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EFFECT OF DIGITALLY COMPUTED DRIVES ON PERFORMANCE OF CONTINUOUS LINEAR SYSTEMS

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16. Abstract

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EFFECT OF DIGITALLY COMPUTED DRIVES ON PERFORMANCE OF CONTINUOUS LINEAR SYSTEMS

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SUMMARY

The dependence of the Bode response upon digital sample rate for first- and second-order linear continuous systems driven by a digital computer is derived. Open-loop lead compensation introduced within the digital computer in order to increase the system bandwidth is also examined in terms of derived Bode responses. This introduction of lead terms within the digital computer (open-loop compensation) is shown to be effective at operating frequencies below the Nyquist frequency. Indeed, in most practical applications, empirical determination of these lead coefficients appears to be a sufficient compensation method.

INTRODUCTION

The introduction of lead terms into the drive equations of an external hardware device in order to extend the bandwidth of the device is a common practice in flight-simulator systems. The devices are generally position-driven servo systems (although the techniques presented herein are not restricted to position-driven systems) used to present visual or motion cues to the pilot of the flight simulator, and compensation is desired to lower the time delays involved. The practice, when carried out in the continuous domain (i.e., within an analog-computer simulation), amounts to open-loop continuous compensation. The compensator has to attempt to act like an inverse plant (the plant being the hardware device). Such compensation had been successfully used in many simulators and was generally understood within the simulation community.

With the advent of digital simulation, the use of lead terms was continued (refs. 1 and 2). However, unlike the use in analog simulation in which the compensation was effective within the linear operating range of the servo, the use in digital simulation was less effective, particularly at low iteration rates. Moreover, the problem is not generally well understood within the simulation community. Indeed, even the case of no compensation (i.e., the effects of a digital drive on an analog-system response) is not well understood.

This paper presents the derivation of the dependence of the Bode response upon digital sample rate for the following cases:

Effects on first-order linear systems

No compensation
First-order compensation

Effects on second-order linear systems

No compensation
First-order compensation
Second-order compensation

Bode plots for each case at various sample rates are also presented. These plots may be used to determine the increased bandwidth obtainable for digitally driven first- or second-order linear systems through the use of open-loop lead compensation at selected sample rates. In actual practice, however, nonlinearities of the servo systems, such as velocity or acceleration limits, are encountered, and the increase in bandwidth is not as great as that of the linear system.

SYMBOLS

A	zero of Z-transfer function for output of first-order system $GG_{OH}(z)$
a	parameter of first-order linear model of position-driven servo, 1/sec
B,C	parameters that determine the zero of Z-transfer function for output of second-order system $GG_{OH}(z)$
c(nT)	system output value at discrete time nT
c(s)	Laplace transform of system output
c(t)	system output as continuous function of time
c(z)	Z-transform of system output
$c_{ss}(nT)$	system output value at discrete time nT during steady state
$c_{ss}(t)$	system output as continuous function of time during steady state
D	zero of Z-transfer function for digitally compensated second-order system output c(z)
D_1	zero of Z-transfer function for first-order digitally compensated second-order system output c(z)
D_2	zero of Z-transfer function for second-order digitally compensated second-order system c(z)
dz	differential of z
E_1	coefficient of sine term of $c_{ss}(nT)$ of first-order system
E_2	coefficient of sine term of $c_{ss}(nT)$ of second-order system
F_1	coefficient of cosine term of $c_{ss}(nT)$ of first-order system

F_2	coefficient of cosine term of $c_{ss}(nT)$ of second-order system
G_1	coefficient of $c_{ss}(nT)$ of first-order system
G_2	coefficient of $c_{ss}(nT)$ of second-order system
$G(s)$	Laplace transfer function for linear model of position-driven servo
$G_N(s)$	Laplace transfer function of notch filter
$G_{OH}(s)$	Laplace transfer function of zero-order hold
$GG_{OH}(z)$	Z-transform of $G(s)G_{OH}(s)$
$H(s)$	Laplace transfer function of continuous compensator
$H(z)$	Z-transform of $H(s)$
$H^o(s)$	Laplace transfer function of optimal compensator
I	imaginary part of complex number
i	index of simple poles of a Z-transfer function
j	$= \sqrt{-1}$
k	number of simple poles of $c(z)$
n	index of sample period
p	dummy variable for z
p_i	simple pole of Z-transfer function
R	real part of complex number
r	parameter related to damping parameter of second-order model of position-driven servo, $\sqrt{1 - \xi^2}$
s	Laplace operator
T	sample period, sec
t	time, sec
$u(s)$	Laplace transform of drive signal without compensation
$uH(z)$	Z-transform of $u(s)H(s)$
x	normalized variable combining sample period and natural frequency of second-order model of position-driven servo

y	normalized frequency variable
$Z\{\ \}$	Z-transform of function within braces
z	Z-transform operator
α,β	dummy variables
ξ	damping parameter of second-order model of position-driven servo
ξ_N,ξ_2	damping parameters of notch filter
ω	frequency of sinusoidal drive signal, rad/sec
ω_N	notch frequency of notch filter, rad/sec
ω_n	natural frequency of second-order model of position-driven servo, rad/sec
ω_2	secondary frequency of notch filter, rad/sec

Abbreviations:

DAC	digital-to-analog converter
ZOH	zero-order hold

SAMPLED DATA MODELS

Figure 1 is a block diagram which presents the general form of the sampled data model used in deriving the steady-state sinusoidal sequence response of the digitally compensated position-driven servo. The drive signal is generated within the digital computer. Examples of such drives are an aircraft altitude signal for a terrain model board, a target image azimuth angle for an image projector gimbal, and a motion base vertical position. Quite often, velocity and acceleration terms exist naturally within the simulation program, and the necessary terms therefore exist naturally for lead compensation. When these terms do not exist, it is possible to derive and include new equations without resorting to differentiation of the position signals. (Intuitively, digital differentiation would not add lead information since the process is dependent only on past and present values of position.)

Once the drive signal is constructed within the digital computer, it is output to the position-driven servo through a digital-to-analog converter (DAC), a sample-hold device, at a fixed iteration rate of $1/T$. In many applications it is necessary to insert an analog notch filter between the DAC and the servo to remove the stair-stepping effect of the DAC (regardless of whether lead compensation is being used or not). The notch filter, with the notch set at a frequency of $1/T$ Hz, will be ignored for the purposes of this paper, even though it has some effect on the servo response. The rationale for ignoring this effect is discussed in a later section.

The position-driven servo is modeled as a linear first- or second-order unity-gain system by ignoring nonlinear characteristics such as amplitude, velocity and acceleration limits, friction, stiction, and so forth.

The compensation method considered here has been the application of a continuous method to a discrete system (i.e., the compensator has attempted to cancel the pole or poles of the continuous system only). Figures 2, 3, and 4 show the specific forms of the compensator for the first- and second-order systems.

Salzer, in reference 3, proves that a sinusoidal input sequence of angular frequency ω (where ω is less than the Nyquist frequency), when operated upon by a discrete transfer function, results in a sinusoidal output sequence which is of the same frequency and which has an amplitude and phase relationship to the input. The derivation of these amplitude and phase relationships for the first- and second-order systems, both without compensation and with compensation, are presented in appendixes A and B.

BODE RESULTS

Bode responses are presented in figures 5 to 12. The amplitudes and phases of the steady-state sequence responses to sinusoidal sequence inputs are given as a function of normalized frequency (normalized to the natural frequency of the servo) for various normalized sample periods (normalized to cycles of the natural frequency of the servo per sample).

First-Order Linear System

Figure 5 presents the Bode plots at various sample periods for the continuous first-order linear system, the digitally driven first-order linear system without compensation, and the first-order compensated system.

No compensation.- The differences in response between the digitally driven system without compensation and the continuous system occur in both amplitude and phase. However, the major differences occur in phase; in fact, the phase lag of a zero-order hold alone is $-180(\omega T/2\pi)$ deg.

First-order compensation.- As may be seen in figure 5, compensation of the digitally driven system can be very effective. Indeed, at iteration rates above five samples per cycle ($aT/2\pi < 0.2$), the compensated response is superior to the continuous system response. However, operating ranges must be restricted to frequencies below the Nyquist frequency $\left(\frac{\omega}{a} < \frac{1}{2}\frac{2\pi}{aT}\right)$.

Second-Order Linear System

Normalization to remove the effects of the damping parameter of the second-order linear system on the Bode response was not possible. Therefore, three damping factors are treated $\left(\zeta = \sqrt{2}/2, 0.85, \text{ and } 0.45\right)$.

Figures 6, 7, and 8 present the Bode plots for the continuous second-order linear system with damping factors of $\sqrt{2}/2$, 0.85, and 0.45, respectively, and for the digitally driven system without compensation and with first-order compensation. The plots are presented at various sample periods.

No compensation.- The differences in response, shown in figures 6, 7, and 8, between the digitally driven system without compensation and the continuous system occur in both amplitude and phase. However, the major differences occur in phase.

If an attempt were to be made to estimate the frequency and damping of a linear second-order system by driving the system sinusoidally from a digital computer in order to obtain the Bode response, the estimates would be low. For example, the estimates that could be obtained from the compensation cases of figure 6 are presented in table I.

First-order compensation.- As may be seen in figures 6, 7, and 8, improvements in phase lag obtained through compensation are accompanied by increased gains. Higher iteration rates (lower values of $\omega_n T/2\pi$) improve the phase response but incur increased gains (greater than unity). It appears from comparing the three figures that first-order compensation of second-order systems will be more effective for highly damped systems, thereby giving more improvement in phase lag with less increase in gain.

Second-order compensation.- Unlike first-order compensation, second-order compensation of the second-order system yields Bode responses that are practically independent of the damping parameter. Figure 9 presents the Bode plots for the second-order digital compensation of the second-order linear system. For the three damping factors selected $\left(\sqrt{2}/2,\ 0.85,\ \text{and}\ 0.45\right)$, the amplitude and phase responses overlap too closely to differentiate among them on the scales of figure 9. As may be seen from figure 9, second-order compensation obtains large improvements in both gain and phase lag for the iteration rates shown at operating frequencies below the Nyquist frequency $\left(\frac{\omega}{\omega_n} < \frac{1}{2}\frac{2\pi}{\omega_n T}\right)$.

NOTCH-FILTER EFFECTS

The insertion of an analog notch filter (ref. 4) between the DAC and the servo has been mentioned previously. The filter is of the form

$$G_N(s) = \left(\frac{\omega_2{}^2}{\omega_N{}^2}\right)^2 \frac{s^2 + 2\xi_N\omega_N s + \omega_N{}^2}{s^2 + 2\xi_2\omega_2 s + \omega_2{}^2}$$

The notch frequency ω_N is set at $2\pi/T$ while the secondary frequency ω_2 is usually set at about $1/3\ \omega_N$. Normalized to the scales of figures 5, 6, 7, 8, and 9, the preceding frequencies become $\omega_N/a = 2\pi/aT$, $\omega_N/\omega_n = 2\pi/\omega_n T$ and $\omega_2/a = 2\pi/3aT$, $\omega_2/\omega_n = 2\pi/3\omega_n T$. Therefore, even the secondary frequency approaches the Nyquist frequency $\left(\frac{1}{2}\frac{2\pi}{aT}\right.$ and $\left.\frac{1}{2}\frac{2\pi}{\omega_n T}\right)$ and makes a thorough analysis difficult and unnecessary. Figure 10 presents the Bode response of the typical notch filter. The effects of the notch filter on the response of the linear servo follow approximately the superposition theorem, and the filter affects both the uncompensated and the compensated system response in the same manner. Thus, the effectiveness of compensation, as discussed from figures 5 to 9, is practically unaffected by the inclusion of notch filters in the system. In any case, total system response may be approximated, if desired, through the use of the superposition theorem.

IMPROVED COMPENSATION

The compensation method considered here has been the application of a continuous method to a discrete system (i.e., the compensator has attempted to cancel the pole or poles of the continuous system only). An intuitively better compensator of the discrete system would be the inverse of the zero-order hold and the plant. In the continuous case,

$$H^o(s) = \frac{1}{G(s)}$$

In the discrete case,

$$H^o(s) = \frac{T}{G_{OH}(s)G(s)}$$

The realization of $H^o(s)$, even within a digital computer, is not possible. However, approximations are available. For example, the two-term series expansion for e^{sT} yields

$$\frac{T}{G_{OH}(s)} = \frac{sT}{1 - \frac{1}{e^{sT}}} = 1 + sT$$

and the three-term series expansion yields

$$\frac{T}{G_{OH}(s)} = \frac{1 + sT + \frac{s^2T^2}{2}}{1 + \frac{sT}{2}}$$

Use of even the two-term approximation increases the complexity of the compensation, however. A first-order plant would require second-order compensation:

$$H^o(s) = 1 + s\left(\frac{1}{a} + T\right) + s^2 \frac{T}{a}$$

and a second-order plant would require third-order compensation:

$$H^o(s) = 1 + s\left(\frac{2\xi}{\omega_n} + T\right) + s^2\left(\frac{1}{\omega_n^2} + \frac{2\xi T}{\omega_n}\right) + s^3\left(\frac{T}{\omega_n^2}\right)$$

Use of the three-term approximation involves even more complexity. For the first-order plant,

$$H^o(s) = \frac{1 + s\left(T + \frac{1}{a}\right) + s^2\left(\frac{T}{a} + \frac{T^2}{2}\right) + s^3\left(\frac{T^2}{2a}\right)}{1 + s\left(\frac{T}{2}\right)}$$

$$= 1 + s\left(\frac{1}{a} + \frac{T}{2}\right) + s^2\left(\frac{T}{2a} + \frac{T^2}{4}\right) + s^3\left(\frac{T^2}{4a} - \frac{T^3}{8}\right) + \ldots$$

and for the second-order plant,

$$H^o(s) = \frac{1 + s\left(T + \frac{2\xi}{\omega_n}\right) + s^2\left(\frac{1}{\omega_n^2} + \frac{T^2}{2} + \frac{2\xi T}{\omega_n}\right) + s^3\left(\frac{T}{\omega_n^2} + \frac{\xi T^2}{\omega_n}\right)}{1 + s\left(\frac{T}{2}\right)}$$

$$= 1 + s\left(\frac{2\xi}{\omega_n} + \frac{T}{2}\right) + s^2\left(\frac{1}{\omega_n^2} + \frac{\xi T}{\omega_n} + \frac{T^2}{4}\right) + s^3\left(\frac{T}{2\omega_n^2} + \frac{\xi T^2}{2\omega_n} - \frac{T^3}{8}\right) + \ldots$$

Examination of the responses shown in figures 5 and 9 for particular values of a, ω_n, and T may reveal, in most practical applications, that these increases in complexity are not warranted. However, the additional lead suggested by these approximations without a change of order, as shown in table II, can be analyzed with the previously developed sample-data methods.

