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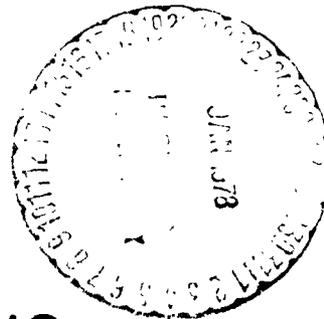
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ORBITAL MOTION OF THE  
SOLAR POWER SATELLITE

**A**NALYTICAL    AND  
**C**OMPUTATIONAL  
**M**ATHEMATICS, INC.



ORBITAL MOTION OF THE  
SOLAR POWER SATELLITE

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# ORBITAL MOTION OF THE SOLAR POWER SATELLITE

by

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## 1.0 Introduction

It has been proposed to put a series of large satellites into geosynchronous orbit for the purpose of collecting solar energy and redirecting it toward the earth via microwave radiation. Preliminary studies are being carried out at JSC on the feasibility of these Solar Power Satellites (SPS).

The large area of the collecting surface (approximately 144 square kilometers) means that solar radiation pressure will cause significant perturbations on the SPS orbit. In fact solar pressure will be as important as gravitational perturbations. This report documents a study on the effects of solar radiation pressure on the SPS orbit. It will be shown that the eccentricity of the orbit can get rather large (.08) even though it is initially zero. This is the primary difference between SPS orbit and other geosynchronous satellite orbits.

The SPS configuration being considered here is described in a study report by the Johnson Space Center (Reference 1). Others are discussed in References 2, 3 and elsewhere. However, the results in this report are applicable to any geosynchronous satellite that resembles a flat surface that continually faces the sun.

The main purpose of this report is to investigate the orbital evolution of the SPS over its expected thirty year lifetime. As a first step, it is assumed that the satellite is in free flight, i.e. there is no active orbit control.

This will make evident the important orbital motions. One of the goals of this study is to describe the motion with analytical formulas. These could then be used as a basis for developing an orbit control theory that will minimize station keeping costs.

The perturbing forces acting on the satellite are discussed in the next section. To a first approximation, three types of forces can be considered separately since they have different effects on the orbit.

- (1) Longitude dependent tesseral terms in the earth's geopotential cause a slow drift of the satellite's mean longitude.
- (2) Sun and moon gravity cause a rotation of the orbital plane.
- (3) Solar radiation pressure will cause an increase in the orbital eccentricity.

Variations in orbital eccentricity  $e$  are discussed in Section 3. Analytical solution methods are used to develop equations for the variation in eccentricity and argument of perigee as a function of time. These equations are valid for arbitrary initial values of eccentricity and inclination. It is shown that  $e$  will have a periodic variation with an amplitude of .04 and period of one year. There is also a linear increase so that  $e$  will grow to .08 within thirty years.

Earth-Sun-Moon gravity will cause long period variations in  $e$ . These effects have been studied with numerical integration methods and are discussed in Section 4. Evolution of the orbital elements is shown for a variety of initial conditions. The maximum value of  $e$  can be reduced by an appropriate choice of initial conditions.

Implications of non-circular, non-equatorial geosynchronous orbits for the SPS are discussed in Section 5. It is shown that the daily variation in longitude is  $2e$  radians.

However, these orbits offer certain advantages for the SPS and should be further evaluated for their impact on the energy collection, transmitting and receiving systems.

## 2.0 Perturbing Forces

The perturbations due to sun-moon gravity and non-spherical earth have been extensively discussed in the literature and only an overview will be given here. Acceleration due to solar radiation pressure will be derived in this section, considering the expected physical dimensions of the SPS.

### 2.1 Non-sphericity of the Earth

This perturbation arises from the fact that the earth is not symmetrical about its spin axis. A slice of the earth perpendicular to its spin axis has an almost elliptical shape. Since the earth rotates once a day and the satellite makes one revolution in approximately one day, these gravitational perturbations act in the same direction over a long period of time. As a result there is a large, long period drift in the geographic mean longitude of the satellite (Reference 4). The other orbital elements are not so severely affected. References 5 and 6 give a good description of this motion.

### 2.2 Luni-Solar Gravity

The luni-solar perturbations have a substantial effect upon the node  $h$  and inclination  $I$  of the orbit. Coupling between the sun, moon and earth's oblateness ( $J_2$ ) can cause large, long period perturbations in  $I$  (Reference 7). Table I shows some representative values of the inclination after two and 26.5 years. If  $I_0 = 0$ , the inclination grows to  $14.7^\circ$  after 26.5 years. An important case is when  $I_0 = 7.3^\circ$  and  $h_0 = 0$ . Then the inclination and node are almost constant.

TABLE I.- VARIATION OF INCLINATION

$I_0$	$h_0$	I (2 yrs.)	Max. I. (26.5 yrs.)
$0^0$	undef.	$1.73^0$	$14.7^0$
$1^0$	$270^0$	$.74^0$	$14.9^0$
$1^0$	$90^0$	$2.00^0$	$15.0^0$
$7.3^0$	$180^0$	$8.00^0$	$29.4^0$
$7.3^0$	$0^0$	$7.30^0$	$7.3^0$ (const.)

### 2.3 Solar Radiation Pressure

The magnitude of the solar radiation pressure depends on the weight and cross sectional area of the satellite. The most important effect is a rotation of the line of apsides and a periodic variation in the eccentricity with a period of about one year. To compute the perturbing acceleration, the following assumptions are made:

- (1) The SPS is a flat plate of 10% reflectivity.  
(Reference 1, Section IV.B.1).
- (2) The flat plate maintains an inertial orientation perpendicular to the satellite-sun line.
- (3) Pressure from the microwave transmission can be neglected.
- (4) The earth's shadow can be neglected.

The solar radiation on a flat plate in the vicinity of the earth is (Reference 8):

$$9.02 \cdot 10^{-6} \text{ N/m}^2 \quad (100\% \text{ reflecting body})$$

$$4.51 \cdot 10^{-6} \text{ N/m}^2 \quad (\text{Blackbody})$$

Thus, for a 10% reflecting body, the solar pressure is

$$4.96 \cdot 10^{-6} \text{ N/m}^2 .$$

Solar array area and weight ranges are given in Reference 1, Figures IV.A.5.2,

$$97 \text{ km}^2 < \text{area} < 136 \text{ km}^2 ,$$

$$48 \cdot 10^6 \text{ kg} < \text{weight} < 123 \cdot 10^6 \text{ kg} .$$

For the analyses carried out in this report, the following "nominal" values were taken:

$$\text{Area} = 143 \text{ km}^2$$

$$\text{Weight} = 82.5 \cdot 10^6 \text{ kg} .$$

Let  $\vec{F}$  be the force due to solar radiation pressure and  $M$  the spacecraft weight. Then the perturbing acceleration is

$$\vec{A} = \frac{\vec{F}}{M}$$

where the magnitude  $A = |\vec{A}|$  is constant. Let  $S$  be the surface area in square meters, then

$$|\vec{F}| = \frac{4.96 \cdot 10^{-6}}{g} g S \quad (2.1)$$

where  $g$  is the acceleration of gravity at the surface of the earth ( $g=9.807 \text{ m}(\text{sec})^{-2}$ ). If  $M$  is expressed in kilograms, then

$$\vec{A} = 5.06 \cdot 10^{-7} \frac{S}{M} \quad (2.2)$$

Note that  $\frac{A}{g}$  is unitless. Using the nominal values for area and weight,

$$\frac{S}{M} = 1.73 \text{ m}^2 \text{ kg}^{-1} \quad , \quad (2.3)$$

and

$$\frac{A}{g} = .875 \cdot 10^{-6} \quad (2.4)$$

Taking into account the expected range in size and weight,  $\frac{A}{g}$  can be in the interval

$$.72 \cdot 10^{-6} < \frac{A}{g} < 1.16 \cdot 10^{-6} \quad . \quad (2.5)$$

One additional comment needs to be made on assumption (4). An equatorial geosynchronous satellite will pass through the earth's shadow once a day during the eleven days before and after the equinoxes. It will remain in the shadow for a maximum of 75 minutes on the day of the equinox. The amount of time in one year that the satellite is in the shadow is small and will not be important in studying the long term effects of solar radiation pressure.

### 3.0 Solar Radiation Pressure Effects on the Orbit

Variations in orbital eccentricity due to the perturbing effects of solar radiation pressure are discussed in this section. The magnitude of the perturbing acceleration was discussed in Section 2.3. An approximate solution is given for the variation of  $e$  as a function of time. This solution is valid for small eccentricities, i.e.  $e < .08$ . Comparison to numerical integration shows that the solution is valid for about eight years. After that time, gravitational effects (discussed in Section 4) become important. However, this solution shows the general nature of the perturbations in

eccentricity and argument of perigee. Also, it could be useful to compute station keeping maneuvers for orbit control purposes.

Musen (Reference 9) did some early work on orbit perturbations due to solar radiation pressure. He was concerned with the orbit of Vanguard I where the rotation of the line of apsides (due to oblateness of the earth,  $J_2$ ) was nearly commensurate with the motion of this sun. This caused large perturbations in the height of perigee. Hori (Reference 10) developed a canonical theory for this resonance problem. Solar pressure was assumed by Musen and Hori to be order of magnitude  $(J_2)^2$ .

The case where solar radiation pressure is large (such as with the SPS) has been treated by Zee (Reference 11), Bosch (Reference 12), Ahmad and Stuiver (Reference 13), and Van der Ha and Modi (Reference 14). The analyses of Bosch and Ahmad and Stuiver are restricted to motion in the ecliptic plane with the sun assumed fixed. Their results are thus valid for only a few revolutions of the satellite. Zee shows that the eccentricity will have a period of one year, but he considers only the case where  $e$  is initially zero, and does not give any quantitative results. Van der Ha and Modi use the two variable expansion procedure to describe the yearly motion of  $e$  for the case where the orbit lies in the ecliptic plane. They use an area to weight ratio of 20, whereas References 1, 2 and 3 indicate a value near 2 or 3 (see equation 2.3). These investigators did not consider the important secular increase in eccentricity or coupling between radiation pressure and gravitational perturbations.

### 3.1 The Solar Radiation Perturbing Function

Let  $\vec{r}$  be the satellite's position vector referenced to an earth-centered coordinate system whose x-axis is in the

direction of the earth's north pole. The x- and y- axes lie, therefore, in the equatorial plane.

The acceleration vector<sup>†</sup> of the satellite is

$$\ddot{\vec{r}} = A \frac{\vec{\rho}}{\rho} + \frac{\partial U^*}{\partial \vec{r}}, \quad (3.1)$$

where  $U^*$  is the gravitational force function<sup>††</sup>.  $\vec{\rho}$  is the vector from the sun to the satellite (Figure 1).

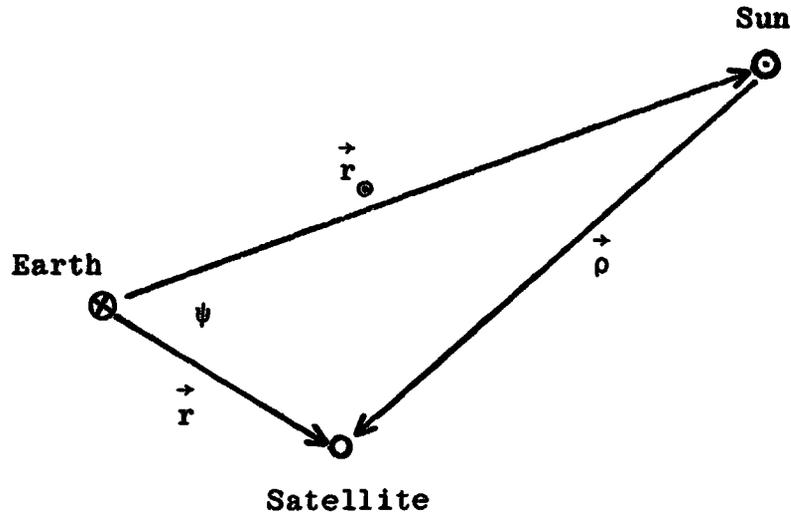


FIGURE 1: Earth-Satellite-Sun Geometry

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<sup>†</sup> Dots refer to derivatives with respect to time, i.e.

$$\frac{d(\quad)}{dt} = (\quad) \dot{\quad} .$$

<sup>††</sup>The sun, moon and earth gravitational effects are included in  $U^*$  .

Let  $\vec{r}_\odot$  be the vector from the earth to the sun. Define the unit vector

$$\hat{\rho} = \frac{\vec{\rho}}{\rho} = -\frac{1}{\rho} (\vec{r}_\odot - \vec{r}) \quad , \quad (3.2)$$

where

$$\rho = |\vec{\rho}| \quad .$$

Also,

$$r_\odot = |\vec{r}_\odot| \quad \text{and} \quad r = |\vec{r}| \quad .$$

For a geosynchronous satellite in a nearly circular orbit,

$$r \approx 42,164 \text{ km.}$$

The earth-sun distance is

$$r_\odot \approx 149.5 \cdot 10^6 \text{ km.}$$

Therefore, the ratio  $\frac{r}{r_\odot}$  will be small, i.e.

$$\frac{r}{r_\odot} \approx 2.8 \cdot 10^{-4} \quad (3.3)$$

The unit vector  $\hat{\rho}$  can be expressed in powers of the small parameter. From the law of cosines (see Figure 1):

$$\rho^2 = r_\odot^2 + r^2 - 2 r_\odot r \cos \psi \quad ,$$

or

$$\frac{1}{\rho} = \frac{1}{r_\odot} \left[ 1 + \left(\frac{r}{r_\odot}\right)^2 - 2\left(\frac{r}{r_\odot}\right) \cos \psi \right]^{-\frac{1}{2}}$$

The above expression can be expanded in powers of  $\frac{r}{r_\odot}$  :

$$\frac{1}{\rho} = \frac{1}{r_\odot} \sum_0^{\infty} \left(\frac{r}{r_\odot}\right)^n P_n(\cos \psi) \quad .$$

where  $P_n(\xi)$  is the Legendre polynomial with argument  $\xi$ .  
The expression for  $\beta$  is then

$$\beta = - \left[ \hat{r}_\ominus - \left( \frac{r}{r_\ominus} \right) \hat{r} \right] \int_0^\psi \left( \frac{r}{r_\ominus} \right)^n P_n(\cos \psi) . \quad (3.4)$$

From the above expression it is seen that the replacement

$$\frac{\vec{\rho}}{\rho} = - \frac{\vec{r}}{r_\ominus} \quad (3.5)$$

involves an error of  $2.8 \cdot 10^{-4}$ . The equations of motion are then

$$\ddot{\vec{r}} = - A \frac{\vec{r}_\ominus}{r_\ominus} + \frac{\partial U^*}{\partial \vec{r}} \quad (3.6)$$

The components of (3.6) in rectangular coordinates are

$$\begin{aligned} \ddot{x} &= - A \frac{x_\ominus}{r_\ominus} + \frac{\partial U^*}{\partial x} , \\ \ddot{y} &= - A \frac{y_\ominus}{r_\ominus} + \frac{\partial U^*}{\partial y} , \\ \ddot{z} &= - A \frac{z_\ominus}{r_\ominus} + \frac{\partial U^*}{\partial z} . \end{aligned} \quad (3.7)$$

Define the new force function

$$\tilde{U} = - A \left[ x \frac{x_\ominus}{r_\ominus} + y \frac{y_\ominus}{r_\ominus} + z \frac{z_\ominus}{r_\ominus} \right] + U^* . \quad (3.8)$$

The differential equations of motion are then

$$\ddot{x} = \frac{\partial \tilde{U}}{\partial x} , \quad \ddot{y} = \frac{\partial \tilde{U}}{\partial y} , \quad \ddot{z} = \frac{\partial \tilde{U}}{\partial z} . \quad (3.9)$$

For perturbation problems, it is desirable to write the force function in the form

$$U = \frac{\mu}{r} (1-V) \quad , \quad (3.10)$$

where

$$V = V_s + V_{\oplus} + V_{\zeta} + V_{\otimes} \quad (3.11)$$

is the "perturbing function".  $\mu$  is the gravitational constant for the earth ( $3.98601 \cdot 10^5 \text{ km}^3 \text{ sec}^{-2}$ ).  $V_s$  is the contribution of solar radiation pressure and can be written as

$$V_s = A \frac{r}{\mu} \left[ x \frac{x_{\oplus}}{r_{\oplus}} + y \frac{y_{\oplus}}{r_{\oplus}} + z \frac{z_{\oplus}}{r_{\oplus}} \right] \quad (3.12)$$

A similar perturbing function was used by Hori (Reference 10).  $V_{\oplus}$  represents the geopotential.  $V_{\zeta}$  and  $V_{\otimes}$  are the gravitational potential functions of the moon and sun, respectively (Reference 7).

### 3 2 Order of Magnitude Considerations

This section considers the magnitudes of the various terms in (3.11). It is shown in Reference 7 that for a geosynchronous satellite in a nearly circular orbit, the magnitudes of the gravitational terms are

$$|V_{\oplus}| \approx 2.5 \cdot 10^{-5}$$

$$|V_{\zeta}| \approx 1.6 \cdot 10^{-5} \quad ,$$

$$|V_{\otimes}| \approx .75 \cdot 10^{-5}$$

From equation (3.12),

$$|V_s| = \frac{a^2}{\mu} A \quad , \quad (3.13)$$

where  $a$  is the semi-major axis.

It ,as shown in Section 2.3 that a typical value of  $A$  is

$$A = .875 \cdot 10^{-6} g \quad .$$

Since

$$g = \frac{\mu}{R_{\oplus}^2} \quad ,$$

where  $R_{\oplus}$  is the radius of the earth, the magnitude of  $V_s$  can be written as

$$|V_s| = \frac{A}{g} \left(\frac{a}{R_{\oplus}}\right)^2 \quad ,$$

or

$$|V_s| = 3.84 \cdot 10^{-5} \quad .$$

Therefore, solar radiation pressure has the order of magnitude of the gravitational terms.

Since  $V_{\oplus}$ ,  $V_{\text{L}}$ ,  $V_{\odot}$  and  $V_s$  have nearly the same magnitudes, it is allowed, for a first approximation, to consider each effect separately in arriving at an analytical solution. The following section will, therefore, investigate the perturbations in the orbital elements due to solar radiation pressure.

### 3.3 Delaunay-Similar Elements

The solution will be developed through the use of canonical element differential equations in an extended phase space. Delaunay-Similar elements in the eccentric anomaly (DSu) have been presented in References 15 and 16. The angular variables are:

u = eccentric anomaly,  
 g = argument of perigee,  
 h = argument of the ascending node,  
 ℓ = time element.

The action variables are:

U = related to the two-body energy,  
 G = total angular momentum magnitude,  
 H = z-component of the angular momentum,  
 L = negative of the total energy.

Differential equations for these variables are<sup>†</sup>

$$\frac{d\alpha_i}{d\tau} = \frac{\partial F}{\partial \beta_i} ,$$

$$\frac{d\beta_i}{d\tau} = - \frac{\partial F}{\partial \alpha_i} , \quad (i=1,2,3,4) \quad (3.14)$$

where the Hamiltonian function is

$$F = U - \frac{\mu}{\sqrt{2L}} + \frac{\mu}{\sqrt{2L}} V \quad (3.15)$$

and V is the perturbing potential function. The time is given in terms of DSu-elements by the equation

$$t = \ell - \frac{U}{2L} e \sin u . \quad (3.16)$$

In unperturbed motion,

$$u = \tau + \text{constant} ,$$

$$\ell = \frac{\mu}{(2L)^{3/2}} \tau + \text{constant} \quad (3.17)$$

---

<sup>†</sup>The following element notation is used:

$$\alpha_1 = u, \quad \alpha_2 = g, \quad \alpha_3 = h, \quad \alpha_4 = \ell, \quad \beta_1 = U, \quad \beta_2 = G,$$

$$\beta_3 = H, \quad \beta_4 = L.$$

and the remaining elements are constants.

The following abbreviations are used (Reference 16):

$$e = \sqrt{1 - \frac{G^2}{U^2}} \quad a = \frac{U}{\sqrt{2L}}$$

$$\frac{r}{a} = 1 - e \cos u \quad , \quad \sin I = \sqrt{1 - \frac{H^2}{G^2}} \quad . \quad (3.18)$$

F is numerically equal to zero, so that U is defined as

$$U = \frac{\mu}{\sqrt{2L}} [1 - V] \quad . \quad (3.19)$$

Therefore, since both U and L depend on V, e and a are slightly different from the instantaneous eccentricity and semi-major axis, respectively, in the case of perturbed motion (V=0).

