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DIGITAL FLIGHT CONTROL SYSTEMS

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I. INTRODUCTION

Under NASA Grant NGR-47-004-116, two major problems have been studied. The first problem studied is the design of stable feedback control laws for sampled-data systems with variable rate sampling. These types of sampled-data systems arise naturally in digital flight control systems which use digital actuators. In these control systems, it is desirable to decrease the number of control computer output commands in order to save wear and tear of the associated equipment. Variable sampling also provides the designer with the capability of a more efficient utilization of the flight control computer than the standard fixed sampling-rate approach. Therefore, a variable sampling approach can also be of value to digital flight control systems using analog actuators. For instance, more time can be devoted to the identification of aircraft parameters or to some other task by reducing the control calculations using variable sampling.

The second major problem studied under NASA Grant NGR-47-004-116 is the design of aircraft control systems which are optimally tolerant of sensor and actuator failures. The first problem to be resolved is the detection of the failed sensor or actuator. If the estimate of the state is used in the control law, then it is also desirable to have an estimator which will give the optimal state estimate even under the failed conditions. Both the detection of sensor and actuator failures and the optimal state estimation with sensor and actuator failures are important control system problems which, if not resolved, can seriously (even fatally) degrade the control system performance of an aircraft.
II. OPTIMAL CONTROL OF SAMPLED-DATA SYSTEMS WITH VARIABLE SAMPLING

At each flight condition, the airplane dynamics can be modeled by a continuous, linear, time-variant, dynamic system [1]. Comparative simulations have indicated that a model-follower scheme in which the error between the model states and the plant states is penalized continuously in time was a more suitable approach in the design of control laws for sampled-data systems with variable sampling than designs based on a minimization of error at only the sampling instants. The aircraft dynamics are continuous and the gust inputs affecting the airplane are continuous random processes, but the measurements are made only at the sampling instants and the control is constrained to be constant between the sampling instants. Thus, the problem can be cast into the format of the stochastic sampled-data regulator problem of linear stochastic optimal control theory. The first problem to be resolved was to find out whether the separation theorem of linear optimal control continues to hold for the stochastic sampled-data regulator problem. This problem has been resolved and the results are reported in reference [2] along with a discrete-time stochastic problem which is equivalent to the stochastic sampled-data regulator problem. A summary of the results follows.

The stochastic sampled-data regulator problem is to find the stochastic optimal control for the dynamical system represented by

\[ \dot{x}(t) = Ax(t) + Bu(t) + w(t) \quad t \in [t_0, t_N] \]  \hspace{1cm} (1)

where \( x \) is the \( n \)-dimensional state vector, \( u \) is the \( r \)-dimensional control vector, and \( w \) is the white Gaussian plant noise vector of dimension \( n \) with \( \mathbb{E}w(t) = 0 \) and \( \mathbb{E}w(t)w'(s) = \mathbb{E}_{\omega} \delta_D(t-s) \) for some positive semidefinite
matrix $\psi$. $E$ denotes the expectation operator and $\delta_D$ is the Dirac delta function. A and B are matrices of appropriate order.

The plant noise is a continuous random process; however, the measurements are available only at the sampling instants:

$$y(t_k) = Cx(t_k) + v(t_k) \quad k = 0, 1, 2, \ldots, N-1$$  \hspace{1cm} (2)

where $y$ is the $m$-dimensional measurement vector, and $v$ is a Gaussian sequence of uncorrelated zero-mean random vectors with $Ev(t_i)v'(t_j) = \psi_{ij}$.

The cost functional penalizes the state and the control continuously in time

$$J = \frac{1}{2} \int_0^T E \left[ x'(t)Q_x x(t) + u'(t)R_u u(t) \right] dt$$  \hspace{1cm} (3)

where $Q_x$ is positive semidefinite and $R_u$ is positive definite. The stochastic sampled-data regulator problem is to find the control sequence \[ \{u(t_k)\}_{k=0}^{N-1} \] with the constraint

$$u(t) = u(t_k) \quad t \in (t_k, t_{k+1}) \quad k = 0, 1, 2, \ldots, N-1$$  \hspace{1cm} (4)

and also the additional constraint that $u(t_k)$ will depend only on the past measurement sequence $y(t_i)$, $i = 1, 2, \ldots, k$ which will minimize the cost functional (3).

In reference [2], it is shown that this stochastic optimal control problem is equivalent to finding the optimal control for the discrete system

$$x_{k+1} = \phi(t_{k+1}, t_k) x_k + \Gamma(t_{k+1}, t_k) u_k + v_k$$

$$y_k = C_k x_k + v_k \quad k = 0, 1, \ldots, N-1$$  \hspace{1cm} (5)

where $x_k = x(t_k)$, $u_k = u(t_k)$, $\phi(t_{k+1}, t_k) = \exp A(t_{k+1} - t_k)$, $\Gamma(t_{k+1}, t_k) =$
\[ f_{k+1}^{\kappa} \Psi(t_{k+1}'s) dsB, \text{ and } w_k \text{ is a zero-mean Gaussian sequence of random vectors with} \]
\[ Ew_k w_j' = [ \int_{t_k}^{t_{k+1}} \Psi(t_{k+1}'s) \Psi'(t_{k+1}'s) ds ] \delta_{kj} \tag{6} \]
with the cost functional
\[ J = \frac{1}{2} \Sigma_{k=0}^{N-1} (x_k Q_k x_k + 2x_k M_k u_k + u_k R_k u_k) \tag{7} \]
where
\[ Q_k = \int_{t_k}^{t_{k+1}} \Psi'(t,t_k') Q_c \Psi(t,t_k') dt \tag{8} \]
\[ M_k = \int_{t_k}^{t_{k+1}} \Psi'(t,t_k') Q_c \Gamma(t,t_k') dt \tag{9} \]
\[ R_k = \int_{t_k}^{t_{k+1}} [R_c + \Gamma'(t,t_k') Q_c \Gamma(t,t_k')] dt \tag{10} \]
The stochastic optimal control is shown to be given by
\[ u^*_i = -[R^{-1}_i M_i' + (R_i + \Gamma_i K_{i+1} R_1')^{-1} \Gamma_i K_{i+1} R_1] \bar{x}_i \tag{11} \]
where \( K_i \) is the solution to the Riccati difference equation
\[ K_i = \bar{\phi}_i' [K_{i+1} - K_{i+1} \Gamma_i (R_i + \Gamma_i K_{i+1} R_1')^{-1} \Gamma_i' K_{i+1} R_1] \bar{\phi}_i + \bar{Q}_i \tag{12} \]
with \( K_N = 0 \) and where \( \Gamma_i = \Gamma(t_{i+1},t_i) \), \( \bar{\phi}_i = \phi(t_{i+1},t_i) - \Gamma_i' R_1^{-1} M_i' \), 
\( \bar{Q}_i = Q_i - M_i R_1^{-1} M_i' \), and \( \bar{x}_i \) is the conditional expectation of \( x_i \) given the observation sequence \( \{y_0, y_1, \ldots, y_i\} \).

Looking at equations (11) and (12), it is seen that the optimal control law for the stochastic sampled-data regulator problem is the same as the deterministic sampled-data regulator problem [3] with \( x_k \) replaced by its estimate \( \bar{x}_k \). Therefore, the separation between estimation and control continues to hold for the stochastic sampled-data regulator problem.
The only modification needed for stochastic control is the use of the derived equivalent discrete plant covariance (6) in the Kalman filter equations for the state estimates.

As a byproduct of this investigation, new results were also obtained concerning the geometric relationship between the optimal solutions to the sampled-data and the continuous regulator problems. The findings are reported in [4]. A brief summary of findings is given below.

Since the dynamical system (1) and the integral cost function (3) used are the same for both continuous and the sampled-data regulators, the optimal sampled-data control must be intuitively an approximation to the continuous one in some sense. The relationship between the two optimal solutions, that is, the sense in which the sampled-data solution is an approximation to the continuous solution, has been obscured due to the separate formulations of these two problems in the control literature. The continuous problem has been solved by using the Pontryagin's minimum principle, by using the Hamilton-Jacobi-Bellman partial differential equation for the optimal cost function, and by some other methods. The sampled-data problem has been solved by converting it into an unconstrained discrete minimization problem through the integration of the cost functional and the system differential equations over each sampling interval and then applying dynamic programming or the discrete minimum principle. In our study, the two problems have been formulated in the same Hilbert space as minimum norm problems. In this geometric formulation, it is shown in reference [4] that the optimal sampled-data control is a "projection" of the optimal continuous control. Specifically, it is shown, that if \([x^*(t), u^*(t)]\) is the optimal continuous regulator solution and \([x^{**}(t), u^{**}(t)]\) is the optimal sampled-data solution to the corres-
ponding deterministic regulator problems, then

\[
[Hx^*(t), u^*(t)] = [H\phi(t, t_o)x(t_o), 0] + \\
\mathcal{P}_N[Hx^*(t) - H\phi(t, t_o)x(t_o), u^*(t)]
\]

where \(H\) is defined by \(H^TH = Q_c\), \(N\) is the sub-Hilbert space of all output-input pairs satisfying the sampled-data constraint with zero initial condition, and \(\mathcal{P}\) denotes the projection operator with respect to the Hilbert space norm

\[
||\langle x, u \rangle || = \frac{1}{2} \int_{t_o}^{t_f} [x'(t)Q_c x(t) + u'(t)R_c u(t)] dt
\]

One of the implications of the result is that if an optimal sampled-data tracking problem is to be solved where the desired trajectory to be tracked is the optimal continuous regulator solution, then the solution obtained would be the same as the solution to the original sampled-data regulator problem. To phrase it precisely, the optimal sampled-data regulator solution is the sampled-data control that minimizes

\[
\frac{1}{2} \int_{t_o}^{t_f} \left[ (x(t) - x^*(t))^\top Q_c (x(t) - x^*(t)) + \\
[u(t) - u^*(t)]^\top R_c [u(t) - u^*(t)] \right] dt
\]

The above equation clearly shows the sense of the approximation of the sampled-data control. Equation (14) shows that the optimal sampled-data solution is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint.

Furthermore, the specific projections have been converted into recursive algorithms to compute the optimal sampled-data control. These algorithms are new; however, they are not necessarily less complicated than the known Riccati equations in the literature.
During the course of the project, a number of sampling interval adaptation control laws have been developed based on minimization of local objective functions. Although these algorithms proved to be successful in various simulations, they suffered from being only locally optimal. A control law that would take into account the changing of the sampling intervals was needed. This was accomplished by modeling the sampling interval sequence as a finite-state Markov chain with known transition probabilities. The finite-state assumption, that is, the constraint that the sampling interval can assume values only from a finite number of sampling intervals, was necessary to avoid an infinite set of equations. Specifically, the system equations are given by

\[ x(t) = Ax(t) + Bu(t) \]  

(16)

where \( x, u, A, B \) are as in equation (1). The sampling interval sequence \( \{T_k, k = 0, 1, 2, \ldots \} \) is a finite-state Markov chain that assumes values \( \{S_1, S_2, \ldots, S_n\} \) with the transition probability

\[ P(T_{k+1} = S_j \mid T_k = S_i) = p_{ij} \]  

(17)

and the initial probability distribution

\[ P(T_0 = S_i) = p_i, \quad i = 1, 2, \ldots, n \]  

(18)

Then, the discrete-time stochastic system is given by

\[ x_{k+1} = \Phi(t_k + T_k, t_k)x_k + \Gamma(t_k + T_k, t_k)u_k, \]

\[ k = 0, 1, \ldots, N-1 \]  

(19)
where $\Phi$, $\Gamma$ are as in equation (5). The observations are modeled as

$$y_k = (x_k, T_k)$$  \hspace{1cm} (20)

That is, at each sampling instant $t_k$, the state vector at $t_k$ is known and the sampling interval $T_k$ to be applied at the instant $t_k$ is known. Note that the sampling intervals to be used at $t_{k+1}$ and further on are uncertain at time $t_k$. Only a statistical knowledge of their uncertainty is known through the transition probabilities. Measurement noise and plant noise are not included in the model to keep the equations simple. The problem is to find the stochastic optimal control

$$\{u_k, k = 0, 1, \ldots, N-1\}$$

that depends only on the past and present measurements $\{y_i, i = 0, 1, 2, \ldots, k\}$ that minimizes

$$J = \frac{1}{2} \sum_{k=0}^{N-1} \mathbb{E} \left[ x_k'Qx_k + u_k'Ru_k + x_k'Hx_k \right]$$  \hspace{1cm} (21)

The stochastic optimal control has been obtained by dynamic programming and is given by

$$u_k = -F(T_k)x_k$$  \hspace{1cm} (22)

where $F(T_k) = F_k,i$ for $T_k = S_i$, given by the recursive equations

$$F_k,i = [R + \Gamma_i'(\sum_{j=1}^{n} p_{ij}^k F_{k+1,j})\Gamma_i]^{-1} \sum_{j=1}^{n} p_{ij}^k (F_{k+1,j} + F_{k,i}^* H F_{k,i})$$  \hspace{1cm} (23)

$$K_k,i = Q + (\phi_i - \Gamma_i F_{k,i})' (\sum_{j=1}^{n} p_{ij}^k F_{k+1,j})(\phi_i - \Gamma_i F_{k,i}) + F_{k,i}^* F_{k,i}$$  \hspace{1cm} (24)

with $K_{N,i} = H$; $i = 1, 2, \ldots, n$; $k = 0, 1, \ldots, N-1$; and $\phi_i = \phi(S_i,0)$, $\Gamma_i = \Gamma(S_i,0)$.

That is, at each sampling instant $t_k$, the control is given by a linear feedback law and the feedback gain at the instant $t_k$ is $F_{k,i}$ where the "$i$" corresponds to the sampling interval $S_i$ at the instant $t_k$. The recursive equations are similar to the standard Riccati equations of
linear optimal control. As it can be seen from equation (24), \( n \) coupled Riccati equations are to be solved recursively. The coupling comes from the terms \( P_{ij} \), corresponding to the transition probabilities for the sampling interval sequence. In the case when \( n = 1 \) and \( P_{11} = 1 \), we get the standard Riccati equation of optimal control.

Steady-state gains can be obtained as \( k \to -\infty \), then \( \lim_{k \to -\infty} k \to -\infty \)

\[ F_{ki} = F_i, \quad i = 1, 2, \ldots, n. \]

The stochastic optimal control for the infinite time problem becomes

\[ u_k = -F_{ki} x_k \]

for all \( k \) such that \( T_k = S_i \).

The infinite time optimization solution is very easy to implement. Steady-state gains, \( \{F_i, \quad i = 1, 2, \ldots, n\} \), where \( n \) is the total number of possible sampling intervals, are calculated off-line and stored. At each sampling instant \( t_k \) the feedback gain \( F_i \) is used where "\( i \)" is determined by the sampling interval \( S_i \) at time \( t_k \). The algorithms for the stochastic optimal control, equations (23) and (24), are not yet available in the literature.

The difficulty of applying these results can be the selection of a sampling interval adaptation law, since the problem formulation does not address that question. However, for any sampling interval adaptation law whose sampling interval sequence can be modeled by a Markov chain with appropriate transition probabilities, the stochastic optimal control derived can be used. Moreover, several examples have been simulated on the digital computer where the sampling interval sequence was chosen arbitrarily and the results have been very successful. A computer program that calculates the stochastic optimal control gains has been written. Given next is an example showing how the results can be applied to a specific problem.
Example

Let the system matrices be given by

\[
A = \begin{bmatrix}
0 & 6.28 \\
-6.28 & -3.14
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\] (25)

The control weighting matrices have been chosen to be

\[
Q = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad R = [1]
\] (26)

Sampling intervals are chosen as

\[
S_1 = .1, \quad S_2 = .01
\] (27)

The transition probabilities have been chosen to be the simplest case

\[
P_{ij} = .5, \quad i,j = 1,2
\] (28)

That is, if the sampling interval at \(t_k\) is \(S_1\), the chance of the next interval being the same and the chance of the sampling interval jumping to \(S_2\) are equally likely.

