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OPTIMUM WALL IMPEDANCE FOR SPINNING MODES -
A CORRELATION WITH MODE CUT-OFF RATIO

by Edward J. Rice
Lewis Research Center
Cleveland, Ohio 44135

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Abstract

A correlating equation relating the optimum acoustic impedance for the wall lining of a circular duct to the acoustic mode cut-off ratio is presented and compared to exact calculations. The optimum impedance was correlated with cut-off ratio because the cut-off ratio appears to be the fundamental parameter governing the propagation of sound in the duct. Modes with similar cut-off ratios respond in a similar way to the acoustic liner. The correlating equation is useful for the design of suppressors for aircraft engine inlets having a steady mean flow with a boundary layer and spinning noise source excitation. The correlation is a semi-empirical expression developed from an empirical modification of an equation originally derived from sound propagation theory in a thin boundary layer. Exact calculations of the optimum wall impedance were made over a wide range of frequency parameters, boundary layer thicknesses and flow Mach numbers to develop and verify the correlation. This correlating equation represents a part of a simplified liner design method, based upon modal cut-off ratio, for multimodal noise propagation.

Symbols

- $R_0$: optimum resistance coefficient
- $R_\pi$: optimum reactance coefficient
- $c$: speed of sound, m/sec
- $D$: duct diameter, m
- $F$: refraction function, see equation (5)
- $f$: frequency, Hz
- $G$: empirical function, see equations (4), (13), and (14)
- $Q$: uniform steady flow Mach number in duct
- $M_m$: spinning mode lobe number (circular or annular)
- $R$: amplitude of eigenvalue $m$
- $R_m$: value of $R$ at the optimum impedance
- $r_m$, $\mu$: radial mode order
- $x$: axial coordinate, m
- $a$: complex radial eigenvalue ($a = Re^{i\theta}$)
- $\delta$: boundary layer thickness for 1/7th power velocity profile, m
- $\zeta$: optimum specific acoustic impedance with a boundary layer
- $\zeta_0$: optimum specific acoustic impedance with zero boundary layer thickness (slip flow)
- $\eta$: frequency parameter, $fd/c$
- $\sigma$: specific acoustic resistance
- $\sigma_m$: optimum specific acoustic resistance
- $\mu$: radial mode order
- $\xi$: mode cut-off ratio, see equation (1)
- $\sigma$: attenuation coefficient
propagation coefficient
\( \phi \) angular coordinate, radians
\( \omega \) phase of eigenvalue, degrees
\( \omega_G \) phase of function \( G \), see equation (14), degrees
\( \Delta \omega_G \) part of \( \omega_G \), see equations (15) and (16), degrees
\( \gamma_m \) phase of eigenvalue at optimum impedance, degrees
\( \gamma_{m, \mu} \) \( \gamma_m \) with modal indices included
\( X \) specific acoustic reactance
\( X_m \) optimum specific acoustic reactance
\( X_{m, 0} \) \( X_m \) with boundary layer neglected
\( \omega \) circular frequency, rad/sec

Sound Propagation Model

The sound propagation theory used for the so-called exact calculations was outlined in Ref. 9 and discussed in detail in Ref. 12. The exact calculation is based on the geometry shown in Fig. 1. The duct is cylindrical with acoustic treatment on the walls and the steady flow contains sheared flow near the walls. The sheared flow profile is the standard 1/7th power profile to a distance \( \delta \) from the wall. A linear profile is placed near the wall. Only inlet conditions (negative Mach number) are considered with the sound propagation being opposed by the steady flow. The shear layer is uniform in the axial direction and no acoustic reflections are considered. The calculation procedure uses the classical plug flow Bessel function solutions in the duct interior matched to a Runge-Kutta integration through the boundary layer.