#### 3.4 Development of F in Terms of Elements

Considering only perturbations due to solar radiation pressure, the hamiltonian is

$$F = U - \frac{\mu}{\sqrt{2L}} + F_s \quad ,$$

where

$$F_s = \frac{\mu}{\sqrt{2L}} V_s \quad .$$

Using (3.12),

$$F_s = \frac{A}{\sqrt{2L}} r \left[ x \frac{x_{\odot}}{r_{\odot}} + y \frac{y_{\odot}}{r_{\odot}} + z \frac{z_{\odot}}{r_{\odot}} \right] \quad (3.20)$$

The coordinates are given in terms of elements by

$$\begin{aligned}
 x &= a \left[ \xi_1 (\cos u - e) + \sqrt{1-e^2} \xi_2 \sin u \right] , \\
 y &= a \left[ \zeta_1 (\cos u - e) + \sqrt{1-e^2} \zeta_2 \sin u \right] , \\
 z &= a \left[ \eta_1 (\cos u - e) + \sqrt{1-e^2} \eta_2 \sin u \right] ,
 \end{aligned} \tag{3.21}$$

where the following abbreviations are used:

$$\begin{aligned}
 \xi_1 &= \cos h \cos g - \sin h \sin g \cos I , \\
 \xi_2 &= \cos h \sin g - \sin h \cos g \cos I , \\
 \zeta_1 &= \sin h \cos g + \cos h \sin g \cos I , \\
 \zeta_2 &= \sin h \sin g + \cos h \cos g \cos I , \\
 \eta_1 &= \sin g \sin I , \\
 \eta_2 &= \cos g \sin I ,
 \end{aligned} \tag{3.22}$$

The direction cosines of the sun are (see Appendix A):

$$\begin{aligned}
 \frac{x_{\odot}}{r_{\odot}} &= \cos(l_{\odot} + g_{\odot}) - e_{\odot} \cos g_{\odot} , \\
 \frac{y_{\odot}}{r_{\odot}} &= C \sin(l_{\odot} + g_{\odot}) - C e_{\odot} \sin g_{\odot} , \\
 \frac{z_{\odot}}{r_{\odot}} &= S \sin(l_{\odot} + g_{\odot}) - S e_{\odot} \sin g_{\odot} ,
 \end{aligned} \tag{3.23}$$

where the following notations are used:

$$\begin{aligned}
 C &= \cos(23^{\circ}27') \quad , \\
 S &= \sin(23^{\circ}27') \quad , \\
 l_{\odot} &= n_{\odot} t + l_{\odot 0} \quad , \\
 n_{\odot} &= 360^{\circ} \cdot (365.2422 \text{ days})^{-1} \quad , \quad (3.24) \\
 e_{\odot} &= .0167 \quad , \\
 g_{\odot} &= 281^{\circ} \quad , \\
 \theta &= n_{\odot} t + l_{\odot 0} + g_{\odot}
 \end{aligned}$$

When expressions for  $x, y, z, r$  and equations (3.23) are inserted into (3.20),  $F_s$  is given in terms of elements:

$$\begin{aligned}
 F_s &= \frac{Aa^2}{\sqrt{2L}} \left\{ \left[ (1+e^2) \cos u - \frac{3}{2} e - \frac{1}{2} e \cos 2u \right] \cdot \right. \\
 &\quad \cdot \left[ \xi_1 (\cos \theta - e_{\odot} \cos g_{\odot}) + N_1 (\sin \theta - e_{\odot} \sin g_{\odot}) \right] \\
 &\quad + (1-e^2) \left[ \sin u - \frac{1}{2} e \sin 2u \right] \cdot \quad (3.25) \\
 &\quad \cdot \left. \left[ \xi_2 (\cos \theta - e_{\odot} \cos g_{\odot}) + N_2 (\sin \theta - e_{\odot} \sin g_{\odot}) \right] \right\}
 \end{aligned}$$

where

$$N_1 = C \zeta_1 + S \eta_1 \quad , \quad N_2 = C \zeta_2 + S \eta_2 \quad . \quad (3.26)$$

The variable  $\theta$  in (3.25) contains the time,

$$j = n_{\odot} t + l_{\odot} + g_{\odot} \quad (3.27)$$

Using the time equation (3.16),  $\theta$  can be considered an abbreviation involving DSu-elements:

$$\theta = n_{\odot} l - \frac{n U}{2L} e \sin u + l_{\odot} + g_{\odot} \quad (3.28)$$

Carrying out the products in (3.25) ,

$$\begin{aligned} F = \frac{Aa^2}{2\sqrt{2L}} & \left\{ \left[ (1+e^2) \xi_1 - (1-e^2) N_2 \right] \cos (\theta+u) + \right. \\ & + \left[ (1+e^2) N_1 + (1-e^2) \xi_2 \right] \sin (\theta+u) + \\ & + \left[ (1+e^2) \xi_1 + (1-e^2) N_2 \right] \cos (\theta-u) + \\ & + \left[ (1+e^2) N_1 - (1-e^2) \xi_2 \right] \sin (\theta-u) - \\ & - 3e \xi_1 \cos \theta - 3e N_1 \sin \theta + \\ & + \left[ \frac{1}{2} e(1-e^2) N_2 - \frac{1}{2} e \xi_1 \right] \cos (\theta+2u) - \\ & - \frac{1}{2} e \left[ N_1 + (1-e^2) \xi_2 \right] \sin (\theta+2u) - \\ & - \frac{1}{2} e \left[ \xi_1 + (1-e^2) N_2 \right] \cos (\theta-2u) - \\ & - \frac{1}{2} e \left[ N_1 - (1-e^2) \xi_2 \right] \sin (\theta-2u) - \\ & - 2 e_{\odot} (\xi_1 \cos g_{\odot} + N_1 \sin g_{\odot}) \left[ (1+e^2) \cos u - \frac{3}{2} e - \frac{1}{2} e \cos 2u \right] - \\ & \left. - 2 e_{\odot} (\xi_2 \cos g_{\odot} + N_2 \sin g_{\odot}) (1-e^2) \left[ \sin u - \frac{1}{2} e \sin 2u \right] \right\} . \end{aligned} \quad (3.29)$$

Since the interest is in long period motion, short period terms (those periodic in  $u$ ) will be eliminated from  $F_s$ . This can be done at the same time that the time equation (3.28) is inserted into  $F_s$ . The terms dependent on time in (3.29) are

$$\frac{\sin}{\cos}(\theta + i u) \quad , \quad i=0, \pm 1, \pm 2,$$

or

$$\frac{\sin}{\cos}(n_{\odot} \ell + \lambda \sin u + l_{\odot\odot} + g_{\odot} + i u), \quad i=0, \pm 1, \pm 2,$$

where

$$\lambda = - \frac{U n_{\odot} e}{2L}$$

The following relations (Reference 17, p.23) are used<sup>†</sup>

$$\sin(\alpha + \beta \sin \gamma) = \sum_{-\infty}^{+\infty} J_j(\beta) \sin(j \gamma + \alpha)$$

$$\cos(\alpha + \beta \cos \gamma) = \sum_{-\infty}^{+\infty} J_j(\beta) \cos(j \gamma + \alpha)$$

Thus,

$$\begin{aligned} \sin(n_{\odot} \ell + \lambda \sin u + l_{\odot\odot} + g_{\odot} + i u) &= \sum_{j=-\infty}^{+\infty} J_j(\lambda) \sin \left[ n_{\odot} \ell \right. \\ &\quad \left. + l_{\odot\odot} + g_{\odot} + (i+j)u \right] \quad . \end{aligned} \quad (3.30)$$

The mean of a function  $f(u)$  with respect to  $u$  is defined as

$$\langle f(u) \rangle_u = \frac{1}{2\pi} \int_0^{2\pi} f(u) du \quad .$$

<sup>†</sup>  $J_j(\beta)$ ,  $j=0, 1, 2, \dots$ , are the Bessel coefficients.

Therefore, using (3.30) ,

$$\langle e^{in(\theta + i u)} \rangle_u = J_{-1}(\lambda) \sin(n_{\odot} \ell + \ell_{\odot\odot} + g_{\odot}). \quad (3.31)$$

Similarly,

$$\langle \cos(\theta + i u) \rangle_u = J_{-1}(\lambda) \cos(n_{\odot} \ell + \ell_{\odot\odot} + g_{\odot}). \quad (3.32)$$

Using (3.29), (3.31) and (3.32), the elimination of short period terms results in

$$\begin{aligned} \langle F_s \rangle_u = \frac{Aa^2}{\sqrt{2L}} & \left\{ \left[ (1-e^2) J_1(\lambda) N_2 - \right. \right. \\ & - \left. \frac{1}{2} e (3J_0 + J_2) \xi_1 \right] \cos(n_{\odot} \ell + \ell_{\odot\odot} + g_{\odot}) \\ & - \left[ (1-e^2) J_1(\lambda) \xi_2 + \right. \\ & + \left. \frac{1}{2} e (3J_0(\lambda) + J_2(\lambda)) N \right] \sin(n_{\odot} \ell + \ell_{\odot\odot} + g_{\odot}) + \\ & \left. + \frac{3}{2} e e_{\odot} (\xi_1 \cos g_{\odot} + N_1 \sin g_{\odot}) \right\}. \quad (3.33) \end{aligned}$$

Note that several terms cancelled because of the identity

$$J_j(\lambda) = (-1)^j J_{-j}(\lambda) .$$

The averaged hamiltonian is now

$$\langle F \rangle_u = U - \frac{\mu}{\sqrt{2L}} + \langle F_s \rangle_u . \quad (3.34)$$

Since  $u$  does not appear explicitly in  $\langle F \rangle_u$  .  $U$  will be a constant. The remaining developments will concern (3.34) only. Therefore, the notation  $\langle \rangle_u$  will no longer be needed.

The hamiltonian in terms of elements is then

$$F = U - \frac{\mu}{\sqrt{2L}} + F_s \quad . \quad (3.35)$$

where  $F_s$  is a function of  $g, h, l, U, G, H, L$  given by the right side of (3.33).

A further simplification of  $F_s$  can be made. Consider the Bessel coefficients appearing in (3.33). The argument  $\lambda$  has been defined as

$$\lambda = \frac{U_n}{2L} e \quad .$$

Since  $L$  is the negative of the total energy,

$$L = \frac{\mu}{2\tilde{a}} \left( 1 - 2 \frac{\tilde{a}}{r} V_s \right) \quad , \quad (3.36)$$

where  $\tilde{a}$  is the instantaneous osculating semi-major axis. From (3.19),

$$U = \frac{\mu}{2L} (1 - V_s) \quad ,$$

Inserting (3.36) into the above expression and expanding in powers of  $V_s$ <sup>†</sup>:

$$U = \sqrt{\mu\tilde{a}} \left[ 1 + O(V_s^2) \right] \quad . \quad (3.37)$$

From (3.36) and (3.37) ,

$$\frac{2L}{U} = \sqrt{\frac{\mu}{\tilde{a}^3}} \left[ 1 + O\left(\frac{\tilde{a}}{r} V_s\right) \right] \quad . \quad (3.38)$$

---

<sup>†</sup> In the unperturbed case,  $U$  is equivalent to the classical Delaunay variable  $L$  .

Since  $n^2 = \sqrt{\frac{\mu}{a^3}}$  is the osculating mean motion, it is seen that

$$\frac{U_n}{2L}$$

is approximately the ratio of the mean motion of the satellite to mean motion of the sun, i.e.

$$\frac{U_n}{2L} \approx \frac{1}{365} \quad (3.39)$$

Therefore,

$$|\lambda| \approx (2.8 \cdot 10^{-3}) e$$

or

$$|\lambda| < 2.8 \cdot 10^{-3}$$

for a geosynchronous satellite. Any term depending on  $\lambda$  will therefore be neglected.

The Bessel coefficients can be expressed as a power series in the argument:

$$J_n(\lambda) = \frac{\lambda^n}{2^n n!} \left[ 1 - \frac{\lambda^2}{2^2 \cdot 1 \cdot (n+1)} + \frac{\lambda^4}{2^4 \cdot 1 \cdot 2 \cdot (n+1) \cdot (n+2)} - \dots \right]$$

Since all powers of  $\lambda$  can be neglected,  $J_1(\lambda)$  and  $J_2(\lambda)$  in  $F_s$  can be set to zero and  $J_0(\lambda)$  can be set to one.

Thus, the expression for  $F_s$  becomes

$$F_s = -\frac{3}{2} G \varepsilon \left[ e \xi_1 (\cos v - e_{\odot} \cos g_{\odot}) + e N_1 (\sin v - e_{\odot} \sin g_{\odot}) \right] \quad (3.40)$$

where

$$\begin{aligned}
 e\xi_1 &= e \cos g \cos h - e \sin g \sin h \cos I \quad , \\
 eN_1 &= C \left[ e \cos g \sin h + e \sin g \cos h \cos I \right] \\
 &\quad + S e \sin g \sin I \quad , \qquad (3.41)
 \end{aligned}$$

$$v = n_{\oplus} k + l_{\oplus} + g_{\oplus} \quad .$$

The small parameter  $\epsilon$  is unitless. (Remember that  $A$  has units of acceleration).  $g$  is the acceleration of gravity at the surface of the earth and  $R_{\oplus}$  the radius of the earth. Therefore,

$$\epsilon = \frac{A}{g} \left( \frac{a}{R_{\oplus}} \right)^2 \frac{\mu}{G\sqrt{2L}} \quad .$$

But from

$$G = U \sqrt{1-e^2} \quad ,$$

and (3.19) ,

$$G\sqrt{2L} = \mu (1-V_s) \sqrt{1-e^2}$$

so that  $\epsilon$  can be expressed as

$$\epsilon = \frac{A}{g} \left( \frac{a}{R_{\oplus}} \right)^2 (1-V_s)^{-1} (1-e^2)^{-\frac{1}{2}} \quad . \qquad (3.42)$$

Since  $\epsilon$  is already small,  $V_s$  and  $e^2$  can be neglected

$$\epsilon = \frac{A}{g} \left( \frac{a}{R_{\oplus}} \right)^2 \quad . \qquad (3.43)$$

Therefore, the small parameter  $\epsilon$  is the order of magnitude of  $|V_s|$  (see section 3.2).

### 3.5 Element Differential Equations

Introduce the non-singular elements

$$p = \sqrt{1 - \frac{G^2}{U^2}} \cos(g+h) \quad , \quad q = \sqrt{1 - \frac{G^2}{U^2}} \sin(g+h) \quad ,$$

$$P = \sqrt{1 - \frac{G}{H}} \cos h \quad , \quad Q = - \sqrt{1 - \frac{G}{H}} \sin h \quad .$$

(3.44)

These elements are defined for zero eccentricity and inclination.

The differential equations for  $p$  and  $q$  are given by the chain rule:

$$\frac{dp}{d\tau} = \frac{\partial p}{\partial G} \frac{dG}{d\tau} + \frac{\partial p}{\partial g} \frac{dg}{d\tau} + \frac{\partial p}{\partial U} \frac{dU}{d\tau} + \frac{\partial p}{\partial h} \frac{dh}{d\tau} \quad ,$$

with a similar equation for  $\frac{dq}{d\tau}$  . Make use of the canonical

differential equations

$$\frac{dg}{d\tau} = \frac{\partial F}{\partial G} \quad , \quad \frac{dG}{d\tau} = - \frac{\partial F}{\partial g} \quad ,$$

$$\frac{dh}{d\tau} = \frac{\partial F}{\partial H} \quad , \quad \frac{dU}{d\tau} = - \frac{\partial F}{\partial u}$$

Also, since  $F$  no longer depends on  $u$  :

$$\frac{\partial F}{\partial u} = 0$$

The necessary partial derivatives of  $p$  and  $q$  are:

$$\begin{aligned} \frac{\partial p}{\partial G} &= -\frac{\eta^2}{Ge} \cos(g+h) , & \frac{\partial q}{\partial G} &= -\frac{\eta^2}{Ge} \sin(g+h) , \\ \frac{\partial p}{\partial g} &= -e \sin(g+h) , & \frac{\partial q}{\partial g} &= e \cos(g+h) , \\ \frac{\partial p}{\partial h} &= -e \sin(g+h) , & \frac{\partial q}{\partial h} &= e \cos(g+h) . \end{aligned} \quad (3.45)$$

The differential equations for  $p$  and  $q$  are then

$$\begin{aligned} \frac{dp}{d\tau} &= \frac{\eta^2}{Ge} \cos(g+h) \frac{\partial F}{\partial g} - e \sin(g+h) \left( \frac{\partial F}{\partial G} + \frac{\partial F}{\partial H} \right) , \\ \frac{dq}{d\tau} &= \frac{\eta^2}{Ge} \cos(g+h) \frac{\partial F}{\partial g} + e \cos(g+h) \left( \frac{\partial F}{\partial G} + \frac{\partial F}{\partial H} \right) . \end{aligned} \quad (3.46)$$

where  $\eta = \sqrt{1-e^2}$  .

It is necessary to develop the right sides of equations (3.46) in terms of  $p, q, P, Q$  . This will be done in the following steps:

- (1) Compute the derivatives of  $F$  with respect to  $g, G, H$  .
- (2) Insert these derivatives into the right sides of equations (3.46).
- (3) Use equations (3.44) to express the right sides of (3.46) in terms of  $p, q, P, Q$  .

These three steps have been carried out in Appendix B. All the necessary partial derivatives are given there. The resulting nonsingular equations are:

$$\frac{dp}{d\tau} = \frac{3}{2} \epsilon \left\{ \left[ \eta^2 P Q + q Q (qP+Qp) \right] (\cos v - e_0 \cos g_0) - \right. \\ \left. - \left[ \eta^2 C (1-P^2) + \eta^2 S P (2-(P^2+Q^2))^{\frac{1}{2}} - \right. \right. \\ \left. - C q P (qP+Qp) + \right. \\ \left. \left. + S q (qP+Qp)(1-(Q^2+P^2)) (2-(P^2+Q^2))^{-\frac{1}{2}} \right] (\sin v - e_0 \sin g_0) \right\} \\ (3.47)$$

$$\frac{dq}{d\tau} = \frac{3}{2} \epsilon \left\{ \left[ \eta^2 (1-Q^2) - p Q (qP+Qp) \right] (\cos v - e_0 \cos g_0) + \right. \\ \left. - \left[ \eta^2 C P Q - \eta^2 S Q (2-(P^2+Q^2))^{\frac{1}{2}} + \right. \right. \\ \left. + C p P (qP+Qp) - \right. \\ \left. \left. - S p (qP+Qp) (1-(Q^2+P^2)) (2-(P^2+Q^2))^{-\frac{1}{2}} \right] (\sin v - e_0 \cos g_0) \right\}$$

Notice that  $\eta$  is a function of  $p$  and  $q$  :

$$\eta = \sqrt{1-(p^2+q^2)} .$$

The differential equations for  $P$  and  $Q$  are developed in a similar manner as that done above. The details are carried out in Appendix B and the equations are shown on the following page:

$$\begin{aligned} \frac{dP}{d\tau} = & -\frac{3}{4} \epsilon \left\{ Q \left[ P (qQ - pP) + 2 p \right] (\cos v - e_{\odot} \cos g_{\odot}) + \right. \\ & + C P \left[ P (qQ - pP) + 2 p \right] (\sin v - e_{\odot} \sin g_{\odot}) + \\ & + S \left[ 1 - (P^2 + Q^2) \right] \left[ 2 - (P^2 + Q^2) \right]^{-\frac{1}{2}} \left[ P (pP - qQ) - \right. \\ & \left. \left. - 2 p \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} \end{aligned} \quad (3.48)$$

$$\begin{aligned} \frac{dQ}{d\tau} = & \frac{3}{4} \epsilon \left\{ Q \left[ Q (pP - qQ) + 2 q \right] (\cos v - e_{\odot} \cos g_{\odot}) + \right. \\ & + C P \left[ Q (pP - qQ) + 2 q \right] (\sin v - e_{\odot} \sin g_{\odot}) + \\ & + S \left[ 1 - (P^2 + Q^2) \right] \left[ 2 - (P^2 + Q^2) \right]^{-\frac{1}{2}} \left[ Q (pP - qQ) + \right. \\ & \left. + 2 q \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} . \end{aligned}$$

The remaining differential equations come from the canonical equations

$$\frac{dl}{d\tau} = \frac{\partial F}{\partial L} \quad , \quad \frac{dL}{d\tau} = -\frac{\partial F}{\partial l} .$$

These are computed in a straightforward manner from (3.35), (3.40) and (3.41):

$$\begin{aligned} \frac{dl}{d\tau} = \frac{\mu}{(2L)^{3/2}} & \left\{ 1 + \frac{9}{2} \epsilon \left[ 1 - (p^2+q^2) \right]^{1/2} \left\{ \left[ p(1-Q^2) - \right. \right. \right. \\ & - qQP \left. \right] (\cos v - e_{\odot} \cos g_{\odot}) + \\ & + \left[ C(q-qP^2-pPQ) + \right. \\ & \left. \left. + S(qP+Qp)(2-(P^2+Q^2))^{3/2} \right] (\sin v - e_{\odot} \cos g_{\odot}) \right\} , \end{aligned} \quad (3.49)$$

$$\begin{aligned} \frac{dL}{d\tau} = \frac{3}{2} n_{\odot} \epsilon & \left[ 1 - (p^2+q^2) \right]^{1/2} \left\{ \left[ p(1-Q^2) - qQP \right] \sin v - \right. \\ & - \left[ C(q-qP^2-pPQ) + \right. \\ & \left. \left. + S(qP+Qp)(2-(P^2+Q^2))^{3/2} \right] \cos v \right\} . \end{aligned}$$

### 3.6 Orbits in the Ecliptic Plane

For orbits that lie in the ecliptic plane<sup>†</sup>,

$$h = 0 \quad , \quad I = \epsilon$$

Therefore,

$$P = \sqrt{1 - \cos \epsilon} \quad , \quad Q = 0 \quad ,$$

and the differential equations are greatly simplified:

$$\frac{dp}{d\tau} = -\frac{3}{2} \epsilon \left[ 1 - (p^2+q^2) \right] (\sin v - e_{\odot} \sin g_{\odot}) \quad . \quad (3.50)$$

<sup>†</sup>  $\epsilon$  is the angle between the equatorial and ecliptic planes.

$$\frac{dq}{d\tau} = \frac{3}{2} \epsilon \left[ 1 - (p^2 + q^2) \right] (\cos v - e_{\odot} \cos g_{\odot}) \quad , \quad (3.51)$$

$$\frac{dP}{d\tau} = 0 \quad , \quad (3.52)$$

$$\frac{dQ}{d\tau} = 0 \quad , \quad (3.53)$$

$$\frac{d\ell}{d\tau} = \frac{\mu}{(2L)^{3/2}} \left\{ 1 + \frac{9}{2} \epsilon \left[ 1 - (p^2 + q^2) \right]^{\frac{1}{2}} \cdot \left[ p (\cos v - e_{\odot} \cos g_{\odot}) + q (\sin v - e_{\odot} \sin g_{\odot}) \right] \right\} \quad (3.54)$$

$$\frac{dL}{d\tau} = \frac{3}{2} n_{\odot} \epsilon \left[ 1 - (p^2 + q^2) \right]^{\frac{1}{2}} \left[ p (\sin v - e_{\odot} \cos g_{\odot}) - q (\cos v - e_{\odot} \sin g_{\odot}) \right] \quad . \quad (3.55)$$

Comments

- (1) Equations (3.52) and (3.53) indicate that the orbital plane ( i.e. inclination) will remain fixed, as expected. This is because there are no out-of-plane perturbations on the orbit.
- (2) The equations no longer depend on  $\frac{\cos \epsilon}{\sin \epsilon}$  .
- (3) The equations for p and q contain secular terms that are proportional to the eccentricity of the sun's orbit.

### 3.7 Solution of the Linearized Equations

When the eccentricity and inclination of the satellite are small, the element differential equations can be simplified by neglecting from equations (3.47), (3.48) and (3.49), the second and higher degree terms in  $p, q, P, Q$ . The resulting equations are:

$$\frac{dp}{d\tau} = -\frac{3}{2} \epsilon \left[ C + S \sqrt{2} P \right] (\sin v - e_{\odot} \sin g_{\odot}) \quad , \quad (3.56)$$

$$\begin{aligned} \frac{dq}{d\tau} = \frac{3}{2} \epsilon \left[ \cos v - e_{\odot} \cos g_{\odot} \right. \\ \left. + S \sqrt{2} Q (\sin v - e_{\odot} \sin g_{\odot}) \right] \quad , \quad (3.57) \end{aligned}$$

$$\frac{dP}{d\tau} = \frac{3}{4} \sqrt{2} S \epsilon p (\sin v - e_{\odot} \sin g_{\odot}) \quad , \quad (3.58)$$

$$\frac{dQ}{d\tau} = -\frac{3}{4} \sqrt{2} S \epsilon q (\sin v - e_{\odot} \sin g_{\odot}) \quad , \quad (3.59)$$

$$\begin{aligned} \frac{dl}{d\tau} = \frac{\mu}{(2L)^{3/2}} \left\{ 1 + \frac{9}{2} \epsilon \left[ p (\cos v - e_{\odot} \cos g_{\odot}) + \right. \right. \\ \left. \left. + C q (\sin v - e_{\odot} \sin g_{\odot}) \right] \right\} \quad , \quad (3.60) \end{aligned}$$

$$\frac{dL}{d\tau} = \frac{3}{2} n_{\odot} \epsilon \left[ p \sin v - C q \cos v \right] \quad (3.61)$$

These equations are not linear since  $v$  contains  $l$  through the equation

$$v = n_{\odot} l + l_{\odot 0} + g_{\odot} \quad .$$

Also,  $\epsilon$  depends on  $L$ . However, notice that the perturbation in total energy (equation (3.61)) will be small because of the coefficient  $n_{\odot} \epsilon$ . It is also periodic. It is therefore allowed to let  $L$  be a constant.