Figure 11 presents the Bode plots for the first-order system with no compensation, and with the three different first-order compensators of table II. Figure 12 presents the comparable results for the second-order linear system with a damping parameter of $\sqrt{2}/2$.

The additional phase improvement presented in figures 11 and 12 is obtained from the additional lead, but at the expense of gain distortion. It is therefore recommended that, rather than attempting to cancel the system poles or to approximate the inverse hold-plant system, one should empirically determine the lead coefficients of the compensator, possibly by using the values of table II as starting values. Empirical determination of the coefficients may allow for response improvements in light of the nonlinearities of the servo system, in addition to improvements above and beyond those obtained from using estimates of the continuous system poles.

CONCLUDING REMARKS

The dependence of the Bode response upon digital sample rate for first- and second-order linear systems, both with and without open-loop compensation, has been derived. Bode plots that may be used to determine the increased bandwidth obtainable for digitally driven first- or second-order systems through lead compensation were then presented. The examination of these plots revealed the following:

The effects of driving a continuous first- or second-order system from a digital computer are manifested chiefly within the Bode response of the system as changes in phase. Attempts to estimate the natural frequency and damping of a second-order system from Bode responses obtained in such a manner (i.e., digitally driven) are likely to result in low estimates.

The introduction of lead terms into the drive equations within the digital computer so as to extend the bandwidth of the system has been shown to be effective at operating frequencies below the Nyquist frequency. Indeed, first-order lead compensation of first-order linear systems, and second-order lead compensation of second-order systems, give excellent improvements in both phase and gain response, particularly at iteration rates above five samples per cycle of the natural frequency of the system. First-order lead compensation of second-order systems gives improvement in phase response that is accompanied by increases in gain response (greater than unity).

The improvements in phase, with accompanying gain changes, may not be fully realized in practical applications since nonlinearities of the servo systems (such as velocity limits, acceleration limits, friction, etc.) often limit the possible increases in system bandwidth.

Attempts to approximate the inverse hold-plant discrete compensator may be unwarranted in terms of improved response. Empirical determination of lead coefficients for the compensator appears to be more practical.

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APPENDIX A

DERIVATION OF BODE RESPONSES FOR FIRST-ORDER LINEAR SYSTEM

This appendix derives Bode responses for the first-order linear system. The first-order system is to be solved for the case of no compensation and the case of first-order compensation.

No Compensation

The Bode response of a first-order open-loop linear system driven by a digital-to-analog converter (DAC) may be obtained directly from the substitution of $z = e^{j\omega T}$ into the Z-transform of the system (ref. 5) as follows:

Given

$$G(s) = \frac{a}{s + a}$$

$$G_{OH}(s) = \frac{1 - e^{-sT}}{s}$$

then

$$GG_{OH}(z) = Z\left\{G(s)G_{OH}(s)\right\} = \frac{z - 1}{z} Z\left(\frac{1}{s} - \frac{1}{s + a}\right) = \frac{1 - e^{-aT}}{z - e^{-aT}}$$

Substitution of $z = e^{j\omega T}$ yields

$$GG_{OH}(e^{j\omega T}) = \frac{(1 - e^{-aT})}{(e^{j\omega T} - e^{-aT})} \frac{(e^{-j\omega T} - e^{-aT})}{(e^{-j\omega T} - e^{-aT})}$$

$$= \frac{(1 - e^{-aT})(\cos \omega T - e^{-aT} \cancel{- j \sin} \omega T)}{1 - 2e^{-aT} \cos \omega T + e^{-2aT}}$$

Since

$$\text{Amplitude} = \sqrt{R^2 + I^2}$$

and

$$\text{Phase} = \tan^{-1}\left(\frac{I}{R}\right)$$

then, finally,

$$\text{Amplitude} = \frac{1 - e^{-aT}}{1 - 2e^{-aT}\cos \omega T + e^{-2aT}} \sqrt{\cos^2 \omega T - 2e^{-aT}\cos \omega T + e^{-2aT} + \sin^2 \omega T}$$

$$= \frac{1 - e^{-aT}}{\sqrt{1 - 2e^{-aT}\cos \omega T + e^{-2aT}}}$$

and

$$\text{Phase} = \tan^{-1}\left(\frac{-\sin \omega T}{\cos \omega T - e^{-aT}}\right)$$

First-Order Compensation

A block diagram of the model to be used in deriving the Bode response of the first-order digital compensation of the first-order linear system is shown in figure 2. The drive signal is a sine wave, and the compensator is the inverse of the plant. The Z-transform of the system output is

$$c(z) = uH(z)GG_{OH}(z)$$

and is derived in the following manner:

Given

$$u(s) = \frac{\omega}{s^2 + \omega^2}$$

$$H(s) = \frac{s + a}{a}$$

then

$$uH(z) = Z\left\{\left(\frac{\omega}{a}\right)\frac{s + a}{s^2 + \omega^2}\right\}$$

Now let $y = \omega/a$ and $x = aT$ so that

$$uH(z) = \frac{yz\left(z - \cos xy + \frac{1}{y}\sin xy\right)}{(z - e^{jxy})(z - e^{-jxy})}$$

Given

$$G(s) = \frac{a}{s + a}$$

$$G_{OH}(s) = \frac{1 - e^{-sT}}{s}$$

then

$$GG_{OH}(z) = \frac{z - 1}{z}\,Z\left\{\frac{a}{s(s + a)}\right\} = \frac{1 - e^{-x}}{z - e^{-x}}$$

Now let $A = -\cos xy + \frac{1}{y}\sin xy$ so that, finally,

$$c(z) = \frac{y(1 - e^{-x})z(z + A)}{(z - e^{-x})(z - e^{jxy})(z - e^{-jxy})}$$

The Bode response for the system will be obtained from the inverse Z-transform of the system output $c(z)$. The method of solution used to solve the no-compensation case, that is, substitution of $z = e^{j\omega T}$ into $H(z)GG_{OH}(z)$, would not yield the correct results for this system; rather, the results would be applicable to this system with an additional sampler placed ahead of $H(s)$. The inverse Z-transform is obtained by use of the relation (from ref. 5)

$$c(nT) = \frac{1}{2\pi j}\oint c(z)z^{n-1}dz = \sum_{i=1}^{k}\left[z^{n-1}(z - p_i)c(z)\right]_{z=p_i}$$

APPENDIX A

Inversion of

$$c(z) = \frac{y(1 - e^{-x})z(z + A)}{(z - e^{-x})(z - e^{jxy})(z - e^{-jxy})}$$

results in

$$c(rT) = y(1 - e^{-x})\left\{\left[\frac{z^n(z + A)}{(z - e^{jxy})(z - e^{-jxy})}\right]_{z=e^{-x}} + \left[\frac{z^n(z + A)}{(z - e^{-x})(z - e^{-jxy})}\right]_{z=e^{jxy}} + \left[\frac{z^n(z + A)}{(z - e^{-x})(z - e^{jxy})}\right]_{z=e^{-jxy}}\right\}$$

$$= y(1 - e^{-x})\left[\frac{e^{-nx}(e^{-x} + A)}{(1 - 2e^{-x} \cos xy + e^{-2x})} + \frac{e^{jnxy}(1 + Ae^{-jxy} - e^{-x}e^{jxy} - Ae^{-x})}{2j \sin xy(1 - 2e^{-x} \cos xy + e^{-2x})} - \frac{e^{-jnxy}(1 + Ae^{jxy} - e^{-x}e^{-jxy} - Ae^{-x})}{2j \sin xy(1 - 2e^{-x} \cos xy + e^{-2x})}\right]$$

$$= y(1 - e^{-x})\left[\frac{e^{-nx}(e^{-x} + A)}{1 - 2e^{-x} \cos xy + e^{-2x}} + \frac{\sin nxy(1 - Ae^{-x}) + A \sin (n - 1)xy - e^{-x} \sin (n + 1)xy}{\sin xy(1 - 2e^{-x} \cos xy + e^{-2x})}\right]$$

$$= y(1 - e^{-x})\left\{\frac{e^{-nx}(e^{-x} + A)}{1 - 2e^{-x} \cos xy + e^{-2x}} + \frac{\sin nxy\left[1 - Ae^{-x} + (A - e^{-x}) \cos xy\right]}{\sin xy(1 - 2e^{-x} \cos xy + e^{-2x})} - \frac{\cos nxy\left[(A + e^{-x}) \sin xy\right]}{\sin xy(1 - 2e^{-x} \cos xy + e^{-2x})}\right\}$$

The first term in the preceding equation is a transient term that is approximately zero at steady state (large values of n). Let

$$E_1 = 1 - Ae^{-x} + (A - e^{-x}) \cos xy$$

$$F_1 = -(A + e^{-x}) \sin xy$$

$$G_1 = \frac{y(1 - e^{-x})}{\sin xy(1 - 2e^{-x} \cos xy + e^{-2x})}$$

Then

$$c_{ss}(nT) = G_1(E_1 \sin nxy + F_1 \cos nxy)$$

where $c_{ss}(nT)$ implies large values of n. By polar conversion, the amplitude and phase lag of $c_{ss}(t)$, the sine-wave reconstruction through the points of $c_{ss}(nT)$, now become

$$\text{Amplitude} = G_1\sqrt{E_1{}^2 + F_1{}^2}$$

and

$$\text{Phase} = 57.3 \tan^{-1} \frac{F_1}{E_1}$$

APPENDIX B

DERIVATION OF BODE RESPONSES FOR SECOND-ORDER LINEAR SYSTEM

This appendix derives Bode responses for the second-order linear system. The second-order system is to be solved for the cases of no compensation, first-order compensation, and second-order compensation.

No Compensation

The Bode response of a second-order open-loop linear system driven by a digital-to-analog converter (DAC) may be obtained directly from the substitution of $z = e^{j\omega T}$ into the Z-transform of the system as follows:

Given

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$G_{OH}(s) = \frac{1 - e^{-sT}}{s}$$

then

$$GG_{OH}(z) = \frac{z - 1}{z} Z\left\{\frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}\right\}$$

Let $x = \omega_n T$ and $r = \sqrt{1 - \xi^2}$ so that

$$GG_{OH}(z) = \frac{z\left(1 - e^{-\xi x} \cos rx - \frac{\xi}{r} e^{-\xi x} \sin rx\right) + \left(e^{-2\xi x} - e^{-\xi x} \cos rx + \frac{\xi}{r} e^{-\xi x} \sin rx\right)}{z^2 - 2ze^{-\xi x} \cos rx + e^{-2\xi x}}$$

Let

$$B = 1 - e^{-\xi x} \cos rx - \frac{\xi}{r} e^{-\xi x} \sin rx$$

and

$$C = e^{-2\xi x} - e^{-\xi x} \cos rx + \frac{\xi}{r} e^{-\xi x} \sin rx$$

then

$$GG_{OH}(z) = \frac{zB + C}{z^2 - 2ze^{-\xi x} \cos rx + e^{-2\xi x}}$$

Substitution of $z = e^{j\omega T}$ yields

$$GG_{OH}(e^{j\omega T}) = \frac{(B \cos \omega T + C) + jB \sin \omega T}{(e^{-2\xi x} + \cos 2\omega T - 2e^{-\xi x} \cos rx \cos \omega T) + j(\sin 2\omega T - 2e^{-\xi x} \cos rx \sin \omega T)}$$

Let

$$\alpha = e^{-2\xi x} + \cos 2\omega T - 2e^{-\xi x} \cos rx \cos \omega T$$

and

$$\beta = \sin 2\omega T - 2e^{-\xi x} \cos rx \sin \omega T$$

As a result,

$$GG_{OH}(e^{j\omega T}) = \frac{\left[(B \cos \omega T + C) + jB \sin \omega T\right]}{\alpha + j\beta} \frac{(\alpha - j\beta)}{(\alpha - j\beta)}$$

$$= \frac{\left[\alpha(B \cos \omega T + C) + \beta B \sin \omega T\right] + j\left[\alpha B \sin \omega T - \beta(B \cos \omega T + C)\right]}{\alpha^2 + \beta^2}$$