The stochastic optimal control gains have been solved by using equations (23) and (24) to obtain the steady-state values

\[
F_1 = [-.0181 \quad .4455]
\]

\[
F_2 = [.01246 \quad .0612]
\]

The stochastic optimal closed-loop system has been simulated by using arbitrary sampling interval sequences. Figure 1 shows one of these runs. For this fixed sampling interval sequence, the deterministic optimal control feedback gains have been found also, and the closed-loop simulation of this system is in Figure 2. An overlay of the two figures would show that the two trajectories are virtually the same. For a
Figure 1. Optimal Stochastic State Trajectories
Figure 2. Optimal Deterministic State Trajectories
fixed sampling interval sequence and the cost function of (21), expectation can be dropped when the sampling interval sequence is fixed because the system becomes deterministic in this case. Since this resulting deterministic time-varying linear quadratic regulator solution is optimal, the closeness of the stochastic and the deterministic trajectories are quite noteworthy. The remarkable aspect of the stochastic optimal control is that it not only matches the ultimate optimal deterministic performance for this fixed sampling interval sequence, but it also matches the performance of several other sampling interval sequences in the simulations. Of course, the deterministic optimal control is not realizable for the problem we are considering since it requires the absolute knowledge of the whole sampling interval sequence.

**Stability of Sampled-Data Systems with Variable Sampling**

The deterministic stability conditions of the stochastic model of the previous section has been investigated jointly with Dr. D. P. Stanford of the College William and Mary. The results have been reported in [5]. From the formulation in the previous section, it is seen that "n" closed-loop discrete system matrices will be obtained

\[ \Phi_1^{-1} F_1, \Phi_2^{-1} F_2, \ldots, \Phi_n^{-1} F_n, \]

where "n" is the total number of possible sampling intervals and the "Fi" are the stochastic optimal steady-state gains obtained through equations (23) and (24). The question of whether any initial condition can be brought to zero by a repeated application of the matrices in (29) in some order has been investigated in the stability analysis [5]. If a sequence of matrices can be found whose terms are selected from the set of matrices in (23) for each initial condition such that the state goes to zero in the
limit when this sequence of matrices are applied, then the set of matrices will be called convergent. In [5], it is shown that "precontractiveness" is a necessary and sufficient condition for a set of matrices to be convergent. It is also shown that "contractiveness" is a sufficient condition for a set of matrices to be convergent. The stochastic optimal feedback gains have been found for a number of examples. It has been found that the closed-loop system matrices resulting from the stochastic optimal control turn out to be contractive in each case. However, an analytical justification has not been found yet.

III. FAILURE ACCOMMODATION IN CONTROL SYSTEMS

The second major problem which has been investigated in NASA Grant NGR-47-004-116 is the design of aircraft control systems which are optimally tolerant of sensor and actuator failures. A design method has been developed, and the results have been reported in Ref. [6]. The method developed is based on Bayesian decision theory.

Each sensor and actuator failure mode (including the normal operation mode) is formulated as one hypothesis. Using M-ary hypothesis testing, the corresponding likelihood ratios are computed for each hypothesis. The computations of likelihood ratios require M different Kalman filters corresponding to M different failure modes of the system where M is the total number of different hypotheses. By comparing the likelihood ratios, the most likely failure mode of the system is selected in the Bayesian sense.

The unique feature of this method is the flexibility of modeling the sensor failures as noise with unknown mean and variance. The mean of each sensor in a failed mode is computed on line by employing maximum likelihood estimation. This estimate for the mean is used in the calcu-
lation of the likelihood ratios which make them generalized likelihood ratios. The advantage of the method is that it is not required for the designer to know how the sensor will fail. This point has been demonstrated in the simulation [7] by the superior performance of the system in the detection of both increased noise type and hard-over type failures. Although the variances of the sensors in failed modes have been fixed in Ref. [6], the same approach for the estimation of the mean can be used for the variance as well. The maximum likelihood estimation of the parameters requires the storage of a moving window of innovations of each Kalman filter.

The applicability of the fault-tolerant system design has been demonstrated by using a real-time hybrid simulation for a space shuttle orbiter developed jointly with Dr. R. C. Montgomery of NASA/Langley Research Center. The failures were identified in two or three sampling periods. The simulations indicate that the use of steady-state Kalman filters were adequate.

Parameter Adaptive Estimation

From the preceding failure detection problem in aircraft control systems research, the additional benefit of obtaining the optimal state estimate under failed conditions resulted. This problem is exceedingly important if state variable feedback is used and if a filter is used to get the state estimates. This estimation problem has been resolved in a Ph.D. thesis [7] and extensions of the work have been reported in [8]. This problem is known as parameter adaptive estimation in the literature. The following is a summary of the work reported in [7] and [8]. Consider the observation model

\[ y(t) = z(\theta, t) + v(t) \]  

(30)
where θ is a random variable and, for each fixed value of θ, z is a random process and v is a white Gaussian random process. The problem is to find the minimum mean-square estimate of the signal z. Detection problems can be easily modeled by this formulation with a suitable choice of the random variable θ [7]. The general case when θ has an arbitrary distribution has been worked in [8]. When the parameter θ has a discrete distribution

\[ P(θ = θ_i) = P_i, \quad i = 1, 2, \ldots; \]  

(31)

it is shown in [8] and [9] that the minimum mean-square estimate of the signal z is given by

\[ \hat{z} = \sum_{i=1}^{∞} \pi_i(t)\hat{z}_i(t) \]  

(32)

where \( \hat{z}_i(t) \) is the estimate of the signal \( z(θ_i, t) \) given the observation \( y(s) = z(θ_i, s) + v(s), \quad 0 \leq s \leq t \) and where \( \pi_i(t) \) is given by

\[ \pi_i(t) = P(θ = θ_i \mid y(s), \quad 0 \leq s \leq t) \]  

(33)

That is, \( \{\hat{z}_i(t)\}_{i=1}^{∞} \) are the parameter conditioned estimates and \( \{\pi_i(t)\}_{i=1}^{∞} \) are the posterior probabilities of the parameter θ.

The posterior probabilities satisfy the stochastic differential equations

\[ d\pi_i(t) = \pi_i(t)[\hat{z}_i(t) - \sum_{j=1}^{∞} \pi_j(t)\hat{z}_j(t)]R^{-1}(t)[dy(t) - \sum_{j=1}^{∞} \pi_j(t)\hat{z}_j(t)] \]  

(34)

with the initial conditions

\[ \pi_i(0) = P_i, \quad i = 1, 2, \ldots \]  

(35)

The parameter conditioned estimator has two parts: A non-adaptive part in which the parameter conditioned estimates are found and an adaptive part in which the posterior probabilities are found. The form of the solution given by (32) and (34) implies that if recursive equations
are known for the parameter conditioned estimates, then a completely recursive solution is found to the problem. The recursive form of the solution is, of course, very advantageous in terms of implementation of the filter. The following example demonstrates how these results can be applied to a nonlinear filtering problem.

**Example:** Consider the random telegraph signal, \( Z(t) \), with values \( \pm 1 \) and transition density \( \alpha \) where \( \alpha \) is a random variable with prior distribution \( P(\alpha = \alpha_i) = P_i, \ i = 1, 2, \ldots, M \). The observation model is given by

\[
y(t) = Z(t) + v(t)
\]

where \( v \) is a unit-variance Gaussian process, from (32) and (34), it is seen that the minimum mean-square estimate of the signal is given by

\[
\hat{Z}(t) = \frac{1}{M} \sum_{i=1}^{M} \pi_i(t) \hat{Z}_i(t)
\]

where the parameter conditioned estimates \( \hat{Z}_i(t) \) are known to be given by [8]

\[
d\hat{Z}_i(t) = 2\alpha_i \hat{Z}_i(t) + [1 - \hat{Z}_i^2(t)](dy(t) - \hat{Z}_i(t)dt)
\]

which, in turn, derives the stochastic differential equation (34)

\[
d\pi_i(t) = \pi_i(t)[\hat{Z}_i(t) - \sum_{j=1}^{M} \pi_j(t)\hat{Z}_j(t)](dy(t) - \sum_{j=1}^{M} \pi_j(t)\hat{Z}_j(t)dt)
\]

with \( \pi_i(0) = P_i, i = 1, 2, \ldots, M \).
Directions for Further Research

In the work on optimal sampled-data regulators, a promising and interesting problem is to find early computable algorithms to perform the projection in equation (13). In this way, optimal sampled-data regulator gains can be obtained as a function of the optimal continuous regulator gains. So far the equations obtained are not any simpler than the known equations available in the literature. However, this study can lead to an understanding of how the sampled-data optimal gains vary with the sampling interval even though the algorithms may turn out to be complicated.

In the work on the stochastic modeling approach to variable sampling, more analytical and experimental study is needed. Analytical conditions that guarantee the convergence of gains for the steady-state gains [equations (23) and (24)] would be very useful. Simulation of a wide range of applications to different problems is needed to see how far the stochastic optimal law can be "stretched." So far, the simulations indicate that the stochastic optimal control law gives virtually the ultimate deterministic performance.

In the work on parameter adaptive estimation, more work is necessary on the following questions. What happens when the parameter does not have the prescribed distribution? In this case, for instance, does the set of posterior probabilities converge to a value that is nearest to the actual value?

Also, additional effort is necessary to understand the stability characteristics of the stochastic differential equations for the posterior probabilities (34).
REFERENCES


A separation theorem for the stochastic sampled-data LQG problem†

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This paper considers the control of a continuous linear plant disturbed by white plant noise when the control is constrained to be a piecewise constant function of time; i.e., a stochastic sampled-data system. The cost function is the integral of quadratic error terms in the state and control, thus penalizing errors at every instant of time while the plant noise disturbs the system continuously. The problem is solved by reducing the constrained continuous problem to an unconstrained discrete one. It is shown that the separation principle for estimation and control still holds for this problem when the plant disturbance and measurement noise are Gaussian.

1. Introduction

Various studies of the deterministic discrete-time regulator problem have been made since the original work of Kalman and Koepcke (1958). A tutorial review of these basic results can be found in the paper by Dorato and Levits (1971). The deterministic sampled-data regulator problem has also been studied by Levits et al. (1971). The stochastic counterparts of the continuous and discrete linear quadratic Gaussian problems result in the separation principle for estimation and control (Joseph and Tou 1964, Gunkel and Franklin 1963, Potter 1964). Various extensions of the separation theorem to include a larger class of continuous and discrete cost functions have also been made (e.g., Streich 1965, Wonham 1968, Halyo and Foukles 1974, and the references therein).

In this paper the control of a stochastic linear sampled-data system (i.e., the stochastic counterpart of the work by Levits et al. (1971)) is considered. A continuous linear plant is disturbed by a continuous Gaussian white plant noise, while the control is constrained to be constant between sampling instants, say \( t_k \), but can change at every sampling instant. Measurements, which can be expressed as linear combinations of the state variables corrupted by Gaussian white noise, are made at the sampling instants \( t_k \). Such systems are often encountered, particularly if the control law is to be implemented on a digital computer. On the other hand, the performance of the system depends on the deviations of the state vector from a steady-state condition (represented as the zero vector) at every instant of time, not only at the sampling instants. Hence, a cost function which penalizes deviations in the state continuously is
more appropriate as a performance criterion† than a discrete cost function. The problem is solved by reducing it to a discrete stochastic regulator where the states are measured with no error.

2. Statement of the stochastic sampled-data problem

Consider the following continuous stochastic dynamical system represented by

\[ x(t) = A(t)x(t) + B(t)u(t) + D(t)v(t), \quad t \in [t_0, t_1] \]  

(1)

where \( x \) is the \( n \)-dimensional state vector, \( u \) is the \( r \)-dimensional control vector, and \( v \) is the white Gaussian plant noise vector of dimension \( p \) with \( E[v(t)]=0 \) and \( E[v(t)v^*(s)] = F(t)\delta_{ts}(t-s) \) for some positive semidefinite matrix \( F \). \( A \), \( B \), and \( D \) are time-varying matrices of compatible order. Equation (1) is interpreted as the following stochastic integral equation (Wong 1971):

\[ x(t) = x(t_0) + \int_{t_0}^{t} A(s)x(s) \, ds + \int_{t_0}^{t} B(s)u(s) \, ds + \int_{t_0}^{t} D(s) \, dW(s) \]  

(2)

where \( W(t) \) is a Wiener process with \( EW(t) = 0 \) and

\[ E(W(t) - W(t_0))(W(s) - W(t_0))' = \int_{t_0}^{\min(t,s)} F(\tau) \, d\tau \]

The plant noise can be considered as the formal time derivative of the Wiener process \( W \).

In the stochastic sampled-data regulator the plant noise is a continuous random process, whereas the measurement noise is a discrete random process. Measurements of some of the linear combinations of states with additive noise are available at the sampling instants

\[ y(t_k) = C_k x(t_k) + e(t_k), \quad k = 0, 1, ..., N, \quad t_0 < t_1 < ... < t_N = t_1 \]  

(3)

where \( y \) is the \( m \)-dimensional measurement vector, \( \{e(t_k)\} \) is the measurement noise vector which is a Gaussian sequence of uncorrelated zero mean random vectors with \( E[e(t_k)e^*(t_j)] = \delta_{kj} \theta_0 \) for some positive definite matrix \( \theta_0 \), and

\( EW(t_k)e^*(t_i) = Ee(t_k)e^*(t_i) = Ee(t_k)W(t_i) = 0 \) for all \( t \) in \( [t_0, t_1] \) and \( i = 0, 1, ..., N \).

The controls are constrained by

\[ u(t_k) = u(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, ..., N-1 \]

(4)

In order for a solution to exist to eqn. (2), the following further assumptions will be made: the elements of \( A(t) \) are bounded and measurable real functions of time, \( B(t) \) is integrable, and \( \int_{t_0}^{t} D(t)F(t)V(t) \, dt \) is finite.

† If the sampling rate can be chosen high enough a discrete cost function may be adequate; however, in many cases this increases the cost of the computer he needs stringent requirements on its speed of operation. This trade-off makes the design of control laws, which do not degrade at low sampling rates, important.
The following cost functional is used to achieve the desired system performance:

\[ J = \frac{1}{2} E \left[ \int_{t_0}^{t_f} \left( x'(t)Q(t)x(t) + u'(t)R(t)u(t) \right) dt + x'(t_f)P(x(t_f)) \right] \]  

(5)

where \( Q(t) \) is positive semidefinite and \( R(t) \) is positive definite on \([t_0, t_f]\).

Now the stochastic optimal control problem can be stated as follows: given the linear stochastic dynamical system (1), and a partition \( t_0, t_1, \ldots, t_N \) of the interval \([t_0, t_f]\), find a control sequence \( (u^*(t_k), k = 0, 1, \ldots, N-1) \) with the constraint (4) and also the additional constraint that \( u^*(t_k) \) will depend only on the past measurement sequence \( (y(t_i), i = 0, 1, \ldots, k-1) \) which will minimize the cost functional (5).

3. Equivalent stochastic discrete-time problem

It will be shown that the constrained continuous stochastic optimization problem can be transformed into an unconstrained discrete stochastic optimization problem by integrating the system differential equations and the cost functional over each sampling interval. The problem will be embedded into the known format of the standard discrete linear quadratic Gaussian regulator problem. Thus, it will be proved that the separation between estimation and control is still valid for this constrained continuous stochastic optimization problem.

Under the assumptions made in § 2, the solution to eqn. (1), for any bounded control, is given by

\[ x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\,ds + \int_{t_0}^{t} \Phi(t, s)D(s)\,dW(s) \]  

(6)

where \( \Phi(t, s) \) is the state-transition matrix of \( \dot{x}(t) \) defined as the solution of matrix differential equation

\[ \frac{d}{dt} \Phi(t, s) = A(t)\Phi(t, s), \quad (t > s) \]

with \( \Phi(s, s) = I \).