The perturbed part of the time element equation (3.60) will be small because it is proportional to the eccentricity. Therefore, let

$$\lambda = \frac{\mu}{(2L)^{3/2}} \tau \quad (3.62)$$

The derivatives of  $P$  and  $Q$  are also proportional to  $e$ . In fact, the effect of solar radiation pressure on the orbital plane is negligible when compared to the gravitational effects. It is shown in Reference 7 that sun-moon-earth gravity cause a motion of the orbital plane that is described by the following expressions:

$$P = \sigma \left[ 1 - \alpha \sin (\omega \tau + \theta) \right] \quad (3.63)$$
$$Q = \frac{\sigma}{\gamma} \alpha \cos (\omega \tau + \theta) \quad ,$$

where

$$\sigma = .0902 \quad , \quad \gamma = 1.015 \quad , \quad \omega = 5.170 \cdot 10^{-5}.$$

$\alpha$  and  $\theta$  are integration constants that depend on the initial values of  $P$  and  $Q$ . The mean values are

$$P = \sigma \quad , \quad Q = 0 \quad , \quad (3.64)$$

and correspond to the equilibrium solution

$$I = 7.31^{\circ} \quad , \quad h = 0 \quad .$$

For most initial condition, P and Q will never be far away from their equilibrium values. Therefore, insert (3.64) into equations (3.56) and (3.57), giving approximate equations for the derivatives of p and q :

$$\frac{dp}{d\tau} = -\frac{3}{2} \epsilon \beta (\sin \theta - e_{\odot} \sin g_{\odot}) , \quad (3.65)$$

$$\frac{dq}{d\tau} = \frac{3}{2} \epsilon (\cos \theta - e_{\odot} \cos g_{\odot}) ,$$

where the additional abbreviations have been introduced:

$$\beta = C + S\sigma \sqrt{2} , \quad \theta = \delta \tau + l_{\odot 0} + g_{\odot} , \quad (3.66)$$

$$\delta = \frac{n_{\odot} \mu}{(2L)^{3/2}} . \quad (3.67)$$

But since  $\frac{(2L)^{3/2}}{\mu}$  is the mean motion of the geosynchronous satellite,

$$\delta = (365.25)^{-1} . \quad (3.68)$$

Equations (3.65) are uncoupled and can therefore be immediately solved:

$$p = \Phi \beta (\cos \theta + \tau \delta e_{\odot} \sin g_{\odot}) + C_1 \quad (3.69)$$

$$q = \Phi (\sin \theta - \tau \delta e_{\odot} \cos g_{\odot}) + C_2 ,$$

where

$$\Phi = \frac{3 \epsilon}{2 \delta} . \quad (3.70)$$

$C_1$  and  $C_2$  are integration constants that depend on the initial values of  $p$  and  $q$ .

### Comments on the Solution

1. The motion can be represented in a plane with  $p, q$  the rectangular coordinates.
2. In all cases,  $p, q$  will describe an ellipse whose center has a linear translation. The motion around the ellipse has a period of one year.
3. The mean eccentricity will have a linear increase or decrease.

### 3.8 Numerical Results

This section will discuss some quantitative and qualitative results of the solution developed in Section 3.7. First, the solution is verified by comparing it to a numerical integration. Then the solution is used to describe the general behavior of orbital eccentricity and longitude of perigee.

#### 3.8.1 Numerical Experiments

It is necessary to demonstrate that the analytical solution and its associated assumptions are valid. This has been done by carrying out comparisons with a numerical solution obtained from the STEPR multirevolution program (References 18 and 19). Since the purpose here is to check out the accuracy of equations (3.69), only solar radiation pressure was included as a force model option in STEPR. The additional effects of gravity will be discussed in Section 4. It will also be shown there that equations (3.69) give a good approximation to the complete problem over a period of a few years.

Comparisons between the analytical solution and STEPR are shown in Tables II(a) and II(b). The area to weight ratio

was assumed to be  $1.73 \text{ m}^2 \text{ kg}^{-1}$  (see Section 2.). The initial epoch was noon January 1, 1980. Eccentricity  $e$  based on the analytical solution was first computed by evaluating  $p_A$  and  $q_A$  from equations (3.69). The algorithm for computing  $p_A$  and  $q_A$  is described in Appendix C.  $e_A$  is then obtained from

$$e_A = (p_A + q_A)^{\frac{1}{2}} \quad (3.71)$$

The location of perigee is obtained from

$$\begin{aligned} \cos \tilde{g}_A &= \frac{e_A}{p_A} \quad , \\ \sin \tilde{g}_A &= \frac{e_A}{q_A} \quad , \end{aligned} \quad (3.72)$$

where  $\tilde{g}$  is the longitude of perigee, defined by

$$\tilde{g} = g + h \quad (3.73)$$

The values from STEPR are denoted by  $e_N$  and  $\tilde{g}_N$ .

TABLE II.- ANALYTIC SOLUTION VERSUS NUMERICAL SOLUTION

(a)  $e_0 = 0$  ,  $\tilde{g}_0$  undefined

Years						Percent Error	
	$e_A$	$e_N$	$\tilde{g}_A$	$\tilde{g}_N$	$e_A$	$\tilde{g}_A$	
9.6	.0481	.0485	132.7 <sup>0</sup>	133.7 <sup>0</sup>	0.8	0.7	
19.5	.0610	.0620	148.8	149.6	1.6	0.5	
30.1	.0511	.0517	-173.8	-174.2	1.2	0.2	

TABLE II. - CONTINUED

(b)  $e_0 = .021$  ,  $\tilde{g}_0 = -80.6^\circ$

Years					Percent Error	
	$e_A$	$e_N$	$\tilde{g}_A$	$\tilde{g}_N$	$e_A$	$\tilde{g}_A$
9.6	.0328	.0337	153.7 <sup>0</sup>	155.0 <sup>0</sup>	2.6	0.9
19.5	.0502	.0519	167.4	168.3	3.3	0.5
30.1	.0548	.0553	-150.9	-151.9	0.9	0.7

Comments

1. The results in Table II show that the analytical solution gives between 2 and 3 digits of accuracy over a period of 30 years. This is sufficiently accurate to describe the general behavior of the orbit.
2. The accuracy of equations (3.69) substantiates the assumptions that were made in the course of their derivation. The same approach can be used to solve the complete problem (including gravitational perturbations).
3. The errors in  $e_A$  and  $\tilde{g}_A$  do not increase with time. Therefore the analytical solution contains all long period effects.
4. Plots of  $e$  versus time are shown in Figures 8 and 9, respectively.

3.8.2 Qualitative Description of the Orbit

The orbital behavior can be described for different initial conditions on eccentricity  $e_0$  and longitude of perigee  $\tilde{g}_0$ . Also, the motion will depend on the epoch of initialization since the problem depends explicitly on time (i e. on the position of the sun in its imagined orbit about the earth). The example cases considered here are

shown in the table below. These cases were chosen so as to demonstrate some of the essential features of the motion and to illustrate some preferred orbits.

TABLE III.- INITIAL CONDITIONS FOR EXAMPLES

Case No.	$e_0$	$\tilde{g}_0$	Initial Epoch
1	.0	undefined	noon, 1 Jan. 1980
2	.0210	$-80.6^\circ$	noon, 1 Jan. 1980
3	.0292	$-34.8$	noon, 1 Jan. 1980
4	.0214	$-67.4$	noon, 1 Jan. 1980
5	.0	undefined	noon, 3 April 1980
6	.0	undefined	noon, 1 Oct. 1980

For each case, the evolution of elements  $p$ ,  $q$  and  $e$  is considered over a period of ten years. This is the maximum time interval for which the analytical solution is valid (see numerical comparisons in Section 4). The area to weight ratio was taken to be  $1.73 \text{ m}^2 \text{ kg}^{-1}$ , as before. Equations (3.69) were programmed on the Hewlett-Packard 9810 micro-computer, using the algorithm described in Appendix C. The HP9810 plotter was used to produce the plots shown in Figures 2 through 9.

Two figures illustrate the results for each case:

(a)  $p$  versus  $q$ : This shows the motion in  $p, q$ -space. Also,  $e$  and  $\tilde{g}$  are the polar coordinates of the point that traces out the curve (see equations (3.44)). The direction of motion is indicated by arrows. When the curve passes through the origin, the limiting value of  $\tilde{g}$  will be the tangent to the curve.

(b)  $e$  versus time: This figure shows the variations in  $e$  as a function of time in years.

Discussion of Results

Case 1 (Figures 2(a) and 2(b))

- (1) The motion begins from the origin in the  $p, q$ -plane and in a direction that is approximately  $90^\circ$  from the sun's initial mean longitude<sup>†</sup>  $\theta_0$ . This can be seen from equations (3.69) with  $\beta = 1$  :

$$e \cos \tilde{g} = \Phi \left[ \cos (\delta \tau + \theta_0) - \cos \theta_0 \right] ,$$

$$e \sin \tilde{g} = \Phi \left[ \sin (\delta \tau + \theta_0) - \sin \theta_0 \right] .$$

Notice that  $e = 0$  for  $\tau = 0$ . Then,

$$\tan \tilde{g} = \frac{\sin (\delta \tau + \theta_0) - \sin \theta_0}{\cos (\delta \tau + \theta_0) - \cos \theta_0} .$$

Using l'Hopital's rule,

$$\lim_{\tau \rightarrow 0} \tan \tilde{g} = - \cot \theta_0 = \tan ( 90^\circ + \theta_0 ) .$$

For this case, noon on January first corresponds to  $\theta_0 = - 80.6^\circ$ . Therefore  $\tilde{g}_0 \approx 9.4^\circ$ .

---

<sup>†</sup> The sun's initial mean longitude is  $\theta_0 = \lambda_{\odot 0} + g_{\odot}$ , where  $g_{\odot}$  is measured from the vernal equinox (x-axis) in the ecliptic plane. See also Appendix C.

- (2) The center of the ellipse in the  $p, q$ - plane is initially at  $(C_1, C_2)$  and moves toward the lower left at the rate of  $\dot{\Phi} \approx 2\pi$  per year. Notice that this rate depends on the area to weight ratio of the satellite by way of the small parameter  $\Phi$ . The direction cosines of the motion are  $(\sin g_\odot, -\cos g_\odot)$ . Using  $g_\odot = 282.5^\circ$ , the direction cosines are  $(-.9763, -.2166)$ .
- (3) The eccentricity is periodic and returns almost to zero after one year. Its maximum in the first year is approximately  $2\Phi$ . The motion of the ellipse in the  $p, q$ - plane is seen as a nearly linear component in the variation of  $e$ .

Case 2 (Figures 3(a) and 3(b))

- (1) The initial values of  $\epsilon_0$  and  $\tilde{g}_0$  were chosen so that constants  $C_1$  and  $C_2$  were both zero. The ellipse is centered initially at the origin. For a while, the eccentricity is nearly constant at the value of  $\Phi$ . But as the ellipse moves away from the origin, the oscillations in  $e$  increase in amplitude. Just as the origin is no longer inside the ellipse, the amplitude will be  $2\Phi$ . After that, the amplitude stays the same, but there is a linear component to the variation of  $e$ . (See also Figure 9.)
- (2) As long as the origin lies inside the ellipse,  $\tilde{g}$  will circulate once a year. If the origin is still inside the ellipse and the path passes near the origin,  $\tilde{g}$  will change very rapidly. When the origin no longer lies inside the ellipse,  $\tilde{g}$  varies about the mean value of  $g_\odot - 90^\circ$ . The amplitude of these variations decreases with time.

Case 3 (Figures 4(a) and 4(b))

- (1) The initial conditions for this case were chosen so that the ellipse would initially pass through the origin and its center would move directly toward the origin.
- (2) The important result of this choice of initial conditions is revealed by the behavior of  $e$  as shown in Figure 4(b). In fact Figure 4(b) is a mirror image of Figure 3(b).

Case 4 (Figures 5(a) and 5(b))

- (1) This case is similar to Case 3 in that the center of the ellipse will pass through the origin. However, the motion is such that the eccentricity is nearly constant for a longer period of time. For example the maximum value of eccentricity will be less than .025 for four and a half years.
- (2) The longitude of perigee  $\tilde{g}$  will circulate.

Case 5 (Figures 3(a) and 6(b))

This case and the next one show the effect of the epoch of initialization on the subsequent motion.  $e_0$  and  $\tilde{g}_0$  are the same as in Case 1, but the epoch is three months later. The ellipse is moving directly away from the origin and the linear growth component of  $e$  is more severe. This case is not desirable when large eccentricities are to be avoided.

Case 6 (Figures 7(a) and 7(b))

Again, the eccentricity is initially zero, but the epoch is six months later than Case 5 and nine months later than Case 1. However, the motion is very similar to Case 3. The center of the ellipse in the  $p, q$ -plane moves directly toward the origin

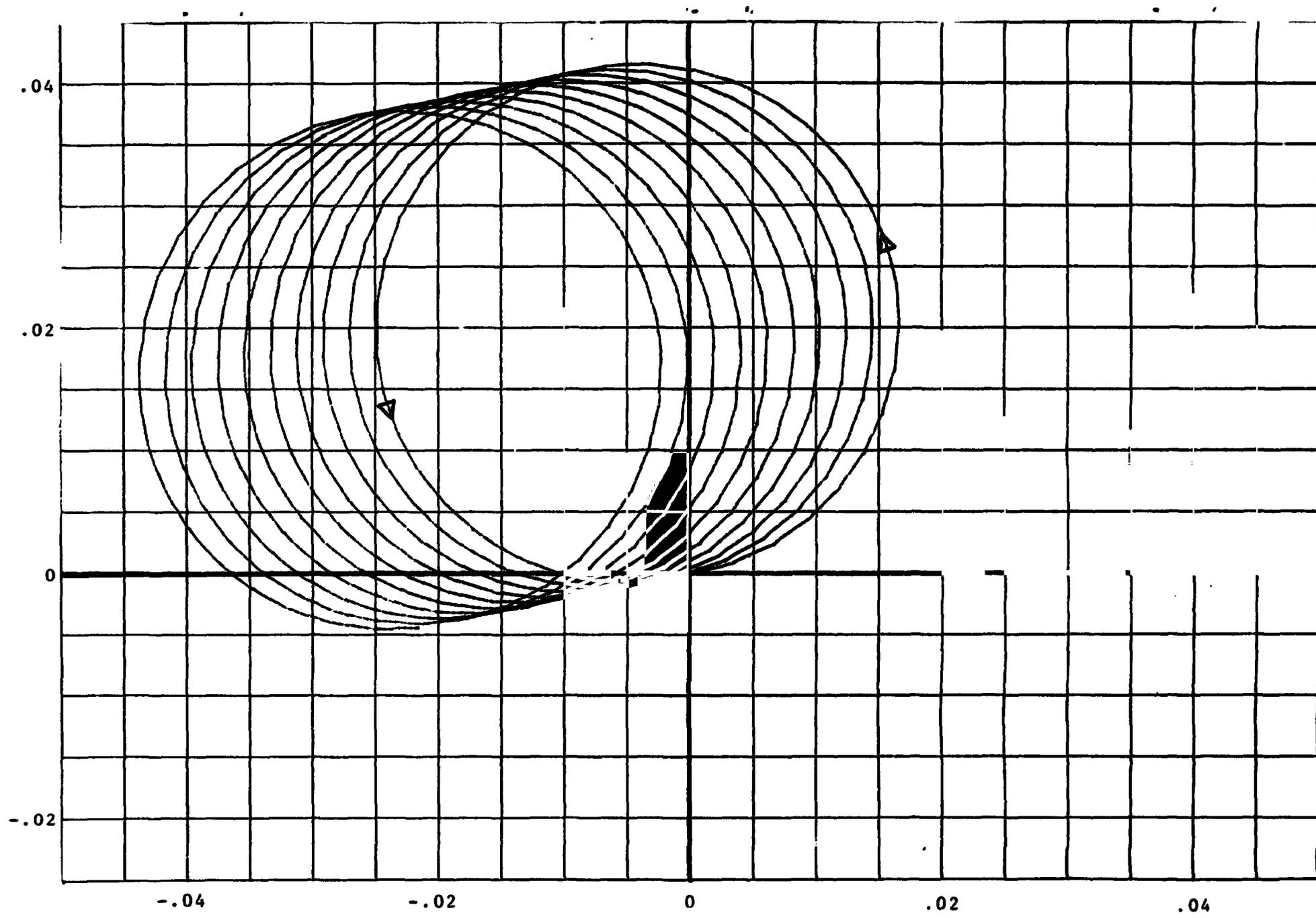


FIGURE 2: Variations in Eccentricity, Case 1  
(a) p versus q

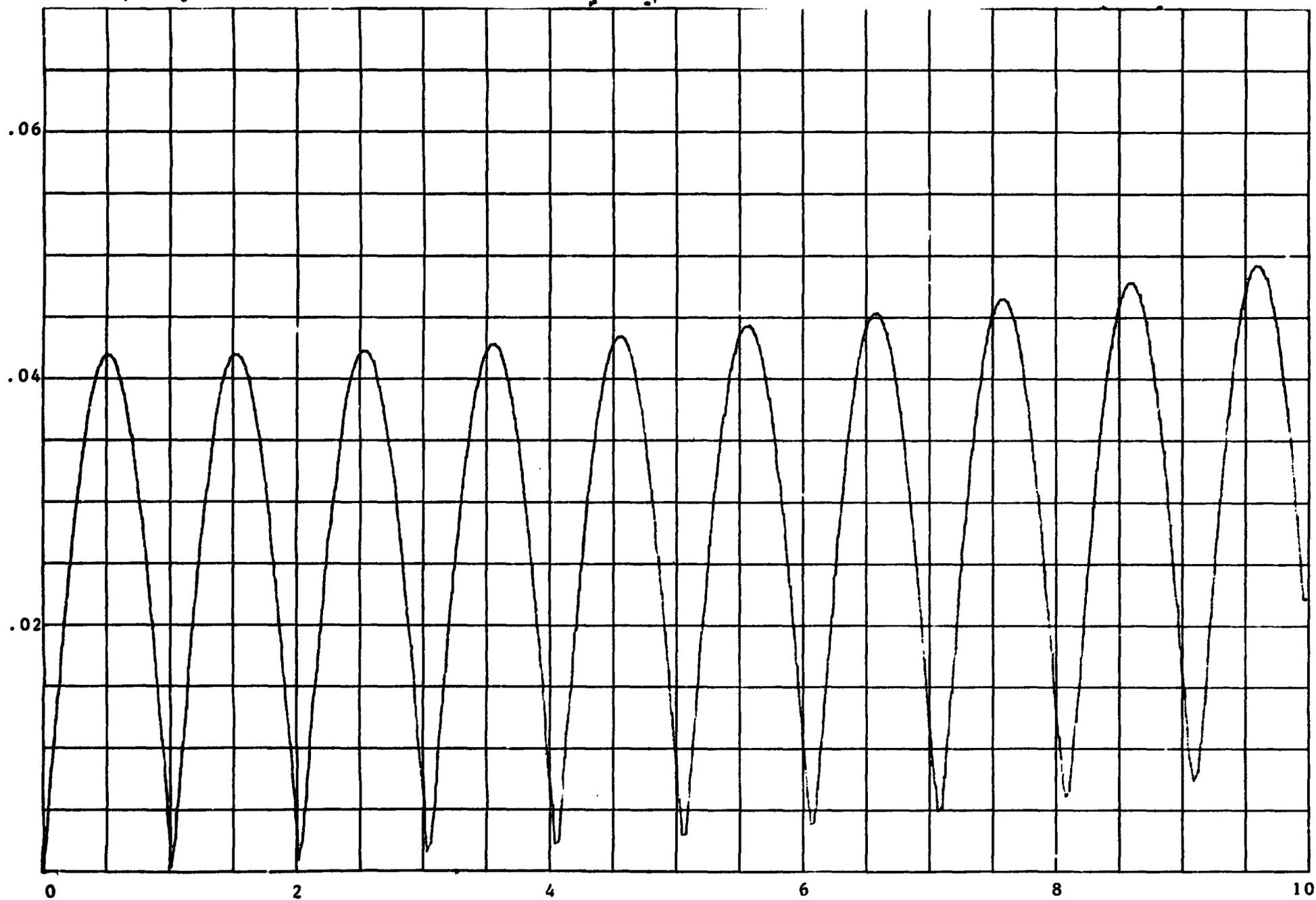


FIGURE 2: Concluded  
(b) e versus Time (years)

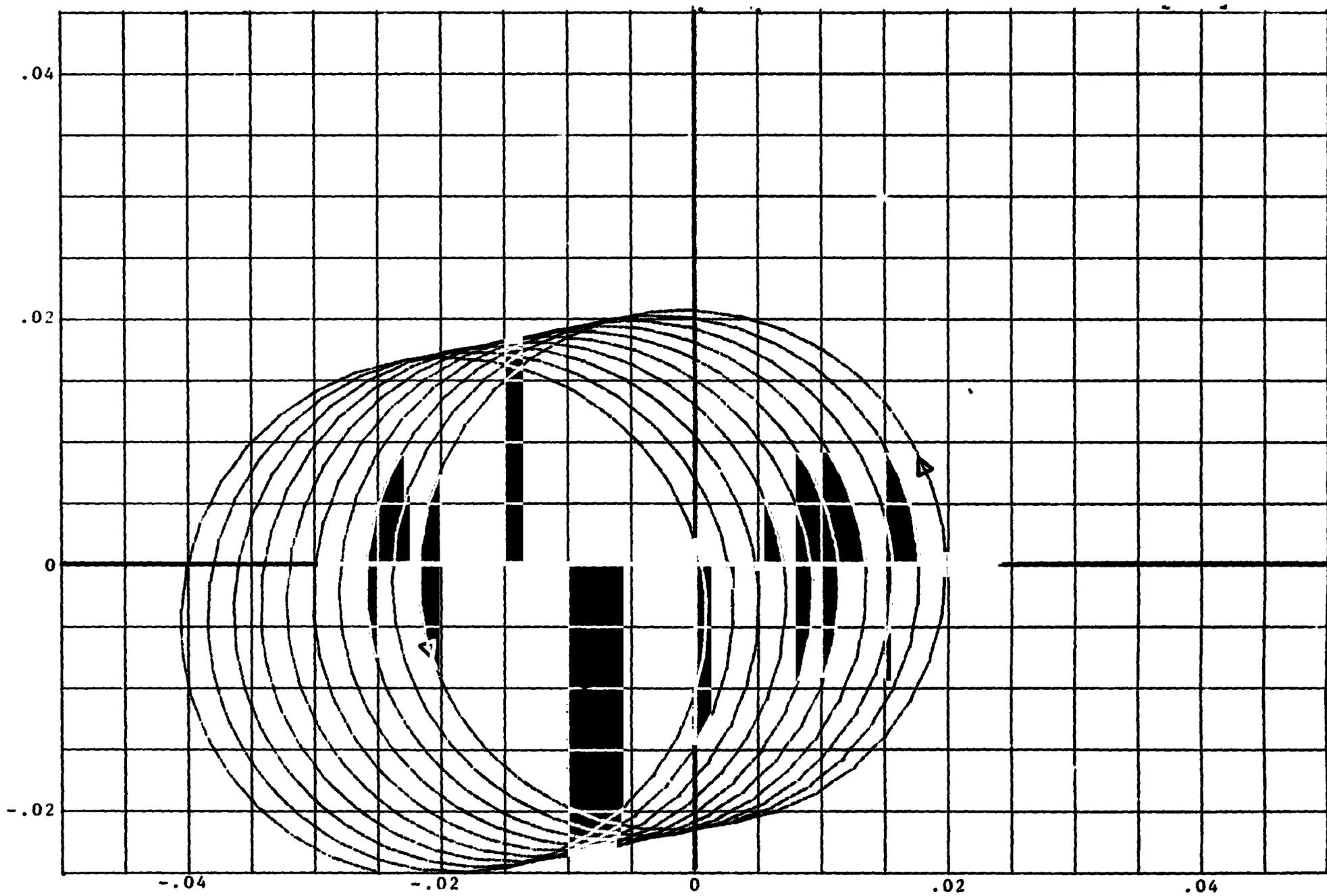


FIGURE 3: Variations in Eccentricity, Case 2  
(a) p versus q

f

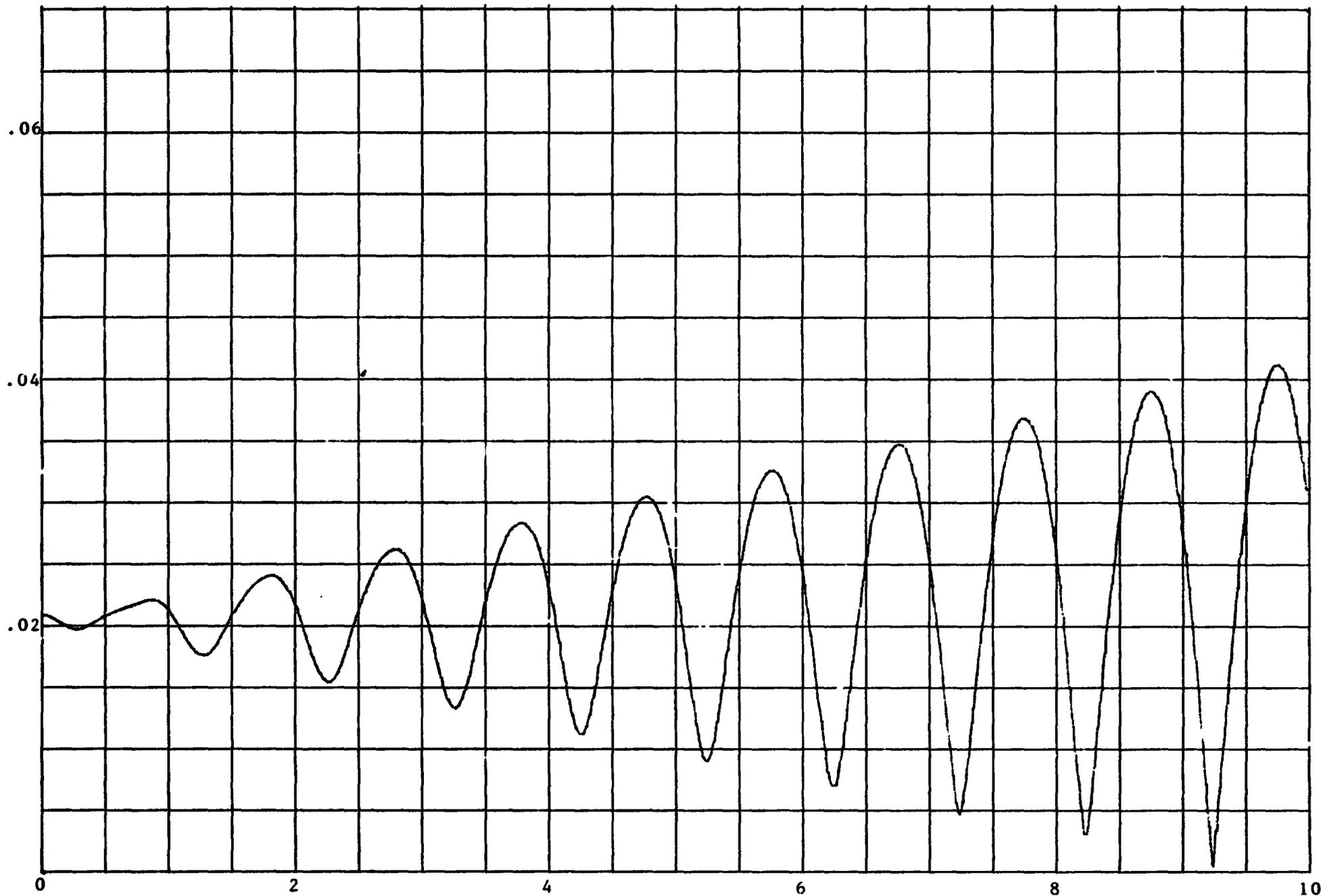


FIGURE 3: Concluded  
(b) e versus Time (years)