Optimum Wall Impedance Definition

The liner optimum acoustic impedance for single mode sound propagation is defined exactly as presented in Ref. 9. Figure 2 illustrates the concept of the optimum impedance for a single mode. The attenuation coefficient (\( \zeta \)) and the propagation coefficient (\( \phi \)) are components of the complex axial wave number where the acoustic pressure varies as \( \exp(i\omega/c(\gamma + i\phi)x) \). Note that contours of constant damping (the solid curves) are closed curves in the wall impedance plane. As the attenuation coefficient is increased in value these contours shrink in size and ultimately converge to a point designated by a plus sign. Further increases in damping produce curves which are not closed in the vicinity of this point and in fact represent large contours associated with the next higher radial mode. This limiting point in Fig. 2 thus represents the impedance with maximum damping for the particular mode under consideration, and this point is designated as the optimum wall impedance for the mode. The associated value of the attenuation coefficient is not considered here but can be found in Refs. 2, 8, and 12. Similar procedures for defining optimum impedance have been published in the literature.

Mode Cut-Off Ratio

The definition of mode cut-off ratio is the same as that used in Ref. 9. This can be expressed as,

\[
\xi = \frac{\omega}{R \sqrt{(1 - \mu_0^2) \cos 2\gamma}}
\]

where

\[
\gamma = \frac{\omega x}{c}
\]

is the frequency parameter and the complex eigenvalue for the mode is given by,

\[
z = \Re \xi
\]

For a hardwall duct equation (1) reduces to the more familiar cut-off ratio expression of Ref. 17. When a spinning mode is highly propagating (\( \xi \) large) the acoustic wave motion is mainly in the axial direction, while near cut-off (\( \xi \approx 1 \)) the wave motion is mainly transverse or circumferential.

Correlation of Optimum Impedance

The starting point for the optimum impedance correlation is an equation derived in Ref. 12. This equation in turn was derived from the thin boundary layer theory of Ref. 18. The expression is,

\[
\zeta = \frac{\omega x}{m} + \frac{1}{X_m} = \frac{(1 + \zeta)^2}{1 - 2 \mu X_0}
\]

where \( \zeta \) is the optimum wall impedance with the boundary layer refraction considered, \( \mu \) is the more simply calculated optimum impedance for plug flow (no boundary layer), \( \zeta \) is the nondimensional boundary layer thickness, and \( F \) in it's simplest form from Ref. 12 is,

\[
F = \frac{\xi \mu X_0}{4}
\]

The term \( G \) in the denominator of equation (1) was not present in the equation from Ref. 12 and has been inserted for additional empirical corrections to be made in this study. A similar procedure was used in Refs. 8 and 9 to provide an empirical correction to make equation (1) valid down to cut-off (\( \xi = 1 \)), but the term comparable to \( G \) was considered to be only a function of cut-off ratio. We will find here that \( G \) must also be a function of Mach number (\( \mu_0 \)), frequency parameter (\( \eta \)), and boundary layer thickness (\( \varepsilon \)) when a broader parameter range in the exact propagation solution is considered.

Optimum Impedance-Without a Boundary Layer

The first step in developing the correlation equation (4) is to generate an expression for the optimum acoustic impedance for the slug flow case which is a function of cut-off ratio. An expression for this optimum impedance is available as,
\[ \tau_0 = \eta_0 + i \chi_0 = \eta_0^2 (\eta_0 - i \chi_0) \]  
(6)

where

\[ B_x = \frac{1.15}{\mu + 0.78 \varepsilon^{2/3} \mu^{1/4}} \]  
(7)

\[ B_x = \frac{0.92}{\mu^2 + 3.17 \varepsilon^{2/3}} \]  
(8)

and

\[ Q = 1 + \eta_0 (\zeta + i \eta) = \frac{1 - \eta_0}{1 - \eta_0^2} \]  
(9)

Equations (7) and (8) are correlations presented in Ref. 10 and \( Q^2 \) is the usual impedance multiplier occurring due to the continuity of displacement boundary condition used at the wall with slip flow.