Finally,

$$\text{Amplitude} = \frac{1}{\alpha^2 + \beta^2}\sqrt{\left[\alpha(B \cos \omega T + C) + \beta B \sin \omega T\right]^2 + \left[\alpha B \sin \omega T - \beta(B \cos \omega T + C)\right]^2}$$

and

$$\text{Phase} = \tan^{-1}\left[\frac{\alpha B \sin \omega T - \beta(B \cos \omega T + C)}{\alpha(B \cos \omega T + C) + \beta B \sin \omega T}\right]$$

APPENDIX B

First-Order Compensation

Figure 3 presents the block diagram of the model to be used in deriving the Bode response of the first-order digital compensation of the second-order linear system. The Z-transform of the system output is

$$c(z) = uH(z)GG_{OH}(z)$$

and is derived in the following manner:

Given

$$u(s) = \frac{\omega}{s^2 + \omega^2}$$

$$H(s) = \frac{2\xi\omega_n s + \omega_n^2}{\omega_n^2}$$

then

$$uH(z) = Z\left\{\frac{\omega}{\omega_n^2}\,\frac{2\xi\omega_n s + \omega_n^2}{s^2 + \omega^2}\right\} = Z\left\{\frac{2\xi\omega}{\omega_n}\left(\frac{s}{s^2 + \omega^2} + \frac{\omega_n/2\xi}{s^2 + \omega^2}\right)\right\}$$

Let $y = \omega/\omega_n$ and $x = \omega_n T$ so that

$$uH(z) = \frac{2\xi yz\left(z - \cos xy + \frac{1}{2\xi y}\sin xy\right)}{(z - e^{jxy})(z - e^{-jxy})}$$

Since

$$GG_{OH}(z) = \frac{zB + C}{z^2 - 2ze^{-\xi x}\cos rx + e^{-2\xi x}} = \frac{zB + C}{(z - e^{-\xi x}e^{jrx})(z - e^{-\xi x}e^{-jrx})}$$

and if D_1 is defined as

$$D_1 = -\cos xy + \frac{1}{2\xi y}\sin xy$$

then, finally,

$$c(z) = \frac{2\xi yz(z + D_1)(zB + C)}{(z - e^{jxy})(z - e^{-jxy})(z - e^{-\xi x}e^{jrx})(z - e^{-\xi x}e^{-jrx})} \tag{1}$$

Second-Order Compensation

The block diagram of the model to be used in deriving the Bode response of the second-order digital compensation of the second-order linear system is shown in figure 4. The Z-transform of the system output is derived in the following manner:

Given

$$u(s) = \frac{\omega}{s^2 + \omega^2}$$

$$H(s) = \frac{s^2 + 2\xi\omega_n s + \omega_n{}^2}{\omega_n{}^2}$$

then

$$uH(z) = Z\left\{\frac{\omega}{\omega_n{}^2}\,\frac{s^2 + 2\xi\omega_n s + \omega_n{}^2}{(s + j\omega)(s - j\omega)}\right\}$$

From Z-transform theory (ref. 5),

$$uH(z) = \sum_{i=1}^{2}\left[(p - p_i)\frac{z}{z - e^{Tp}}\,uH(p)\right]_{p=p_i}$$

Therefore,

$$uH(z) = \frac{z}{z - e^{j\omega T}}\,\frac{\omega}{\omega_n{}^2}\,\frac{(j\omega)^2 + 2j\xi\omega_n\omega + \omega_n{}^2}{2j\omega}$$

$$+ \frac{z}{z - e^{-j\omega T}}\,\frac{\omega}{\omega_n{}^2}\,\frac{(-j\omega)^2 - 2j\xi\omega\omega_n + \omega_n{}^2}{-2j\omega}$$

Let $y = \omega/\omega_n$ and $x = \omega_n T$. Then

$$uH(z) = \frac{z\left[(1 - y^2) + j2\xi y\right](z - e^{-jxy}) - z\left[(1 - y^2) - j2\xi y\right](z - e^{jxy})}{2j(z - e^{jxy})(z - e^{-jxy})}$$

$$= \frac{z\left[z(1 - y^2) - (1 - y^2)e^{-jxy} + j2\xi yz - j2\xi ye^{-jxy}\right]}{2j(z - e^{jxy})(z - e^{-jxy})}$$

$$- \frac{z\left[z(1 - y^2) - (1 - y^2)e^{jxy} - j2\xi yz + j2\xi ye^{jxy}\right]}{2j(z - e^{jxy})(z - e^{-jxy})}$$

$$= \frac{z\left[1 - y^2)(2j \sin xy) + j4\xi yz - j4\xi y \cos xy\right]}{2j(z - e^{jxy})(z - e^{-jxy})}$$

$$= \frac{2\xi yz\left(z - \cos xy + \dfrac{1 - y^2}{2\xi y} \sin xy\right)}{(z - e^{jxy})(z - e^{-jxy})}$$

Let

$$D_2 = -\cos xy + \frac{1 - y^2}{2\xi y} \sin xy$$

As a result,

$$uH(z) = \frac{2\xi yz(z + D_2)}{(z - e^{jxy})(z - e^{-jxy})}$$

Since $G(s)$ and $G_{OH}(s)$ are unchanged from the first-order compensation case,

$$GG_{OH}(z) = \frac{zB + C}{(z - e^{-\xi x}e^{jrx})(z - e^{-\xi x}e^{-jrx})}$$

Finally,

$$c(z) = \frac{2\xi yz(z + D_2)(zB + C)}{(z - e^{jxy})(z - e^{-jxy})(z - e^{-\xi x}e^{jrx})(z - e^{-\xi x}e^{-jrx})} \qquad (2)$$

Inverse Z-transform of Second-Order Linear System

The difference between equation (1) and equation (2) lies wholly in the difference (for a given x, y, and ξ) in the constants D_1 and D_2. Therefore the Bode response can be derived from the equation for either case, with the other case being obtained by a mere change of subscript. Let $D = D_1$ or D_2, depending on the desired compensation; then

$$c(z) = \frac{2\xi yz(z + D)(zB + C)}{(z - e^{jxy})(z - e^{-jxy})(z - e^{-\xi x}e^{jrx})(z - e^{-\xi x}e^{-jrx})}$$

Inversion of $c(z)$ results in

$$c(nT) = \left[z^{n-1}(z - e^{jxy})c(z)\right]_{z=e^{jxy}} + \left[z^{n-1}(z - e^{-jxy})c(z)\right]_{z=e^{-jxy}}$$

$$+ \left[\text{Transient terms that are (functions} \times e^{-n\xi x})\right]$$

At steady state,

$$c_{ss}(nT) = 2\xi y\left[\frac{e^{jnxy}(e^{jxy} + D)(e^{jxy}B + C)}{2j \sin xy(e^{jxy} - e^{-\xi x}e^{jrx})(e^{jxy} - e^{-\xi x}e^{-jrx})}\right.$$

$$\left. - \frac{e^{-jnxy}(e^{-jxy} + D)(e^{-jxy}B + C)}{2j \sin xy(e^{-jxy} - e^{-\xi x}e^{jrx})(e^{-jxy} - e^{-\xi x}e^{-jrx})}\right]$$

Let

$$G_2 = \frac{2\xi y}{\sin xy(e^{jxy} - e^{-\xi x}e^{jrx})(e^{-jxy} - e^{-\xi x}e^{jrx})(e^{jxy} - e^{-\xi x}e^{-jrx})(e^{-jxy} - e^{-\xi x}e^{-jrx})} \qquad (3)$$

Then

$$c_{ss}(nT) = G_2\left[\frac{e^{jnxy}}{2j}(e^{jxy} + D)(e^{jxy}B + C)(e^{-jxy} - e^{-\xi x}e^{jrx})(e^{-jxy} - e^{-\xi x}e^{-jrx})\right.$$

$$\left. - \frac{e^{-jnxy}}{2j}(e^{-jxy} + D)(e^{-jxy}B + C)(e^{jxy} - e^{-\xi x}e^{jrx})(e^{jxy} - e^{-\xi x}e^{-jrx})\right]$$

$$= \frac{G_2}{2j}\Big(\left[e^{jnxy}B + (DB + C)e^{j(n-1)xy} + DCe^{j(n-2)xy}\right]$$

$$- e^{-\xi x}\Big\{Be^{j[(n+1)xy+rx]} + (DB + C)e^{j(nxy+rx)} + DCe^{j[(n-1)xy+rx]}$$

$$+ Be^{j[(n+1)xy-rx]} + (DB + C)e^{j(nxy-rx)} + DCe^{j[(n-1)xy-rx]}\Big\}$$

$$+ e^{-2\xi x}\left[Be^{j(n+2)xy} + (DB + C)e^{j(n+1)xy} + DCe^{jnxy}\right]\Big)$$

$$- \frac{G_2}{2j}\Big(\left[e^{-jnxy}B + (DB + C)e^{-j(n-1)xy} + DCe^{-j(n-2)xy}\right]$$

$$- e^{-\xi x}\Big\{Be^{-j[(n+1)xy+rx]} + (DB + C)e^{-j(nxy+rx)} + DCe^{-j[(n-1)xy+rx]}$$

$$+ Be^{-j[(n+1)xy-rx]} + (DB + C)e^{-j(nxy-rx)} + DCe^{-j[(n-1)xy-rx]}\Big\}$$

$$+ e^{-2\xi x}\left[Be^{-j(n+2)xy} + (DB + C)e^{-j(n+1)xy} + DCe^{-jnxy}\right]\Big)$$

$$= G_2\Big(\left[B \sin nxy + (DB + C) \sin (n - 1)xy + DC \sin (n - 2)xy\right]$$

$$- e^{-\xi x}\Big\{B \sin \left[(n + 1)xy + rx\right] + B \sin \left[(n + 1)xy - rx\right]$$

$$+ (DB + C) \sin (nxy + rx) + (DB + C) \sin (nxy - rx)$$

$$+ DC \sin \left[(n - 1)xy + rx\right] - DC \sin \left[(n - 1)xy - rx\right]\Big\}$$

$$+ e^{-2\xi x}\left[B \sin (n + 2)xy + (DB + C) \sin (n + 1)xy + DC \sin nxy\right]\Big)$$

(Equation continued on next page)

$$= G_2\Big\{\sin nxy\left[B + (DB + C)\cos xy + DC\cos 2xy\right]$$

$$- \cos nxy\left[(DB + C)\sin xy + DC\sin 2xy\right]$$

$$- 2e^{-\xi x}\cos rx\left[\sin nxy(B\cos xy + DB + C + DC\cos xy)\right.$$

$$\left.- \cos nxy(DC\sin xy - B\sin xy)\right]$$

$$+ e^{-2\xi x}\sin nxy\left[B\cos 2xy + (DB + C)\cos xy + DC\right]$$

$$+ e^{-2\xi x}\cos nxy\left[B\sin 2xy + (DB + C)\sin xy\right]\Big\}$$

Finally,

$$c_{ss}(nT) = G_2(E_2\sin nxy + F_2\cos nxy)$$

where

$$E_2 = \left[B + (DB + C)\cos xy + DC\cos 2xy\right]$$

$$- 2e^{-\xi x}\cos rx\left[(B + DC)\cos xy + (DB + C)\right]$$

$$+ e^{-2\xi x}\left[B\cos 2xy + (DB + C)\cos xy + DC\right]$$

$$F_2 = -\left[(DB + C)\sin xy + DC\sin 2xy\right] - 2e^{-\xi x}\cos rx\sin xy(B - DC)$$

$$+ e^{-2\xi x}\left[B\sin 2xy + (DB + C)\sin xy\right]$$

and, from equation (3),

$$G_2 = \frac{2\xi y}{\sin xy\left[1 - 2e^{-\xi x}\cos(xy + rx) + e^{-2\xi x}\right]\left[1 - 2e^{-\xi x}\cos(xy - rx) + e^{-2\xi x}\right]}$$

Therefore, the amplitude and phase lag of $c_{ss}(t)$, the sine-wave reconstruction through the sample points of $c_{ss}(nT)$, are then

$$\text{Amplitude} = G_2\sqrt{E_2^2 + F_2^2}$$

and

$$\text{Phase} = 57.3 \tan^{-1} \frac{F_2}{E_2}$$

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TABLE I.- ESTIMATES OF FREQUENCY AND DAMPING FOR A CONTINUOUS SECOND-ORDER LINEAR SYSTEM

Sample period	Frequency estimate	Damping estimate
0.05	0.9	0.65
.1	.825	.61
.2	.715	.58
Continuous	1.0	.7071

TABLE II.- COMPARISON OF COMPENSATOR TRANSFER FUNCTIONS

Compensation method	H(s)	
	First-order linear system	Second-order linear system
Plant poles	$1 + s\left(\frac{1}{a}\right)$	$1 + s\left(\frac{2\xi}{\omega_n}\right) + s^2\left(\frac{1}{\omega_n^2}\right)$
Plant poles + two-term approximation of hold	$1 + s\left(\frac{1}{a} + T\right)$	$1 + s\left(\frac{2\xi}{\omega_n} + T\right) + s^2\left(\frac{1}{\omega_n^2} + \frac{2\xi T}{\omega_n}\right)$
Plant poles + three-term approximation of hold	$1 + s\left(\frac{1}{a} + \frac{T}{2}\right)$	$1 + s\left(\frac{2\xi}{\omega_n} + \frac{T}{2}\right) + s^2\left(\frac{1}{\omega_n^2} + \frac{\xi T}{\omega_n} + \frac{T^2}{4}\right)$