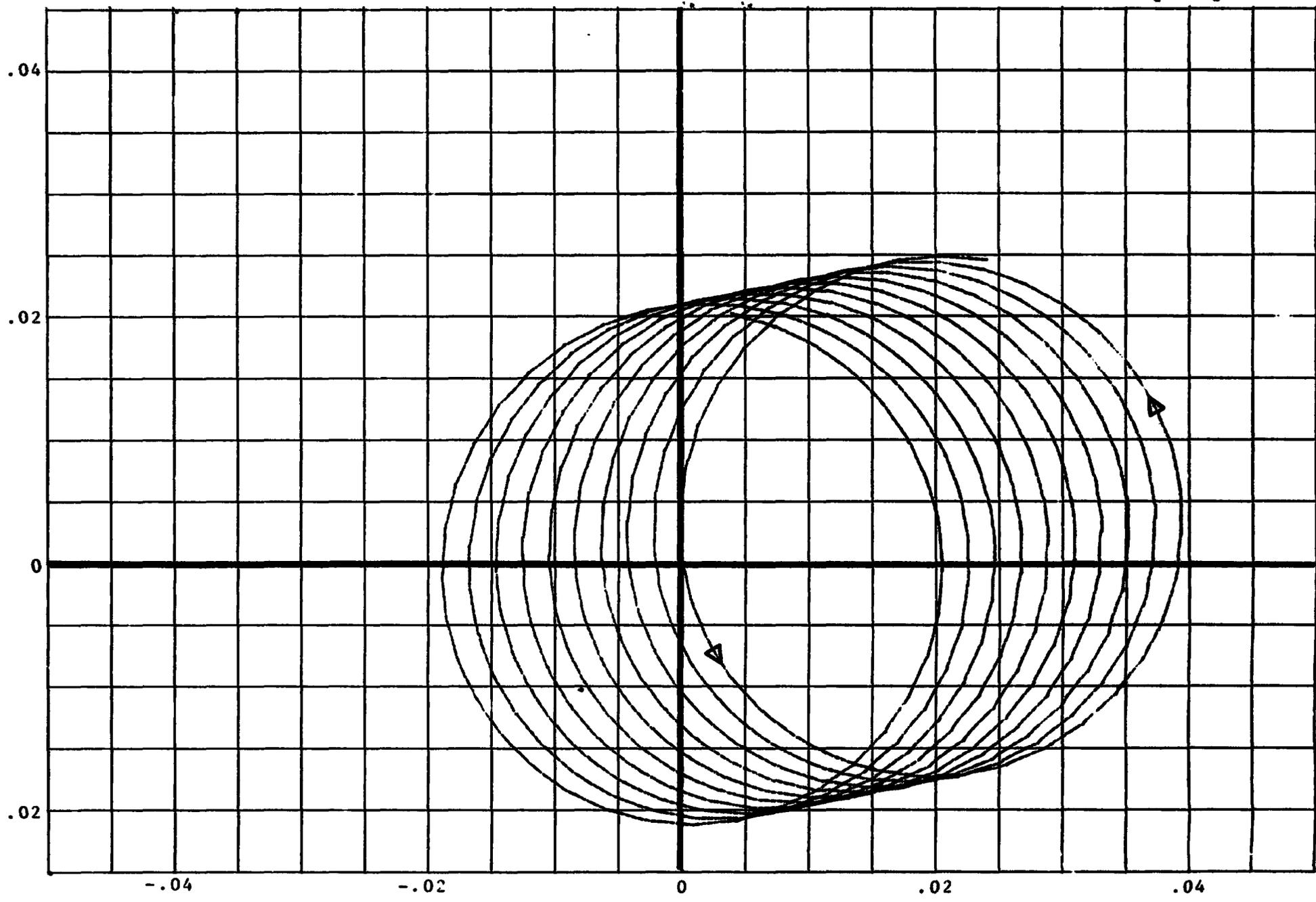


FIGURE 4: Variations in Eccentricity, Case 3  
(a) p versus q

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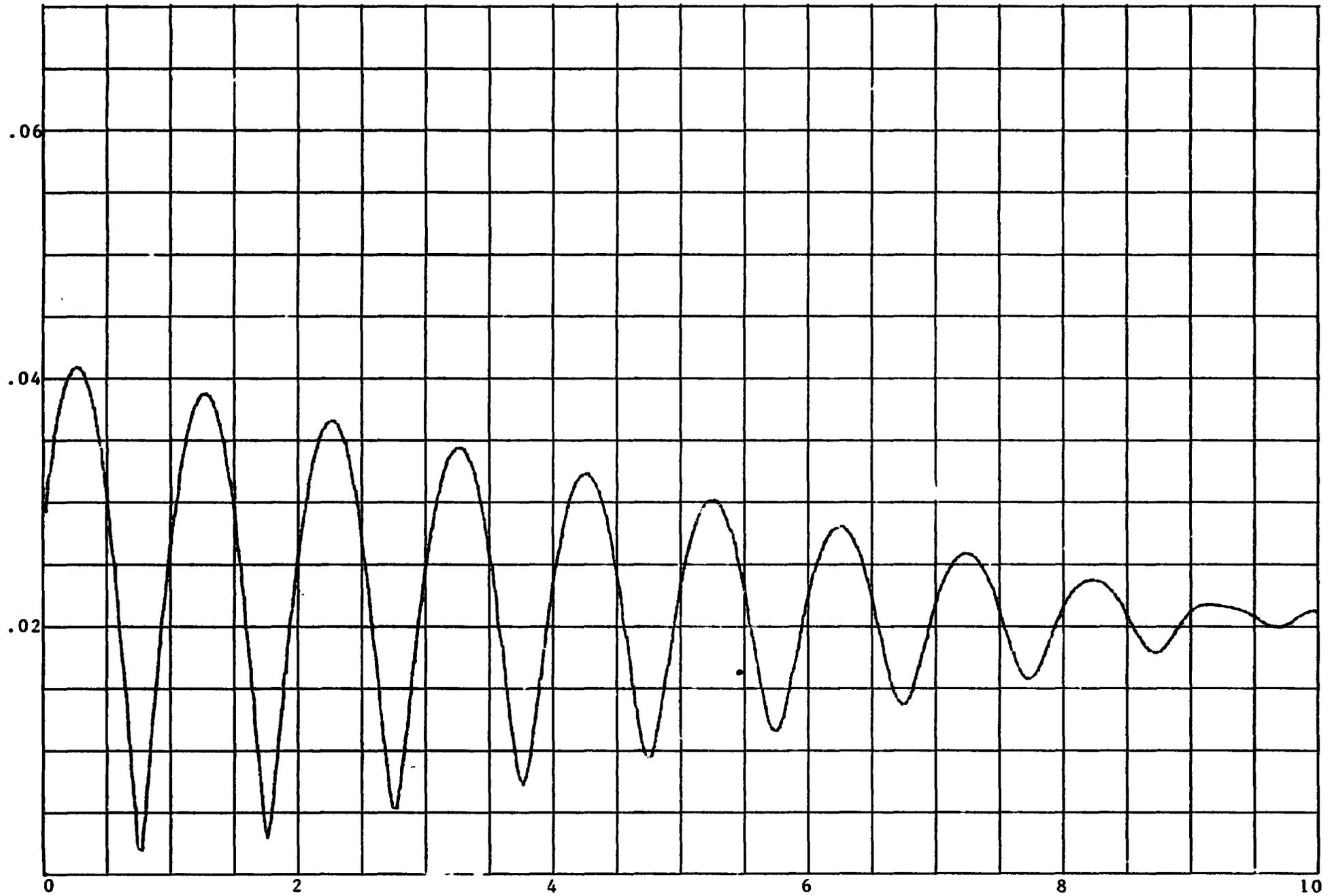


FIGURE 4: Concluded  
(b) e versus Time (years)

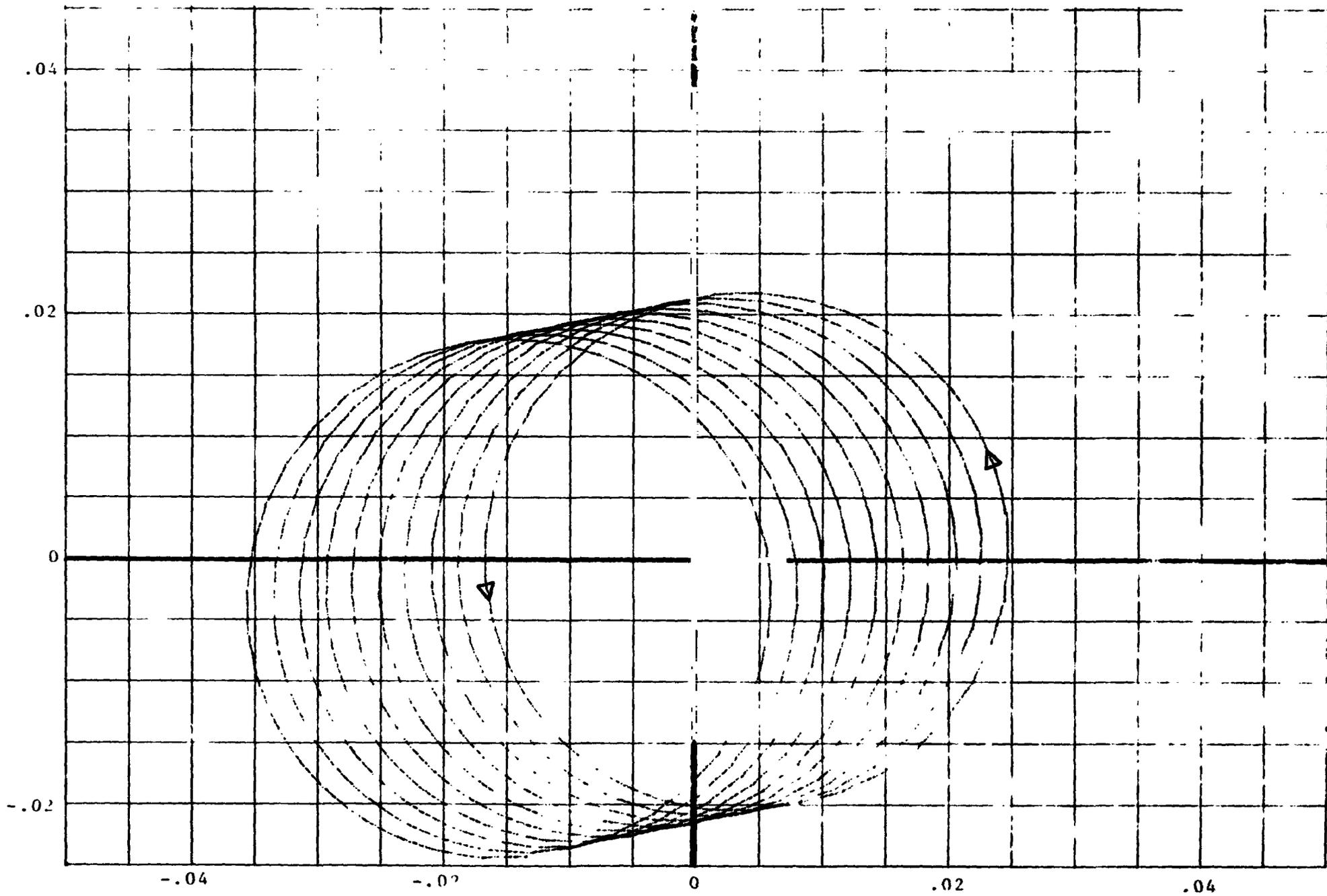


Figure 5: Variations in Eccentricity, Case 4  
(a)  $p$  versus  $q$

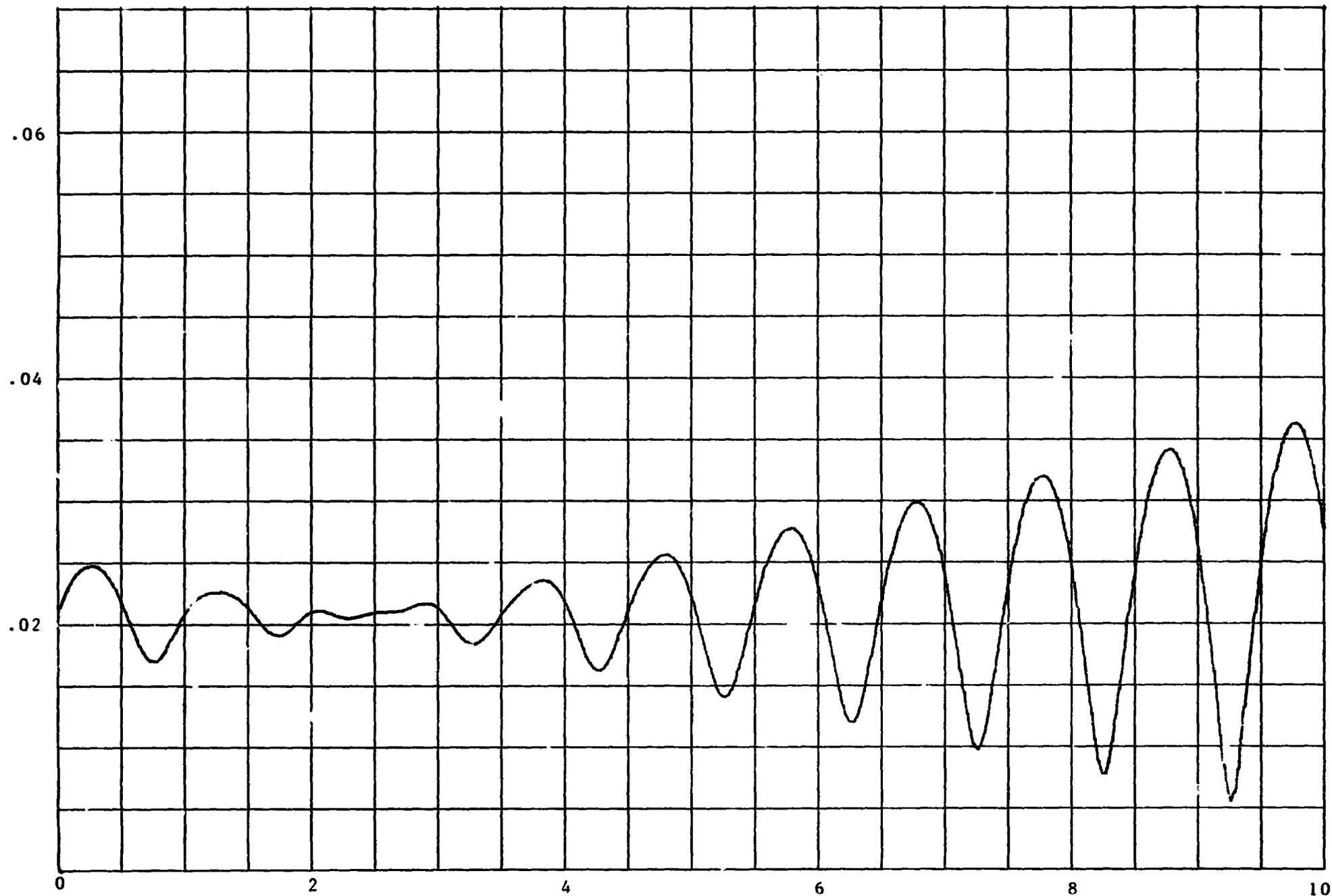


FIGURE 5: Concluded  
(b) e versus Time (years)

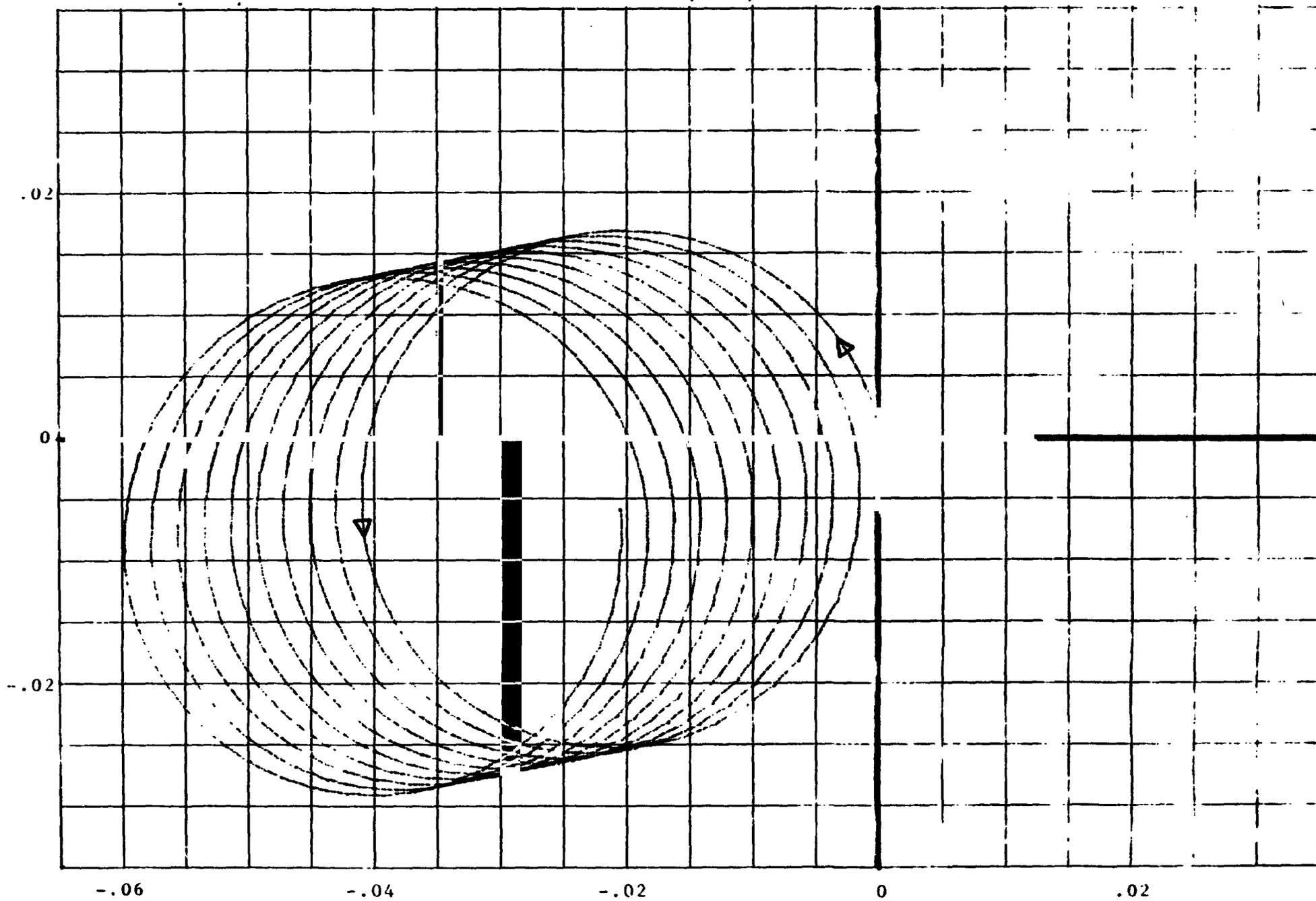


FIGURE 6: Variations in Eccentricity, Case 5  
 (a)  $p$  versus  $\alpha$

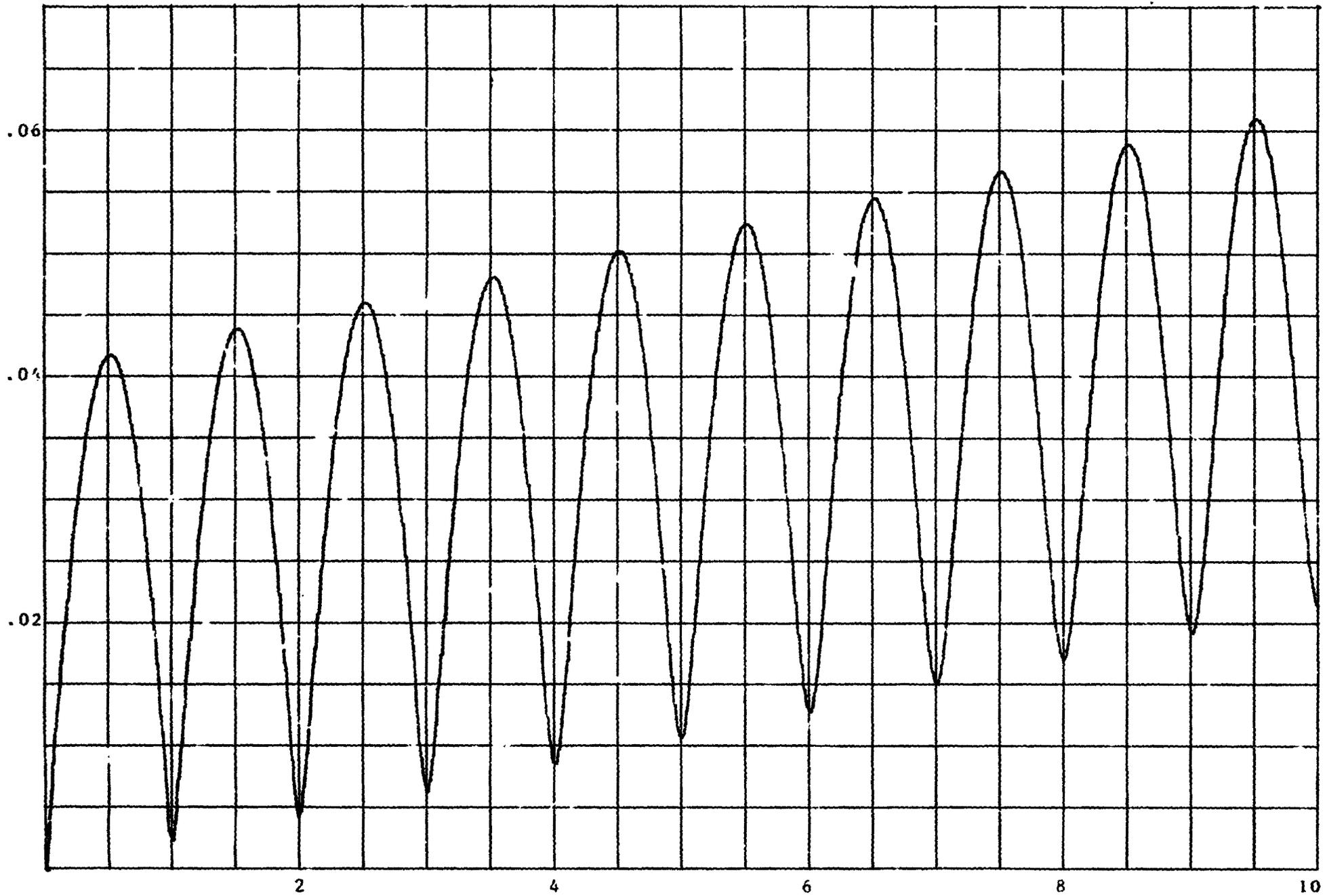


FIGURE 6: Concluded  
(b) e versus Time (years)