Using (6), the state for \( t_k \leq t < t_{k+1} \) can be computed from the state at time \( t_k \):

\[ x(t) = \Phi(t, t_k)x_k + \Gamma(t, t_k)u_k + \xi(t, t_k) \]  

(7 a)

where \( x_k = x(t_k), u_k = u(t_k) \), and

\[ \Gamma(t, t_k) = \int_{t_k}^{t} \Phi(t, s)B(s)\,ds \]  

(7 b)

\[ \xi(t, t_k) = \int_{t_k}^{t} \Phi(t, s)D(s)\,dW(s) \quad \text{for} \quad t_k < t < t_{k+1} \]  

(7 c)

The term given by eqn. (7 c) shows that the plant noise corrupts the state in a continuous fashion, but the resulting discrete-time system will be given by

\[ x_{k+1} = \Phi(t_{k+1}, t_k)x_k + \Gamma(t_{k+1}, t_k)u_k + \xi \]  

(8 a)
where \( \xi_k = \xi(t_{k-1}, t_k) \) is a zero mean white Gaussian sequence of random vectors with (Jazwinski 1970)

\[
E \xi_k \xi_j^T = \left[ \int \Phi(t_{k-1}, s) D(s) F(s) F^T(s) \Phi^T(t_{k-1}, s) \, ds \right] \mathcal{S}_{kj} = F_k \mathcal{S}_{kj} \quad (8b)
\]

The assumptions made about the independence of plant noise and \( w(t_k), x(t_k) \) in § 2 will still be valid for the new discrete plant noise \( \xi(t_k) \).

The cost integral can also be written as the sum of \( N \) integrals as in the deterministic case. The control at time \( t_k \) must only depend on the measurements made until \( t_{k-1} \) and the initial estimate \( \hat{x}(t_0) \). This can be expressed by restricting \( u(t_k) \) to be measurable with respect to \( Y_k \), where \( Y_k \) is the minimal \( \sigma \)-algebra generated by the measurement sequence \( (y(t_i), i=0, 1, ..., k-1) \) and the initial estimate \( \hat{x}(t_0) \). The usual choice of \( \hat{x}(t_0) \) as the mean of \( x(t_0) \) satisfies this condition.

Using (7a), the cost functional (5) can be put into the following form:

\[
J = E \left[ \frac{1}{2} x_N^T R_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q_k x_k + 2x_k^T M_k u_k + u_k^T R_k u_k) + \int_0^{t_k} \Phi'(l, t_k) Q(t) \Phi(l, t_k) \, dl + \int_0^{t_k} \xi'(l, t_k) \mathcal{S}_{kj} \Phi(l, t_k) \, dl \right] \quad (9a)
\]

where \( Q_k, M_k, R_k \) are given by

\[
Q_k = \int_0^{t_k} \Phi'(l, t_k) Q(t) \Phi(l, t_k) \, dl \quad (9b)
\]

\[
M_k = \int_0^{t_k} \Phi'(l, t_k) \mathcal{S}_{kj} \Phi(l, t_k) \, dl \quad (9c)
\]

\[
R_k = \int_0^{t_k} (H(t) + \mathcal{Y}(l, t_k) Q(t) \mathcal{Y}(l, t_k)) \, dl \quad (9d)
\]

Note that \( R_k \) will be positive definite since \( R_k \) is positive definite. Similarly, \( Q_k \) will be positive semidefinite.

We shall now show that the last two terms in eqn. (9a) can be ignored as far as the minimization of the cost functional is concerned:

**Lemma 1**

The control sequence \( (u_k, k=0, 1, ..., N-1) \) minimizes the cost functional \( J \) given by eqn. (9a) if, and only if, it minimizes the following cost functional \( J_1 \):

\[
J_1 = E \left[ \frac{1}{2} x_N^T R_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^T Q_k x_k + 2x_k^T M_k u_k + u_k^T R_k u_k \right] \quad (10)
\]
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**Proof**

From (7.1), note that \( \xi(t, t_k) \) depends only on the increments of \( W(s) \) in the interval \([t_k, t]\), which are independent of \( \xi(t_0) \) and \( W(t) \); hence, so is \( \xi(t, t_k) \) of \( W(s), t_0 \leq s \leq t_k \) and \( x(t_0) \). On the other hand, \( x(t_k) \) depends only on \( W(s), t_0 \leq s \leq t_k \) and \( x(t_0) \), so that \( \xi(t, t_k) \) and \( x(t_k) \) are independent whenever \( t \geq t_k \). Thus,

\[
E x_k' \int_t^{t_k} \Phi'(t, t_k) Q(t) \xi(t, t_k) dt = E(x_k') \int_t^{t_k} \Phi'(t, t_k) Q(t) E(\xi(t, t_k)) dt = 0
\]

since \( \xi(t, t_k) \) has zero mean. Furthermore, the last term in (9 a) is a constant which does not depend on the control sequence; hence, it can be excluded from the minimization. So the lemma follows.

At this stage, in the case of the deterministic regulator, a preliminary feedback of the form \( u_k = -R_k^{-1} M_k x_k + \hat{u}_k \) would reduce the problem into the standard regulator problem. However, as \( x_k \) is not available, the same cannot be done in this problem. Instead, we now show that minimization of the cost functional \( J_1 \) is equivalent to the minimization of the same expression with \( x_k \)'s replaced by their conditional expectations \( \hat{x}_k \).

**Lemma 2**

The control sequence \( \{u_k, k = 0, 1, ..., N - 1\} \) minimizes the cost functional \( J_2 \), given by (10), if, and only if, it minimizes the following cost functional \( J_2 \):

\[
J_2 = E \left[ \frac{1}{2} \hat{x}_N' H \hat{x}_N + \frac{1}{2} \sum_{k=0}^{N-1} (\hat{x}_k' Q_k \hat{x}_k + 2 \hat{x}_k' M_k u_k + u_k' R_k u_k) \right]
\]

(11)

where \( \hat{x}_k = E(x_k | Y_k) \).

**Proof**

Since \( u_k \) is measurable with respect to \( Y_k \), using well-known theorems on conditional expectations,

\[
E(x_k' M_k u_k) = E(E(x_k' M_k u_k | Y_k)) = E(E(x_k' | Y_k) M_k u_k) = E(x_k' M_k u_k)
\]

(12)

Similarly, by letting \( \hat{x}_k = x_k - \hat{x}_k \):

\[
E(x_k' Q_k v_k) = E(\hat{x}_k' Q_k v_k + 2 \hat{x}_k' Q_k \hat{x}_k + \hat{x}_k' Q_k \hat{x}_k)
\]

Using the fact that \( E(\hat{x}_k' Q_k \hat{x}_k) = 0 \), we get

\[
E(x_k' Q_k v_k) = E(\hat{x}_k' Q_k \hat{x}_k + E(\hat{x}_k' Q_k \hat{x}_k | Y_k))
\]

(13)

Since the \( \sigma \)-algebras generated by \( \hat{x}_k \) and \( Y_k \) are independent, it follows that

\[
E(FQ_k \hat{x}_k | Y_k) = E(FQ_k \hat{x}_k) \quad \text{a.e.}
\]

(14)
Note that the right-hand side of (14) does not vary with \( u_k \). Substituting (12), (13) and (14) into (11), we get

\[
J_1 = E \left[ \frac{1}{2} \tilde{x}_N' H \tilde{x}_N + \frac{1}{2} \sum_{k=0}^{N-1} (\tilde{x}_k'Q_k \tilde{x}_k + 2 \tilde{x}_k' R_k u_k + u_k' R_k u_k) \right] + \frac{1}{2} E(\tilde{x}_N' H \tilde{x}_N) + \frac{1}{2} \sum_{k=0}^{N-1} E(\tilde{x}_k'Q_k \tilde{x}_k) \tag{15}
\]

Since the second term in (15) does not depend on \((u_k, k = 0, 1, \ldots, N-1)\), it may be dropped as far as the minimization with respect to \((u_k, k = 0, 1, \ldots, N-1)\) is concerned. So \( J_1 \) reduces to \( J_2 \) given by (11); this completes the proof of the lemma.

Thus, to find the optimal control for the original cost functional \( J \), it is sufficient to find the control sequence which minimizes the cost functional \( J_2 \) of (11). Note that the cost \( J_2 \) depends only on the estimates of the state at the sampling instants. Looking back at eqns. (8 a), (8 b) and (3), it is clear that these estimates will be given by the well-known Kalman filter equations (Jazwinski 1970, Kalman 1960). We can now introduce a preliminary feedback of the form \( u_k = -R^{-1} M_k' \tilde{x}_k + \bar{u}_k \) and embed the problem into the standard discrete linear quadratic Gaussian problem. We sum up these results in the next theorem.

**Theorem**

Consider the stochastic optimal control problem described in § 2. A unique control sequence \((u^*_k, k = 0, 1, \ldots, N-1)\) which minimizes the cost functional \( J \) of (4) exists and is given by

\[
u^*_k = - (R_i^{-1} M_i' + (R_i + \Gamma_i' K_{i+1} \Gamma_i)^{-1} \Gamma_i' K_{i+1} \Phi_i) \tilde{x}_k \tag{16}
\]

where \( K_i \) is the solution to the Riccati difference equation

\[
K_i = \Phi_i' (K_{i+1} - K_{i+1} \Gamma_i (R_i + \Gamma_i' K_{i+1} \Gamma_i)^{-1} \Gamma_i' K_{i+1}) \Phi_i + Q_i \tag{17}
\]

with the boundary condition \( K_N = H \)

where \( \Gamma_i = \Gamma(i, i+1), \quad \Phi_i = \Phi(i+1, i), \quad \Gamma_i = \Gamma_i' H_i^{-1} M_i', \quad Q_i = Q_i - M_i H_i^{-1} M_i', \) and \( i = 0, 1, \ldots, N-1 \).

**Proof**

Introducing the preliminary feedback (Payne and Silverman 1973, Thomasson and Cook 1973)

\[
u_k = - R_k^{-1} M_k' \tilde{x}_k + \bar{u}_k
\]
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the cost functional becomes

\[ J_2 = k \left[ \frac{1}{2} \dot{x}_k' H \dot{x}_k + \frac{1}{2} \sum_{k=0}^{N-1} (x_k' (Q_k + M_k R_k + M_k') x_k + u_k' R_k u_k) \right] \]  (18)

The estimates \( \hat{x}_k \) can be obtained by the Kalman filter equations (Jazwinski 1970, Kalman 1960)

\[ \dot{\hat{x}}_{k+1} = A \hat{x}_k + (1 - \lambda_k) u_k \]  (19)

where \( \nu_k \) is the white innovation sequence.

Hence, the original constrained continuous problem is reduced to minimizing \( J_2 \) as given in (18) with the dynamics of (19) where \( u_k \) may depend on \( \dot{x}_k \). However, this is a discrete LQG problem and its solution is given by (16) and (17); e.g. see Kushner (1971).

Looking at eqns. (16) and (17), it is seen that the optimal control law for this stochastic optimization problem is the same as the deterministic sampled-data regulator problem (Levis et al. 1971) with \( x_k \) replaced by its estimate \( \hat{x}_k \). Therefore, the separation between estimation and control continues to hold for this constrained continuous stochastic optimization problem.

4. Conclusions

In this paper it is shown that the separation principle between estimation and control continues to hold for the stochastic sampled-data regulator. The problem is solved by reducing the constrained continuous stochastic optimization problem into an unconstrained discrete stochastic one. The results are obtained by embedding the problem into the standard discrete stochastic regulator problem.

Using similar techniques to the one described here, it can also be shown that the separation of estimation and control continues to hold for linear systems using \( n \)-th-order holds and having computational delays in the control loop.

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ON THE RELATION BETWEEN THE SAMPLLED-DATA AND THE CONTINUOUS OPTIMAL LINEAR REGULATOR PROBLEMS

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Abstract

In this paper, the geometric relationship between the optimal solutions to the sampled-data and continuous linear quadratic regulator problems is investigated in a Hilbert space framework. It is shown that the optimal sampled-data solution, excluding the response due to the initial condition, is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint. That is, the optimal sampled-data solution is an optimal approximation to the continuous solution. In fact, it is shown that the sampled-data solution can be obtained by solving a sampled-data tracking problem with the continuous solution as the desired trajectory.

1. Introduction

This paper is concerned with the relationship between the optimal continuous and the optimal sampled-data linear quadratic regulator problem [1], [2]. Since the dynamical system and the integral cost function used are the same for both the continuous and the sampled-data regulator, intuitively, the optimal sampled-data control must be an approximation to the optimal continuous one in some sense. In this paper the precise relationship between these two optimal solutions will be investigated in a vector space setting.

In the control literature, the continuous and the sampled-data regulator problems have usually been treated using different methods: The continuous regulator problem has been solved by using Pontryagin's minimum principle, by using the Hamilton-Jacobi-Bellman partial differential equation for the optimal cost function, and by a few other methods. The sampled data problem has been solved by converting it into an unconstrained discrete optimization problem through the integration of the cost functional and the system differential equations over each sampling interval and then applying dynamic programming or the discrete minimum principle. The relationship between the two optimal solutions is, thus, obscured due to these separate formulations. In this paper, the two problems are formulated in the same framework by using a Hilbert space approach. This geometric formulation reduces the problem to one of finding the element of minimum norm in two linear varieties in a Hilbert space. (This approach has been suggested for a simple control problem in [3]). It is shown that, excluding the unforced response of the system due to the initial condition, the sampled-data control and the resulting state trajectory is the projection of the optimal continuous control and its corresponding state trajectory; that is, the optimal sampled-data solution is, in fact an optimal approximation of the continuous solution with respect to an appropriate Hilbert space norm.

2. A Minimum Norm Theorem in a Hilbert Space

Before proceeding to formulate the problem in a Hilbert space setting, we shall prove a general theorem concerning the elements of minimum norm of two linear varieties in a Hilbert space which will be required in the following derivation. We now state the projection theorem and its extension to linear varieties for ease of reference; the proofs can be found in [3].

Lemma 2.1 (Projection Theorem) Let H be a Hilbert space and M be a closed subspace of H. Corresponding to any vector x in H, there exists a unique vector m* in M such that m* is the closest element in M to x in the sense of the inner product norm. Furthermore, a necessary and sufficient condition that m* be this unique vector is that x - m* be orthogonal to M. (We will denote the projection operator onto M by P_M; i.e., m* = P_M(x).

Corollary 2.2 Let H and M be as in Lemma 2.1. Let x be a fixed element in H and let V be the linear variety x + M. Then there exists a unique vector v* in V of minimum norm. Furthermore, a necessary and sufficient condition that v* be this unique vector is that v* be orthogonal to the subspace M.

The following theorem describes the relationship between the elements of minimum norm of two linear varieties in a Hilbert space where the generator subspace of one variety is a subset of the other corresponding subspace.

Theorem 2.3 Let H be a Hilbert space and let M and N be closed subspaces of H such that N is a subset of M. Let x be a fixed element in H and let V and W be the linear varieties defined by...
\[ x_0 + M \text{ and } x_0 + N \text{ respectively. Then there exist a unique vector } v^* \text{ in } V \text{ and a unique vector } w^* \text{ in } W \text{ of minimum norm. Furthermore, } u^* \text{ can be obtained from } v^* \text{ by}
\]
\[ w^* = x_0 + P N m^* \tag{1} \]

where \( m^* \) is the unique element of \( M \) defined by the relation
\[ v^* = x_0 + m^* \tag{2} \]

**Proof:** By Corollary 2.2, there exists a unique element \( v^* \) in \( V \) and a unique element \( w^* \) in \( W \) of minimum norm. Since \( v^* \) is in \( V \), it must have a representation of the form \( v^* = x_0 + m^* \) where \( m^* \) is an element of \( M \). Clearly, \( m^* \) must be unique. Similarly, there exists a unique element \( n^* \) in \( N \) such that \( w^* = x_0 + n^* \). Since \( N \) is a sub-Hilbert space of \( M \), there exists a direct sum decomposition of \( M \) [4] given by
\[ N = N + N_L \tag{3} \]

where \( N \) is the orthogonal complement of \( N^L \) with respect to \( N \), and + denotes the direct sum. In order to prove that \( w^* = x_0 + P N m^* \), we have to show
\[ < x_0 + P m^*, n > = 0 \text{ for all } n \in N \tag{4} \]

where \( < \cdot, \cdot > \) denotes the inner product. Since \( P N m^* \) is identically zero, we have
\[ < x_0 + P m^*, n > = < x_0 + P m^* + P N m^*, n > \tag{5} \]

Since \( m^* \) has the direct sum decomposition \( P m^* + P N m^* \), we get
\[ < x_0 + P m^*, n > = < x_0 + m^*, n > = \]
\[ < v^*, n > = 0 \tag{6} \]

The last equality follows from Corollary 2.2 and the proof is complete.