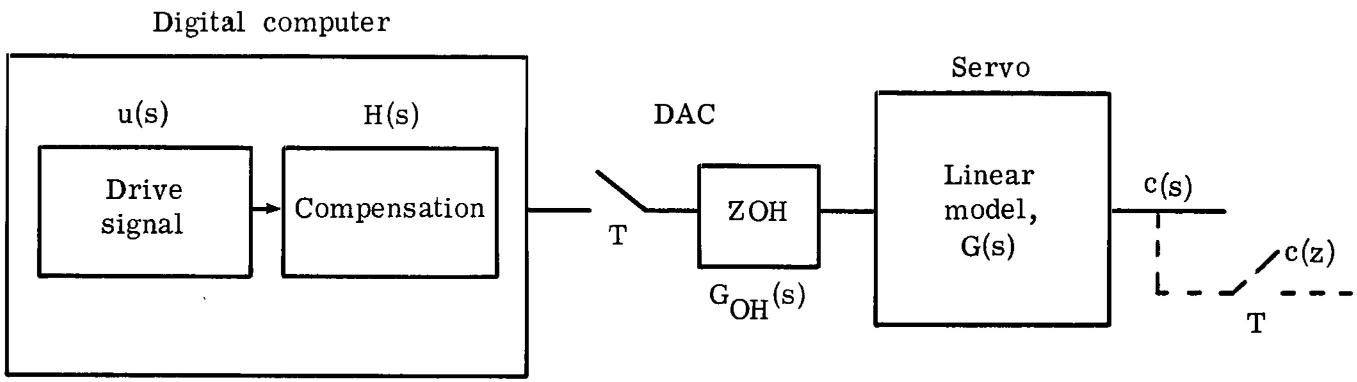


Figure 1.- General form of sampled data model.

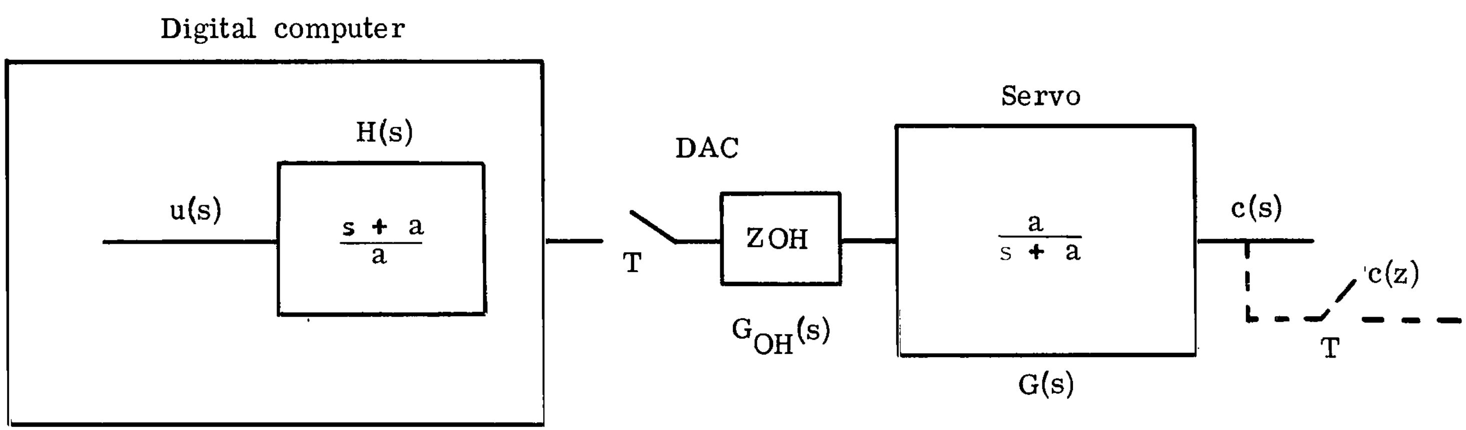


Figure 2.- First-order digital compensation of first-order linear system.

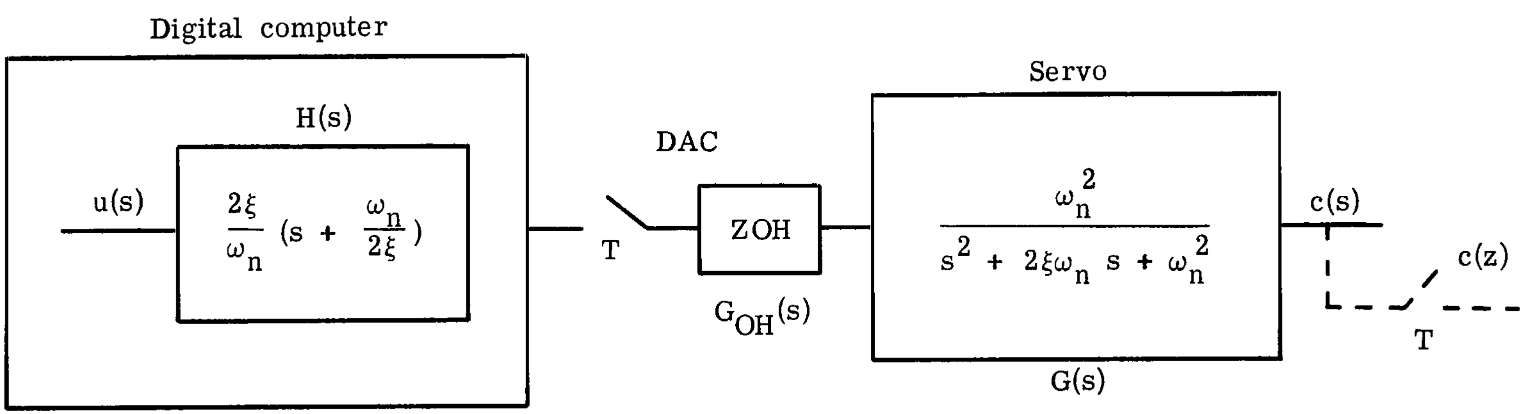


Figure 3.- First-order digital compensation of second-order linear system.

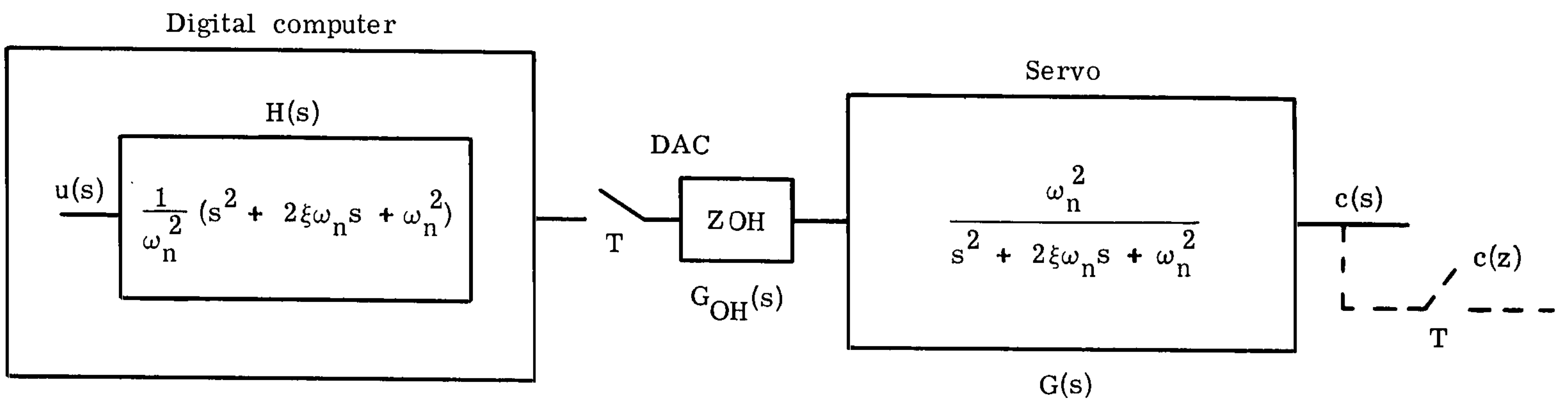
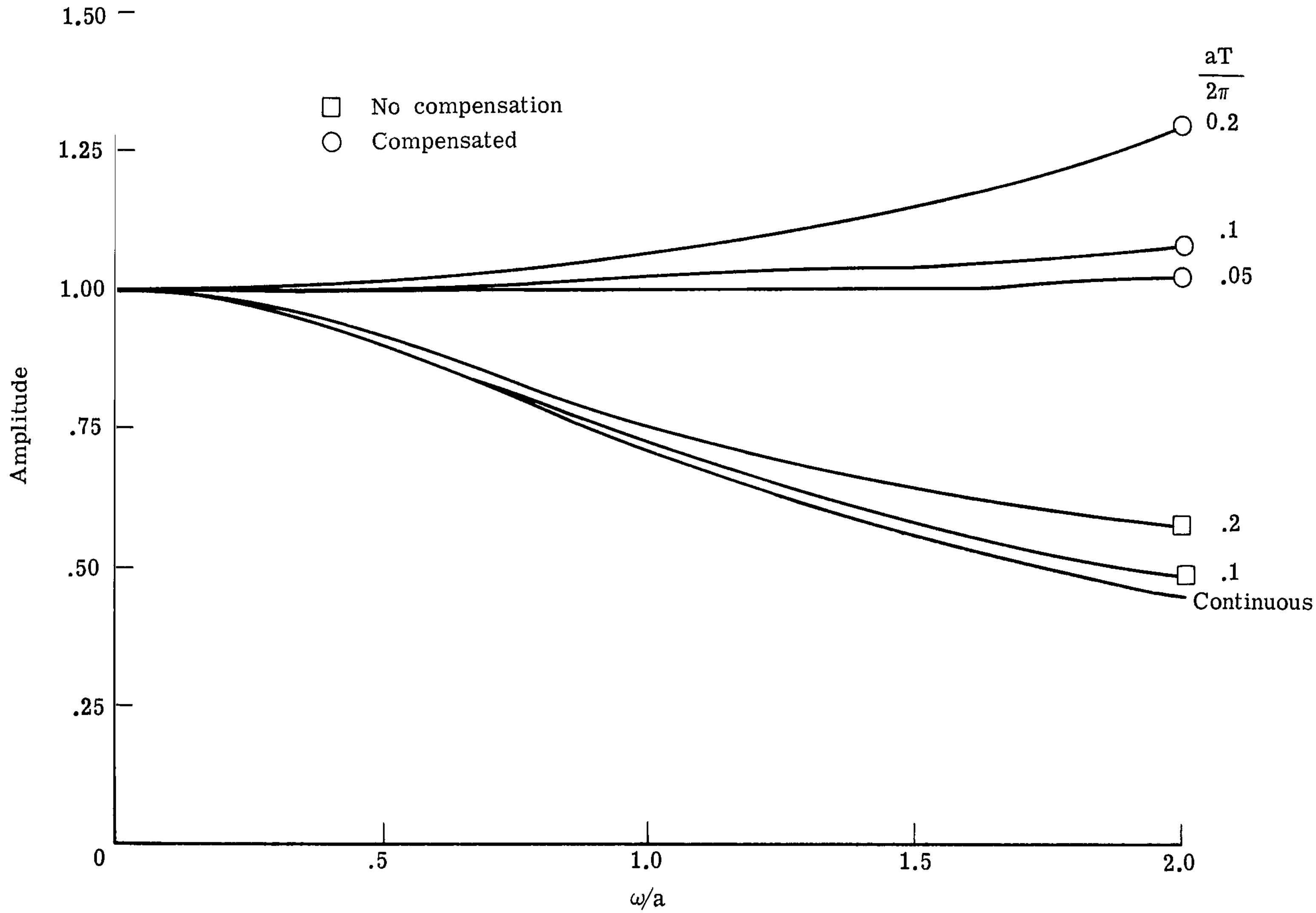
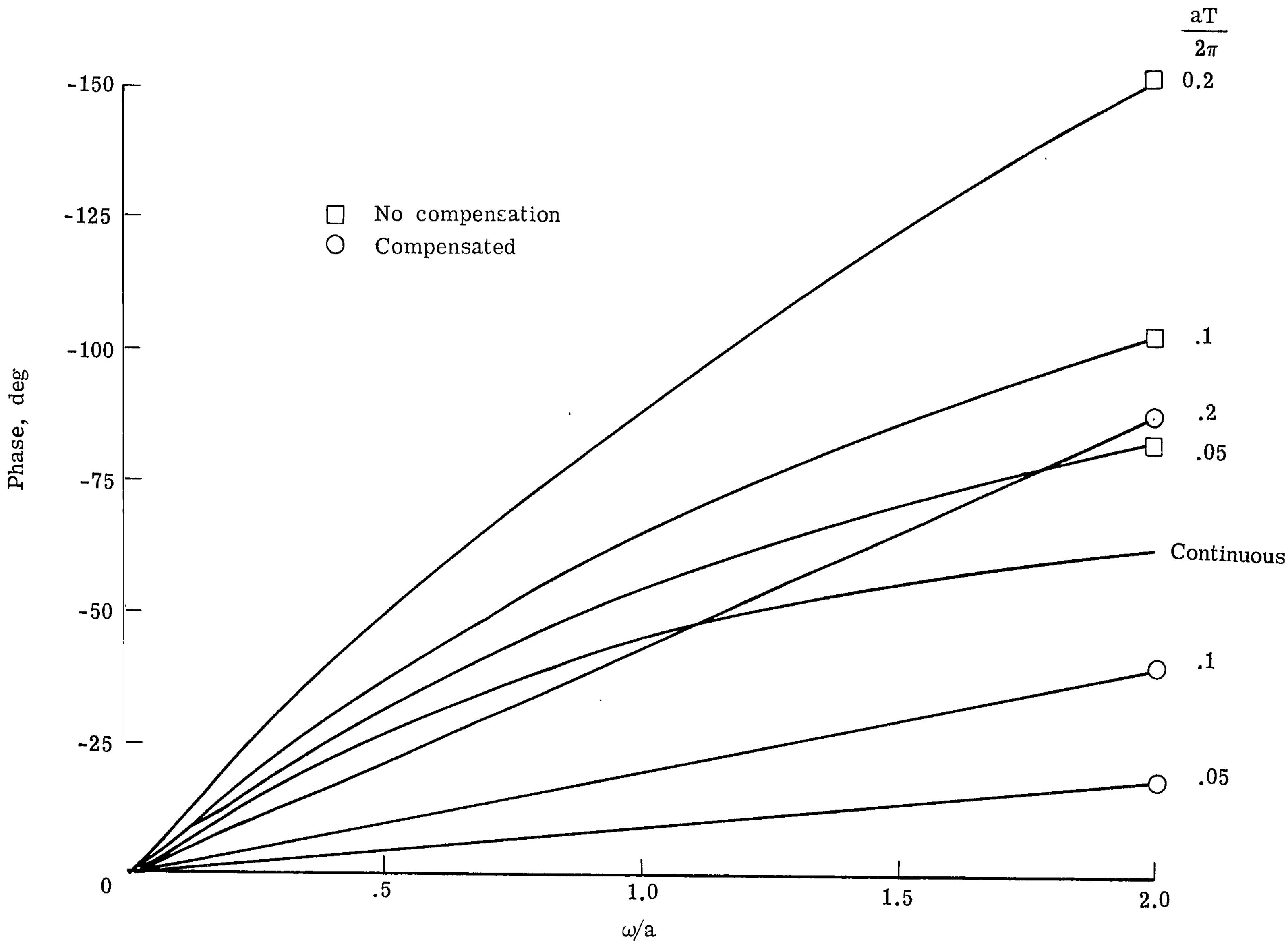