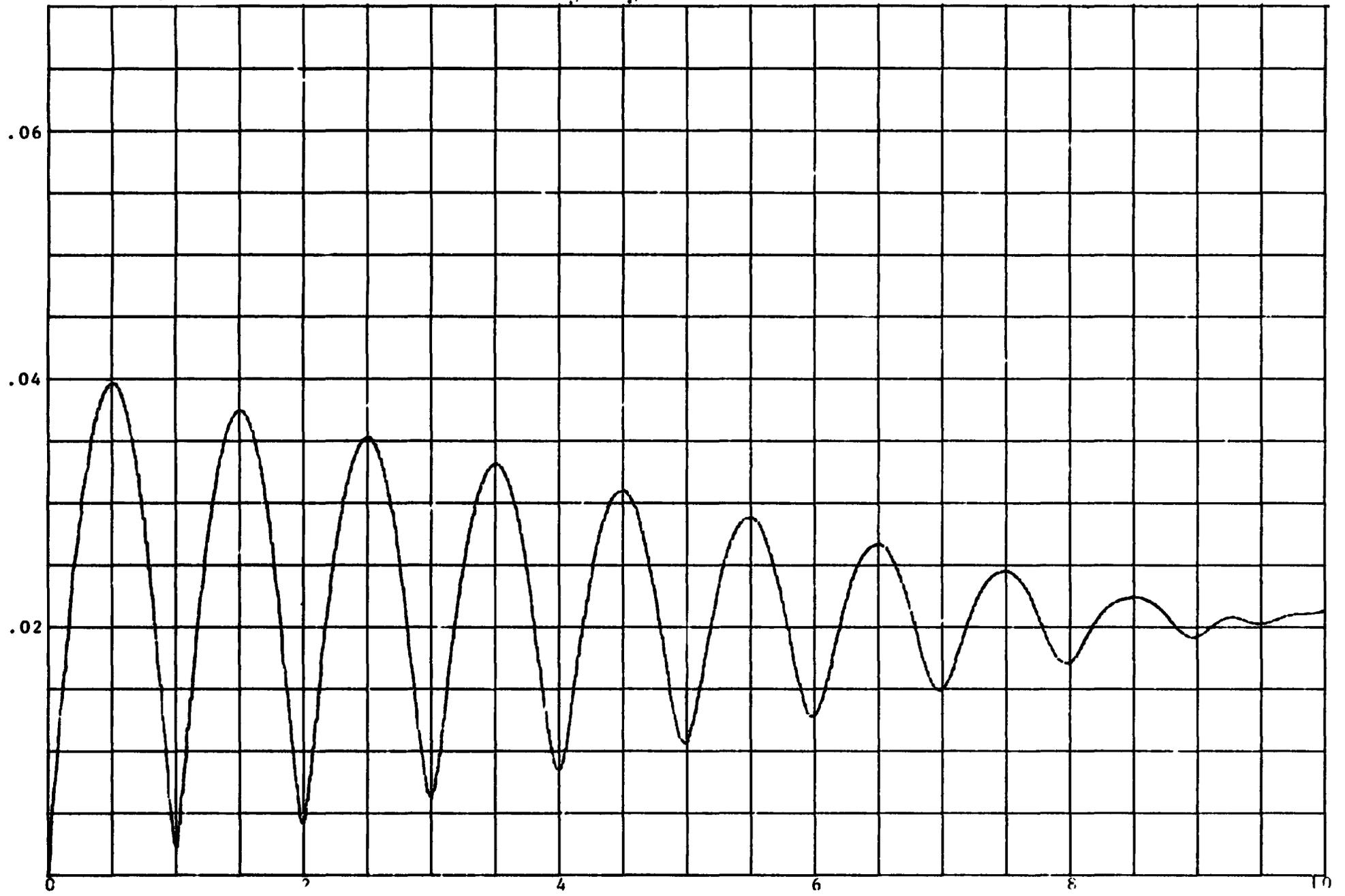


FIGURE 7: Concluded  
(b)  $e$  versus Time (years)

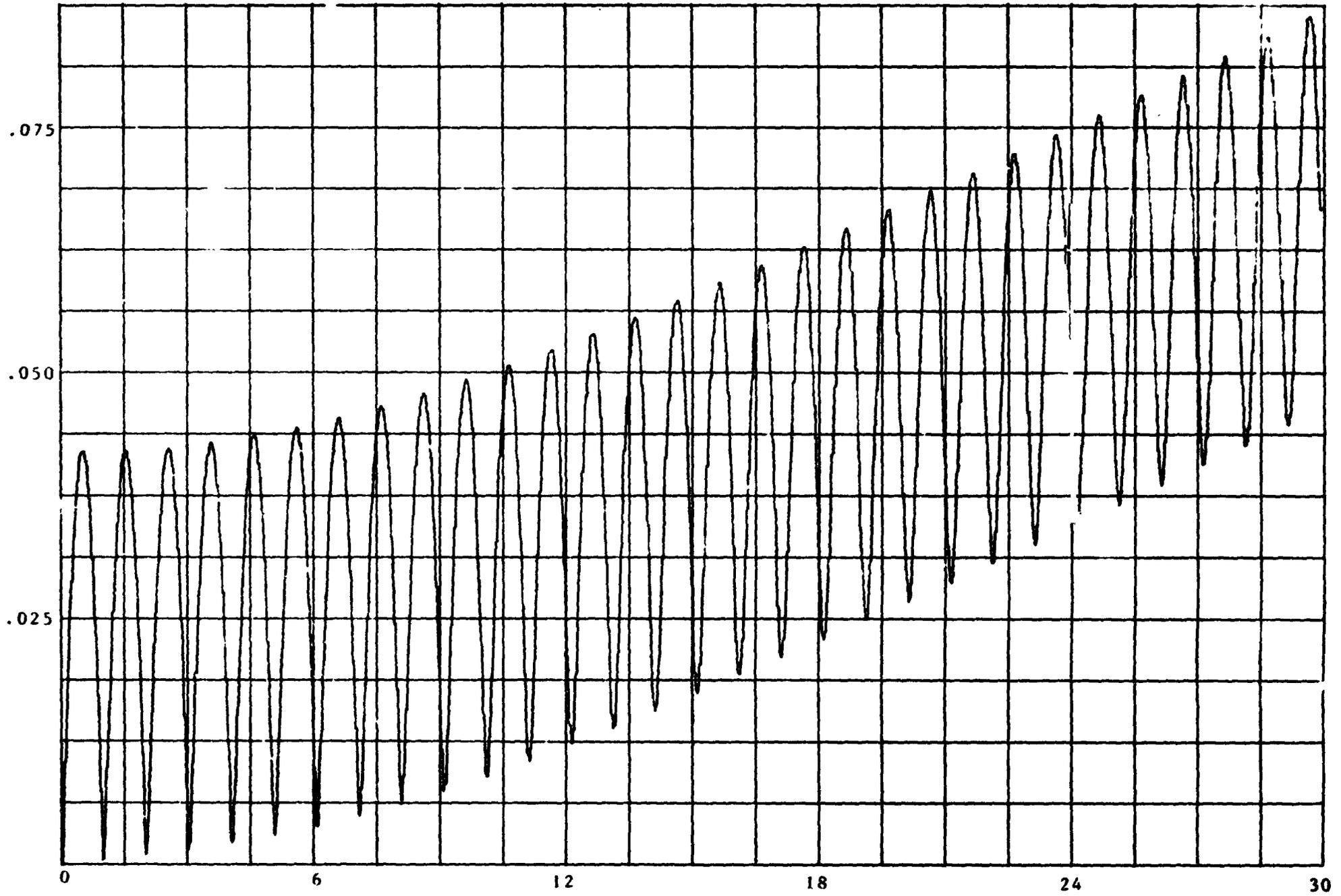


FIGURE 8: Long Term Variation in Eccentricity, Case 1

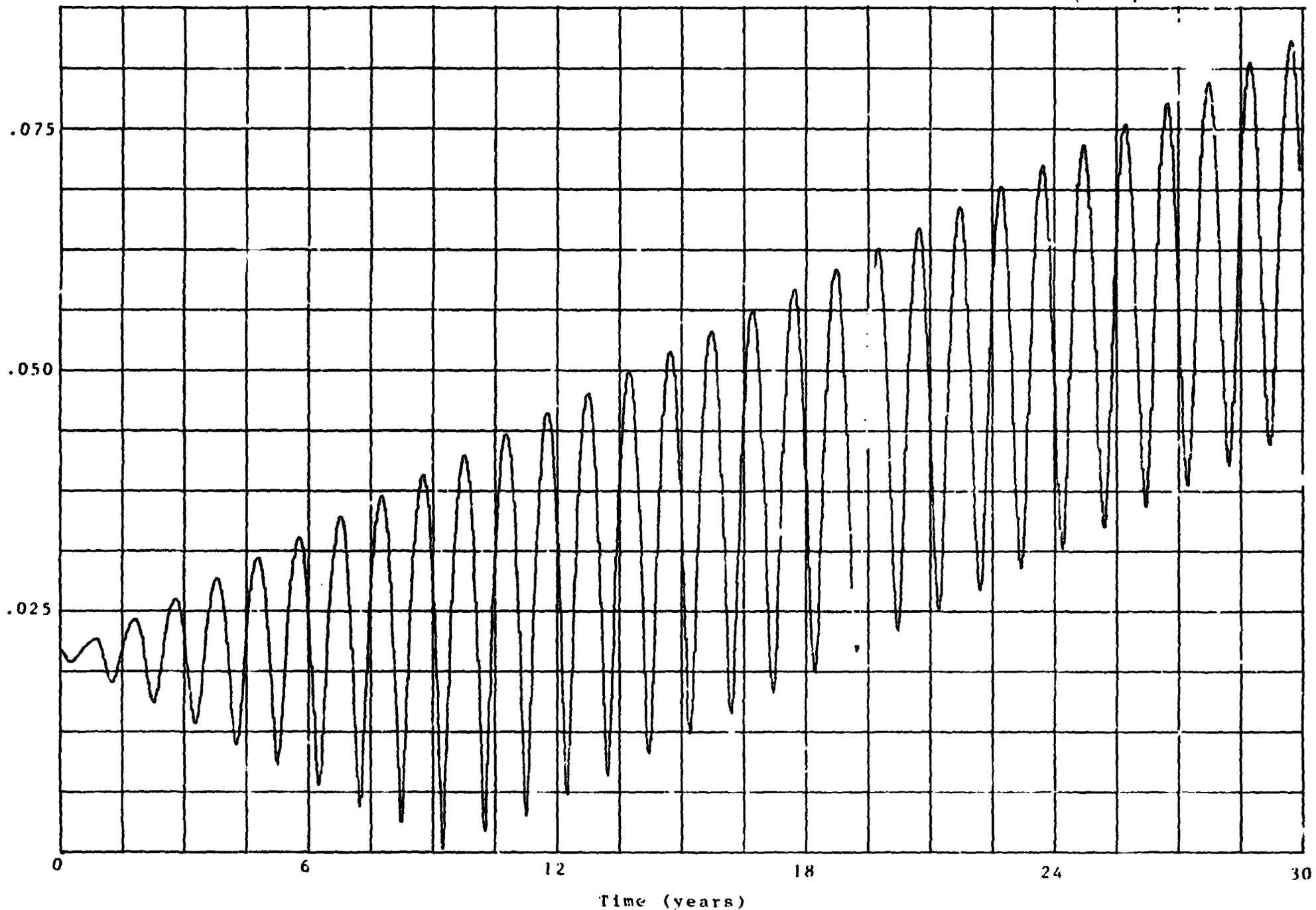


FIGURE 9: Long Term Variation in Eccentricity, Case 2

and the amplitude of oscillations in  $e$  decreases to zero after ten years.

Figures 8 and 9 show the long term variations in  $e$  for Cases 1 and 2, respectively. Of interest here are the effects of the sun's orbital eccentricity  $e_{\odot}$  over 30 years. In fact, this secular increase becomes the dominant effect on the motion. However, the numerical studies that are discussed in the next section show that gravitational perturbations eventually become significant and can actually limit the secular growth in  $e$  due to  $e_{\odot}$ .

#### 4.0 Long Period Variations in Eccentricity and Inclination

The long period changes in the shape and orientation of the SPS geosynchronous orbit are discussed in this section. Gravitational and solar radiation perturbations are included in the analysis. An analytical solution does not yet exist when the problem contains all perturbations simultaneously. Therefore, the results discussed in this section are based on a numerical integration of the satellite equation of motion.

As indicated in Sections 1 and 2, the perturbing effects can generally be separated as follows:

(1) *Rotation of the orbital plane*

The combined effects of sun and moon gravity and the oblateness of the earth cause large, long period changes in the inclination. If initially zero, the inclination will increase at the rate of .859 degrees per year. Solar radiation pressure has a very small effect on the inclination.

(2) *Variation in the orbital eccentricity*

The eccentricity can have large changes, primarily due to solar radiation pressure. However,

gravitational perturbations due to luni-solar gravity and earth's oblateness can be important over a long period of time.

(3) *Drift of the satellite's mean longitude*

This effect is caused primarily because the satellite's orbital revolutions are in resonance with the daily rotation of the earth. Luni-solar gravity has a small direct effect on the drift in longitude. There is an indirect influence due to solar radiation pressure (  $e$  varies ) and luni-solar gravity (  $I$  varies ).

This section is concerned with the long period (30 years) changes in  $e$  and  $I$  . Since these two motions are independent (so long as  $e$  and  $I$  are relatively small) they will be considered separately for several types of orbits.

All numerical results discussed in this section were obtained with the STEPR multirevolution integration program (References 18 and 19). Gravitational perturbation included geopotential terms up to order and degree of 6. The sun and moon were treated as point masses. The model for solar radiation pressure is described in Section 2.3.

#### 4.1 Eccentricity Variations

The variations in  $e$  have been thoroughly discussed in Section 3. However, as mentioned there, that analysis did not take into account the coupling between solar radiation pressure and gravity. This effect produces a long term variation in  $e$  .

The accuracy of equation (3.69) compared to a numerically integrated solution containing gravity is shown in Table IV. The orbits compared are the same as in Table II.

$e_A$  and  $\tilde{g}_A$  are the values obtained from the analytical solution.  $e_S$  and  $\tilde{g}_S$  are the numerical solutions obtained from the STEPR program. This is a comparison of the force model used in Section 3, not of the analytical solution method.

TABLE IV.-ANALYTICAL SOLUTION VERSUS STEPR SOLUTION

(a)  $e_0 = 0$  ,  $\tilde{g}_0$  undefined

Years	$e_A$	$e_S$	$\tilde{g}_A$	$\tilde{g}_S$	Percent Error	
					$e_A$	$\tilde{g}_A$
0.5	.0418	.0431	97.3 <sup>0</sup>	99.0 <sup>0</sup>	3.0	1.6
1.4	.0391	.0392	86.0	90.9	0.3	5.3
2.6	.0405	.0435	129.2	137.2	7.9	5.8
4.6	.0422	.0460	136.9	151.0	8.2	9.3
9.6	.0481	.0438	132.7	165.9	9.8	20.0

(b)  $e_0 = .0210$  ,  $\tilde{g}_0 = -80.6^0$

Years	$e_A$	$e_S$	$\tilde{g}_A$	$\tilde{g}_S$	Percent Error	
					$e_A$	$\tilde{g}_A$
0.5	.0209	.0222	95.2 <sup>0</sup>	95.7 <sup>0</sup>	6.1	0.5
1.4	.0193	.0202	71.6	73.1	4.6	2.1
2.6	.0247	.0250	154.0	154.1	1.4	0.1
4.6	.0286	.0278	163.3	166.6	2.9	2.0
9.6	.0328	.0243	153.7	168.5	35.0	8.9

Comments on Table IV

1. The analytical solution is accurate to within a few percent within the first few years. Therefore, the figures shown in Section 3 will be accurate to within a few percent and describe the general character of the motion.
2. Long period effects due to gravity become important after about eight or nine years.
3. The analytical solution could be useful for orbit prediction and control over a few years.

Long term variations in  $e$  have been studied for the cases shown in Table V. Figures 10 through 13 show  $e$  versus time for 30 years. For each case, the inclination was approximately  $7.3^\circ$ . However, the motion of  $e$  was essentially the same as for the case when the inclination was initially zero.

TABLE V. -INITIAL CONDITIONS FOR EXAMPLES

<u>Case No.</u>	<u><math>e_0</math></u>	<u><math>\tilde{g}_0</math></u>	<u>Initial Epoch</u>
1	.0	undefined	noon, 1 Jan. 1980
2	.0210	$-80.6^\circ$	noon, 1 Jan. 1980
3	.0214	$-67.4$	noon, 1 Jan. 1980
4	.0	undefined	noon, 3 April 1980

The plots shown in Figures 10 through 13 were output from the STEPR program, using the CALCOMP plotter hardware and software that is available on the Johnson Space Center Univac 1110 computer system.

## Discussion of Results

### Case 1 (Figure 10)

- (1) The curve doesn't being exactly at  $e=0$  since there is no output from STEPR until after 44 revolutions. This is because the extrapolation table needs to be built before the first multirevolution step is taken.
- (2) The long term gravitational effects can be seen by comparing Figure 10 with Figure 8. (Note that the scales are different). Both curves have the same general shape. The linear trend of Figure 8 has been moderated in Figure 10.
- (3) Large values of  $e$  can occur for an uncorrected SPS orbit. Such large values are probably unacceptable to the spacecraft and ground systems.

### Case 2 (Figure 11)

This case can be compared with Figure 9. The almost linear increase in amplitude levels off after about 20 years. However, the curves are very similar for the first ten years.

### Case 3 (Figure 12)

This cases corresponds to Case 4 (Figure 5(b)) in Section 3 where the value of  $e$  is limited during the first few years. Also, if Figure 12 is shifted about an inch to the left along the time axis, it is almost identical to Figure 11. This can be seen by overlaying the two figures. Initial conditions of the orbit can be manipulated so that the nearly constant  $e$  phase occurs anywhere along the time axis. This may be done to achieve certain desirable results for station keeping purposes. It must be remembered, however, that since the problem depends on time, the epoch of initialization must also be taken into account. (Compare Case, 1, 5 and 6 in Section 3).