### 3. Formulation of the Problem

In this section, we shall formulate the two regulator problems in a Hilbert space framework so that we can apply the theorem of the last section to find the precise mathematical sense in which the optimal sampled-data solution is an approximation of the continuous solution. Consider the linear dynamical system represented by the differential equation
\[ x(t) = A(t) x(t) + B(t) u(t) \tag{3.1} \]
\[ y(t) = C(t) x(t) \]

where \( x(t) \) is the \( n \)-dimensional state vector, \( u(t) \) is the \( r \)-dimensional control vector, \( y(t) \) is the \( r \)-dimensional measurement vector, and \( x(a) \) is the initial condition. \( A(t), B(t), \) and \( C(t) \) are matrices of appropriate order with continuous elements. The following cost functional will be used to achieve the desired system performance:
\[ J(x(t), u(t)) = \int_a^b (y^T(t)y(t) + u^T(t)R(t)u(t))dt \tag{3.2} \]

where \( R(t) \) is a positive definite matrix with continuous elements. We will denote the \( \mathbb{R}^r \)-valued functions on \([a, b]\) that are square integrable in the Lebesgue sense by \( L^2_{\mathbb{R}}(a, b) \), i.e.,
\[ L^2_{\mathbb{R}}(a, b) = \{ u(t): \int_a^b u^T(t)u(t)dt < \infty \} \tag{3.3} \]

Similarly, let \( L^2_{\mathbb{R}}(a, b) \) be the space of \( \mathbb{R}^r \)-valued square integrable functions on \([a, b] \), and we will formulate the two problems in the Hilbert space \( H = L^2_{\mathbb{R}}(a, b) \times L^2_{\mathbb{R}}(a, b), \) the Cartesian product of \( L^2_{\mathbb{R}}(a, b) \) and \( L^2_{\mathbb{R}}(a, b), \) with the inner product defined by
\[ < (y_1(t), u_1(t)), (y_2(t), u_2(t)) > = \int_a^b (y_1^T(t)y_2(t) + u_1^T(t)R(t)u_2(t))dt \tag{3.4} \]

so that the norm induced by the inner product 3.4 gives the desired cost functional of 3.2. Since the solution of the differential equation 3.1 is given by
\[ x(t) = \xi(t, a) x(a) + \int_a^t \xi(t, s)B(s)u(s)ds \tag{3.5} \]
\[ y(t) = C(t)x(t) \]

where \( \xi(t, s) \) is the state transition matrix of \( A(t), \) we will be concerned with the ordered pairs \((y(t), u(t))\) in \( H \) that satisfy the integral constraint 3.5 for the continuous regulator. It is easy to show that the subset \( M \) of \( H \) defined by
\[ M = \{ (y(t), u(t)) \in L^2_{\mathbb{R}}(a, b) \times L^2_{\mathbb{R}}(a, b): y(t) = \int_a^t \xi(t, s)u(s)ds \}
\[ \xi(t, s) = C(t)x(s)B(s) \tag{3.6} \]

is a subspace, so that we can define the linear variety in \( H \) by
\[ V = (C(t)x(a), 0) + M \tag{3.7} \]

Thus, the continuous regulator problem is one of finding the element of minimum norm in \( V \). To apply the Hilbert space theory of the previous section, we have to show that \( M \) is a closed subspace. To this end, let us define the linear transformation \( L \) from \( L^2_{\mathbb{R}}(a, b) \) into \( L^2_{\mathbb{R}}(a, b) \) by
\[ L(u(t)) = \int_a^t \xi(t, s)u(s)ds \tag{3.8} \]

Using Fubini's theorem, it can easily be shown that for all \( y \) in \( L^2_{\mathbb{R}}(a, b) \)
\[ < Lu, y > = < u, \int_a^b \xi^T(s, t)y(s)ds > \tag{3.9} \]

which proves that \( L \) is a bounded and, therefore, continuous linear transformation. (See Theorem 5 in [5], p. 48). We also note the linear transformation \( L^* \) from \( L^2_{\mathbb{R}}(a, b) \) defined by
\[ L^*(y(t)) = \int_a^t \xi^T(s, t)y(s)ds \tag{3.10} \]

is the adjoint of \( L \). So if \((y_n, u_n)\) is a sequence of elements in \( M \) converging to an element \((y, u)\) in \( M \), we have
\[ \lim_n (y_n, u_n) = \lim_n (\int_a^b \xi^T(t, s)y(s)ds, u_n) = \]
\[ (\int_a^b \xi^T(t, s)y(s)ds, \lim_n u_n) \in M \]

This follows since \( L \) is a continuous operator and, therefore, \( M \) is closed.
In the sampled-data problem, we will be given a partition \((a = t_0, t_1, \ldots, t_N = b)\) of \([a, b]\) such that over each sampling interval the control remains constant, i.e., \(u(t) = u(t_k)\) for \(t_k \leq t < t_{k+1}\).

In this case, the subspace of interest will be
\[
N = \{(y, u) \in L^2_{a} \times L^2_{b} : y(t) = \int_{t_0}^{t} T(t, \tau) u(\tau) d\tau, u(t) = u(t_k) \text{ for } t_k \leq t < t_{k+1}\}
\] (3.11)

Clearly, \(N\) is a subset of \(V\). To show that \(N\) is closed, we note that the same linear operator \(L\) defined in 3.8 is used except now its domain is a subset of \(L^2_{a}\). So we only have to show that this domain is closed in \(L^2_{a}\). Clearly, the set of all functions that are constant over each sampling interval is isometrically isomorphic to \(R^m \times R^m \times \cdots \times R^m\) (\(N\) times) so that \(N\) is closed.

So the sampled-data regulator problem is one of finding the element of minimum norm in the linear variety \(W\) defined by
\[
W = (C(t)\theta(t)a x(a), 0) + N
\] (3.12)

Thus, we have cast the two regulator problems into the vector space format of the previous section.

4. Optimal Sampled-Data Solution as an Approximation to the Optimal Continuous Solution

In this section, we will first show that the Hilbert space formulations of the sampled-data and the continuous regulator problems in the previous section do indeed give the standard results. Then, using the theorem concerning two linear varieties in Section 2., we will state the precise sense in which the optimal sampled-data solution is an approximation of the optimal continuous solution. We will also show that the optimal sampled-data solution can be obtained by taking the appropriate projection of the optimal continuous one.

Consider the continuous regulator problem formulation described in the previous section by equations 3.1-3.7. A necessary and sufficient condition for \((y^*, u^*)\) to be the element of minimum norm in \(V\) by Corollary 2.2, is that \((y^*, u^*)\) be orthogonal to \(M\), which implies
\[
\langle (y^*, u^*), (y, u) \rangle = 0 \text{ for all } (y, u) \text{ in } N
\] (4.1)

Since the inner product on \(L^2_{a} \times L^2_{b}\) is the sum of inner products on \(L^2_{a}\) and \(L^2_{b}\)
\[
\langle y^*(t), u(t) \rangle = \int_{t_0}^{t} T(t, \tau) u(\tau) d\tau + u^*(t) T(t, t_0) y^*(t) = 0
\] for all \(u\) in \(L^2_{a}(a, b)
\] (4.2)

By using 3.9, we get
\[
\langle y^*(t), u^*(t) \rangle = \int_{t_0}^{t} T(t, \tau) y^*(\tau) d\tau + u^*(t) T(t, t_0) y^*(t) = 0
\] for all \(u\) in \(L^2_{a}\)
\] (4.3)

It follows that
\[
R(t) u^*(t) + \int_{t_0}^{t} T(t, \tau) y^*(\tau) d\tau = 0
\] (4.4)

Therefore, the optimal continuous solution is given by
\[
u^*(t) = -R^{-1}(t) \int_{t}^{b} T'(s,t) y^*(s) ds
\] (4.5)
or, equivalently
\[
u^*(t) = -R^{-1}(t) B'(t) \int_{t}^{b} \psi'(s,t) C'(s) C(s) x^*(s) ds
\] (4.6)

It is easy to show that there exists a positive semidefinite matrix \(K(t)\) such that
\[
\int_{t}^{b} \psi'(s,t) C'(s) C(s) x^*(s) ds = K(t) x^*(t)
\] for \(t\) in \([a, b]\)

where \(K(t)\) is the unique positive semidefinite solution of
\[
K(t) = -A'(t) K(t) - K(t) B(t) R^{-1}(t) B(t)^T K(t) - C'(t) C(t) \text{ with } K(b) = 0
\] (4.8)

which is the Riccati equation of the linear continuous optimal control.

Consider now the formulation of the sampled-data regulator problem described by equations 3.1-3.5 and 3.11-3.12. A necessary and sufficient condition that \((y^{**, u^{**}})\) be the optimal solution is by Corollary 2.2 that
\[
\langle (y^{**, u^{**}}), (y, u) \rangle = 0 \text{ for all } (y, u) \text{ in } N
\] (4.9)

which implies (after some manipulation)
\[
\int_{t}^{b} (x^{**, t})' C'(t) C(t) \Gamma(t, t_0) + u^{**, t} R(t)) dt = 0
\] (4.10)

for all vectors \((u_0, u_1, \ldots, u_{N-1})\) in \(R^m\)
where \(\Gamma(t, t_1) = \int_{t_1}^{t} \theta(t) R(t) \theta(t) dt\). This implies that the terms in the brackets in (4.10) must be zero. Starting from the last term in 4.10 and using the fact that \(x^{**, t} = (\Gamma(t, t_{N-1}) x^{**, t_{N-1}} + \Gamma(t, t_{N-1}) u_{N-1}) u_{N-1}\), it can be shown that the optimal sampled-data control law is given by
\[
u^{**, t} = -R^{-1}(t) \int_{t}^{b} T'(s,t) y^{**, s} ds
\] (4.11)

where \(\Gamma(t, t_1) = \int_{t_1}^{t} \theta(t) R(t) \theta(t) dt\) and
\[
\int_{t_1}^{t} \theta(t) R(t) \theta(t) dt = \int_{t_1}^{t} \theta(t) R(t) \theta(t) dt
\] (4.12)

which is the Riccati equation of the linear continuous optimal control.
The sampled-data and continuous linear regulator problems are formulated in the same Hilbert space as minimum norm problems. The geometric relationship between the two optimal solutions is investigated. It is shown that the optimal sampled-data solution is an optimal approximation to the continuous regulator solution in an appropriate Hilbert space norm. Specifically, it is shown that, excluding the response due to the initial condition, the optimal sampled-data solution is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint. It should be also noted that extensions to the case where $[a, b]$ is an infinite interval can be easily obtained with this method with slight modifications.

References


Failure Accommodation in Digital Flight Control Systems by Bayesian Decision Theory

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Introduction

The most striking impact of new technology in aircraft flight control stems from the advent of the modern, high-speed, digital computer. Control concepts previously considered untractable can now be considered because of the flexibility and speed of information processing made available by this new technology. One important new potential that exists is the ability of digital system to reorganize itself to accommodate for failures in sensors and actuators. This reorganization is possible, provided there is enough duplication of function between the actuators or the sensors in a given control system. This paper presents a design method for digital flight control systems that will be optimally tolerant of sensor failures.

Modern control methods allow one to determine the part of the state space of an aircraft that can be dynamically influenced by a given actuator (the controllability subspace) and the part of the state space that a given sensor can produce information about using state estimator theory (observability subspace). Reference 1 provides a good treatment of theoretical considerations involved in determination of these subspaces. Redundancy is provided in either sensors or actuators when there is overlapping of the subspaces of the various sensors or actuators in a given system. For example, consider the longitudinal dynamics of an airplane. If there are three sensors on the aircraft, say an accelerometer, to measure normal acceleration, a pitch-rate gyro, and an elevator position transducer, and if the aircraft state is completely observable from outputs of either sensor, then it is possible, using say a minimum order observer, to estimate the behavior of one sensor based on the output of another one. Redundancy, in that situation, does exist and can be used by cross-checking state estimates obtained by one sensor with those obtained from another one.

Theoretical considerations for determining the absolute level of redundancy that exists in a given system were developed in Ref. 3. Reference 3 also presented a failure detection filter designed to make use of the system redundancy. One limitation of that work was that no consideration of the practical noise environment of the sensors was made and failure detection depended on observing a steady-state bias in an error plane in a state space for aircraft applications, however, a design process is desired that enables rapid detection of failures during maneuvering transients and accounts for the normal operational noise environment of the aircraft and the control system actuators and sensors.

A design method is presented for resolving both problems in that it accounts for noise in sensors and is capable of determining hardover as well as increased noise-type failures during maneuvering transients. Incorporation of failure detection and recovery into an aircraft control system design is a joint detection, estimation, and control problem. The design method presented here produces a decision for detecting system failures which is optimal in the Bayesian sense. In addition, because of the theoretical development, one is able to account for uncertainty in the aircraft’s stability derivatives, mass, inertia, and geometric characteristics. Although the method developed can be applied to both sensor and actuator failures, only sensor failure detection and recovery are considered.

The approach taken here uses M-ary hypothesis testing with generalized likelihood ratios. The elements of this theory were originally developed at the close of World War II for a binary hypothesis testing problem of determining whether a radar return signal represented a target or not. In that case there are clearly two hypotheses—either there is a target or there is not. Theoretically, one can assign a cost to either failing to detect a real target or creating a false alarm. A performance index can be constructed which expresses the cost of making a decision based on a given radar return. This index can be minimized by selection of threshold points for decision whether or not the return represents a target. Elements of this problem are outlined in Ref. 4, which also contains a brief description of the M-ary hypothesis testing and generalized likelihood ratios. In this paper the set of hypotheses used is, first, that all sensors are functioning properly and, then, M - 1 further hypotheses stating that the ith sensor group has failed (i = 1, 2, . . ., M - 1). In the next section the theory for applying M-ary hypothesis testing to self-reorganizing systems is presented. Then, it is applied to an example aircraft problem.

Sensor Failure Accommodation Using M-ary Hypothesis Testing

Consider the equations of motion of an aircraft to be represented by
\[ x = Ax + Bu + w \]  

where \( x \) is an \( n \)-dimensional state vector, \( u \) is an \( m \)-dimensional control vector, \( w \) is a zero mean Gaussian white noise process with a covariance matrix \( W \delta(t - r) \). In Eq. (1) matrices \( A \) and \( B \) are determined from the aircraft’s \( \psi \), velocity and control derivatives, its mass and inertia characteristics, and its geometric characteristics. The variable \( w \) may, but need not, represent turbulence. It may represent uncertainty in the designer’s knowledge of the characteristics of the aircraft. Basically, it can be thought of as representing the error constrained to be constant with sampling interval \( T \), that is in calculation of \( ^{\dot{\Phi}} \), given \( x \) and control derivatives, its mass and inertia characteristics, dimensional control vector, \( w \) is a zero mean Gaussian white noise process with a covariance matrix \( \Sigma \).  

\[ \gamma = \int_0^T \Phi(s) \, ds \]  

and \( w(k) \) is a zero mean, white Gaussian sequence with covariance \( E(w(k)w'(j)) = Q \delta_{kj} \) where 

\[ Q = \int_0^T \Phi'(s)W\Phi(s) \, ds \]  

Let us assume that the control system has \( M-1 \) sensor failure modes for each mode 

\[ y(k) = C_i x(k) + v_i(k) \quad i = 1, 2, ..., M-1 \]  

where \( v_i(k) \) is a Gaussian white noise sequence where 

\[ E[v_i(k)] = 0, \quad E[v_i(k)v'_j(j)] = R_i \delta_{ij} \]  

The quantity \( m_i \) is an unknown (nonrandom) parameter vector.  

We shall solve the problem as if \( m_i \) were known and then use the maximum likelihood estimate of \( m_i \) under the \( i \)th hypothesis. This procedure is known as generalized likelihood ratio approach in the communication literature. This approach to failure modeling enables the designer to compensate for hardover failures of arbitrary magnitude. Increased sensor noise-type failures can be modeled by appropriate selection of the noise variances \( R_i \).  