Figure 4.- Second-order digital compensation of second-order linear system.



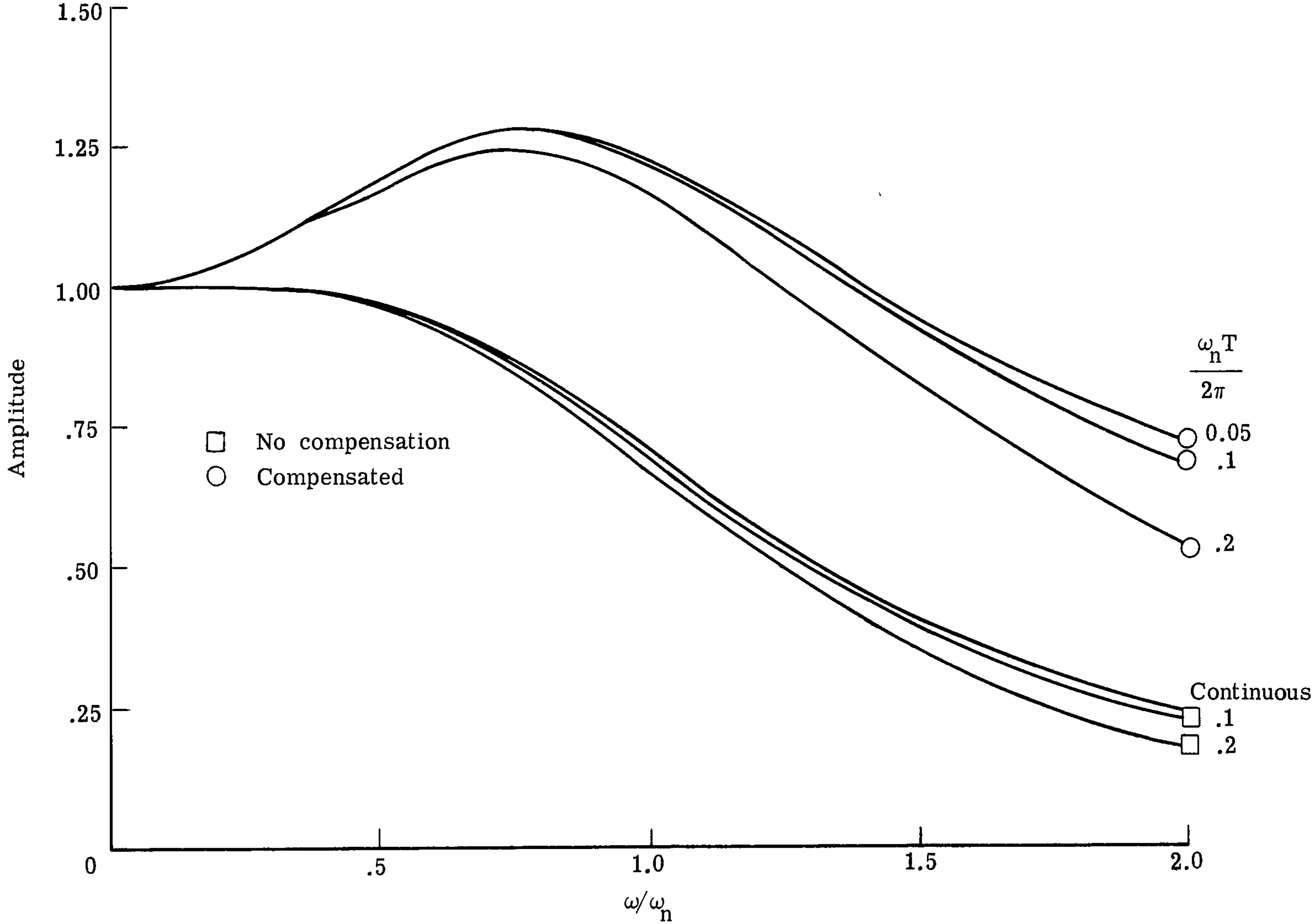
(a) Amplitude.

Figure 5.- Amplitude and phase of first-order linear system.



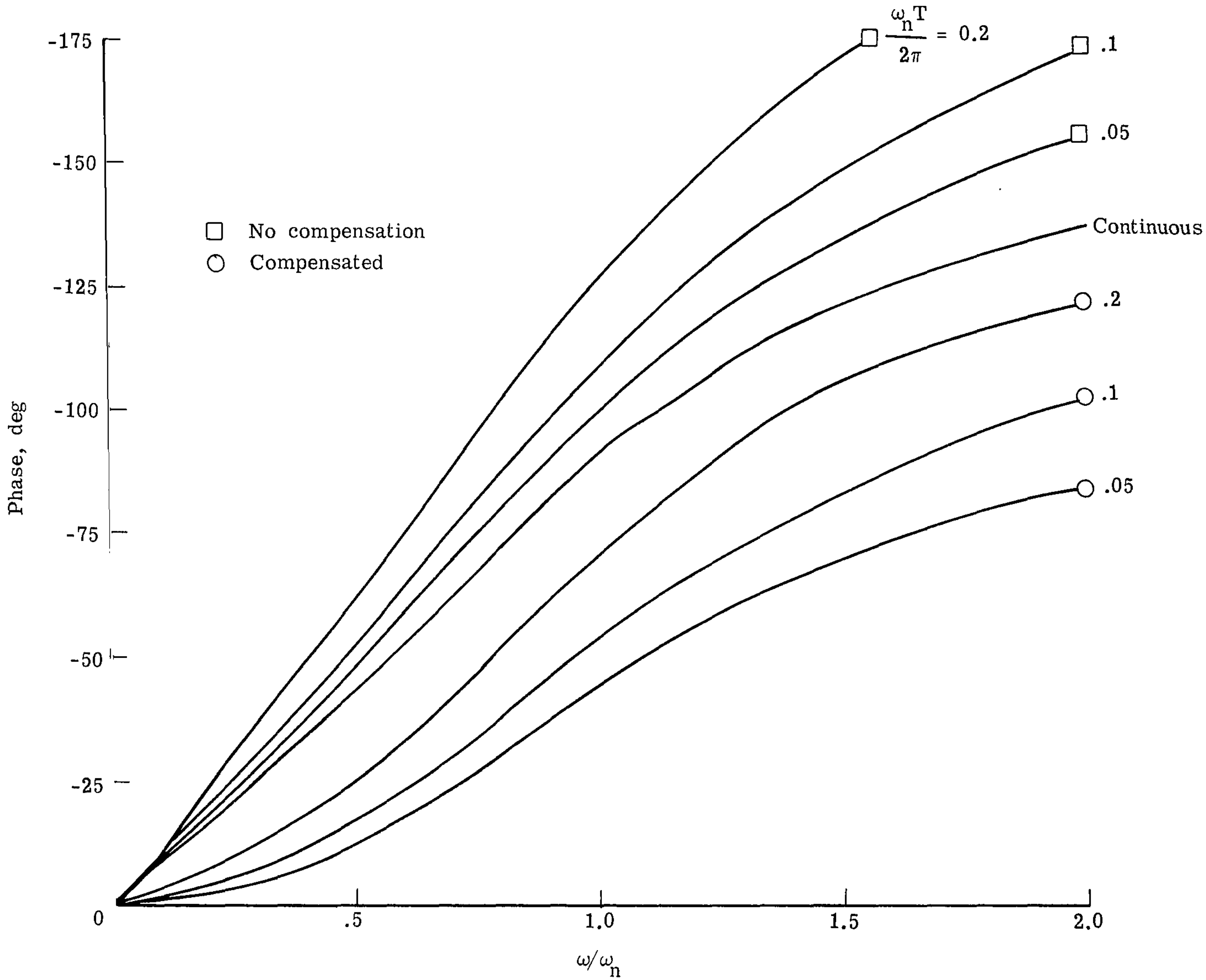
(b) Phase.

Figure 5.- Concluded.



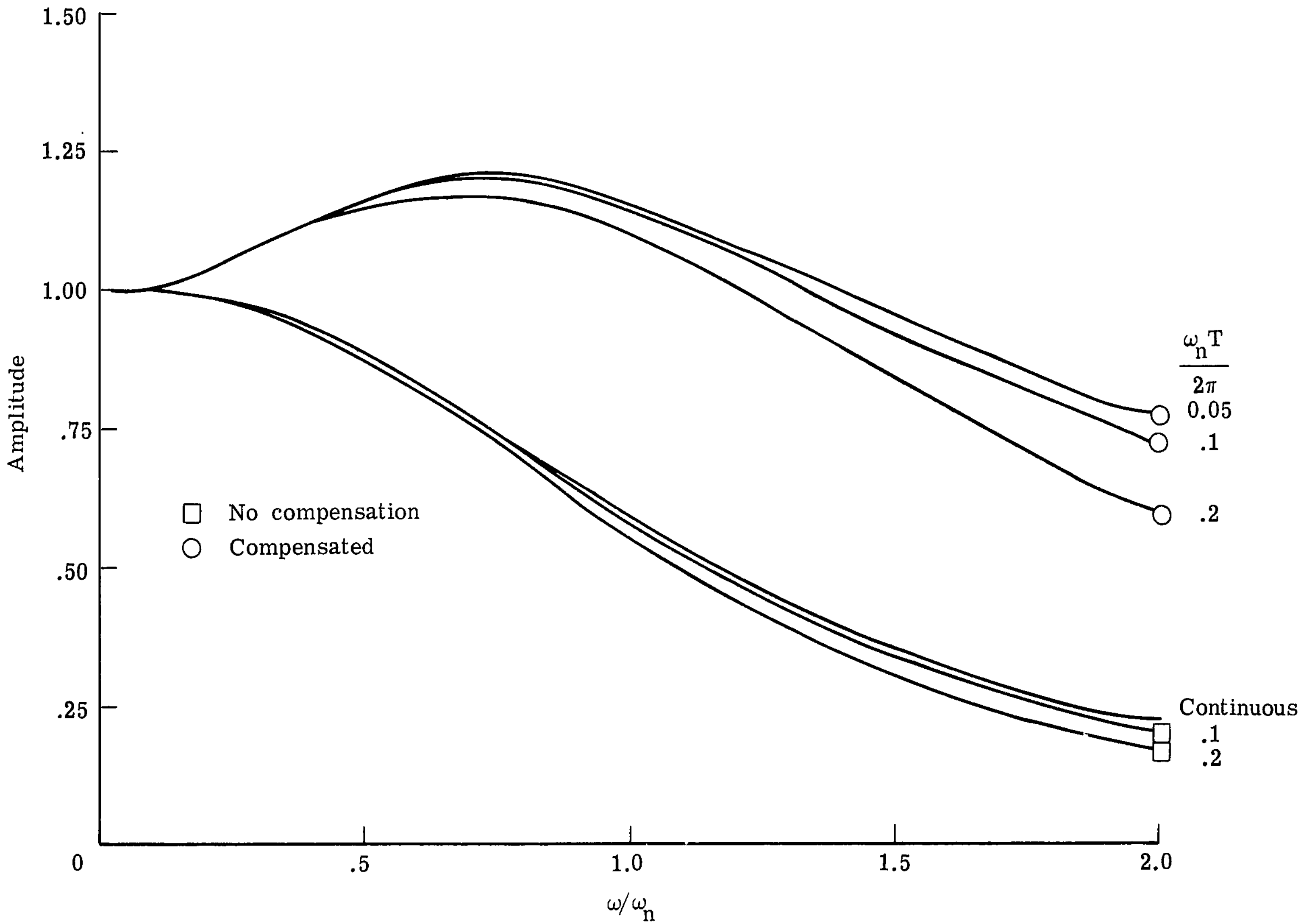
(a) Amplitude.