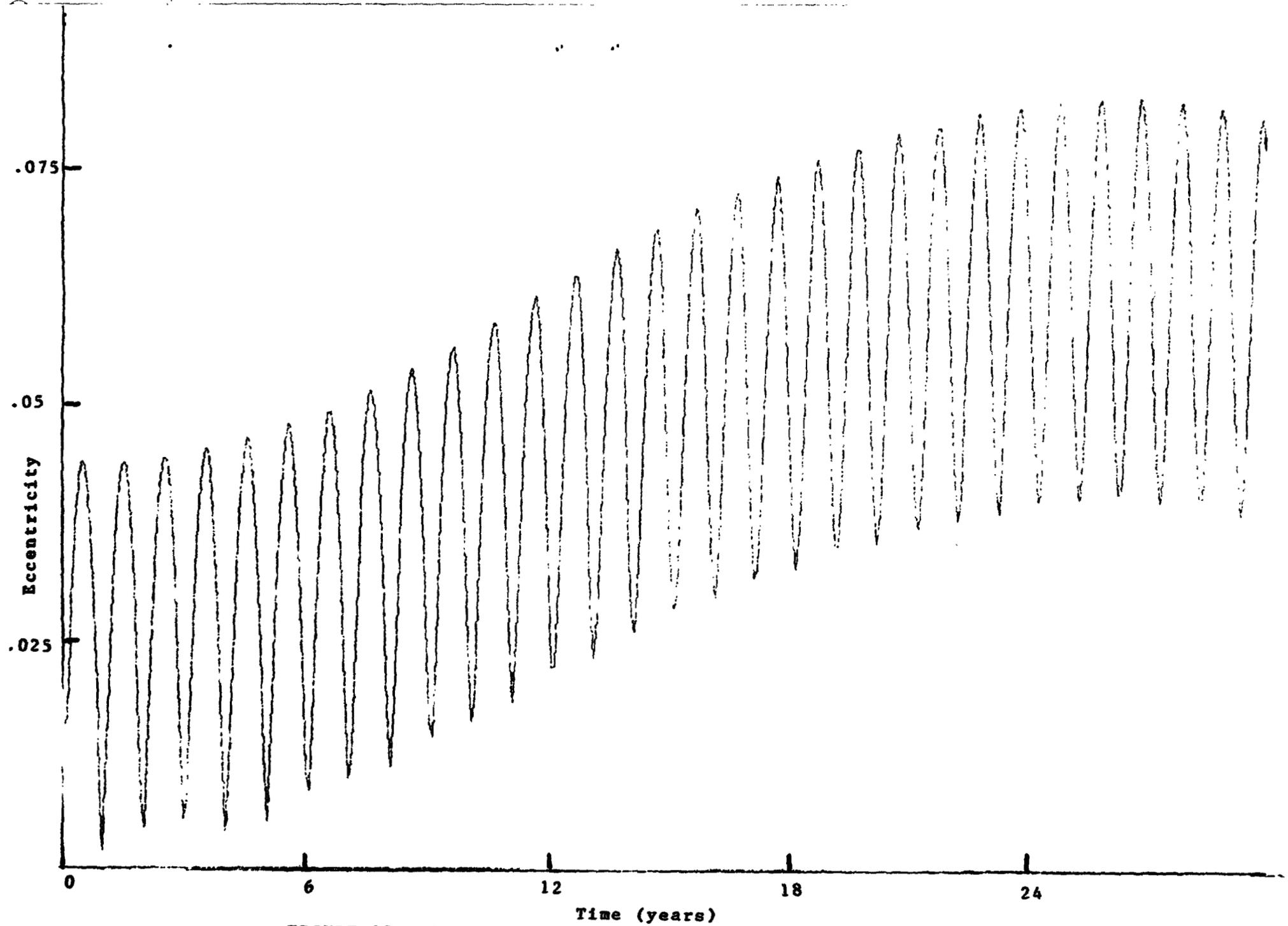


FIGURE 10: Long period Variations in Eccentricity, Case 1

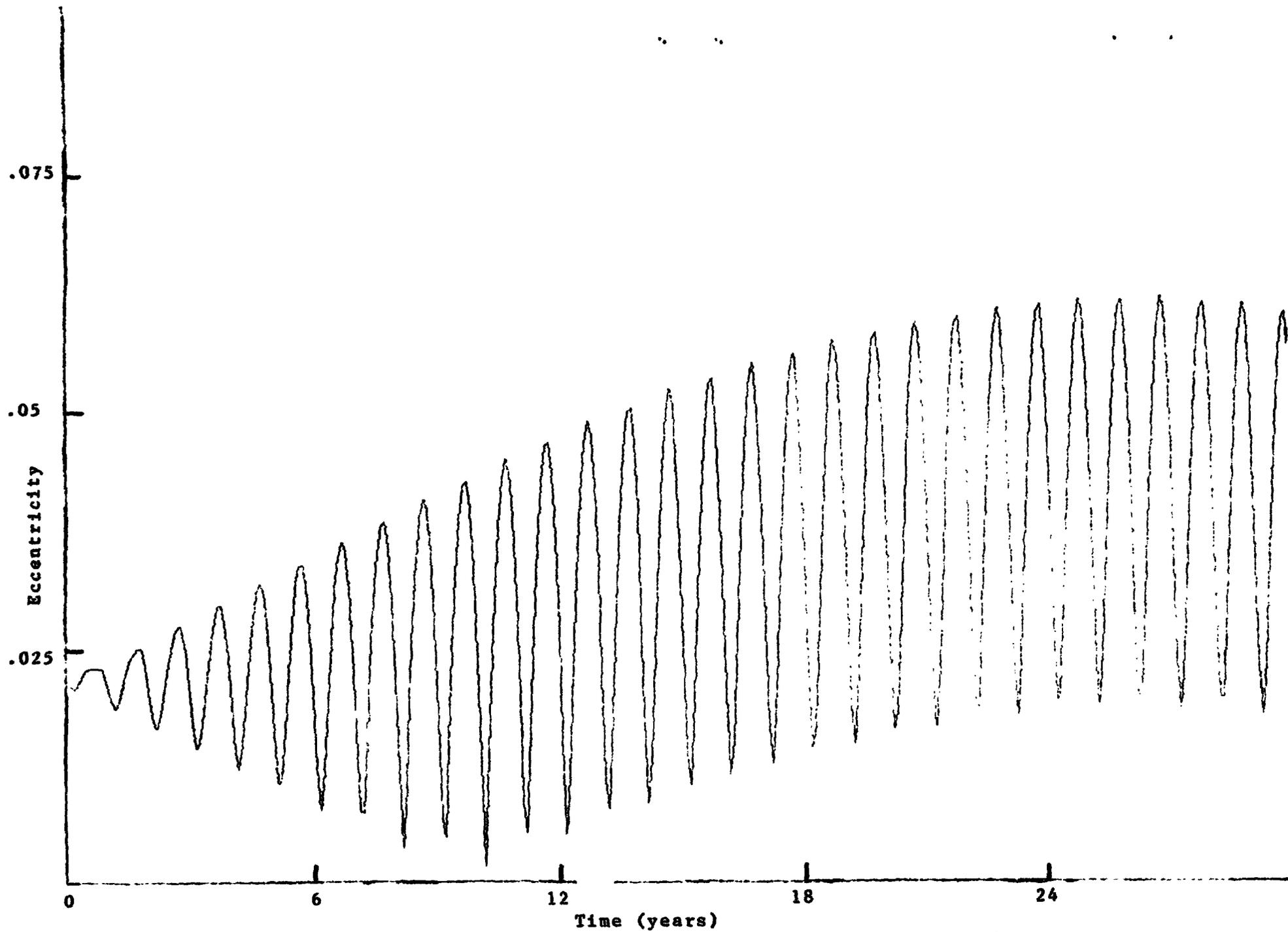


FIGURE 11: Long Period Variations in Eccentricity, Case 2

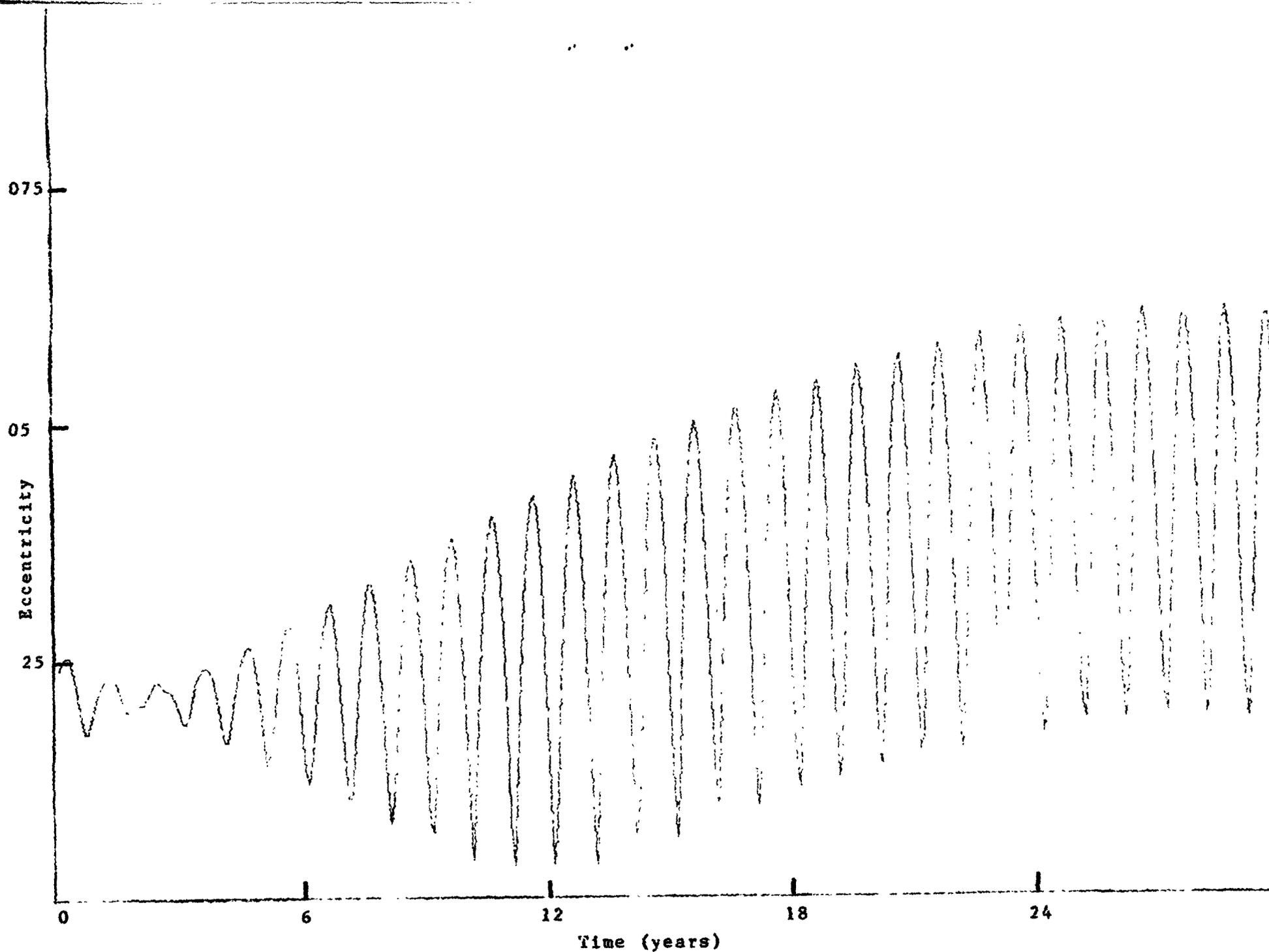


FIGURE 12: Long Period Variations in Eccentricity, Case 3

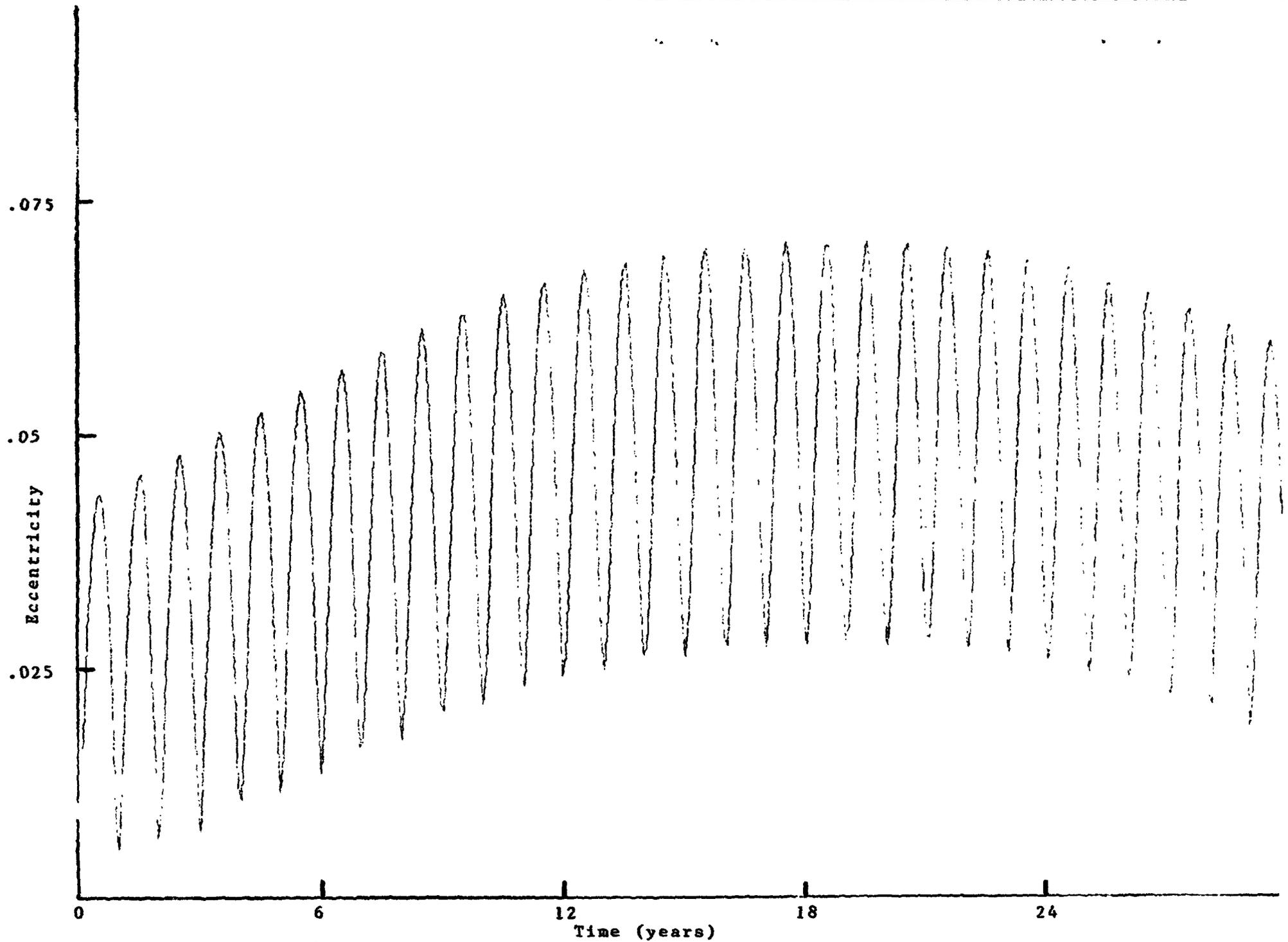


FIGURE 13: Long Period Variations in Eccentricity, Case 4

Case 4 (Figure 13)

The initial conditions are the same as for Case 1 except that the epoch is three months later. (Compare Case 1 and Case 5 in Section 3.) The linear component in  $e$  is evident over the first ten years. However, the long period effects eventually cause a decrease in eccentricity so that the maximum  $e$  for this case is less than the maximum  $e$  for Case 1. This example shows the importance of the long period gravitational terms.

4.2 Inclination Variations

The motion of the orbital plane has been thoroughly discussed in Reference 7. The discussion given here presents a numerically integrated solution and some plots of inclination as a function of time. Table VI shows the cases that are discussed here.

TABLE VI. -INITIAL CONDITIONS FOR EXAMPLES

---

<u>Case No.</u>	<u><math>I_0</math></u>	<u><math>h_0</math></u>	<u>Initial Epoch</u>
1	$0^0$	undefined	noon, 1 Jan. 1980
2	$7.3^0$	$0.0$	noon, 1 Jan. 1980
3	2.0	270.0	noon, 1 Jan. 1980

---

For each case, the eccentricity was small and had no appreciable effect on inclination or node.

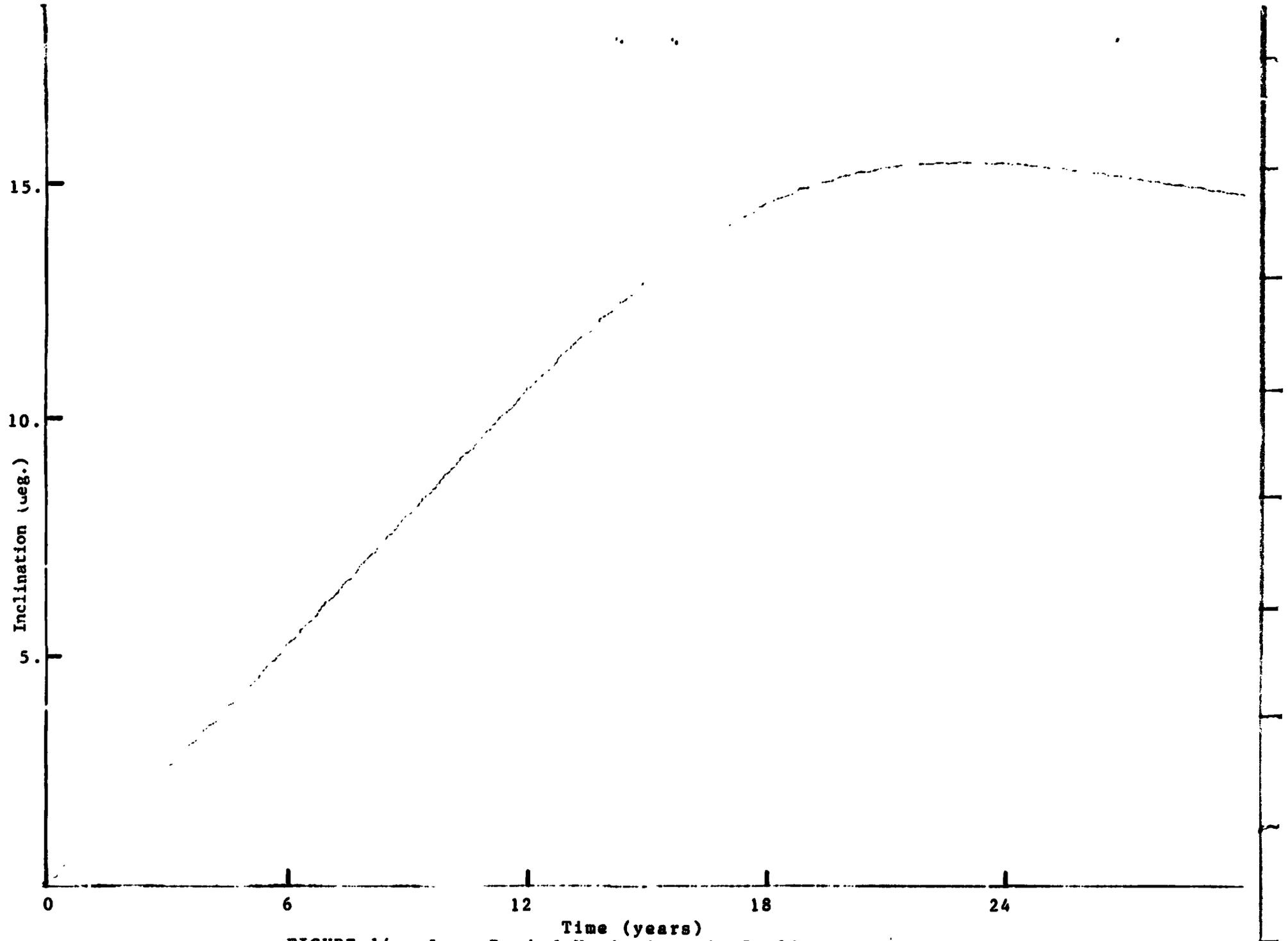


FIGURE 14: Long Period Variations in Inclination, Case 1

15.

10.

5.

0

6

12

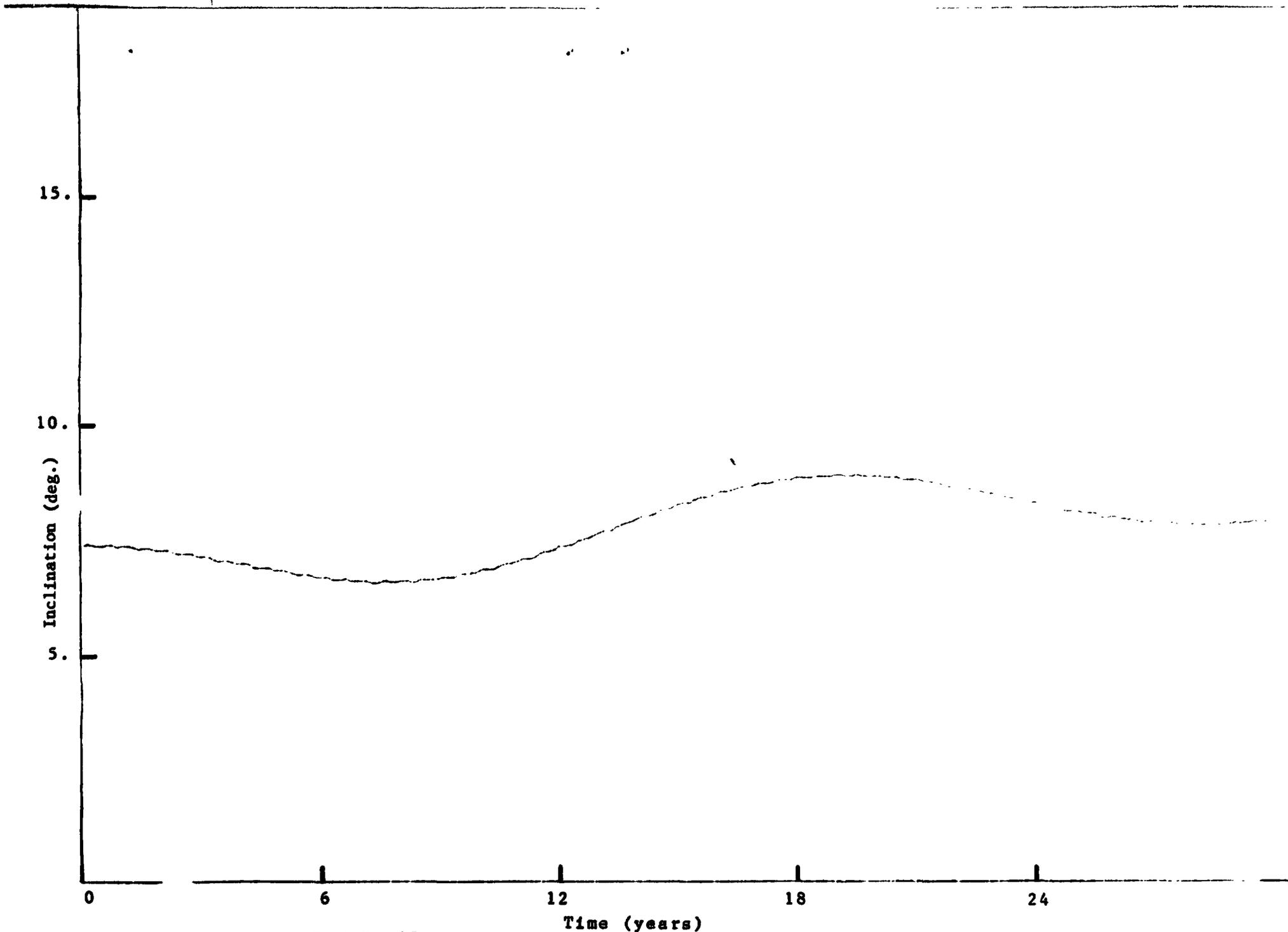
18

24

Inclination (deg.)

Time (years)

FIGURE 15: Long Period Variations in Inclination, Case 2



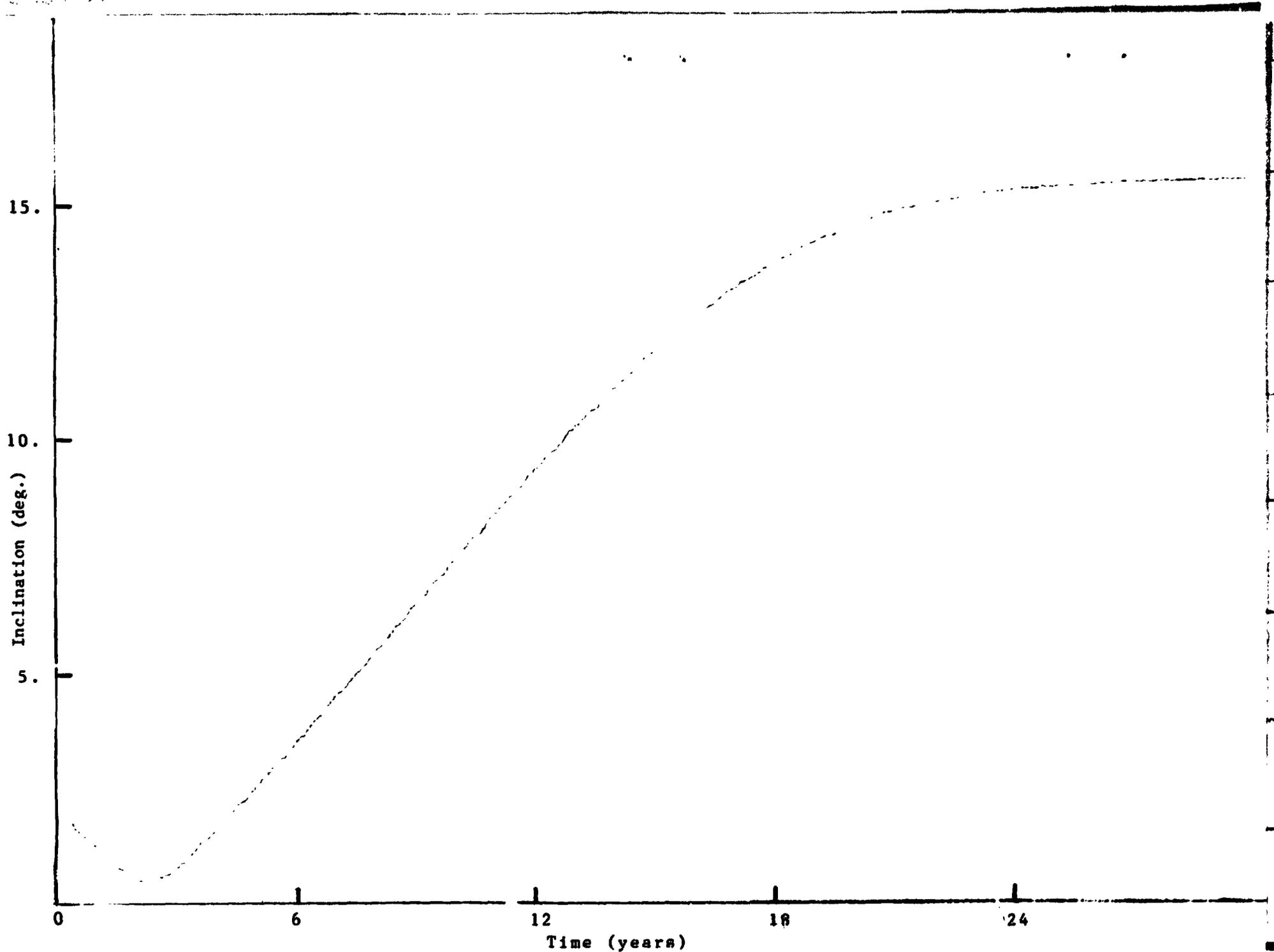


FIGURE 16: Long Period Variations in Inclination, Case 3

Case 1 (Figure 14)

If initially zero, the inclination increases to about  $15^\circ$  after 26 years. For the first few years, the increase is almost linear at the rate of .859 degrees per year.