The terms \( B_0 \), \( B_x \), and \( Q \) must be converted to functions of mode cut-off ratio. This will be done by introducing the eigenvalue into the expressions for \( B_0 \) and \( B_x \) such that cut-off ratio may be reinserted. The denominators of equations (7) and (8) are roughly proportional to eigenvalue and eigenvalue squared, respectively, as shown by the curve fits in Fig. 3. Note that Fig. 3 is plotted for \( M_o = 0 \) in which case \( Q = 1 \), and from equation (6) \( \eta_0/\eta = B_0 \) and \( \chi_0/\eta = -B_x \). The plotted points for \( m = 1, 7, \) and 20 use \( \mu = 1, 2, 3, \) and 10 while for \( m = 0 \) and 50, only \( \mu = 1 \) was available. By using the curve fits of Fig. 3 and the cut-off ratio definition of equation (1), equation (6) can be written as,

\[ \tau_0 = \eta_0^2 \sqrt{1 - \eta_0^2} \left[ 1 - \frac{10.84}{\eta} \sqrt{1 - \eta_0^2} \right] \]  
(10)

The quantity \( Q \) given by equation (9) must now be expressed as a function of cut-off ratio. Using equations (1) and (3), equation (9) can be expressed as,

\[ Q = \frac{1 - \eta_0}{1 - \eta_0^2} \sqrt{1 - \eta_0^2} \left( 1 + \frac{\sin 2\varphi}{\cos 2\varphi} \right) \]  
(11)

The quantity \( \sin 2\varphi/\cos 2\varphi \) is still a function of the particular mode under consideration, but it is generally quite small compared to one. Rather than discard this quantity completely an average value based on many modes was calculated. Exact calculations of optimum impedance were compared to equation (10) for modes near cut-off and the phase angle of the second quantity in the radical of equation (11) was estimated. Equation (10) was used since it was anticipated that \( \varphi = 0 \) in equation (4) for \( M_o > 0.5 \) and thus \( \tau_0 = \eta_0 \). This average phase angle was found to be twelve degrees (thus \( \varphi = 6^\circ \)) and equation (11) can be expressed as,

\[ Q = \frac{1 - \eta_0}{1 - \eta_0^2} \sqrt{1 - \eta_0^2} \left( 1 + \frac{\sin 2\varphi}{\cos 2\varphi} \right) \]  
(12)

With this value of \( Q \) inserted into equation (10) the optimum impedance correlation is seen to contain modal information only in the form of mode cut-off ratio.

**Optimum Impedance With a Boundary Layer**

With \( \tau_0 \) expressed in terms of cut-off ratio alone the function \( G \) in equation (4) can now be correlated as a function of cut-off ratio. A multitude of exact calculations of optimum impedance were made using the calculation procedure as outlined in a previous section (Sound Propagation Model). These calculations included many modes from the first propagating spinning mode \( (m = 1, \mu = 1) \) to the highest lobe number which would propagate. A wide range of frequency parameters \( (\eta = 5 \text{ to } 30) \), Mach numbers \( (M_o = -0.1 \text{ to } -0.7) \), and boundary layer thickness \( (\varepsilon = 0.005 \text{ to } 0.1) \) were considered. The exact values of optimum impedance were compared to equation (4) and the value of the function \( G \) was calculated to obtain equality. The amplitude and phase of \( G \) were then correlated with the input parameters to obtain the following correlations. The amplitude of \( G \) can be expressed as,

\[ |G| = \frac{1.1 \eta - 5.41}{1 + 1.5 \varepsilon^2} \]  
(13)

and the phase as

\[ \varphi_G = \frac{50}{\xi} - \omega \]  
(14)

where for \( |M_o| < 0.5 \)

\[ \omega_G = \frac{80}{\eta \varepsilon^2 |M_o|} \]  
(15)

and for \( |M_o| > 0.5 \)

\[ \omega_G = \frac{640 H_o^2}{\eta \varepsilon^2} \]  
(16)

Note that \( \varphi_G \) is expressed in degrees.

Near cut-off the exponential in equation (13) is very small and \( G \approx 0 \) and \( \tau_0 = \eta_0 \). Note that \( \tau_0 \) (derived for plug flow) does not contain boundary layer thickness, and thus near cut-off the optimum impedance is not a function of boundary layer thickness.