For the normal unfailed condition we will assume 

\[ y(k) = C_0 x(k) + v_0(k) \]  

where \( E[v_0(k)] = 0 \) and \( E[v_0(k)v'_0(j)] = R_0 \delta_{ij} \). Hence, for a system with three failure modes, as considered in the next section, we have four hypotheses to consider 

\[ H_0: y(k) = C_0 x(k) + v_0(k) \]  

\[ H_i: y(k) = C_i x(k) + v_i(k) \quad i = 1, 2, 3 \]  

\[ H_{i'}: y(k) = C_0 x(k) + v_0(k) \]  

where \( C_i, (i = 1, 2, 3) \) is \( C_0 \) matrix with the rows corresponding to the \( i \)th group of sensors replaced by zeros.  

We will be concerned with the selection of the most probable hypothesis, based on a finite set of measurements, \( Y(K) = \{y(1), y(2), y(2)...y(K)\} \). To do this we construct a Bayesian cost function for the \( M \)-ary problem 

\[ \delta = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} P_{H_i} C_y \int_{t_j}^{t_{i+1}} P_{Y|H_j}(\alpha|H_i) \, d\alpha \]  

subject to 

\[ \sum_{i=0}^{M-1} P_{H_i} = 1 \]  

and where the sets \( Z_i, i = 0, 1, ..., M-1 \), are disjoint and their union represents the entire observation space. \( P_{H_i} \) is the a priori probability of hypothesis \( H_i \) being true, \( C_y \) is the cost of selecting \( H_i \) when \( H_i \) is true, and \( P_{Y|H_j}(\alpha|H_i) \) is the conditional probability density of the measurement sequence \( Y \) given that \( H_j \) is true. The symbol \( \delta \) implies that the integral is carried over the decision region \( Z_i \) in the observation space. Decision regions \( Z_i \) are subsets of observation space such that if \( Y \) is in \( Z_i \) then the hypothesis \( H_i \) is to be selected. Note that the integral in Eq. (4) represents nothing more than the probability of making the incorrect decision of selecting hypothesis \( H_i \) when \( H_i \) is true for \( i \neq j \). So the Bayes risk \( \delta \), represents the sum of probabilities corresponding to different decisions weighted by the a priori probabilities \( P_{H_i} \) and the design weights \( C_y \). The problem is to choose the boundaries of decision regions \( Z_i \) that will result in minimum Bayes risk. These boundaries are, in effect, switching hypersurfaces for the decision logic in the measurement space.  

The minimization of Bayes risk can be performed easily by rewriting the cost function (4) in the form 

\[ \delta = \sum_{i=0}^{M-1} \int_{Z_i} \psi_i(\alpha) \, d\alpha \]  

where 

\[ \psi_i(\alpha) = \sum_{j=0}^{M-1} P_{H_j} C_y P_{Y|H_j}(\alpha|H_j) \]  

The Bayes risk is minimized by selecting \( H_i \) at each point \( \alpha \) in the observation space such that \( \psi_i(\alpha) \) is the smallest of \( M \) possible values of \( \psi_i(\alpha) \) \( (k=0, 1, ..., M-1) \). Hence, the optimal decision regions are 

\[ Z_i(\alpha|\psi_i(\alpha) = \min_{k=0, 1, ..., M-1} \psi_i(\alpha), \quad 0 \leq k \leq M-1 \]  

From a computational point of view, it is convenient to introduce a dummy hypothesis \( H_M \) with a priori probability \( P_{H_M} = 0 \) with \( H_M: y(k) = v_0(k) \). Then, an equivalent decision criterion can be given in terms of likelihood ratios, \( \Lambda_i(\alpha) \) 

\[ \Lambda_i(\alpha) = P_{Y|H_i}(\alpha|H_i)/P_{Y|H_j}(\alpha|H_j) \quad i = 0, 1, ..., M-1 \]  

Dividing each \( \psi_i \) in Eq. (6) by the probability density of \( Y(K) \) under \( H_M \), we get an equivalent decision criterion in terms of the likelihood ratios 

\[ \Lambda_i(\alpha) = \sum_{j=0}^{M-1} P_{H_j} C_y \Lambda_j(\alpha) \]  

Then 

\[ Z_i(\alpha|\Lambda_i(\alpha) = \min_{0 \leq k \leq M-1} \Lambda_i(\alpha), \quad 0 \leq k \leq M-1 \]  

The advantage of using likelihood ratios is that the boundaries of the decision regions are linear hyperplanes and not general hypersurfaces in the likelihood ratio space \( \Lambda_1, \Lambda_2,... \)
From the chain rule of probability densities and the Gaussian density of the observations, it can be shown that the likelihood ratio for the problem considered is given by

$$\Lambda = \frac{\prod_{i=0}^{M-1} (\det R_i)^{1/2}}{\left(\frac{1}{2\pi \det Q_i} \right)^{1/2} - \sum_{i=0}^{M-1} \lambda^i}.$$  

where $r_i(k)$ is the innovation of the measurements under the $i$th hypothesis given by

$$r_i(k) = y(k) - C_i x_i(k | k-1) - m_i(k)$$  

with $x_i(k | k-1) = E \{ x(k) | Y(k-1), H_i \}$. The matrix $Q_i(k)$ in Eq. (11) is given by

$$Q_i(k) = C_i V_{A_i} C_i^T + R_i.$$  

In Eq. (12), the true value of $m_i(k)$ should be used to get the exact likelihood ratio. Since this is not available, we will use the sample mean of $m_i(k)$, that is, the maximum likelihood estimate of $m_i$ at the $k$th instant under the $i$th hypothesis. That makes $\Lambda$, a generalized likelihood ratio.

To compute $\dot{x}_i(k | k-1)$ and $V_{A_i}(k | k-1)$, $M$ Kalman filters are required. A bank of Kalman filters operating in parallel has been used for parameter adaptive control in Ref. 8. The filter equations are listed as follows for completeness

$$\dot{x}_i(k | k-1) = \dot{x}_i(k | k-1) + K_i(k) r_i(k)$$

$$x_i(k | k-1) = F_i x_i(k | k-1) + G_i u(k)$$

where $\dot{x}_i(k)$ is the estimate of the aircraft state under the $i$th hypothesis defined by

$$\dot{x}_i(k | k-1) = E \{ x(k) | Y(k), H_i \}.$$  

The filter gain $K_i(k)$ in Eq. (13) can be calculated recursively from the algorithm

$$K_i(k) = V_{A_i}(k | k-1) C_i^T Q_i^{-1}$$

where $Q_i(k)$ is given by Eq. (13) and the prediction error variance $V_{A_i}(k | k-1)$ is given by

$$V_{A_i}(k | k-1) = \Phi V_{A_i}(k-1) \Phi^T + Q$$

where $V_{A_i}(k)$ is the filter error variance given by

$$V_{A_i}(k) = (I - K_i(k) C_i) V_{A_i}(k | k-1).$$

Considerable simplification occurs if one considers $C_j = C_{ij} = 1, (j \neq i)$ and $C_j = 0$. Ramifications of this assumption are discussed in the example to follow. Under those conditions the equations for $\Lambda$, may be modified without loss of generality to select the maximum of

$$\Lambda = \sum_{i=0}^{M-1} \frac{1}{\sqrt{(2\pi \det Q_i)^{-1}} - \sum_{j=0}^{M-1} \lambda^i}.$$  

where $K$ is the total number of measurements used to make the decision. If the steady-state Kalman filter is used, we can select the largest of

$$\ln P_{H_i} - \frac{1}{2} \ln Q_i - \frac{1}{2} \sum_{j=0}^{K} r_i(j) Q_i^{-1} r_i(j),$$

$i=0,1,...,M-1$

Also, if the a priori probabilities of $H_i$ are equal, without loss of generality, we may take

$$\tau_i[ Y(k) ] = \frac{1}{2} \ln Q_i + \frac{1}{2} \sum_{j=0}^{K} r_i(j) Q_i^{-1} r_i(j),$$

and select the hypothesis $H_i$ corresponding to the smallest $\tau_i$, $i=0,1,...,M-1$. The next section demonstrates the application of this method to a practical problem.

**Application to Aircraft Flight Control**

The theory developed in the previous section has been applied to the design of a control system for one space shuttle orbiter configuration at a Mach number of 5 and an altitude of 120,000 ft. Taking the state to be defined as $x = p, \theta, r, \beta$' and the only effective control $u = v$, the aircraft equations of motion can be written as

$$x' = \begin{bmatrix} -0.0580 & 0 & 0.0170 & -5.791 \\ 1.0 & 0 & 0.5773 & 0 \\ -0.0029 & 0 & -0.0085 & -0.7438 \\ 0.5 & 0.0055 & -0.8660 & -0.0009 \end{bmatrix} x + \begin{bmatrix} 2.256 \\ 0 \\ 0.0553 \\ 0 \end{bmatrix} u + w$$
Table 1: Evaluation of $W$

<table>
<thead>
<tr>
<th>$W$ component</th>
<th>Level of certainty</th>
<th>Error scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1 (p$ component)</td>
<td>2</td>
<td>0.05 rad/sec</td>
</tr>
<tr>
<td>$W_2 (\phi$ component)</td>
<td>0</td>
<td>1 rad</td>
</tr>
<tr>
<td>$W_3 (r$ component)</td>
<td>2</td>
<td>0.01 rad/sec</td>
</tr>
<tr>
<td>$W_4 (\beta$ component)</td>
<td>3</td>
<td>0.001 rad/sec</td>
</tr>
</tbody>
</table>

$^a$ 0 implies absolute certainty, 1 implies a high level of certainty, 2 implies only moderate certainty, and 3 implies not too sure.

The selection of the variance $W$ for the last equation involves consideration of: 1) the uncertainty that we, as designers, feel related to our knowledge of the equations of motion, 2) the relative scales of the variables, and 3) the environment, with regard to turbulence, under which the vehicle must operate. We will only consider the first two items here. Table 1 shows the authors' interpretation of the level of certainty and scale considerations. Concerning the level of certainty, it was felt that the $\phi$ equation was well understood since it represents a well-known kinematic relationship. A high level of certainty was assigned to the $\phi$ equation. On the other hand, the $p$ and $r$ equations were felt to be better defined than the $\beta$ equation. Turning to scale considerations we have, in effect, equated an error of $1/0.05$ in the computation of $p$ to one of $1/0.001$ in the computation of $\beta$. The $W$ matrix selected is constructed from the elements of Table 1 as follows:

$W = \text{diag} \begin{bmatrix} 2(0.05)^2, 0(1)^2, 2((1.01)^2), 3(0.001)^2 \end{bmatrix}$

The discretized equations of motion using a zero-order-hold with a sampling interval of 0.1 sec is

$x(k+1) = x + u(k) + w(k)$

and the discrete variance matrix for the process $w(k)$ is

$Q = \begin{bmatrix} 0.4757 & 0.04757 & -0.0066 & 0.0236 \\ 0.04757 & 0.00654 & 0.00100 & 0.00309 \\ -0.006 & 0.00100 & 0.02015 & -0.00185 \\ 0.0236 & 0.00309 & -0.00185 & 0.00211 \end{bmatrix} \times (10)^3$

which was evaluated using Eq. (3). Note that, because of the sampling, even though the $\phi$ equation was considered absolutely certain, uncertainty does result in the $\phi$ equation of the discrete model. Also, the components of the plant noise vector are correlated in the discrete model.

For illustration, consider that the vehicle has three sensors: a roll-rate gyro, a yaw-rate gyro, and a sideslip indicator. Therefore, there will, therefore, be four hypotheses to consider, as follows:

$H_0: y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_0$

$H_1: y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_1$

$H_2: y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_2$

$H_3: y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + v_3$

Variances of the measurement error are taken, consistent with current technological capability, to be

$R_0 = \text{diag} \begin{bmatrix} (0.05)^2, (0.01)^2, (0.01)^2 \end{bmatrix}$

Failure covariances are assumed to be larger than the unfailed ones. (The behavior of the resulting system will, however, be illustrated for both statistical failures—increased variance—and for hardover failures. This capability is a direct result of not assuming a zero-mean measurement error in the failure states.) The values of $R_1$, $R_2$, and $R_3$ used are

$R_1 = \text{diag} \begin{bmatrix} 0.025, 0.0001, 0.0001 \end{bmatrix}$

$R_2 = \text{diag} \begin{bmatrix} 0.0025, 0.001, 0.0001 \end{bmatrix}$

$R_3 = \text{diag} \begin{bmatrix} 0.0025, 0.0001, 0.01 \end{bmatrix}$

For each hypothesis the state is observable but, given the measurement errors and uncertainties in the vehicle equations of motion, each hypothesis has a different capability of estimating the state of the aircraft. Hence, embedded in the theory is the consideration of the capability of any given sensor group, corresponding to each hypothesis, to estimate the state of the aircraft. This is reflected in the error covariance matrix elements of each hypothesis. As an example, $E[ (\hat{p} - p)^2 ]$ under each hypothesis is indicated as the $(1, 1)$ element of the error covariance matrix and is $0.00075, 0.0015, 0.000082, 0.000087$ for $H_p, H_1, H_2$, and $H_3$, respectively. As expected, $H_0$ has the smallest value of $E[ (\hat{p} - p)^2 ]$, indicating that this hypothesis, if true, can produce the best estimate of $p$. Also indicated, however, is the fact that $H_1$ produces the worst estimate. Again, this is expected since $H_1$ corresponds to deletion of roll-rate gyro information.

In this example, the Bayesian risk weights $C_i$ are taken as $C_i = 1$ for $i \neq j$ and $C_j = 1$. Also, steady-state Kalman filters are used so that Eq. (18) is applicable. For the example here, Eq. (18) becomes (using a memory size of five samples)

$\tau_0 = -\frac{5}{2} (22.508) + \frac{1}{2} \sum_{k=1}^{5} r_{ij}(k)$
During control system operation the scalars \( r_i \) should be using the innovations \( r_i \) of the Kalman filter bank stored over the past five samples. Then, the hypothesis corresponding to the minimum \( r_i \) should be selected.

The behavior of the system has been studied using a hybrid computer facility in which the equations of motion of the vehicle were programmed on an analog computer and the control system was mechanized in a digital computer. Figure 2 illustrates the unaugmented step response of the vehicle to an aileron input. This aircraft is a nonminimum phase system indicated by roll reversal. Also, the aircraft possesses a large coupling of the Dutch roll into the aileron response. Digital feedback was employed at a cycle time of 0.1 sec using feedback gains \((-4.9, 0.4, 14.5, -6)\) for \((p, \theta, r, \beta)\) respectively, to the aileron. The gains were selected to be constrained to a control system operating with only roll control. Figure 3 shows the response of the closed-loop aircraft to the same pilot step input when state variable feedback (perfect measurement of each state) is employed. Considerable improvement in flying qualities could be obtained if yaw control were available. Figure 4 illustrates the same step response using noisy measurements and accepting \(H_0\). No actuators and sensors have been failed in Fig. 4. In Fig. 5 the responses of the system are indicated for the case where \(H_2\) is true but for each hypothesis being accepted at different times. The failure mode considered in Fig. 5 is an increase in measurement noise. Note that at the start of the record \(H_0\) is selected and produces poor characteristics, as can be seen by comparing the \(H_0\) true portion of the roll-rate trace of Fig. 5 with that of Fig. 4. Had there been no failure, those traces would be almost identical. When \(H_2\) is selected at approximately 5 sec, poor characteristics are still produced. However, when \(H_3\) is selected at approximately 10 sec the system moves to a normal operation, only to return to its poor characteristics when \(H_4\) is selected at approximately 15 sec. This figure illustrates the effect of accepting hypothesis \(H_4\) when \(H_4\) is true. It indicates the effect of cost selection of the \(C_i\) terms in the Bayesian risk function. Figure 6 shows the fault tolerant system in operation when failures of increased noise type are introduced. By looking at the \((p, r, \theta)\) measurements, it can be seen that the following sensor failure modes have been simulated: \([H_0, H_1, H_2, H_3]\). The plot showing the hypothesis accepted indicates the performance of the detector logic. Figure 7 deals with the detection of hardover failures. A hardover failure in the beta sensor has been simulated. Note that, although detection logic is able to detect sensor failures in all cases quite rapidly, the detector logic takes a longer time to reject a failure hypothesis when the
Fig. 4 Response of closed-loop system with noisy measurements under normal operation.

Fig. 5 Response of closed-loop system demonstrating effects of accepting hypothesis $H_0$, $H_2$, $H_2$, when $H_2$ is true.