Figure 6.- First-order compensation of second-order linear system with $\xi = \sqrt{2}/2$.



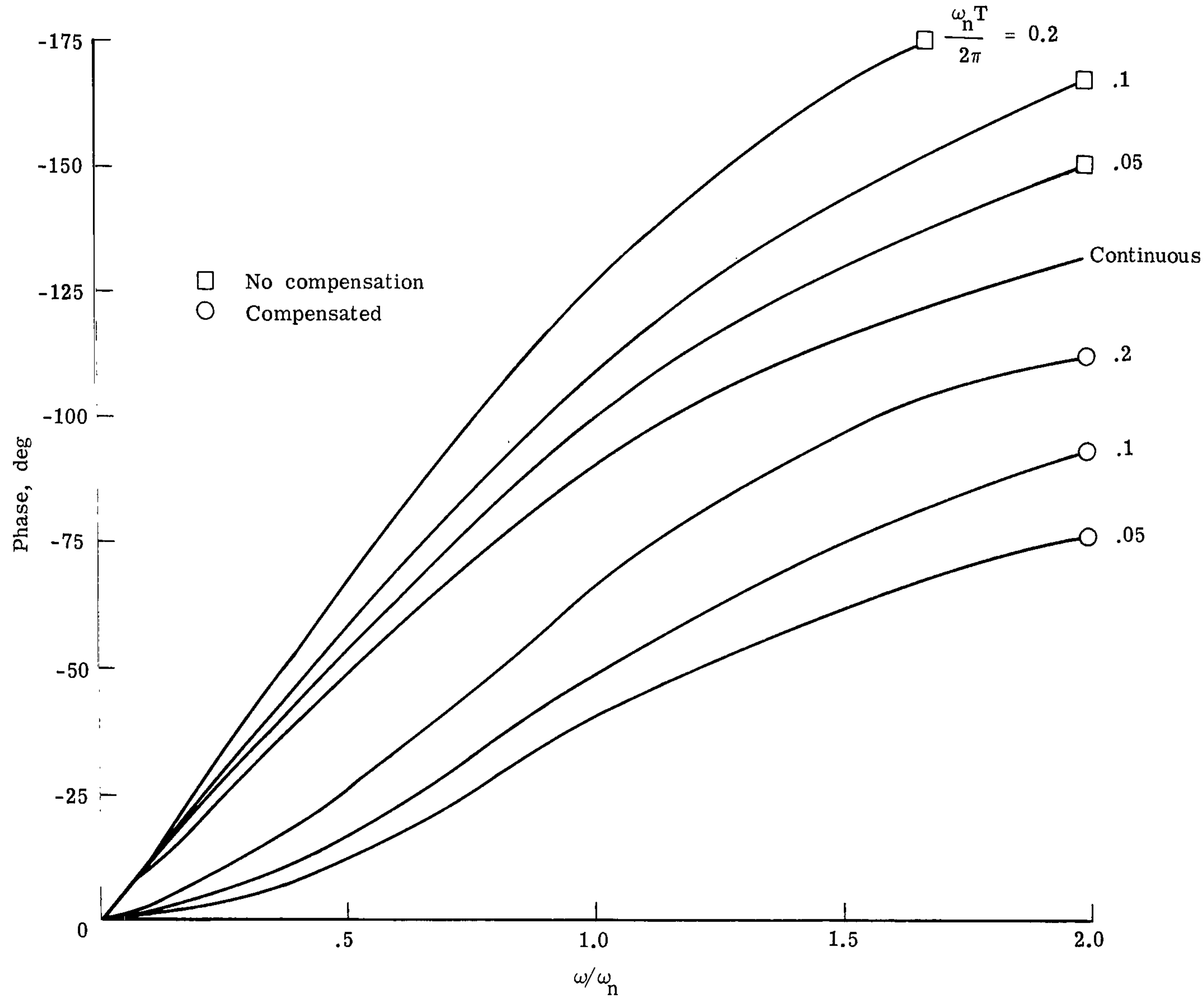
(b) Phase.

Figure 6.- Concluded.



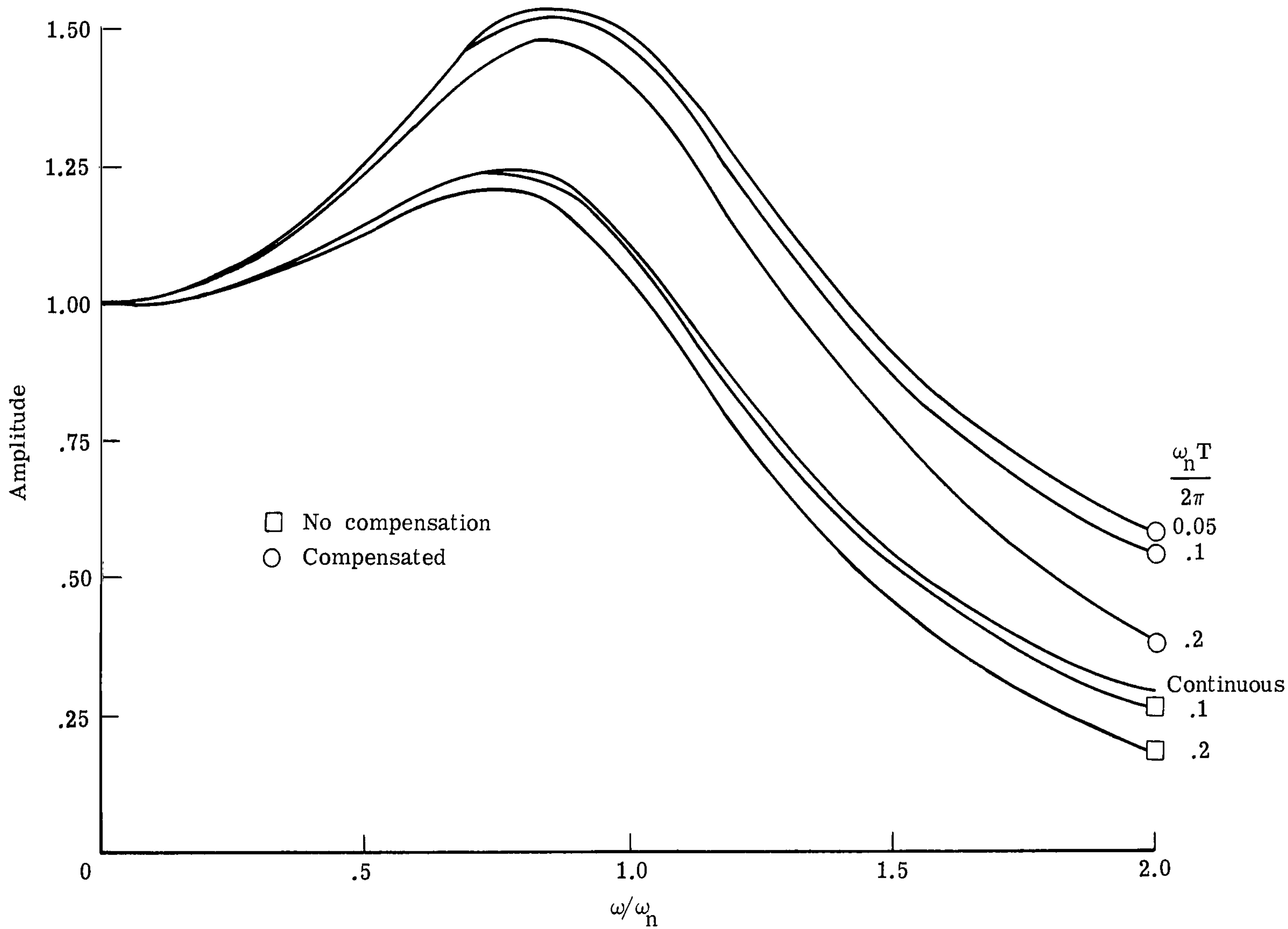
(a) Amplitude.

Figure 7.- First-order compensation of second-order linear system with $\xi = 0.85$.



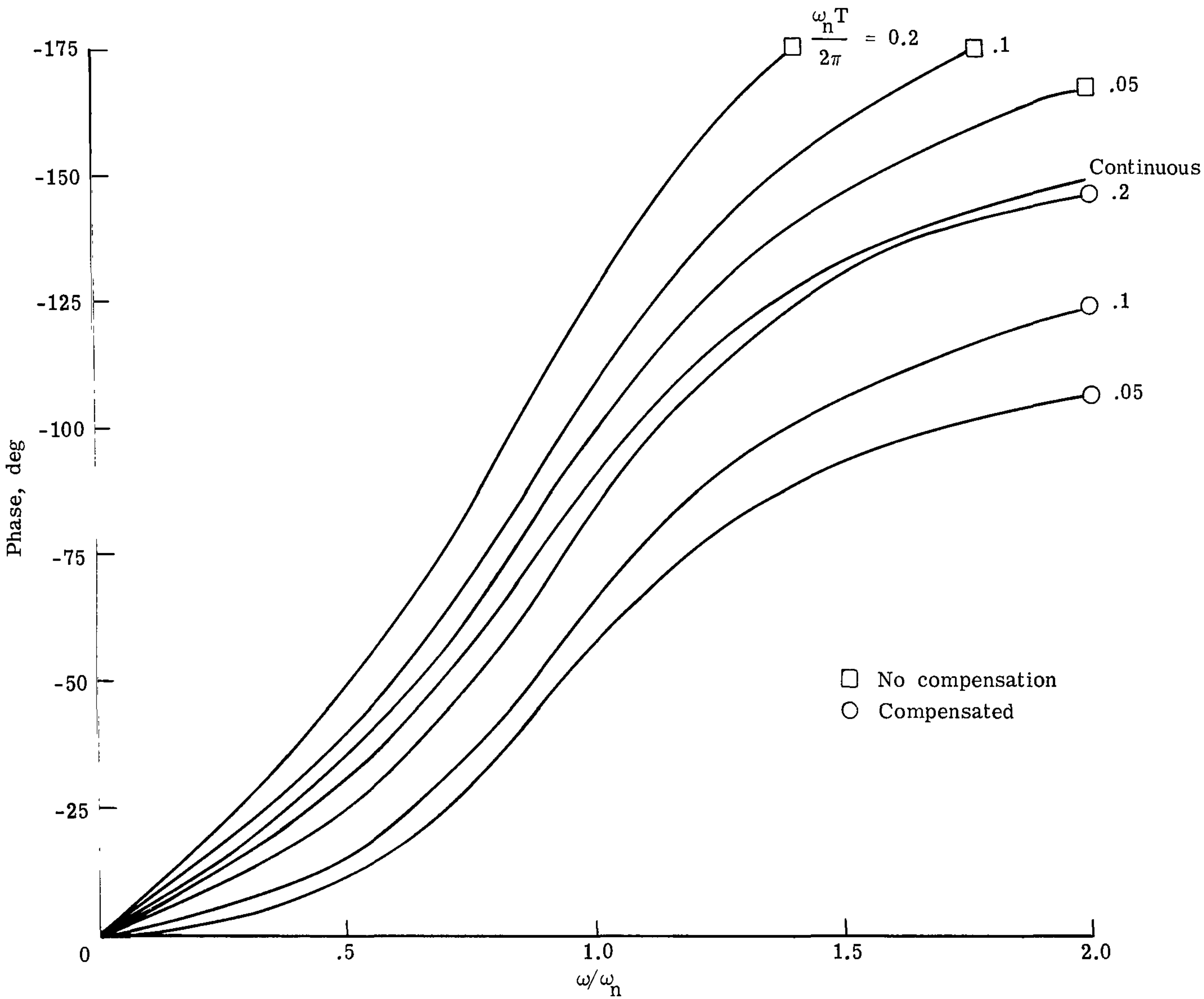
(b) Phase.

Figure 7.- Concluded.