Case 2 (Figure 15)

The inclination is nearly constant. The long period oscillation observed in this figure is caused by the precession of the moon's orbital plane on the ecliptic. This motion has a period of 18.6 years and depends on the epoch of initialization.

Case 3 (Figure 16)

When the node is initially near 270 degrees, the inclination will decrease to almost zero and then increase. With this approach, inclination can be kept small over a longer period of time. The advantage is that out of plane station keeping maneuvers will be reduced or eliminated. The effectiveness of this procedure depends to some extent on the orientation of the moon's orbit and therefore the epoch.

5.0 Daily Effects Due to Nonzero Eccentricity and Inclination

The changes in eccentricity and inclination that were described in the previous sections will cause a daily motion of the satellite, as observed from the rotating earth. It will wander north and south of the equator as well as east and west of the mean longitude. The ground track may be a circle, ellipse, figure eight or some other shape depending on the values of  $e$  and  $I$ . Usually there are restrictions on the allowed latitude and longitude deviations of a geosynchronous satellite, because of requirements on satellite

and ground systems. This section will show some typical ground tracks for the case of small values of eccentricity and inclination.

### 5.1 Development of Equations

The frame of reference will be a rectangular coordinate system with x- and y-axes in the equatorial plane and z-axis toward the north pole. The x-axis is directed toward the vernal equinox. Figures 17 and 18 define the symbols to be used in this development.  $\phi_{\odot}$  is the angle between the Greenwich meridian and the x-axis.

Equations will be developed here that give latitude  $\psi$  and longitude  $\lambda$  as a function of time. Begin with the expressions for rectangular coordinates in terms of elements,

$$\begin{aligned} x &= r (\cos \Omega \cos v - \sin \Omega \sin v \cos I) \quad , \\ y &= r (\sin \Omega \sin v + \cos \Omega \sin v \cos I) \quad , \quad (5.1) \\ z &= r \sin v \sin I \quad , \end{aligned}$$

where  $v = \omega + f$  (5.2)

Since  $\frac{z}{r} = \sin \psi$  ,

$$\sin \psi = \sin v \sin I \quad (5.3)$$

Note that  $\psi$  is uniquely determined by (5.3).

Development of equations for the longitude  $\lambda$  are somewhat more difficult. First, notice that expressions must be found for both  $\sin \lambda$  and  $\cos \lambda$ . From Figure 17,

$$\frac{x}{\rho} = \cos \phi \quad \frac{y}{\rho} = \sin \phi \quad , \quad (5.4)$$

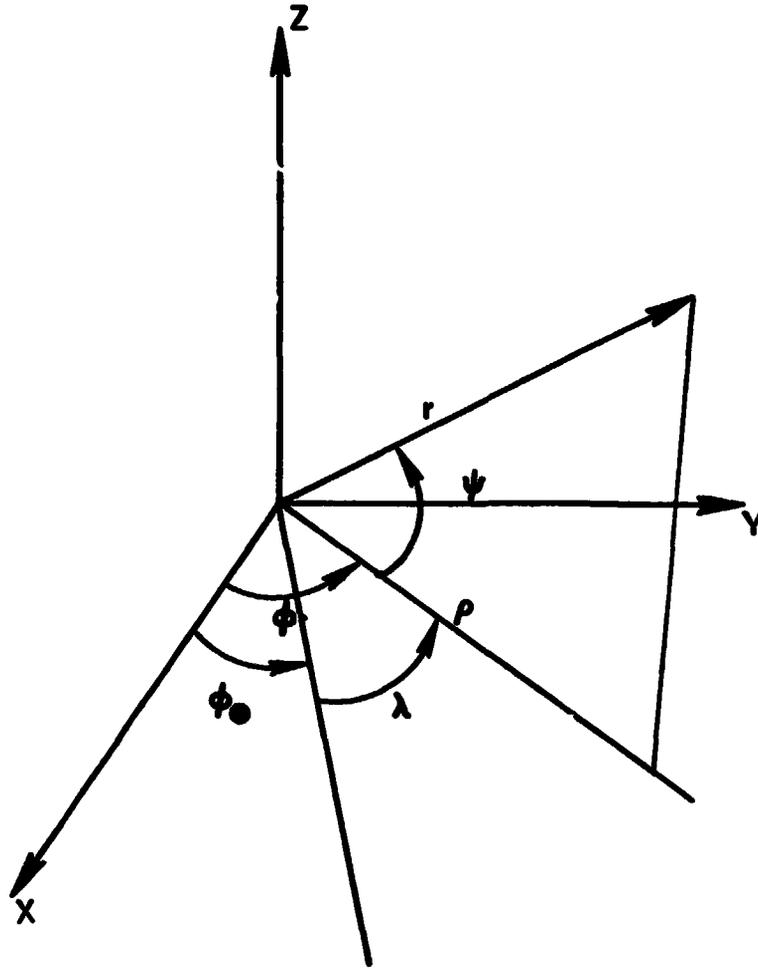


FIGURE 17: Polar Coordinate System

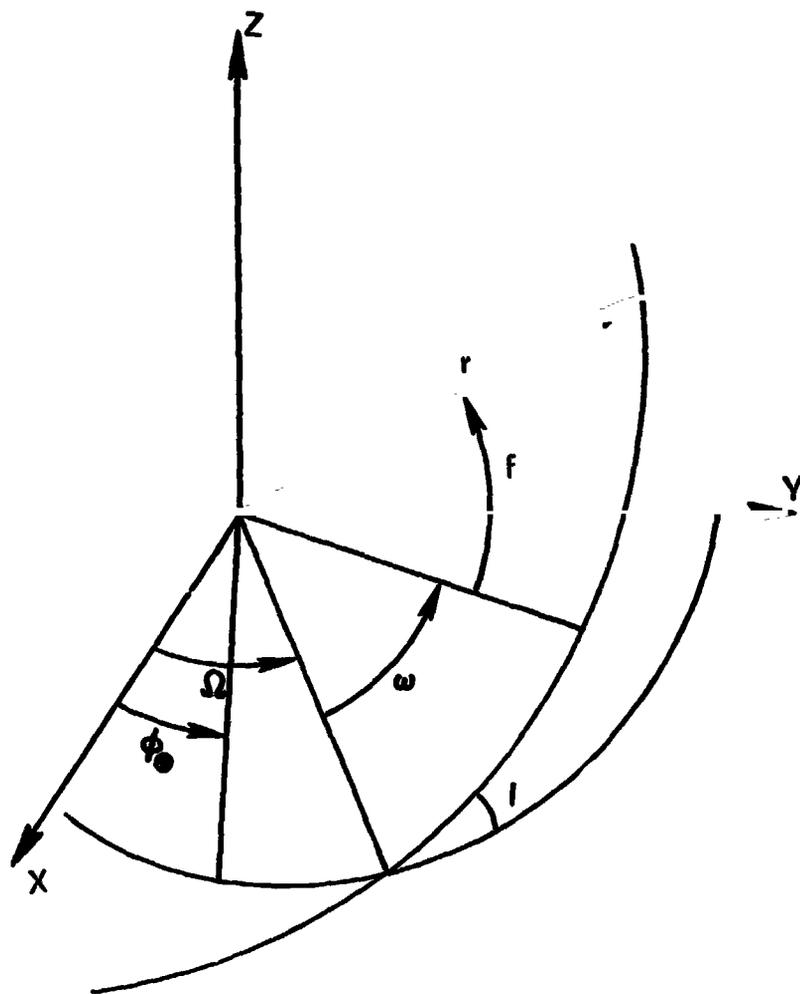


FIGURE 18: Orbital Elements

so that

$$\begin{aligned} \sin \phi &= \frac{r}{\rho} (\sin \Omega \cos v + \cos \Omega \sin v \cos I) \\ \cos \phi &= \frac{r}{\rho} (\cos \Omega \cos v - \sin \Omega \sin v \cos I) \end{aligned} \quad (5.5)$$

The longitude can be expressed in terms of  $\phi$  and  $\phi_{\odot}$  by

$$\lambda = \phi - \phi_{\odot} \quad ,$$

so that

$$\begin{aligned} \sin \lambda &= \sin \phi \cos \phi_{\odot} - \cos \phi \sin \phi_{\odot} \quad , \\ \cos \lambda &= \cos \phi \cos \phi_{\odot} + \sin \phi \sin \phi_{\odot} \quad . \end{aligned} \quad (5.6)$$

Insert (5.5) into (5.6), collect terms and make use of trigonometric identities. After introducing the small parameter

$$\alpha = \frac{1}{2} (1 - \cos I) \quad , \quad (5.7)$$

the results are

$$\begin{aligned} \sin \lambda &= \frac{r}{\rho} \left[ (1 - \alpha) \sin (v + \Omega - \phi_{\odot}) - \alpha \sin (v - \Omega + \phi_{\odot}) \right] \\ \cos \lambda &= \frac{r}{\rho} \left[ (1 - \alpha) \cos (v + \Omega - \phi_{\odot}) + \alpha \cos (v - \Omega + \phi_{\odot}) \right] \end{aligned} \quad (5.8)$$

$$\text{where} \quad \frac{r}{\rho} = (1 - \sin^2 I \sin^2 v)^{-\frac{1}{2}} \quad . \quad (5.9)$$

The true anomaly  $f$  is given as a function of time by the implicit equation

$$nt - l_0 = E - e \sin E \quad (5.10)$$

and

$$f = 2 \tan^{-1} \left[ \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right] \quad (5.11)$$

The keplerian elements  $n$  (mean motion),  $e$ ,  $I$ ,  $\omega$ ,  $\Omega$ , can be assumed constant over a few days.

It is desirable to measure the longitude relative to some "mean" value. Define the new variables

$$\delta = \lambda - \theta \quad (5.11)$$

and

$$\theta = l_0 + \Omega - \phi_{\oplus 0} + \omega \quad (5.12)$$

The rotation of the earth is expressed as

$$\phi_{\oplus} = \omega_{\oplus} t + \phi_{\oplus 0}$$

For geosynchronous satellites,

$$\omega_{\oplus} = n \quad ,$$

so that

$$\phi_{\oplus} = nt + \phi_{\oplus 0}$$

Making use of the notation

$$l = nt + l_0 \quad ,$$

we have

$$\phi_{\oplus} = l + \phi_{\oplus 0} - l_0 \quad (5.13)$$

Inserting (5.13) into (5.8) and using (5.12),

$$\sin \lambda = \frac{r}{\rho} \left[ (1-\alpha) \sin (f - l + \theta) - \alpha \sin (f + l + 2\omega - \theta) \right] \quad , \quad (5.14)$$

$$\cos \lambda = \frac{r}{\rho} \left[ (1-\alpha) \cos (f - l + \theta) + \alpha \cos (f + l + 2\omega - \theta) \right] \quad .$$

Since  $|\delta|$  will be less than  $90^\circ$ , it can be defined by  $\sin \delta$ . Make use of

$$\sin \delta = \sin \lambda \cos \theta - \cos \lambda \sin \theta \quad (5.15)$$

Substitute (5.14) into (5.15) and collect terms. The result is

$$\sin \delta = \frac{r}{p} \left[ (1 - \alpha) \sin (f - \ell) - \alpha \sin (f + \ell + 2\omega) \right] \quad (5.16)$$

Finally, equations (5.3), (5.9), (5.10), (5.11) and (5.16) define the ground track as a function of time.

## 5.2 Small Eccentricity and Inclination

It is desirable to express latitude  $\psi$  and longitude  $\delta$  as explicit functions of time, i.e. as functions of  $\ell$ . This can be done for small  $e$  and  $I$  by making use of power series expansions. Assume that

$$e \leq .064 \quad , \quad I \leq 7.3^\circ$$

Define

$$b = \frac{1}{2} \sin I \quad , \quad (5.17)$$

then

$$b \approx e$$

Power series expansions will be carried out in terms of  $e$  and  $b$ , keeping terms of order  $e^2$ ,  $b^2$  and  $eb$ .

5.2.1 Longitude Equation

From equation (5.16),  $\sin \delta$  is expressed in terms of  $f$  and  $l$ . Consider

$$\frac{r}{p} = \left[ 1 - \sin^2 I \sin^2 (f + \omega) \right]^{-\frac{1}{2}} .$$

This expression can be rewritten in terms of  $b$ ,

$$\frac{r}{p} = \left[ (1 - 2b^2) + 2 b^2 \cos 2 (f + \omega) \right]^{-\frac{1}{2}} .$$

Expanding the above expression with the aid of the binomial theorem,

$$\frac{r}{p} = (1 - 2 b^2) - b^2 \cos 2 (f + \omega) + O(b^4)$$

Also,

$$\alpha = \frac{1}{2} (1 - \cos I) = b^2 + O(b^4)$$

Therefore, after truncating terms of order  $b^4$ , the expression for  $\sin \delta$  is

$$\begin{aligned} \sin \delta &= (1 - 3b^2) \sin (f - l) \\ &\quad - \frac{1}{2} b^2 \sin (f + l + 2\omega) \\ &\quad - \frac{1}{2} b^2 \sin (3f - l + 2\omega) \end{aligned}$$

Notice that  $3b^2$  can be dropped from the first term since  $\sin (f-l) = O(e)$ . Also,  $f$  can be replaced by  $l$  in the last two terms, for similar reasons. Then,

$$\sin \delta = \sin (f - l) - b^2 \sin 2 (l + \omega). \quad (5.18)$$

Making use of the *Equation of the Center*,

$$f - l = 2e \sin l + \frac{5}{16} e^2 \sin 2l + \dots, \quad (5.19)$$

and the expansion for sine,

$$\sin \theta = \theta - \frac{1}{6} \theta^3 + \dots, \quad (5.20)$$

the explicit expression for longitude is

$$\begin{aligned} \delta = & 2e \sin l + \frac{5}{16} e^2 \sin 2l \\ & - b^2 \sin 2 (l + \omega) + O(e^3, b^3) \end{aligned} \quad (5.21)$$

### 5.2.2 Latitude Equation

The sine of latitude is

$$\sin \psi = 2b \sin (f + \omega)$$

Using (5.19),

$$\sin (f + \omega) = \sin (l + \omega + \chi)$$

where

$$\chi (e, l) = 2e \sin l + \frac{5}{16} e^2 \sin 2l + \dots \quad (5.22)$$

Using the power series expansions for sine and cosine, and the binomial theorem,

$$\cos \psi = 1 - e^2 + e^2 \cos 2 \ell + \dots ,$$

$$\sin \psi = 2e \sin \ell + \frac{5}{16} e^2 \sin 2\ell + \dots$$

Then  $\sin \psi$  can be written as

$$\begin{aligned} \sin \psi &= 2b \sin (\ell + \omega) \\ &+ 2b e \left[ \sin (2\ell + \omega) - \sin \omega \right] + \dots \end{aligned}$$

The latitude is then

$$\begin{aligned} \psi &= 2b \sin (\ell + \omega) \\ &+ 2b e \left[ \sin (2\ell + \omega) - \sin \omega \right] + O(e^3, b^3) . \end{aligned} \tag{5.23}$$

### 5.3 Ground Track

To an error of less than one half degrees, the second degree terms can be neglected in equations (5.21) and (5.23),

$$\delta = 2e \sin \ell , \tag{5.24}$$

$$\psi = 2b \sin (\ell + \omega) \tag{5.25}$$

An approximate equation for the ground track can be obtained by eliminating  $\ell$  from the above equations. From (5.25)

$$\frac{\psi}{2b} - \sin \ell \cos \omega = \cos \ell \sin \omega$$

Squaring both sides,

$$\frac{\psi^2}{4b^2} - \frac{\psi}{b} \sin l \cos \omega + \sin^2 l \cos^2 \omega = \cos^2 l \sin^2 \omega \quad .$$

But

$$\sin l = \frac{\delta}{2e} \quad ,$$

so that

$$\frac{\psi^2}{4b^2} - \frac{\delta \psi \cos \omega}{2b e} + \frac{\delta^2}{4e^2} = \sin^2 \omega \quad (5.26)$$

Equation (5.26) is an ellipse, so that, in general, the ground track will be nearly an ellipse. Consider the special cases:

(1)  $\omega = 90^\circ$

Then the ground track equation becomes

$$\frac{\delta^2}{4e^2} + \frac{\psi^2}{4b^2} = 1$$

This is an ellipse whose axes lie on the equator and a meridian.

(2)  $e = b$  ,  $\omega = 90^\circ$

The ground track is a circle of radius  $e$ .

(3)  $\omega = 0$

The ground track is a straight line with slope

$$\frac{\psi}{\delta} = \frac{b}{e}$$

Actual ground tracks are shown in Figures 19(a), 19(b), 20(a), 20(b) for different values of  $e$  and  $I$ . These were produced on the Hewlett-Packard 9810 by using equations (5.3), (5.9), (5.11) and (5.16). Thus, there are no approximations made for small  $e$  and  $I$ . It is observed that the figures resemble ellipses.

## 6.0 Conclusions

The analysis developed in this report shows that the orbital eccentricity of the SPS can get relatively large. However, for certain cases, the eccentricity can be reduced when proper choices on initial conditions are made.

An analytical solution for the motion of eccentricity and longitude of perigee has been derived. This solution is valid for eight to ten years. It could be used for prediction and control of the SPS orbit.

In order for the analytical solution to be valid for longer periods of time, the gravitational effects must be included. It has been shown by numerical integrations that gravitational perturbations on the eccentricity become important for time intervals longer than ten years. The complete analytical solution is feasible through use of the methods developed in this report.

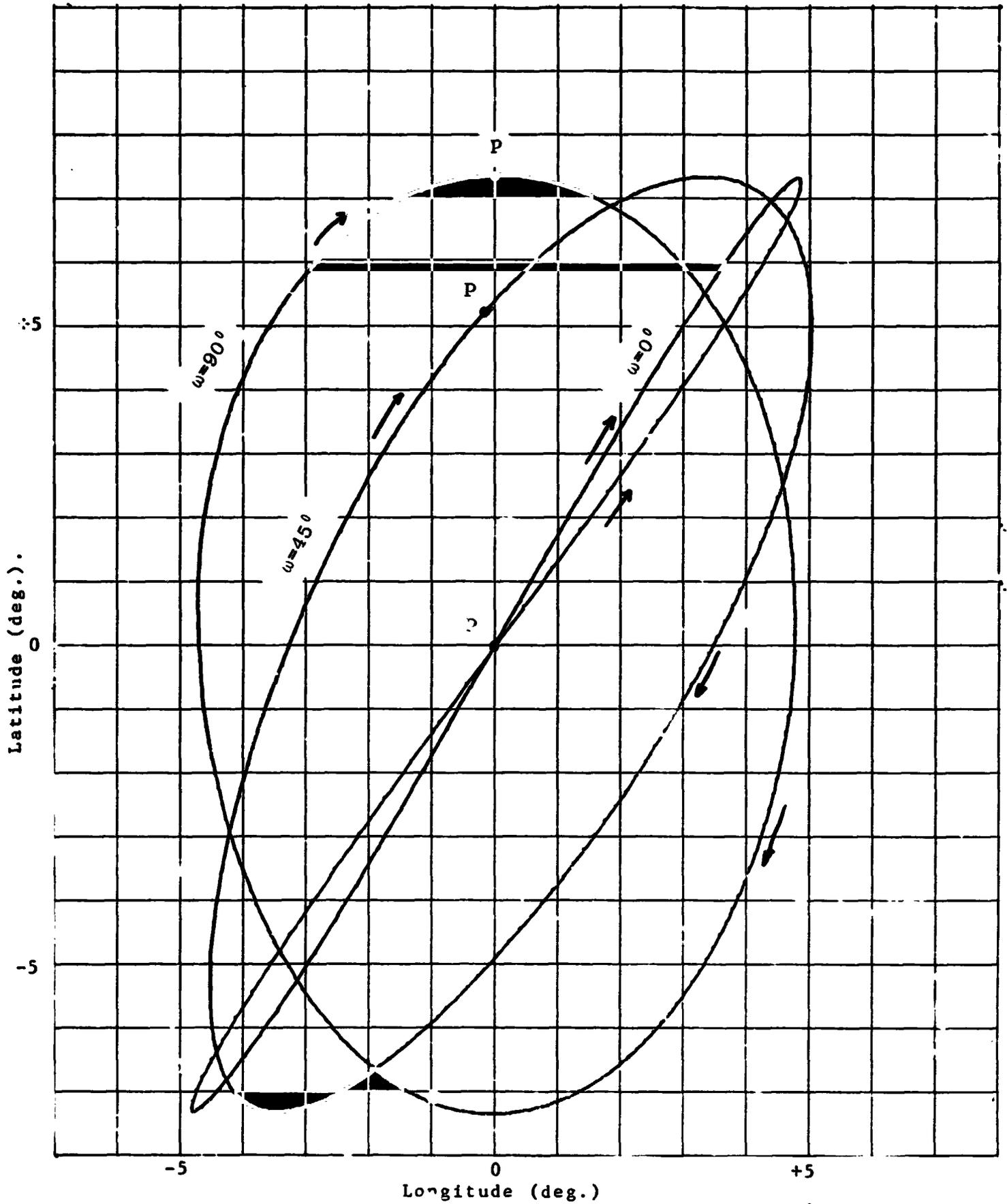
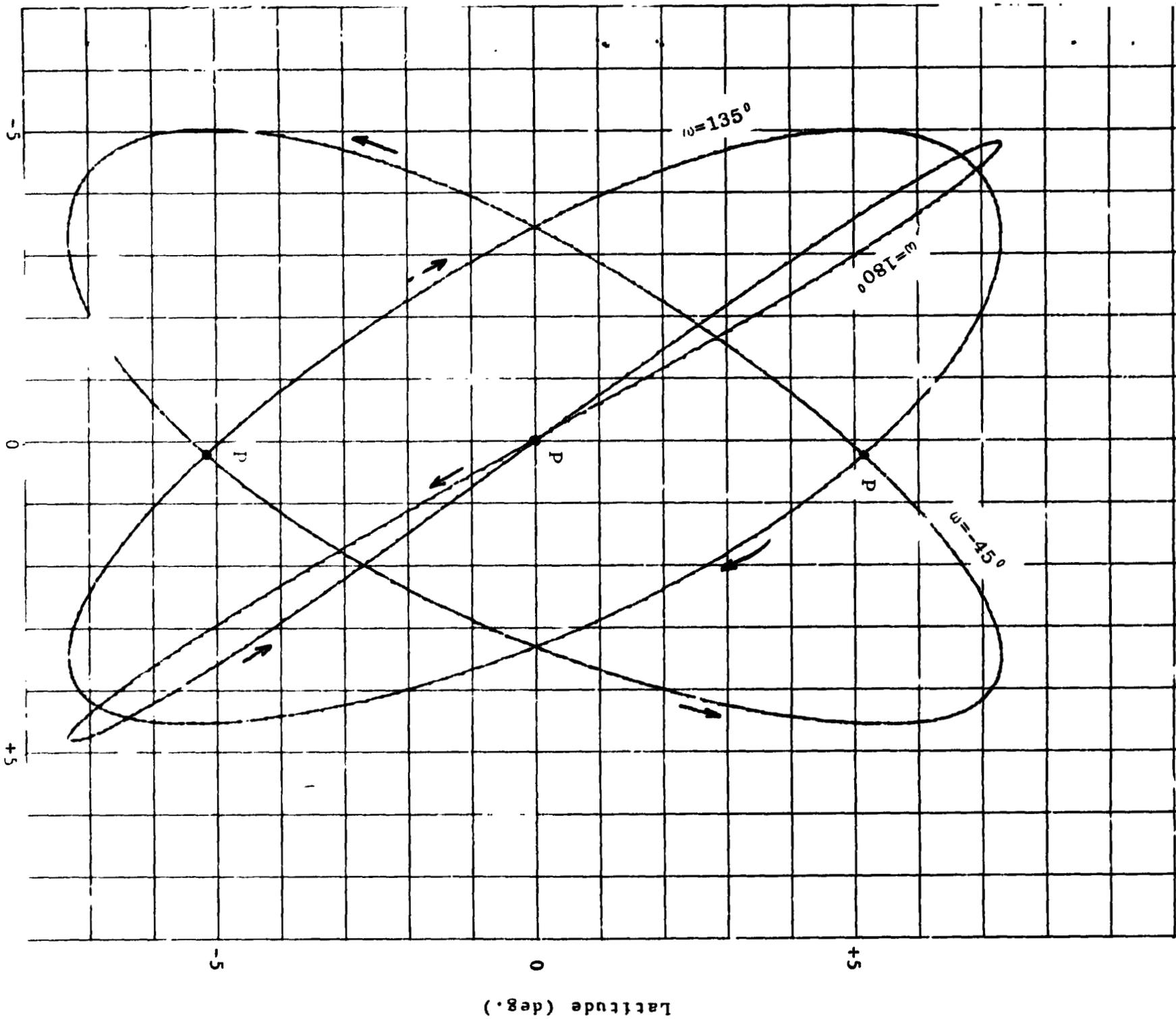