Fig. 6 Operation of fault tolerant system during failures resulting in increased noise.

Fig. 7 Operation under saturation hardover failure.

system is already in one. Further, note also that only the steady-state Kalman filters are used and overall performance may be improved using time-varying Kalman filters.

Conclusions

A digital fault tolerant control system design that accommodates for aircraft sensor failures has been presented.

Each sensor failure mode and the normal operation of sensors are modeled as $M$ different hypotheses. Then, using the Bayesian $M$-ary hypothesis testing approach, a detection logic is developed that results in a bank of $M$ Kalman filters. The
decision logic, which uses $M$ generalized likelihood ratios, selects the hypothesis that minimizes the cost of making a wrong decision in the Bayesian sense. The likelihood ratios are calculated from a moving window of the innovations in each of the Kalman filters. The estimate of the state corresponding to the hypothesis selected by the detection logic is used in the control system. The design system is capable of identifying increased noise type and hardover-type sensor failures. These capabilities are demonstrated using a real-time hybrid simulation for a space shuttle vehicle lateral dynamics.

References
PARAMETER ADAPTIVE ESTIMATION OF RANDOM PROCESSES

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Abstract

This paper is concerned with the parameter adaptive least squares estimation of random processes. The main result is a general representation theorem for the conditional expectation of a random variable on a product probability space. Using this theorem along with the general likelihood ratio expression, the least squares estimate of the parameter is given. In section 4, the results are specialized to the case when the parameter has a discrete distribution. The results can be used to construct an implementable recursive estimator for certain types of nonlinear filtering problems. This is illustrated by some simple examples.

1. Introduction

This paper is concerned with the parameter adaptive estimation of random processes corrupted by additive noise. The problem may be briefly stated as follows. From a given (possibly uncountable) collection of random processes with known distributions, one process is observed with additive noise. The a priori probability, that a specific random process in this collection is observed, is specified for each one in the collection. The problem is to find the least squares estimate of the parameter corresponding to the observed signal process.

There is a large class of physical problems that can be treated in this context. For instance, joint detection and estimation [9], [18], estimation under uncertainty [19], joint estimation and identification [10], parameter adaptive self-organizing control [15], [16], [17].

Parameter adaptive estimation of discrete Gaussian processes with linear dynamic models has been treated by Bastia [12]. Lainiotis [16] has investigated the parameter adaptive estimation problem for continuous Gaussian processes with linear dynamic models. Other related work along these lines can be found in [18] - [21]. The first systematic treatment of joint detection-estimation in a general setting was done by Middleton and Esposito [9].

In the mathematical theory of probability, related works can be outlined as follows. Parameter adaptive estimation for a specific jump Markov process with a finite number states has been studied by Wonham [22]. The same problem for a two-dimensional Markov process with the nonobservable component a jump Markov process and the observable component, a diffusion process, has been resolved by Shiryaev [23]. Related work on the estimation of arbitrary stochastic systems can be found in the work by Kallianpur and Striebel [5] and for Markov processes in [24].

This paper generalizes the results of [10], [9]. The approach taken is along the lines of [13], [5]. The problem is formulated in the usual Bayesian decision theoretic framework. In the second section, a mathematical statement of the problem is given. In section 3, a general representation theorem is proved for the conditional expectation of a random variable on a product probability space using the properties of Radon-Nikodym derivatives. This theorem also yields a Radon-Nikodym representation for the a posteriori probability which can be thought of as a generalization of the Bayes theorem. By using the representation theorem along with the general likelihood ratio expression of Kallath-Duncan [6], [7], the parameter adaptive estimation problem of random processes is solved in section 4. The main theorem of this section can be thought of as a generalized partition theorem. In section 5, by using simple stochastic calculus, the stochastic differential of the a posteriori density corresponding to the a posteriori probability and the stochastic differential equation for the a posteriori density are found. In section 6, the results are specialized to the case when the parameter has a discrete distribution. The results of this section include the case of finite parameter space such as M-ary hypotheses. Section 7 includes some illustration and applications.

2. Statement of the Problem

We will be concerned with two underlying probability spaces \( \Omega_1 \) and \( \Omega_2 \) over which \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) are defined. The parameter space \( \Theta \) will correspond to \( \Omega_1 \) and \( \Omega_2 \) will constitute the observation space. Elements of \( \Theta \) and \( \Omega_2 \) will be denoted as
and \(n\), respectively. Expectation with respect to measures \(P_1\) and \(P_2\) will be denoted by \(E_1\) and \(E_2\). We shall assume that there exists a vector Wiener process \(v(t, n)\) on \((\Omega_2, A_2, P_2)\) with zero mean and increments with covariances \(E_2 dv(t) dv'(t) = R(t) dt\). For each \(n\), we will be given two vector random processes \(z(t, n)\) and \(y(t, n)\) such that the future increments of the Wiener process \(v(t, n)\) will be independent of the past of \(v(t, n)\) and \(z(t, n)\) with

\[
y(t, n) = \int_0^t z(s, n) ds + v(t, n) \quad t \in [0, T]. \tag{2.1}
\]

So that the observation model becomes

\[
y(t, n) = s_x(t, n) ds + v(t, n) \quad t \in [0, T]. \tag{2.2}
\]

The problem is to find the conditional expectation of \(z(t, n)\) given a sample function of \(y(t, n)\) in terms of the conditional expectation of \(z(t, n)\) conditioned on the sample function of \(y(t, n)\). The implicit assumption in parameter adaptive estimation is that the conditional estimates are easily computable.

Given two arbitrary sets \(\Omega_1\) and \(\Omega_2\), we shall denote the product of \(\Omega_1\) and \(\Omega_2\) by \(\Omega_1 \times \Omega_2\), which is, by definition, the set of all pairs \((\omega_1, \omega_2)\) where the points \(\omega_1\) and \(\omega_2\) run through \(\Omega_1\) and \(\Omega_2\), respectively. If \(Y\) is an arbitrary mapping of \(\Omega_1 \times \Omega_2\) into an arbitrary space, we shall denote \(Y_{(\cdot)}\), the section of \(Y\) at \(t\), by the mapping defined on \(\Omega_2\) by \(Y_{(\cdot)}(\omega_2) = Y(x_1, \omega_2)\). If \(A\) is an arbitrary set of \(\Omega_1 \times \Omega_2\), we denote \(A_{\omega_1}\), the section of \(A\) at \(\omega_1\), to be the subset of \(\Omega_2\) defined by \(A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}\).

Now the problem can be stated as follows: Find the conditional expectation of \(z(t, n, \theta)\) given a sample function of \(y(t, n, \theta)\) in terms of the conditional expectation of \(z(t, n, \theta)\) conditioned on the sample function of \(y(t, n, \theta)\) at \(\theta\).

We shall use the following definition of conditional expectation [3]. Let \(g\) be a random variable on the probability space \((\Omega, A, P)\) and \(Y\) be a measurable transformation from \((\Omega, A, P)\) into a measurable space \((\mathbb{V}, B)\). If \(E_0\) exists, then

\[
E_0(g|Y=y) = \frac{\int_{y^{-1}(B)} g(\omega) d\Pi(\omega)}{\Pi^{-1}(B)}
\]

for every \(B \in B\), where \(Y^{-1}(B) = \{\omega : Y(\omega) \in B\}\) and \(\Pi^{-1}(\cdot)\) is the measure on \(\mathbb{R}\) defined by \(\Pi^{-1}(\omega) = \Pi(Y^{-1}(\omega))\) and \(E_0\) is the Borel sets of \(\mathbb{R}\). In the parameter adaptive estimation we will be concerned with two different conditional expectations. If \(Y\) is the set of all \(\mathbb{R}\) valued continuous functions on \([0, T]\) and \(G\) is the Borel sets on \(Y\), then we can define the measurable mapping \(Y\) from \((\Omega_1 \times \Omega_2, A_1 \times A_2, P_1 \times P_2)\) into \((\mathbb{V}, B)\) such that

\[
Y : \Omega_1 \times \Omega_2 \times A_1 \times A_2 \times P_1 \times P_2 \rightarrow (\mathbb{V}, B)
\]

by

\[
y(0, n) = \{y(t, n, \theta) : 0 \leq t \leq T\}.
\]

Similarly, we can define the measurable mapping \(Y_\theta\) where \(\theta\) is some fixed point in the parameter space \(\Omega_1\)

\[
y_\theta : \Omega_1 \times \Omega_2 \times A_1 \times A_2 \times P_1 \times P_2 \rightarrow (\mathbb{V}, B)
\]

by

\[
y_\theta(0, n) = \{y(t, n, \theta) : 0 \leq t \leq T\}.
\]

That is, \(Y_\theta\) is the section of \(Y\) at \(\theta\). Now in this setting we can talk about

\[
\hat{z}(t, y) = \frac{\int_{Y_\theta^{-1}(\{y\})} z(t, \theta, n) dP_1 \theta}{\int_{Y_\theta^{-1}(\{y\})} dP_1 \theta}
\]

and, for a fixed \(\theta \in \Omega_1\),

\[
\hat{z}_\theta(t, y) = \frac{\int_{Y_\theta^{-1}(\{y\})} z(t, \theta, n) dP_1 \theta}{\int_{Y_\theta^{-1}(\{y\})} dP_1 \theta}
\]

The problem is to find \(\hat{z}(t, y)\) in terms of \(\hat{z}_\theta(t, y)\) \(\theta\) runs through the parameter space \(\Omega_1\).

3. Characterization of Conditional Expectation on the Product Probability Space

In this section, we will prove a general theorem concerning parameter adaptive estimation. Namely, a representation will be found for the conditional expectation of a random variable on a product probability space using the properties of Radon-Nikodym derivatives. This theorem also gives a representation for the a posteriori probability. The results will be repeatedly used in the rest of the paper. The representation theorems are generalized Bayes theorems.

Theorem 3.2 Let \((\Omega_1, A_1, P_1)\) and \((\Omega_2, A_2, P_2)\) be two probability spaces and \((\mathbb{V}, B)\) be a measurable space where \(B\) is generated by a countable class of sets. Denote the product probability space \((\Omega_1 \times \Omega_2, A_1 \times A_2, P_1 \times P_2)\) by \((\Omega, A, P)\). Let \(Y\) be a measurable transformation from \((\Omega, A, P)\) into \((\mathbb{V}, B)\) and \(g\) be a random variable on \((\Omega, A, P)\) whose expectation exists. \((g\) is quasi-integrable in Revuë’s sense [2]). Let \(u_0\) denote the measure induced on \((\mathbb{V}, B)\) by the section of \(Y\) at \(\theta \in \Omega_1\), i.e.

\[
u_0(B) = \int_B \nu_0(0) dP_1(0)
\]

for every \(B \in B\).

If there exists a \(c\)-finite measure \(u_0\) on \((\mathbb{V}, B)\) and a net \(N_1\) in \(A_1\) of \(P_1\)-measurable zero such that \(u_0 \ll u_0\) for \(\theta \in \Omega_1\), then the conditional expectation of \(g\) given \(Y\) is

\[
E(g(0, n)/Y(0, n) = \int_{\Omega_1} \int_B \frac{du_0}{\Omega_1 dP_1(0)} \nu_0(0) \frac{1}{\nu_0(0)} \int_B \frac{du_0}{\Omega_1 dP_1(0)}
\]

\[
\int_{\Omega_1} \frac{du_0}{\Omega_1 dP_1(0)} \nu_0(0)
\]

\[
\int_{\Omega_1} \frac{du_0}{\Omega_1 dP_1(0)} \nu_0(0)
\]

\[
(3.2)
\]
where \( \mu(\cdot) = P(Y^{-1}(\cdot)) \), \( \frac{d\mu}{du} \) is the Radon-Nikodym derivative of \( \mu \) w.r.t. \( \nu_0 \), \( R \) denotes expectation w.r.t. the measure \( P \) (i.e. \( P_1 \times P_2 \)), and \( [u] \) denotes almost everywhere with respect to the measure \( \nu \).

Proof: See Appendix.

Corollary 3.3 Under the assumptions of Theorem 3.2, for \( P_1 \) - almost all \( \theta \in \Theta_1 \) and \( \nu_0 \) \( \ll \) \( \mu \), the R-N derivative is given by

\[
\frac{d\mu}{du}(y) = \frac{d\mu_\theta}{d\nu_\theta}(y) \cdot [\nu].
\]

(3.3)

So that

\[
E(g(\theta, n) | Y(\theta, n) = y) = \int_{\text{p}1}(\theta) \cdot E_2(X(\theta, n) | Y(\theta) = y)
\]

\[
\frac{d\mu_\theta}{d\nu_\theta}(y) \cdot dP_1(\theta).
\]

(3.4)

Also

\[
P(A_1 \times \Omega_2 | Y = y) = \int_{\text{p}1}(\theta) \cdot dP_1(\theta).
\]

(3.5)

Proof: See Appendix.

Note that \( P(A_1 \times \Omega_2 | Y = y) \) is the a posteriori probability of the event \( A_1 \) in the parameter space conditioned on \( Y \). \( P(A_1 \times \Omega_2 | Y = y) \) defines a measure \( \pi \) on \( \Lambda_1 \)-sets for each \( y \in Y \) by

\[
\pi(\Lambda_1, y) = (A_1 \times \Omega_2 | Y = y)
\]

(3.6)

We also note that the representation for the a posteriori probability is a generalized Bayes theorem. To see this, let \( \Omega_1 \) be countable, \( \Omega_1 = \{ \theta_1, \theta_2, \ldots, \theta_\infty \} \), and \( \Omega_2 \) = \( \mathbb{R} \) and let \( \nu_0 \) be Lebesgue measure on \( \mathbb{R} \), then by 3.3 and 3.8

\[
\frac{d\mu_\theta}{d\nu_\theta}(y) = \frac{P_1(\theta)}{\pi(\theta, y)}
\]

(3.7)

Note that \( \frac{d\mu_\theta}{d\nu_\theta}(y) \) is by definition, the conditional probability density of \( Y \) given \( \theta \) so that (3.7) is one form of the celebrated theorems of Bayes.

We shall now apply the results of the Theorem 3.2 and Corollary 3.3, to the parameter adaptive estimation of random processes.

4. Parameter Conditioned Estimation of Random Processes

In this section we will apply the results of the previous section to parameter adaptive estimation of random processes. The representation theorem for the conditional expectation of the last section will be used along with the general likelihood ratio expression of Kalat and Bucum [6], [7], [26], [37]. To now state the general likelihood ratio results in the next lemma which is a slightly different restatement of Theorem 2 in [6].

Lemma 4.1 Let \( \Omega_1, \Lambda_1, P_1 \) and \( \Omega_2, \Lambda_2, P_2 \) be two probability spaces. Assume that for each \( \theta \in \Theta_1 \), we are given three measurable vector random processes (taking their values in \( \mathbb{R}^p \)) \( x_0(t, n), y_0(t, n), \) and \( v(t, n) \) with \( n \in \mathbb{N} \) and \( t \in [0, T] \) such that \( v(t, n) \) is a zero mean Wiener process with incremental covariance

\[
E_1 dv(t) \cdot dv(t) = R(t) \cdot dt
\]

and the three random processes are related by the equation

\[
y_0(t, n) = f_1 x_0(s, n) \cdot ds + v(t, n).
\]

(4.1)

Let \( Y_1 \) be the space of all \( \mathbb{R}^n \)-valued continuous functions on \([0, t] \) and \( E_1 \) be the Borel sets of \( Y_1 \). Define the measurable transformations \( Y_0(t, n) \) and \( Y_1(t, n) \) from \( \Omega_1, \Lambda_1, P_1 \) into \( Y_1, E_1 \) by

\[
Y_0(t, n) = (x_0(s, n), 0 \leq s \leq t)
\]

(4.2)

\[
Y_1(t, n) = (y_0(s, n), 0 \leq s \leq t).
\]

(4.3)

Let \( P_1, Y_0^{-1} \) and \( P_2, Y_1^{-1} \) be the measures induced on \( (Y_1, E_1) \) by

\[
P_2 Y_0^{-1}(B = \{ y \in Y_0^{-1}(B) \}) \quad B \in E_1
\]

(4.4)

\[
P_2 Y_1^{-1}(B = \{ y \in Y_1^{-1}(B) \}) \quad B \in E_1.
\]

(4.5)

If, for some \( \theta \in \Theta_1 \),

\[
E_2 \int_0^T \frac{\partial}{\partial \theta} y_0'(t, n) \cdot ds \frac{\partial}{\partial \theta} y_0'(t, n) \cdot dt < 
\]

(4.6)

and

\[
E_2 \int_0^T \frac{\partial}{\partial \theta} y_0'(t, n) \cdot ds \frac{\partial}{\partial \theta} y_0'(t, n) \cdot dt < 
\]

(4.7)

and the future increments \( v(t, n) - v(s, n) \) are independent of the \( \sigma \)-algebra generated by \( y(t, n), 0 \leq t \leq s \), then \( P_2 Y_0^{-1} \ll P_2 Y_1^{-1} \) and the R-N derivative is given by

\[
\frac{dP_2 Y_0^{-1}}{dP_2 Y_1^{-1}}(y) = \exp \left( \int_0^T \frac{\partial}{\partial \theta} y_0'(t, n) \cdot ds - \int_0^T \frac{\partial}{\partial \theta} y_0'(t, n) \cdot ds \right)
\]

(4.8)

where

\[
y_0(s, n) = E_2(z_0(s, \cdot)) \cdot Y_0(s, n) = y_0(s, n), 0 \leq 0 \leq s.
\]

(4.9)

Proof: See Theorem 1 and also Theorem 1 in [7].