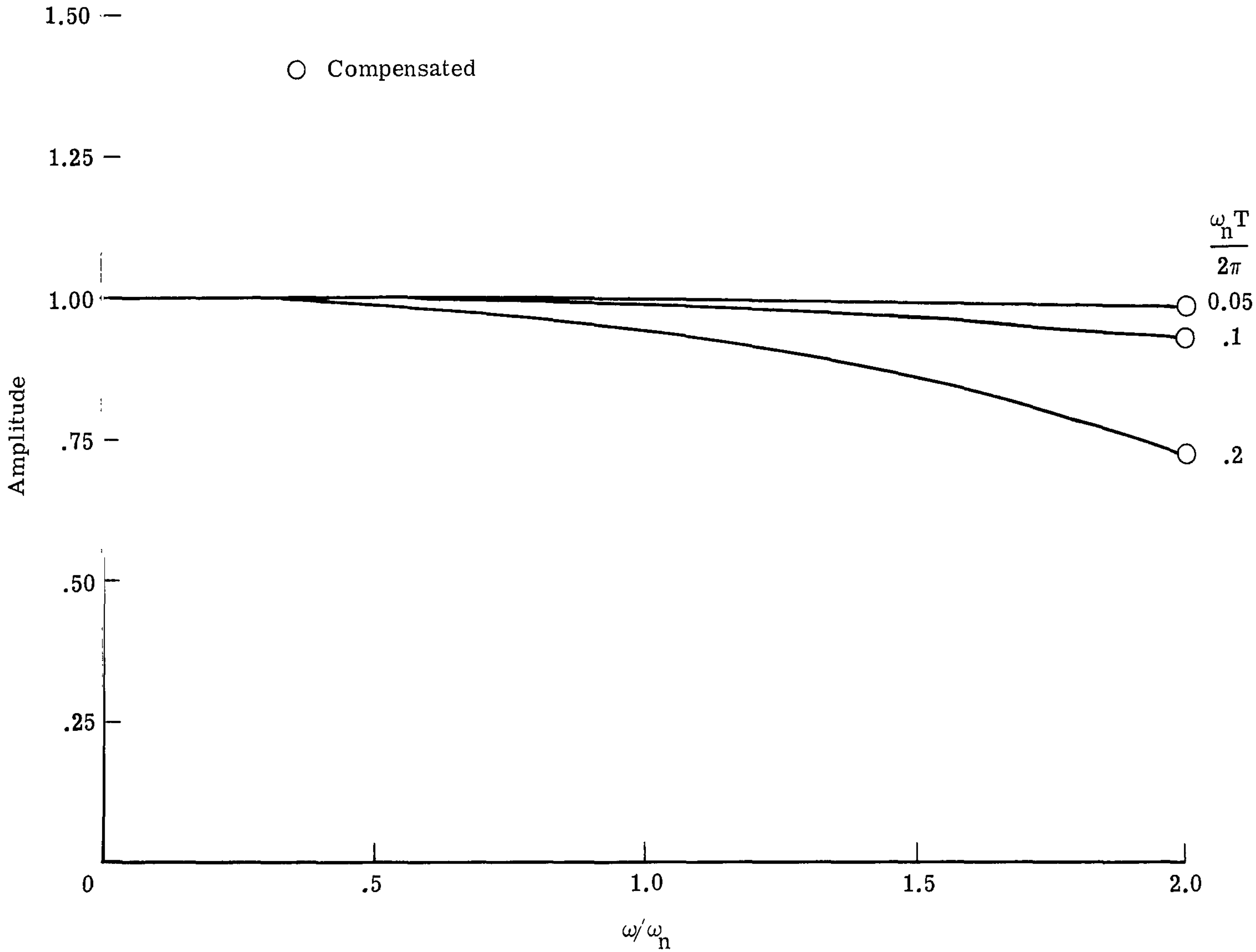
(a) Amplitude.

Figure 8.- First-order compensation of second-order linear system with $\xi = 0.45$.



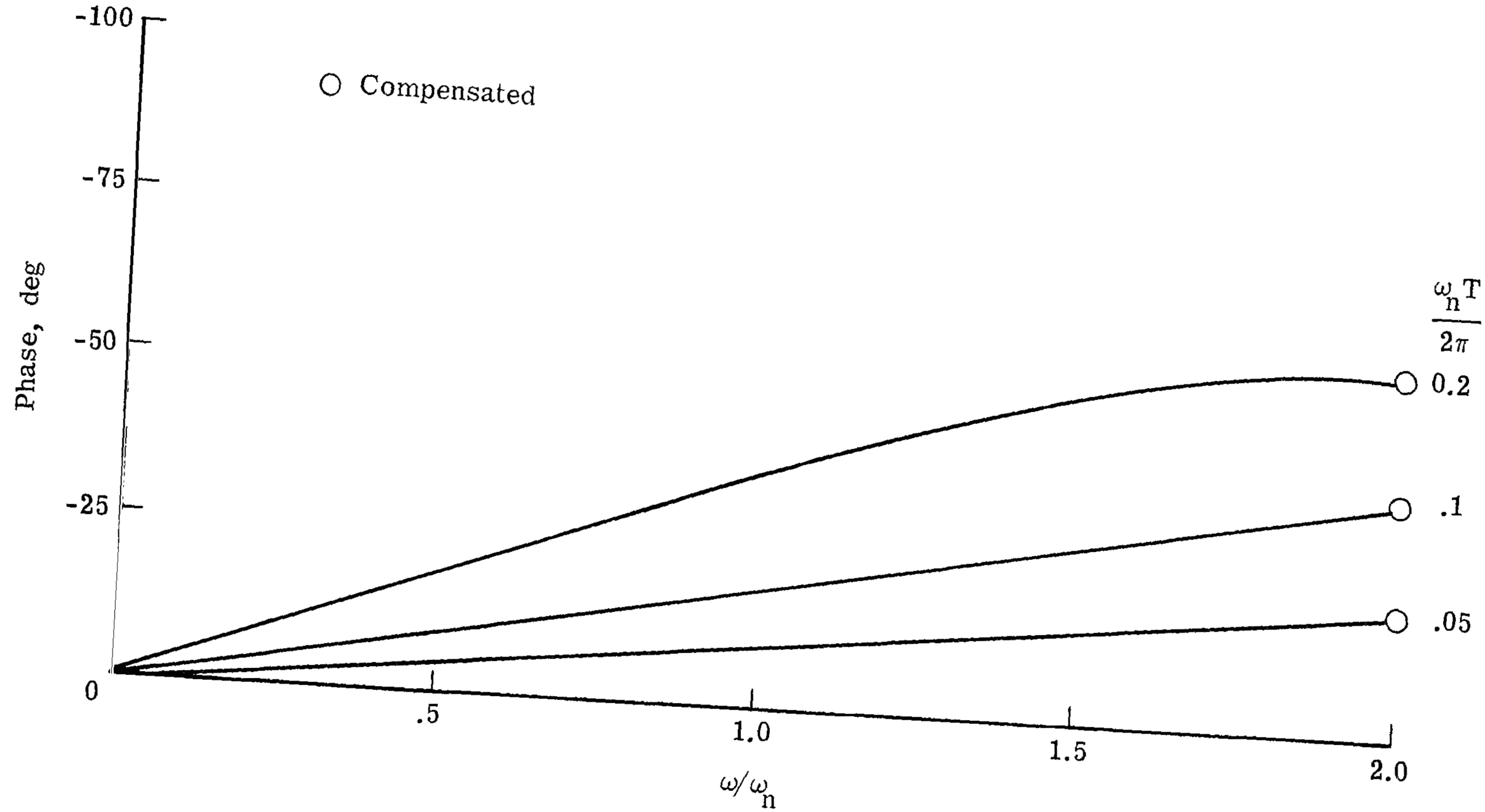
(b) Phase.

Figure 8.- Concluded.



(a) Amplitude.

Figure 9.- Second-order compensation of second-order linear system with
$\xi = 0.45$, $\sqrt{2}/2$, and 0.85.



(b) Phase.

Figure 9.- Concluded.

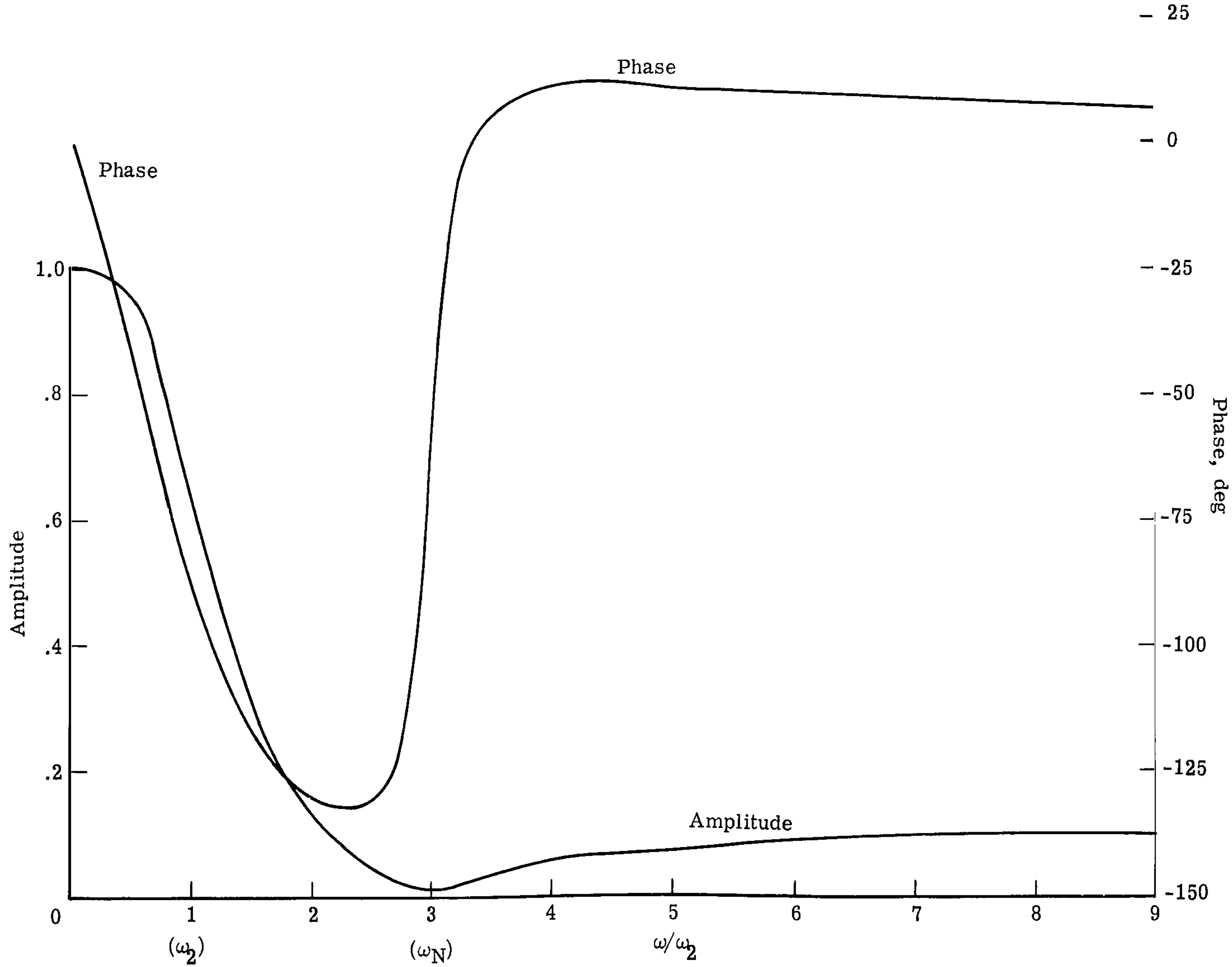
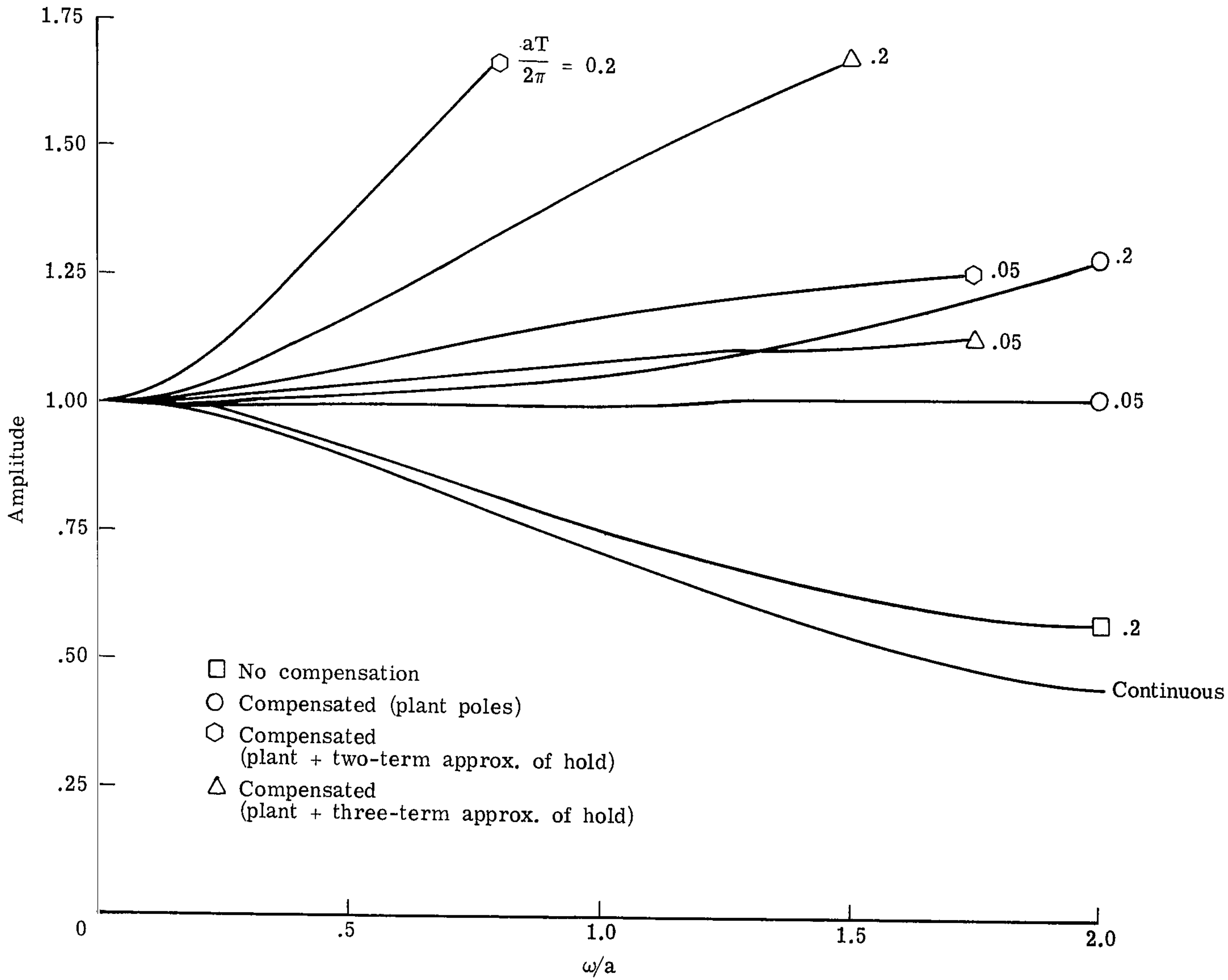
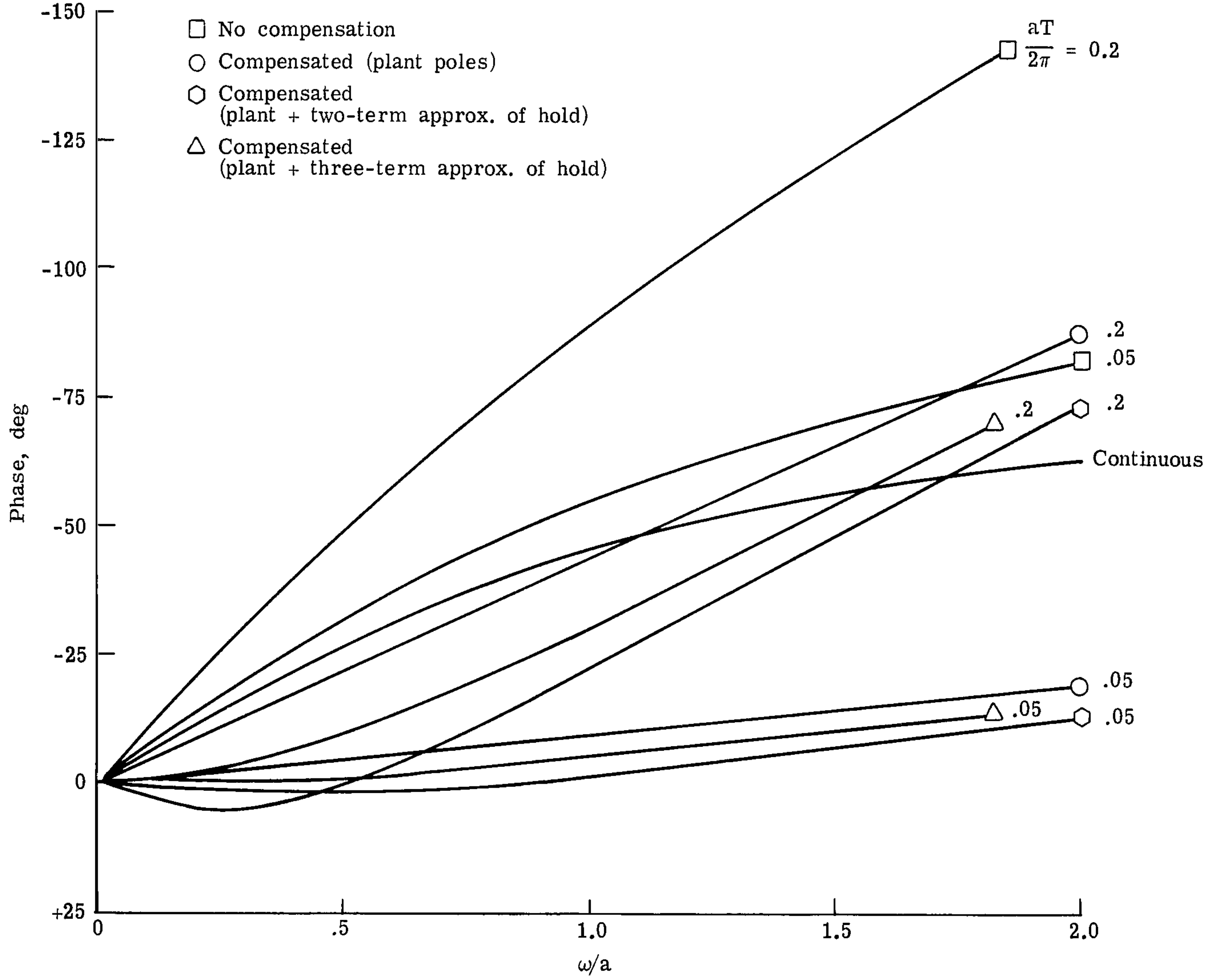


