the original AMPM process, equations (7) through (9). The transition processes of the amplitude and mean value components are defined by their transition probability density functions, for example, equation (65).

The transition process has an associated set of transition moments.

$$E[w_{tr}^{\ell}] = \int_{-\infty}^{\infty} w_2^{\ell}\, p(w_2|s_1,m_1)\, dw_2 \tag{69}$$

The transition moments are obtained from the definition of the transition process, equation (68), by expressing the transition moments in terms of the moments of the independent component processes. The first four transition moments are listed in table 1. The transition moments are functions of time since they are functions of the autocorrelation functions of the amplitude and mean value components. The transition properties allow the possibility of a non-zero values of the odd order moments, which are zero for the fully developed form of the AMPM process. The transition moments show the two limiting cases of the AMPM process for small and large time values. In the limit of small time values, the mean value of the transition process approaches the initial value of the mean value component. In the limit of large time values, the transition moments approach the moments of the fully developed process, which are independent of the initial values of the amplitude and mean value components. All odd order transition moments become zero in the limit of large time values.

The transition properties can also be examined by using the associated flatness factors, which are obtained from the transition moments. In the limit of small time values, the fourth order flatness factor is

$$\lim_{\tau \to o} M_{4,tr}(\tau) = \frac{3A^4 s_1^4 + 6A^2 s_1^2 m_1^2 + m_1^4}{(A^2 s_1^2 + m_1^2)^2} \qquad (70)$$

This factor has a minimum value of one and a maximum value of three. In the former case the moments are dominated by the initial value of the mean value component. In the latter case the initial value of the amplitude modulation process is dominant; the flatness factor has the Gaussian value of three, which results from the fully developed R component process. In the limit of large time values, the fourth order flatness factor becomes that of the fully developed AMPM process, equation (16).

TABLE 1. Transition Moments of the AMPM Process

$$E[m_{tr}] = \rho_m m_1$$

$$E[m_{tr}^2] = (1 - \rho_m^2)c^2 + \rho_m^2 m_1^2$$

$$E[s_{tr}] = \rho_s s_1$$

$$E[s_{tr}^2] = (1 - \rho_s^2)b^2 + \rho_s^2 s_1^2$$

$$E[w_{tr}] = E[m_{tr}]$$

$$E[w_{tr}^2] = A^2 E[s_{tr}^2] + E[m_{tr}^2]$$

$$E[w_{tr}^3] = \{3A^2 E[s_{tr}^2] + 3E[m_{tr}^2] - 2E^2[m_{tr}]\} \, E[m_{tr}]$$

$$E[w_{tr}^4] = 9A^4 E^2[s_{tr}^2] - 6A^4 E^4[s_{tr}] + 6A^2 E[s_{tr}^2] \, E[m_{tr}^2] + 3E^2[m_{tr}^2] - 2E^4[m_{tr}]$$

The transition random process has an associated exceedance expression, which also shows the nonstationary effects of the development of the amplitude and mean value components. The transition exceedance expression is developed from the relations for the second order transition probability density function, which is similar to equation (64), from which the joint density function of the transition process and its first derivative can be developed. Using this joint probability density function and the quasi-steady approximation, the relation for the transition exceedance expression is

$$N(w_2|s_1,m_1) = \iint_{-\infty}^{\infty} N(w_2|s_2,m_2)\ p(s_2|s_1)\ p(m_2|m_1)\ ds_2\ dm_2 \tag{71}$$

All of the required quantities are known: the conditional exceedance expression follows from equation (52) and from the conditional density functions of the amplitude and mean value processes. Thus the transition exceedance expression can be determined. However it can not be obtained in explicit form since the double integration of equation (71) is apparently intractable. The transition exceedance expression shows the limiting cases of the transition process. In the limit of small time values, the exceedance expression becomes that of the quasi-steady conditional process at a single time value, equation (52). In the limit of large time values, the exceedance expression becomes that of the fully developed process, equations (53) and (54).

SUMMARY OF RESULTS

The mathematical properties of a random process formed by the product of a local and an amplitude process, and the sum of that product with a mean value process are examined. The probabilistic structure of the resulting amplitude-modulated-plus-mean (AMPM) process is developed from the structures of the three component processes (the local, amplitude, and mean value components), which are specified to be stationary and Gaussian. The development includes the first and second order properties, and the characteristic function of general order.

The quasi-steady form of the AMPM process is derived by omitting the dynamic properties of the amplitude and mean value components. A method for the analysis of the response of linear dynamic systems to the quasi-steady form of the AMPM process is developed. The system response is shown to be an AMPM process also, at least to the accuracy of the quasi-steady approximation. The probabilistic structure of the system response can however be different from that of the excitation process, depending on the properties of the dynamic system. An analytical relation is developed for the exceedance expression of the quasi-steady form of the AMPM process. The exceedance expression shows a combination of the exponential form of the amplitude modulated process and the Gaussian form of the mean value component process.