Theorem 4.2 Let the assumptions be as in Lemma 4.1. Also assume that \( x_0(t, n) \) is a measurable function in \( (t, n, \theta) \) so that we can now define the measurable random processes \( z(t, n, \theta) \) and \( y(t, n, \theta) \) on the product probability space \( (\Omega, \Lambda, P) = (\Omega_1 \times \Omega_2, \Lambda_1 \ast \Lambda_2, P_1 \ast P_2) \) by

\[
z(t, n, \theta) = z_0(t, n).
\]

(4.10)

\[
y(t, n, \theta) = y_0(t, n).
\]

(4.11)

The three random processes are related by

\[
y(t, n, \theta) = \int_0^t z(s, n, \theta) \cdot ds + v(t, n, \theta)
\]

(4.12)
Let \( Y_t \) be the measurable transformation
\[
Y_t : (\Omega, A, P) \rightarrow (\hat{Y}_t, \hat{B})
\]
\( Y_t(\omega, n) = (y(s, 0, n), 0 \leq s \leq t). \) \tag{4.13}

Note that \( Y_t \) is the section \( Y_t \) at \( t \). If there exists a set \( \hat{B} \) in \( A_1 \) of \( P \) measure zero such that for all \( \omega \in \hat{B} \), the assumptions in Lemma 4 are satisfied, and \( E(\omega(t, n, \theta)) \) exists, then
\[
E(\omega(t, n, \theta)) = \int_{\Omega_1} \lambda_0(t, n) \, dP_1(\omega) \tag{4.14}
\]
where, \( \lambda_0(t, n) \) is given by
\[
\lambda_0(t, n) = \int_{\Omega_1} \lambda_0(t, n) \, dP_1(\omega)
\]
and the R-N derivative \( \lambda_0(t, n) \) is given by
\[
\lambda_0(t, y) = \frac{dP_1 Y_{t+1}}{dP_1 Y_{t}}(y) = \exp \{ \int_{0}^{t} \hat{z}(s, y) R^{-1}(s) \, ds \}
\]
\[
\int_{\Omega_1} \lambda_0(t, y) \, dP_1(\omega) = [PY^{-1}] \tag{4.15}
\]
Proof: For all \( 0 \in \Omega_1 = \Omega \), by Lemma 4.1, we have
\[
P_2 Y_{t+1} \leq P_2 Y_{t} \leq P_2 Y_{t}^{-1}
\]
and the R-N derivative is given by
\[
\lambda_0(t, y) = \frac{dP_1 Y_{t+1}}{dP_1 Y_{t}}(y) \tag{4.16}
\]
So that
\[
E(\omega(t, n, \theta)) = \int_{\Omega_1} \lambda_0(t, n) \, dP_1(\omega)
\]
\[
\int_{\Omega_1} \lambda_0(t, y) \, dP_1(\omega) = [PY^{-1}] \tag{4.17}
\]
Also
\[
P(\lambda_1 \cap \Omega_2 \mid Y_t = Y) = \int_{\lambda_1} \frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) \, dP_1(\omega)
\]
\[
\int_{\Omega_1} \lambda_0(t, y) \, dP_1(\omega) = [PY^{-1}] \tag{4.18}
\]
Proof: Follows from Corollary 3.3.

Corollary 4.4 Under the assumptions of Theorem 4.2, we have the following representation for the R-N derivative
\[
\frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) = \lambda_0(t, y)
\]
where
\[
\lambda_0(t, y) = \frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) = \exp \{ \int_{0}^{t} \hat{z}(s, y) R^{-1}(s) \, ds \}
\]
\[
\frac{1}{2} \int_{0}^{t} \hat{z}(s, y) R^{-1}(s) \, ds \tag{4.21}
\]
Proof: By Corollary 4.3, we have
\[
\frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) = \lambda_0(t, y)
\]
By Remark 4 in the proof of Theorem 3.2
\[
\int_{\Omega_1} \lambda_0(t, y) \, dP_1(\omega) = \int_{\Omega_1} \lambda_0(t, y) \, dP_1(\omega)
\]
The result now follows from Lemma 4.1.

Comments: The R-N derivative \( \lambda_0(t, y) \) is the likelihood ratio for the detection problem (with \( \theta \) fixed)
\[
h_0: y_0(t, n) = \int_{0}^{t} \hat{z}(s, y) ds + v(t, n)
\]
\[
h_0: y_0(t, n) = v(t, n)
\]
\( A(t, y) \) in Corollary 4.4 is the L.R. for the composite hypothesis testing problem
\[
h: y(t, n, 0) = \int_{0}^{t} \hat{z}(s, n, 0) ds + v(t, n)
\]
\[
h: y(t, n, 0) = v(t, n)
\]
Note that
\[
\lambda_0(t, y) = \frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) = \lambda_0(t, y)
\]
represents a normalized likelihood ratio. This normalized LR is the ratio of the conditional probability density of the parameter conditioned on the observation and the probability density of the parameter; that is, \( \lambda_0(t, y) \) is the normalized conditional density conditioned on the observation.

Note also that \( P(\lambda_1 \cap \Omega_2 \mid Y_t = y) \) is the posterior probability of the event \( \lambda_1 \) in the parameter space conditioned on the observation. Since
\[
P(\lambda_1 \cap \Omega_2 \mid Y_t = y) = \int_{\lambda_1} \frac{dP_2 Y_{t+1}}{dP_2 Y_{t}}(y) \, dP_1(\omega),
\]
we have proved the existence of the conditional probability density of the parameter.

Theorems 3.4 and 4.2 unify a number of known results in the literature. For instance, in the case, where the parameter space contains two points, each one corresponding to the presence or the absence of a signal, we obtained the relationship between the optimal estimate of the signal under uncertainty and the optimal estimate of sure signal in the least squares sense derived in reference [9]. In the case, where the signal and the measurement noise are independent and the signal is a Gaussian signal with a linear dynamic model we get the results in [10].

The expression derived for the conditional expectation in this section is useful in the case of fixed observation time interval. If the data is coming continuously, a stochastic differential equation implementation is more practical. In the next section, by using simple Ito calculus on the representations obtained in this section, we shall find the stochastic differential equations that the a posteriori probability and probability density satisfy.
5. Stochastic Differential Equations for the A Posteriori Probability

The results of the previous section are useful when the observation interval is fixed. Since in most technological applications the data is obtained continuously, the differential form of the results is more practical to implement. In this section, we shall find the stochastic differential equations for the \( a \) posteriori probability density and a stochastic differential representation for the \( a \) priori probability by a simple application of Ito's differentiation rule on the normalized conditional probability density representation in Corollary 4.4.

**Theorem 5.1** If the assumptions of Corollary 4.4 hold, then normalized conditional probability density

\[
\frac{dP_2Y_{t-1}}{dP Y_{t-1}}(y) = \lambda_0(t, y)
\]

exists \( P \) \( I \)-almost everywhere and is the unique solution of the following stochastic differential equation

\[
d\lambda_0(t, y) = \lambda_0(t, y)\left(\hat{z}_0(t,y) - \hat{z}(t,y)\right)'R^{-1}(t)(dy(t) - \hat{z}(t,y)dt)
\]

with the initial condition unity.

**Proof:** From Corollary 4.4.

\[
\lambda_0(t, y) = \exp\left(\int_0^t \hat{z}_0'(s, y) R^{-1}(s) dy(s) - \int_0^t \hat{z}'(s, y) R^{-1}(s) dy(s) - \frac{1}{2} \int_0^t \hat{z}_0'(s, y) R^{-1}(s) \hat{z}_0(s, y) ds\right)
\]

\[
\frac{1}{2} \int_0^t \hat{z}'(s, y) R^{-1}(s) \hat{z}(s, y) ds.
\]

Let \( R \) be the identity matrix with no loss of generality.

Since the required partials exist and are continuous, by Ito's differentiation rule \([23]\) we get

\[
d\lambda_0(t, y) = \lambda_0(t, y)\left(\hat{z}_0(t,y) - \hat{z}(t,y)\right)'R^{-1}(t)(dy(t) - \hat{z}(t,y)dt)
\]

\[
+ \frac{1}{2} \int_0^t \hat{z}_0'(s, y) R^{-1}(s) \hat{z}_0(s, y) ds + \frac{1}{2} \int_0^t \hat{z}'(s, y) R^{-1}(s) \hat{z}(s, y) ds
\]

\[
- \hat{z}(t,y) dt.
\]

To show uniqueness, let \( \lambda_1 \) and \( \lambda_2 \) be two solutions of 5.3, then

\[
d(\lambda_1/\lambda_2) = \frac{\lambda_2 d\lambda_1 - \lambda_1 d\lambda_2}{\lambda_2^2}.
\]

By 5.3 it is clear that \( \lambda_2 d\lambda_1 - \lambda_1 d\lambda_2 = 0 \). By using 5.1 and stochastic calculus,

\[
\lambda_1 d\lambda_2 dt = \lambda_1 \hat{z}_0'(t,y) - \hat{z}(t,y)\left(\hat{z}_0(t,y) - \hat{z}(t,y)\right)'(\hat{z}_0(t,y) - \hat{z}(t,y)) dt.
\]

We, therefore, have

\[
d(\lambda_1/\lambda_2) = 0 \Rightarrow \frac{\lambda_1(t)}{\lambda_2(t)} - \frac{\lambda_1(0)}{\lambda_2(0)} = 0 \quad \text{for all } t \in [0, T],
\]

or

\[
\lambda_2(t) = \lambda_2(0) \lambda_1(t)
\]

for all \( t \in [0, T] \).

So if \( \lambda_1(0) = \lambda_2(0) \), we must have \( \lambda_1(t) = \lambda_2(t) \) for all \( t \in [0, T] \). Detailed arguments can be found in \([13]\).

From the stochastic differential equation that the normalized conditional probability density satisfies in Theorem 5.1, it is now easy to find a stochastic differential representation for the \( a \) posteriori probability in \( P(A_1 \times A_2 \mid Y = y) \).

**Theorem 5.2** Let the assumptions of Corollary 4.3 hold. For a fixed parameter set \( A_1 \), the \( a \) posteriori probability

\[
r_t(\lambda_1, y) = \frac{\pi(\lambda_1, y)}{\pi(\lambda_1, y) + \pi(\lambda_2, y)}
\]

admits the following differential

\[
dr_t(\lambda_1, y) = \frac{\lambda_1(t)}{\lambda_2(t)} - \frac{\lambda_1(0)}{\lambda_2(0)} = 0 \quad \text{for all } t \in [0, T],
\]

\[
\lambda_2(t) = \lambda_2(0) \lambda_1(t)
\]

for all \( t \in [0, T] \).

Proof: See appendix.

In the next section, we shall specialize the results to the case where the parameter space is countably infinite.

6. Discrete Parameter Case

The real advantage of parameter conditioned approach to estimation occurs when the parameter space is finite or countably infinite since the solution may then be readily implemented on a digital computer. This will be the case when the parameter has a discrete distribution or it has been suitably quantized to be put on a digital computer. The parameter adaptive approach will of course be more rewarding when the parameter conditioned estimates can be easily obtained. We now give the results for the discrete parameter case.

**Theorem 6.1**

Let the assumptions of Theorem 4.2 hold. If the parameter space is countably infinite (or finite), let

\[
\Omega = \{0, \alpha_2, ..., \alpha_n, ...\}
\]

then

\[
\lambda d\lambda_1 d\lambda_2 = \frac{\lambda_1(t)}{\lambda_2(t)} - \frac{\lambda_1(0)}{\lambda_2(0)} = 0 \quad \text{for all } t \in [0, T],
\]

or

\[
\lambda_2(t) = \lambda_2(0) \lambda_1(t)
\]

for all \( t \in [0, T] \).
where
\[ z_0(t, \gamma) = E(x(\gamma) | \gamma, \theta) \]
and the R-N derivative is given by
\[ z_1(t, \gamma) = \frac{d}{dt} E(x(\gamma) | \gamma, \theta) \]
So that
\[ dP = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]

**Proof:** Follows from Theorem 4.2 by integrating with respect of \( P_1 \) over \( \theta \).

**Corollary 6.6** Under the assumption of Theorem 6.1, for all \( \theta_j \in \Omega, P_1 T^j \ll P Y^1, \) and the R-N derivative is given by
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]
\[ \frac{dP}{dP} = \frac{dP}{dP} \]

**Theorem 6.3**
Under the assumptions of Theorem 5.1 and Theorem 6.1 the a posteriori probability for a fixed value \( \theta \) of the parameter admits the following stochastic differential for
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]
\[ \frac{d}{d\gamma} (\theta, \gamma) = P(\theta_1, P_1) \frac{dP}{dP} \]