FIGURE 19: Satellite Ground Track,  $e=.042$ ,  $I=7.3^\circ$   
 (a)  $\omega=0^\circ, 45^\circ, 90^\circ$



Longitude (deg.)  
 FIGURE 19: Concluded  
 (b)  $\omega = -45^\circ, 135^\circ, 180^\circ$

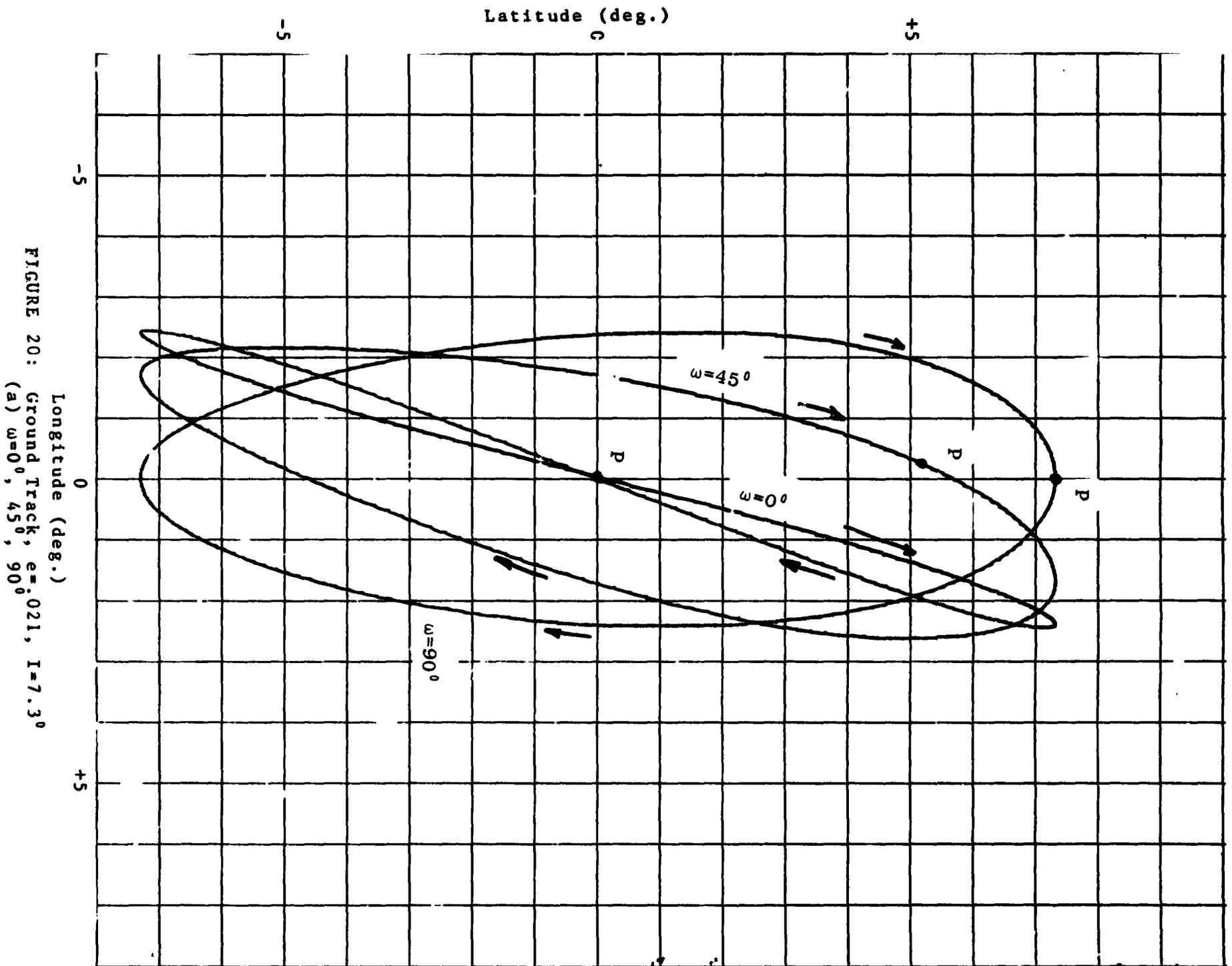
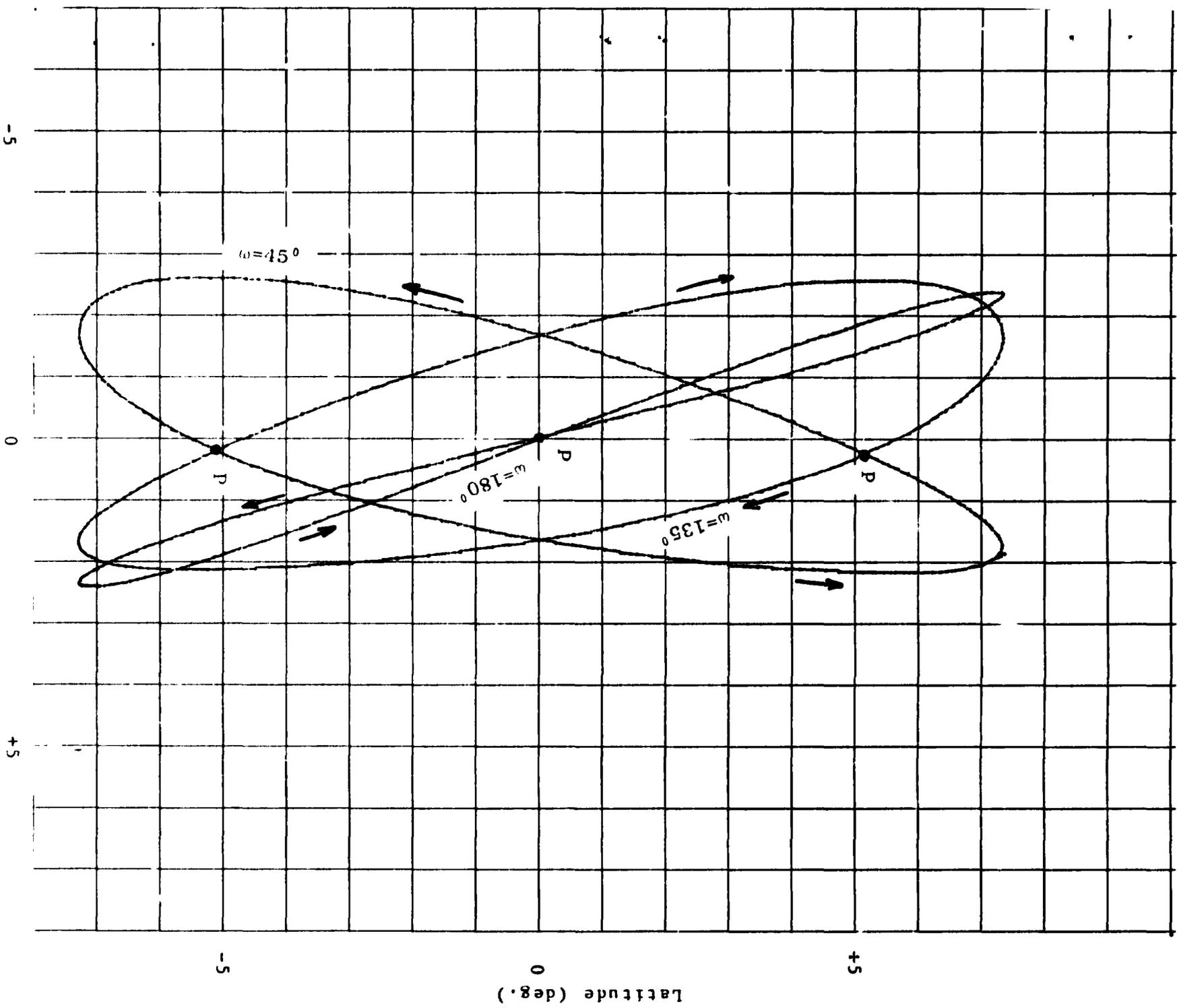


FIGURE 20: Ground Track,  $e=.021$ ,  $I=7.3^\circ$   
 (a)  $\omega=0^\circ$ ,  $45^\circ$ ,  $90^\circ$

Longitude (deg.)  
FIGURE 20: Concluded  
(b)  $\omega = -45^\circ, 135^\circ, 180^\circ$



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**APPENDIX A - POSITION OF THE SUN IN EQUATORIAL COORDINATES**



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APPENDIX A

POSITION OF THE SUN IN EQUATORIAL COORDINATES

It will be assumed that the sun moves on an elliptical orbit about the earth. Using the coordinate system described in Section 3.1, the direction cosines of the sun are:

$$\frac{x_{\odot}}{r_{\odot}} = \cos (f_{\odot} + g_{\odot}) \quad ,$$

$$\frac{y_{\odot}}{r_{\odot}} = C \sin (f_{\odot} + g_{\odot}) \quad , \quad (A1)$$

$$\frac{z_{\odot}}{r_{\odot}} = S \sin (f_{\odot} + g_{\odot}) \quad ,$$

where  $C = \cos \epsilon$  ,  $S = \sin \epsilon$  ,  
and  $\epsilon$  is the angle between the equatorial plane and the ecliptic plane ( $\epsilon = 23^{\circ}27'$ ).  $f_{\odot}$  and  $g_{\odot}$  are the true anomaly and argument of perigee, respectively, of the sun.

Equation (A1) needs to be expressed in terms of the sun's mean anomaly. This will be done by using power series expansions in the eccentricity  $e_{\odot}$  of the sun's orbit. Since  $e_{\odot}$  is small ( $e_{\odot} = .0167$ ), only first degree terms are needed:

$$\cos f_{\odot} = - e_{\odot} + \cos l_{\odot} + e_{\odot} \cos 2l_{\odot} \quad (A2)$$

$$\sin f_{\odot} = \sin l_{\odot} + e_{\odot} \sin 2l_{\odot}$$

Using (A2) in (A1):

$$\frac{x_{\ominus}}{r_{\ominus}} = \cos (l_{\ominus}+g_{\ominus}) - e_{\ominus} \cos g_{\ominus} + e_{\ominus} \cos (2l_{\ominus}+g_{\ominus}) \quad ,$$

$$\frac{y_{\ominus}}{r_{\ominus}} = C \sin (l_{\ominus}+g_{\ominus}) - C e_{\ominus} \sin g_{\ominus} + C e_{\ominus} \sin (2l_{\ominus}+g_{\ominus}) \quad ,$$

$$\frac{z_{\ominus}}{r_{\ominus}} = S \sin (l_{\ominus}+g_{\ominus}) - S e_{\ominus} \sin g_{\ominus} + S e_{\ominus} \sin (2l_{\ominus}+g_{\ominus}) \quad .$$

Terms that are periodic in  $l_{\ominus}$  with coefficient  $e_{\ominus}$  will not be significant and can be neglected. The final expressions are then

$$\frac{x_{\ominus}}{r_{\ominus}} = \cos (l_{\ominus}+g_{\ominus}) - e_{\ominus} \cos g_{\ominus} \quad ,$$

$$\frac{y_{\ominus}}{r_{\ominus}} = C \sin (l_{\ominus}+g_{\ominus}) - C e_{\ominus} \sin g_{\ominus} \quad , \quad (A4)$$

$$\frac{z_{\ominus}}{r_{\ominus}} = S \sin (l_{\ominus}+g_{\ominus}) - S e_{\ominus} \sin g_{\ominus}$$



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**APPENDIX B - NON-SINGULAR DIFFERENTIAL EQUATIONS**

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APPENDIX B

NON-SINGULAR DIFFERENTIAL EQUATIONS

The differential equations (3.47) for  $p$  and  $q$  will be derived from the chain rule (3.46) and the expression (3.40) of  $F_g$  as a function of the DSu variables. The following derivatives are needed:

$$\begin{aligned} \frac{\partial F_s}{\partial G} = \frac{3}{2} \epsilon \left\{ \left[ \frac{\eta^2}{e} (\cos h \cos g - \sin h \cos I \sin g) \right. \right. \\ \left. \left. - \sin h \cos I e \sin g \right] (\cos v - e_\odot \cos g_\odot) \right. \\ \left. + \left[ C \frac{\eta^2}{e} (\sin h \cos g + \cos h \cos I \sin g) \right. \right. \\ \left. \left. + S \frac{\eta^2}{e} \sin I \sin g \right. \right. \\ \left. \left. + C \cos h \cos I e \sin g + S \frac{\cos^2 I}{\sin I} e \sin g \right] (\cos v - \right. \\ \left. \left. - e_\odot \cos g_\odot) \right\} \end{aligned} \tag{B1}$$

$$\begin{aligned} \frac{\partial F_s}{\partial H} = \frac{3}{2} \epsilon \left\{ \sin h e \sin g (\cos v - e_\odot \cos g_\odot) \right. \\ \left. - \left[ C \cos h e \sin g - S \frac{\cos I}{\sin I} e \sin g \right] (\sin v - \right. \\ \left. \left. - e_\odot \sin g_\odot) \right\} \end{aligned} \tag{B2}$$

$$\begin{aligned}
 \frac{\partial F_s}{\partial g} = \frac{3}{2} G \epsilon \left\{ \left[ e \sin g \cos h + e \cos g \sin h \cos I \right] (\cos v - \right. \\
 \left. - e_{\odot} \cos g_{\odot}) + \right. \\
 \left. + \left[ C (e \cos g \sin h - e \cos g \cos h \cos I) - \right. \right. \\
 \left. \left. - S e \cos g \sin I \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} . \tag{B3}
 \end{aligned}$$

Inserting these derivatives into (3.46) and collecting terms, the following expressions are obtained

$$\begin{aligned}
 \frac{dp}{d\tau} = \frac{3}{2} \epsilon \left\{ \left[ -\eta^2 \cos h \sin h (1 - \cos I) \right. \right. \\
 \left. \left. + (\cos I - 1) e^2 \sin g \sin (g+h) \sin h \right] (\cos v - \right. \\
 \left. - e_{\odot} \cos g_{\odot}) \right. \\
 \left. - \left[ C \eta^2 (1 + \cos h (\cos I - 1)) + S \eta^2 \sin I \cos h - \right. \right. \\
 \left. \left. - e^2 \sin (g+h) (\cos I - 1) (C \cos h \sin g - \right. \right. \\
 \left. \left. - S \frac{\cos I}{\sin I}) \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} \tag{B4}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dq}{d\tau} = \frac{3}{2} \epsilon \left\{ \left[ \eta^2 (1 - (1-\cos I) \sin^2 h) - \right. \right. \\
 \left. \left. - e^2 \cos (g+h)(\cos I-1) \sin h \sin g \right] (\cos \nu - e_0 \cos g_0) \right. \\
 \left. + \left[ \eta^2 C \sin h \cos h (1-\cos I) \right. \right. \\
 \left. \left. - \eta^2 S \sin I \sin h \right. \right. \\
 \left. \left. + e^2 \cos (g+h)(\cos I-1)(C \cos h \sin g - \right. \right. \\
 \left. \left. - S \frac{\cos I}{\sin I} \sin g) \right] (\sin \nu - e_0 \sin g_0) \right\}
 \end{aligned} \tag{B5}$$

The above equations can be expressed in terms of  $p, q, P, Q$  by making use of the following expressions:

$$\begin{aligned}
 p &= e \cos (g+h) & q &= e \sin (g+h) \\
 P &= \sqrt{1-\cos I} \cos h & Q &= -\sqrt{1-\cos I} \sin h \\
 \cos (g+h) &= \frac{p}{\sqrt{p^2+q^2}} & \sin (g+h) &= \frac{q}{\sqrt{p^2+q^2}} \\
 \cos h &= \frac{P}{\sqrt{P^2+Q^2}} & \sin h &= \frac{-Q}{\sqrt{P^2+Q^2}}
 \end{aligned} \tag{B6}$$

$$\begin{aligned}
 \sin I \cos h &= P \left[ 2 - (P^2+Q^2) \right]^{\frac{1}{2}} \\
 \sin I \sin h &= -Q \left[ 2 - (P^2+Q^2) \right]^{\frac{1}{2}} \\
 \cos g &= (pP-qQ) \left[ (p^2+q^2) (P^2+Q^2) \right]^{-\frac{1}{2}} \\
 \sin g &= (qP+pQ) \left[ (p^2+q^2) (P^2+Q^2) \right]^{-\frac{1}{2}}
 \end{aligned}$$

The resulting expressions are given in (3.47).

The chain rule applied to the derivatives of P and Q result in

$$\frac{dP}{dt} = \frac{\cos h}{2G\sqrt{1-\cos I}} \left[ \frac{\partial F}{\partial h} - \cos I \frac{\partial F}{\partial g} \right] - \sin h \sqrt{1-\cos I} \frac{\partial F}{\partial H} \quad , \quad (B7)$$

$$\frac{dQ}{dt} = \frac{-\sin h}{2G\sqrt{1-\cos I}} \left[ \frac{\partial F}{\partial h} - \cos I \frac{\partial F}{\partial g} \right] - \cos h \sqrt{1-\cos I} \frac{\partial F}{\partial H}$$

The following additional partial derivative is needed:

$$\frac{\partial F_s}{\partial h} = \frac{3}{2} \epsilon G \left\{ \left[ e \cos g \sin h + e \sin g \cos I \cos h \right] (\cos v - e_{\odot} \cos g_{\odot}) + C \left[ e \sin g \cos I \sin h - e \cos g \cos h \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} \quad (B8)$$

Inserting (B2), (B3) and (B8) into (B7):

$$\begin{aligned} \frac{dP}{dt} = & \frac{3}{4} \epsilon \left\{ \sqrt{1-\cos I} \left[ (1+\cos I) \cos h e \cos g - \right. \right. \\ & \left. \left. - 2 \sin h e \sin g \right] \cdot \left[ \sin h (\cos v - e_{\odot} \cos g_{\odot}) - \right. \right. \\ & \left. \left. - C \cos h (\sin v - e_{\odot} \sin g_{\odot}) \right] + \right. \\ & \left. + S \cos I \left[ \cos h \sqrt{1+\cos I} e \cos g - \right. \right. \\ & \left. \left. - \frac{2 \sin h}{\sqrt{1+\cos I}} e \sin g \right] (\sin v - e_{\odot} \sin g_{\odot}) \right\} \quad , \end{aligned}$$

0-2

$$\begin{aligned} \frac{dQ}{d\tau} = \frac{3}{4} \epsilon \left\{ \sqrt{1-\cos \Gamma} \left[ (1+\cos I) \sin h e \cos g \right. \right. & \quad (B9) \\ & + 2 \cos h e \sin g \left. \right] \\ & \cdot \left[ C \cos h (\sin v - e_{\odot} \sin g_{\odot}) - \sin h (\cos v \right. \\ & \left. - e_{\odot} \cos g_{\odot}) \right] - S \cos I \left[ \sin h \sqrt{1+\cos \Gamma} e \cos g \right. \\ & \left. + \frac{2 \cos h}{\sqrt{1+\cos \Gamma}} e \sin g \right] (\sin v - e_{\odot} \sin g_{\odot}) \left. \right\} \end{aligned}$$

The above equations can now be expressed in terms of  $p, q, P, Q$  by the use of (B6). The resulting non-singular expressions are given in (3.48).



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**APPENDIX C - COMPUTATIONAL ALGORITHM FOR THE ANALYTICAL SOLUTION**



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APPENDIX C

COMPUTATIONAL ALGORITHM FOR THE ANALYTICAL SOLUTION

The approximate analytical solution for p and q was derived in Section 3.7 and is written as

$$\begin{aligned} p &= \Phi \beta (\cos \theta + \tau \delta e_{\circ} \sin g_{\circ}) + C_1 \\ q &= \Phi (\sin \theta - \tau \delta e_{\circ} \cos g_{\circ}) + C_2 \end{aligned} \tag{C1}$$

The computational sequence for evaluating p and q at any time is shown below.

(1) Values of constants

$$\delta e_{\circ} \sin g_{\circ} = (365.25)^{-1} (0.01675) \sin (281^{\circ}.0)$$

$$\delta e_{\circ} \cos g_{\circ} = (365.25)^{-1} (0.01675) \cos (281^{\circ}.0)$$

$$\beta = 0.9679 \quad , \quad \delta = (365.25)^{-1}$$

(2) Value of small parameter

$$\epsilon = (6.611)^2 \frac{S}{M} (5.06 \cdot 10^{-7})$$

S = cross-sectional surface area in square meters

M = weight in kilograms

$$\Phi = \frac{3}{2} (365.25) \epsilon$$

(3) Compute  $C_1$  and  $C_2$

$$C_1 = e_0 \cos \tilde{g}_0 - \Phi \beta \cos (l_{\odot 0} + g_{\odot})$$

$$C_2 = e_0 \sin \tilde{g}_0 - \Phi \beta \sin (l_{\odot 0} + g_{\odot})$$

$e_0$  = initial value of eccentricity

$\tilde{g}_0$  = initial longitude of perigee

$$l_{\odot 0} = 358^{\circ} 28' 33'' + 1295\ 96579'' T \quad ,$$

$$g_{\odot} = 281^{\circ} 13' 15'' + 6189'' T \quad ,$$

$T$  = Julian centuries of 36525 ephemeris days ,  
referenced to 1900 January 0.5 ,

(4) Compute  $p(\tau)$  and  $q(\tau)$  using equation (C1), where

$\tau = (2\pi)$  (number of revolutions)

$$\theta = (365.25)^{-1} \tau + l_{\odot 0} + g_{\odot}$$

It can be assumed that one revolution is equivalent to one day.