The transition properties of the AMPM process are examined. The transition properties show the nonstationary aspects of the process, which are due to the development in time of the amplitude and mean value component processes. The basic relations for the transition probability density and characteristic functions are derived. The transition properties show the development of the AMPM random process from the initial Gaussian form of the process with given values of the amplitude and mean value to the non-Gaussian form of the fully developed process.

REFERENCES

1. Press, Harry; Meadows, May T.; and Hadlock, Ivan: A Reevaluation of Data on Atmospheric Turbulence and Airplane Gust Loads for Application in Spectral Calculations. NACA Rep. 1272, 1956. (Supersedes NACA TN 3362 by Press, Meadows, and Hadlock and TN 3540 by Press and Meadows.)
2. Press, Harry; and Steiner, Roy: An Approach to the Problem of Estimating Severe and Repeated Gust Loads for Missile Operations. NACA TN 4332, 1958.
3. Airplane Strength and Rigidity — Flight Loads. Mil. Specific. MIL-A-008861A(USAF), Mar. 31, 1971.
4. Recommendation 6/1. Amendment of the Airworthiness Technical Manual. Report on Agenda Item 6, Doc. 9065-Air C-10, Airworthiness Committee, Tenth Meeting International Civil Aviation Organization (Montreal), May-June 1973, pp. 6-8 - 6-13.
5. Houbolt, John C.; Steiner, Roy; and Pratt, Kermit G.: Dynamic Response of Airplanes to Atmospheric Turbulence Including Flight Data on Input and Response. NASA TR R-199, 1964.
6. Hoblit, Frederic M.; Paul, Neil; Shelton, Jerry D.; and Ashford, Francis E.: Development of a Power-Spectral Gust Design Procedure for Civil Aircraft. Tech. Rep. ADS-53 (Contract FA-WA-4768), FAA, Jan. 1966.
7. Houbolt, John C.: Design Manual for Vertical Gusts Based on Power Spectral Techniques. Tech. Rep. AFFDL-TR-70-106, U.S. Air Force, Dec. 1970.

8. Sidwell, Kenneth: A Mathematical Examination of the Press Model for Atmospheric Turbulence. NASA TN D-8038, Oct. 1975.

9. Reeves, P. M.; Campbell, G. S.; Ganzer, V. M.; and Joppa, R. G.: Development and Application of a Non-Gaussian Atmospheric Turbulence Model for Use in Flight Simulators. NASA CR-2451, Sept. 1974.

10. Reeves, Paul M.; Joppa, Robert G.; and Ganzer, Victor M.: A Non-Gaussian Model of Continuous Atmospheric Turbulence Proposed for Use in Aircraft Design. AIAA Paper 75-31, Jan. 1975.

11. Reeves, Paul Miller: A Non-Gaussian Model of Continuous Atmospheric Turbulence for Use in Aircraft Design. Ph. D. Thesis, Univ. of Washington, 1974.

12. Reeves, Paul M.; Joppa, Robert G.; and Ganzer, Victor M.: A Non-Gaussian Model of Continuous Atmospheric Turbulence for Use in Aircraft Design. NASA CR-2639, Jan. 1976.

13. Abramowitz, Milton; and Stegun, Irene A., eds.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. NBS Appl. Math. Ser. 55, U.S. Dep. Comm., June 1964.

14. Papoulis, Athanasios: Probability, Random Variables, and Stochastic Processes. McGraw-Hill Book Co., Inc., c.1965.

15. Lighthill, M. J.: Introduction to Fourier Analysis and Generalised Functions. Cambridge Univ. Press, 1958.

16. Gerlach, O. H.; van de Moesdijk, G. A. J.; and van der Vaart, J. C.: Progress in the Mathematical Modeling of Flight in Turbulence. Flight in Turbulence, AGARD-CP-140, 1973, pp. 5-1 - 5-38.

17. Moyal, J. E.: Stochastic Processes and Statistical Physics. J. Royal Statist. Soc., ser. B, vol. 11, 1949, pp. 150-210.

18. Rice, S. O.: Mathematical Analysis of Random Noise. Bell System Tech. J., Pts. III and IV, vol. XXIV, no. 1, Jan. 1945, pp. 46-156 (Reprinted in: Wax, Nelson, ed.: Selected Papers on Noise and Stochastic Processes. Dover Publ., Inc., c.1954.)

19. Lin, Y. K.: Probabilistic Theory of Structural Dynamics. McGraw-Hill Book Co., Inc., c.1967.

20. Parzen, Emanuel: Stochastic Processes. Holden-Day, Inc., 1962.