**Example 1** Consider the nonlinear stochastic dynamic system described by the following stochastic differential equations
\[ dx = \sigma \frac{d}{d\gamma} (\theta, \gamma) dt + dw \]
\[ dy = \sigma \frac{d}{d\gamma} (\theta, \gamma) dt + dw \]

where \( \sigma_0(t) \) is a Gaussian random variable with mean \( \mu \) and variance \( \sigma_0(t) \) and \( \sigma(0) \) is an arbitrary random variable with a discrete distribution \( \nu \). The Wiener processes and \( v \) and \( u \) have zero mean and unit variance. Also, \( \nu(t), \nu(v), \nu(t), \) and \( \nu(t) \) are independent. By applying Theorems 6.1 and 6.4, with \( \sigma(0) \) as the parameter we get the least squares estimate for \( \nu(t) \)
\[ \nu(t) = \nu \]

with the observation model
\[ dy = \sigma \frac{d}{d\gamma} (\theta, \gamma) dt + dw \]

where \( \sigma_0(t) \) is a Gaussian random variable with mean \( \mu \) and variance \( \sigma_0(t) \) and \( \sigma(0) \) is an arbitrary random variable with a discrete distribution \( \nu \). The Wiener processes and \( v \) and \( u \) have zero mean and unit variance. Also, \( \nu(t), \nu(v), \nu(t), \) and \( \nu(t) \) are independent. By applying Theorems 6.1 and 6.4, with \( \sigma(0) \) as the parameter we get the least squares estimate for \( \nu(t) \)
\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Kalman-Bucy filter

\[ \frac{d}{d\gamma} \nu(t) = \frac{d}{d\gamma} \nu(t) + \frac{d}{d\gamma} \nu(t) \]

where \( \nu(t) \) and \( \nu(0) \) are independent normal variables with a distribution
\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Riccati equation with \( \nu(t) = \nu \) and \( \nu(0) \) is an arbitrary random variable with a discrete distribution \( \nu \). The Wiener processes and \( v \) and \( u \) have zero mean and unit variance. Also, \( \nu(t), \nu(v), \nu(t), \) and \( \nu(t) \) are independent. By applying Theorems 6.1 and 6.4, with \( \sigma(0) \) as the parameter we get the least squares estimate for \( \nu(t) \)
\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Kalman-Bucy filter

\[ \frac{d}{d\gamma} \nu(t) = \frac{d}{d\gamma} \nu(t) + \frac{d}{d\gamma} \nu(t) \]

where \( \nu(t) \) and \( \nu(0) \) are independent normal variables with a distribution
\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Riccati equation with \( \nu(t) = \nu \) and \( \nu(0) \) is an arbitrary random variable with a discrete distribution \( \nu \). The Wiener processes and \( v \) and \( u \) have zero mean and unit variance. Also, \( \nu(t), \nu(v), \nu(t), \) and \( \nu(t) \) are independent. By applying Theorems 6.1 and 6.4, with \( \sigma(0) \) as the parameter we get the least squares estimate for \( \nu(t) \)
\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Kalman-Bucy filter

\[ \frac{d}{d\gamma} \nu(t) = \frac{d}{d\gamma} \nu(t) + \frac{d}{d\gamma} \nu(t) \]

where \( \nu(t) \) and \( \nu(0) \) are independent normal variables with a distribution
\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Riccati equation with \( \nu(t) = \nu \) and \( \nu(0) \) is an arbitrary random variable with a discrete distribution \( \nu \). The Wiener processes and \( v \) and \( u \) have zero mean and unit variance. Also, \( \nu(t), \nu(v), \nu(t), \) and \( \nu(t) \) are independent. By applying Theorems 6.1 and 6.4, with \( \sigma(0) \) as the parameter we get the least squares estimate for \( \nu(t) \)
\[ \nu(t) = \nu \]

where \( \nu(t) \) is the solution of the Kalman-Bucy filter

\[ \frac{d}{d\gamma} \nu(t) = \frac{d}{d\gamma} \nu(t) + \frac{d}{d\gamma} \nu(t) \]

where \( \nu(t) \) and \( \nu(0) \) are independent normal variables with a distribution
\[ \nu(t) = \nu \]
\[
\begin{align*}
\text{Example 2: In this example, we shall consider the} \quad \text{random telegraph signal} \quad z(t), \text{with values } \pm 1 \text{ and transition density, a jump/second where } a \text{ is a} \\
\text{discrete random variable with a range of values } \{a_1, a_2, \ldots, a_M\} \text{ and a priori probability distribution} \\
(P_1(a_1), P_2(a_2), \ldots, P_M(a_M)). \text{ The observation model is taken to be} \\
\end{align*}
\]

\[
\begin{align*}
dy(t) &= z(t)dt + dv \\
\text{where } v \text{ is a unit Wiener process. By applying} \\
\text{Theorem 6.1 and 6.4 we get} \\
\end{align*}
\]

\[
\begin{align*}
\text{Remark 1} \quad v(B) &= \int v_0(B) dP_1(0). \quad (A-6) \\
\text{This follows from Fubini's theorem and the fact} \quad \text{that } \quad v_0(B) = (Y^{-1}(B), v(B) \text{ on } (V, \mathcal{B}) \text{ by} \\
\end{align*}
\]

\[
\begin{align*}
\text{Remark 2} \quad y_\theta(n) | Y_{\theta}(n) &= y \quad (A-3) \\
\text{Remark 3} \quad E_\theta(y_\theta(n) | Y_{\theta}(n) &= y) \quad (A-4) \\
\text{So we are to prove that} \\
\end{align*}
\]

\[
\begin{align*}
\text{E_\theta(y_\theta(n) | Y_{\theta}(n) &= y) \quad (A-4) \\
\text{Since } \quad \eta_\theta(0) &= P_1(a_0). \\
\text{Appendix} \\
\end{align*}
\]

\[
\begin{align*}
8. \text{Conclusions} \\
\text{The problem of optimal parameter adaptive esti-} \\
\text{mation for random processes is formulated in the} \\
\text{Bayesian framework. A general R-N derivative repre-} \\
\text{sentation for the least squares estimate of a ran-} \\
\text{dom variable on a product space is derived. The} \\
\text{representation theorem is applied to the optimal} \\
\text{parameter adaptive estimation problem for random} \\
\text{processes to find the least squares estimate of the} \\
\text{observed signal and the a posteriori probability} \\
\text{of parameter conditioned estimates. The stochastic} \\
\text{differential equation for the a posteriori proba-} \\
\text{bility is derived. The results are specialized to} \\
\text{the case where the parameter has a discrete distri-} \\
\text{bution. The approach is illustrated by simple ex-} \\
\text{amples.} \\
\end{align*}
\]

\[
\begin{align*}
\text{Appendix} \\
\text{The appendix contains the proofs that are not} \\
\text{given in the text. To prove Theorem 3.2, we shall} \\
\text{need the following lemma.} \\
\text{Lemma 1.1 Let } (\Omega_1, A_1, P_1) \text{ be a probability space} \\
\text{and } (\Omega_2, A_2, P_2) \text{ be a measurable space where } S \text{ is gen-} \\
\text{erated by a countable class of sets. Suppose that} \\
\text{for each } \theta \text{ in } \Omega_1 \text{ we are given a measure } \nu_\theta(y) \text{ on} \\
\text{each } \Omega_2 \text{ such that, for fixed } B \in A_2, \nu_\theta(B) \text{ is mea-} \\
\text{surable in } \theta. \text{ If there exists a set } N_1 \text{ in } A_1 \text{ of} \\
\text{measure zero such that, for all } \theta \text{ in } \Omega_1 - N_1, \ldots, \nu_\theta(y) \text{ on } (\Omega_2, A_2), \text{ then} \\
\text{there is an } A_1 \times S \text{ measurable function } f \text{ such that} \\
\text{f(0, y) = } \frac{d\nu_\theta}{d\nu_0} (y) \quad (A-9) \\
\text{where } \nu_0(y) \text{ is the measure on } (\Omega_2, A_2), \text{ then} \\
\text{it follows that} \\
\text{0 < } \frac{d\nu_0}{d\nu_0} (y) < \nu_0(y) \quad (A-9) \\
\text{So} \\
\end{align*}
\]
At \( R \) measurable, so that \( \frac{dv}{du} \) can be chosen to be a \( A_1 \times B \) measurable function.

**Remark 3**

\[
\frac{dv}{du} (y) = f_{\theta_1} E_2 (g_0 (n) | Y_0 (n) = y) \quad \frac{du}{dv} (y) \quad dP_1 (n). \tag{A-11}
\]

From Remark 1 we have

\[
v (B) = f_{\theta_1} \left[ f_{\theta_2} (g_0 (n) | Y_0 (n) = y) \right] \quad dP_2 (n) \quad dP_1 (0). \tag{A-12}
\]

By Lemma 1.1 and Remark 2, the terms \( E_2 (g_0 | Y_0 = y) \), \( E_2 (g_0 | Y_0 = y) \), and \( \frac{du}{dv} \) can be chosen to be \( A_1 \times B \) measurable. By applying Fubini's theorem for non-negative functions, + we can write that \( v (B) \)

\[
= f_{\theta_1} \left[ f_{\theta_2} (g_0 (n) | Y_0 (n) = y) \right] \quad dP_2 (n) \quad dP_1 (0).
\]

Since the difference is well-defined, at least one of the integrals should be finite and, consequently, the integral corresponding to that integral must be finite valued almost everywhere \( P_1 \times \omega_0 \). So the following term is well-defined:

\[
E_2 (g_0 | Y_0 = y) \quad \frac{du}{dv} (y) = E_2 (g_0 | Y_0 = y) \quad \frac{du}{dv} (y).
\]

From Theorem 6.5.2 in [3] this expression

\[
= E_2 (g_0 | Y_0 = y) \quad \frac{du}{dv} (y). \tag{A-18}
\]

So that

\[
v (B) = f_{\theta_1} \left[ f_{\theta_2} (g_0 (n) | Y_0 (n) = y) \right] \quad dP_2 (n) \quad dP_1 (0) \quad d_{\omega_0} (y).
\]

Since the term in brackets is \( B \)-measurable by Fubini's theorem and since the \( R \)-derivative is unique a.e.,

\[
\frac{dv}{du} = f_{\theta_1} E_2 (g_0 | Y_0 = y) \quad \frac{du}{dv} (y) \quad dP_1 (0) \quad [\omega_0]. \tag{A-20}
\]

**Remark 4**

\[
\frac{dv}{du} (y) = f_{\theta_1} \frac{du}{dv} (y) \quad dP_1 (n). \tag{A-21}
\]

This follows from Remark 3 by setting \( \omega = \nu (1) = \nu_1 \times \omega_0 (0, n) \) where \( \omega_1 \times \omega_2 \) is the characteristic function of the set \( \Omega \times \Omega_2 \). This can be seen as follows

\[
v (B) = f_{\theta_1} (B) \quad \frac{du}{dv} (y) \quad dP_2 (n) \quad dP_1 (0), \tag{A-22}
\]

Now since \( v < u \) and \( u < \omega_0 \) (Remark 4), we have

\[
\frac{dv}{du} = \frac{du}{dv} \quad [\omega_0], \tag{A-23}
\]

Since

\[ 0 < \frac{dv}{du} \leq [\omega_0], \tag{A-24} \]

we have

\[
\frac{dv}{du} = \frac{dv}{du} \quad [\omega_0]. \tag{A-25}
\]

The result now follows from Remark 3 and Remark 4.

**Proof of Corollary 3.3**

Let \( h_0 (y) \) denote the right hand side of 3.3. From Theorem 3.2 we have

\[
E (g | Y = y) = f_{\theta_1} E_2 (g_0 | Y_0 = y) \quad h_0 (y) \quad dP_1 (0). \tag{A-26}
\]

With \( g = 1_1 A_1 \times \omega_2 (0, n) \) where \( A_1 \subset A_1 \)

\[
E (1_1 A_1 \times \omega_2 | Y = y) = f_{\theta_1} E_2 (g_0 | Y_0 = y) \quad h_0 (y) \quad dP_1 (0). \tag{A-27}
\]

This proves the last assertion in the statement of the corollary.

Integrating both sides w.r.t. \( Y^{-1} \)

\[
f_B E (1_1 A_1 \times \omega_2 | Y = y) \quad d_{\omega_1} (y) = f_{\theta} \left[ f_{\theta_1} h_0 (y) \quad dP_1 (0) \right] \quad d_{\omega_1} (y). \tag{A-28}
\]

Applying the definition of the conditional expectation to the L.H.S. and Fubini's theorem to the right hand side \( h_0 (y) \) is \( A_1 \times B \) measurable and non-negative, we get

\[
\int f_Y (y) \quad 1_1 A_1 \times \omega_2 \quad dF = \int f_{\theta_1} h_0 (y) \quad dP_1 (0). \tag{A-29}
\]

Using Fubini's theorem on the left hand side gives

\[
\int f_{\theta_2} (Y^{-1} (B)) \quad dF (0) = \int f_{\theta_1} h_0 (y) \quad dP_1 (0). \tag{A-30}
\]

So for \( P_1 \)-almost everywhere on \( \Omega_1 \) we have

\[
h_0 (y) = f_{\theta_2} (Y^{-1} (B)) \quad [\omega]. \tag{A-31}
\]

Since \( h_0 (y) \) is \( B \)-measurable for \( P_1 \)-almost all \( 0 \), it must be that

\[
\frac{dv}{du} = h_0 (y). \tag{A-32}
\]

**Proof of Theorem 5.2**

There is no loss of generality in assuming that \( R \) is the identity matrix. We shall first show that the following equality holds almost surely.
\[ f_{\Omega_1} \left[ \int_0^t \lambda_0(s,y) \dot{z}_0(s,y) \, dy(s) \right] \, dP_1(0) \]

(A-33)

By applying the differential rule on the representation of \( \lambda_0 \) in Theorem 4.2, we have

\[ \lambda_0(t,y) - \lambda_0(0,y) = \int_0^t \lambda_0(s,y) \dot{z}_0(s,y) \, dy(s). \]  

(A-34)

Integrating with respect to \( P_1 \) over \( \Omega_1 \) we get

\[ \Lambda(t,y) - \Lambda(0,y) = f_{\Omega_1} \left[ \int_0^t \lambda_0(s,y) \dot{z}_0(s,y) \, dy(s) \right] \, dP_1(0), \]  

(A-35)

On the other hand, by an application of the differential rule on the representation of \( \Lambda(t,y) \) in Corollary 4.4 we get

\[ \Lambda(t,y) - \Lambda(0,y) = \int_0^t \Lambda_0(s,y) \dot{z}_0(s,y) \, dy(s). \]  

(A-36)

Also, by Theorem 4.2, we have

\[ f_{\Omega_1} \dot{z}_0(s,y) \, dP_1(0) \]

(A-37)

So that the right hand sides of A-36 and A-35 must be equal almost surely which proves A-33.

Proof of Theorem 6.4: The existence and uniqueness of this type of infinite order stochastic differential equation has been studied by Rozovski and Shiryaev [28]. To prove Theorem 6.4, we shall apply Theorem 1 in [28]. To this end, we have to show that \( \pi_t(0_1,y) \) is in class M. It is clear that, for each \( i \), \( \pi_t(0_1,y) \) is continuous with probability one. This follows from the continuity property of the Ito integral which states that the points of continuity of the random process defined by the Ito integral are the points of continuity of the Wiener process with respect to which the integral is defined [4]. Measurability conditions are satisfied due to the definition of \( \pi_t(0_1,y) \).

Also

\[ P \left\{ \int_0^t \sum_{i=1}^{\infty} \pi_t(0_i,y) \pi_t(0_i,y) \right\} = 1 \]

Therefore, \( \pi_t(0_1,y) \) is in class M. By Theorem 1 in [28], the system of equations 6.9 has a unique solution in the sense if \( \pi \) and \( \psi \) are two solutions with \( \pi_t(0_1,y) = \psi_t(0_1,y) \), \( i = 1, 2, \ldots \), then

\[ P \left\{ \sup_{t} \sup_{0 \leq s \leq t} \mid \pi_t(0_i,y) - \psi_t(0_i,y) \right\} = 0. \]

References


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7. **Abstract**
   Current adaptive sampling schemes that can be used in sampled-data systems with variable rate sampling do not guarantee the stability of the resulting closed-loop system. Therefore, this study has been undertaken with the objective of finding a stable control law for multirate sampled-data systems. The problem is formulated such that the sampling interval is selected from a set of fixed number of sampling intervals. A necessary and sufficient condition under which these types of systems can be stabilized is given. For a certain subclass of these types of systems, a sampling selection algorithm is given which results in a stable closed-loop system.

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OPTIMAL PARAMETER ADAPTIVE ESTIMATION OF STOCHASTIC PROCESSES

by

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(ABSTRACT)

This study is concerned with the simultaneous detection and least squares estimation of vector random processes. The problem is formulated in the following context: A random process, out of a countably infinite collection of (not necessarily Gaussian) vector random processes with known distributions, is observed with additive white Gaussian noise. The a priori probability, that a specific random process will be observed, is specified for each one in the collection. The least squares estimate of the random process that is being observed is to be found in terms of the hypothesis conditioned estimates.

It is shown that the best estimate is the linear combination of the hypothesis conditioned estimates weighted by the a posteriori probabilities of the hypotheses conditioned on the observations. A Radon-Nikodym derivative representation is derived for the a posteriori probability by using the specific structure of the product probability measure for this problem. It is shown that this Radon-Nikodym derivative can be expressed in terms of the Radon-Nikodym derivatives of measures induced by the random processes in the collection with respect to Wiener measure. By using the recent results on likelihood functions, an expression for the a posteriori probability is found in terms of the conditioned estimates. In this connection, an extended version of the partition theorem of parameter adaptive estimation
is proved. The unique stochastic differential equation, that each a posteriori probability satisfies with its associated a priori probability as the initial condition, is derived for the case of finitely many hypotheses along with an expression for the conditional error covariance in terms of the hypothesis conditioned error covariances.

The results are applied to the parameter adaptive estimation problem in linear continuous and discrete stochastic dynamic systems. In the continuous case, the solution is also obtained through an alternate approach using nonlinear filtering theory. An application of the theory to the design of a digital flight control system which is tolerant of sensor failures is presented with real-time hybrid computer simulation results. A review of random processes and statistical decision theory is also included.