Figure 10.- Bode plot of notch filter. $\omega_N = 3\omega_2$; $\xi_N = 0.05$; $\xi_2 = 0.7$.



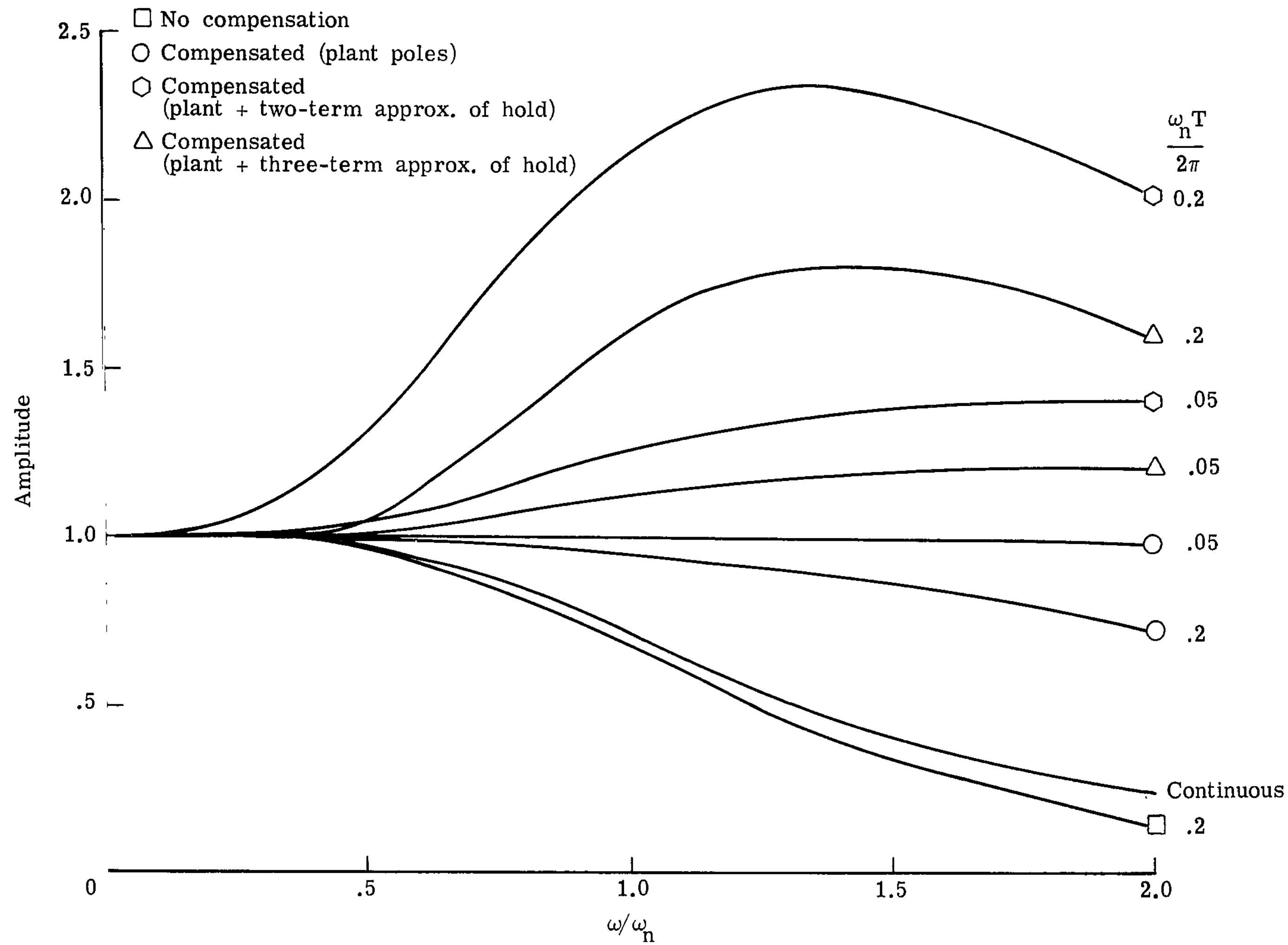
(a) Amplitude.

Figure 11.- Compensation comparisons for first-order linear system.



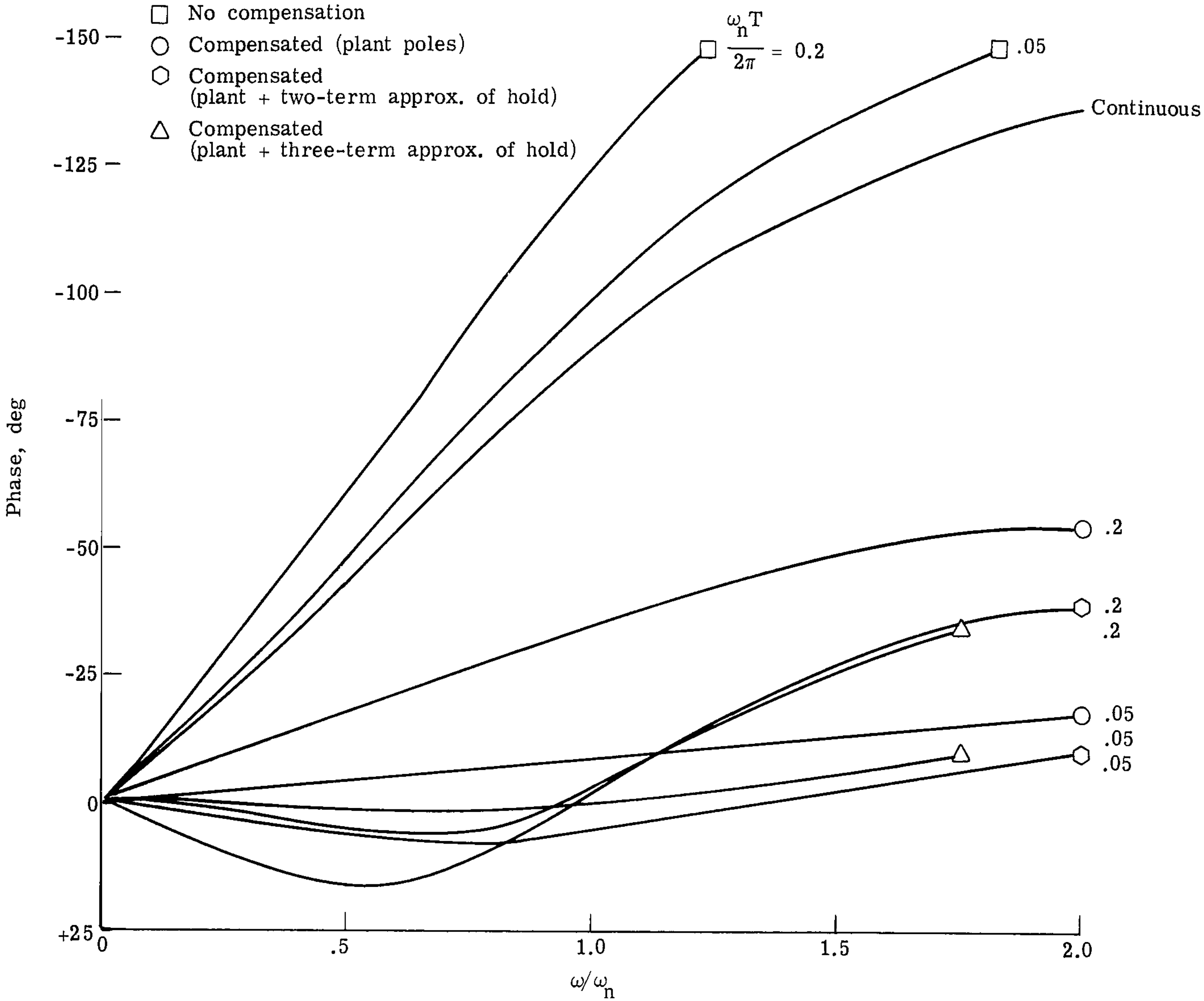
(b) Phase.

Figure 11.- Concluded.



(a) Amplitude.

Figure 12.- Compensation comparisons for second-order linear system.



(b) Phase.

Figure 12.- Concluded.

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