**Summary of the Use of the Correlating Equations**

The procedure for using the correlating equations will be summarized in this section. First the method of calculation for generating the locus of optimum impedance versus cut-off ratio is presented. This procedure would be followed in an engine inlet design if multimodal noise excitation is considered.
The input parameters $\eta$, $M_\infty$, and $\epsilon$ are determined from the sound frequency, duct size, steady flow rate, and an estimate of the boundary layer thickness. A cut-off ratio is selected for calculation and the complex impedance $Z_0$ is calculated using equations (10) and (12). The magnitude and phase of the function $G$ are calculated using equations (13), (14), and (5) along with equation (15) or (16) depending upon the magnitude of the Mach number. The two complex quantities $Z_0$ and $G$ are used in equation (6) (again using equation (5)) to calculate the optimum acoustic impedance components $\rho_m$ (resistance) and $\chi_m$ (reactance). Another value of cut-off ratio is then selected and the procedure is repeated.

If the optimum impedance of a single mode is to be estimated, the cut-off ratio must be first calculated from equation (1) using the mode eigenvalue at the optimum. The eigenvalue can be calculated from correlations given in Ref. 10 which are repeated here for convenience. The eigenvalue amplitude is,

$$R_{m,\mu} \approx R_{m,1} + (\mu - 1)^2 + 0.076 m (\mu - 1)$$

(17)

where

$$R_{m,1} = m + 2.247 m^{1/3} + 1.521 m^{-1/3}$$

(18)

The eigenvalue phase is,

$$\varphi_{m,\mu} = \varphi_{m,1} + 2(\mu + m/7)$$

(19)

where

$$\varphi_{m,1} = \frac{35.15}{(m + 2)^{0.6}}$$

(20)

Equation (18) is valid for $m \neq 0$. If $m = 0$, use $R_{m,1} = 3.278$. These correlations were derived from exact calculations of the optimum eigenvalues for $m = 1$ to 20 and $\mu = 1$ to 10. After the cut-off ratio is calculated in the procedure in the first part of this section is followed. Some caution should be exercised when using equation (4) for single mode optimum impedances. As will be shown in a later section, for large lobe numbers ($m$) and small radial orders ($\mu$) equation (4) must be considered as only an approximation to the exact optimum.

Comparison of Correlation with Exact Calculations

Three input parameters will be varied to illustrate the accuracy of the optimum impedance correlation. These are the frequency parameter ($\eta$), steady flow Mach number ($M_\infty$) and the boundary layer thickness ($\epsilon$). A base case of $\eta = 15$, $M_\infty = 0.4$, and $\epsilon = 0.05$ will be used with one parameter at a time being varied from this base.

Figures 4 and 5 compare the correlation with exact calculations for the resistance and reactance, respectively, when boundary layer thickness - resistance above cut-off and the correlation is seen to hold quite well. For $\eta = 5$, $M_\infty = 0.2$, and $\epsilon = 0.05$ the correlation is seen to hold quite well. For $\eta = 20$, $M_\infty = 0.7$, and $\epsilon = 0.1$ the reactance correlation is seen to be good (Fig. 11) but the resistance correlation (Fig. 10) is seen to be predicted somewhat low. This is a case of extreme refraction in the boundary layer giving a rapid fall-off of resistance above cut-off and the correlation somewhat over predicts this effect. For a low frequency, high Mach number, thick boundary layer case
A semi-empirical correlation for the optimum acoustic impedance for spinning modes in a circular duct has been presented. An engine inlet flow condition with a boundary layer at the wall and a uniform velocity profile in the central core was considered. The correlation is part of an acoustic linear design method being developed for multimodal sound propagation from an aircraft engine through the acoustically lined inlet duct and out into the far-field. The philosophy of this approach was summarized in Ref. 8 and the far-field radiation concepts were reported in Ref. 11. The key to this approach is the distribution of acoustic power as a function of mode cut-off ratio. The elements yet to be developed are the power distribution shifts caused by modal scattering in the acoustic liner and possibly at the duct termination.

The optimum acoustic impedance equation presented here represents an improvement upon earlier published forms. A wide range of input parameters were considered here in contrast to that used for the equation's earlier form. The correlation can be used to quickly estimate optimum impedance and eliminates the elaborate computer programs and considerable computer running time needed for exact calculations.

The use of the correlation for design estimates requires some knowledge of the modal properties of the noise source. Rough estimates of the gross modal properties can be made. For example, in a stationary engine or fan test installation a considerable amount of the acoustic power occurs in the modes near cut-off. An efficient liner might be expected using a near cut-off liner design. For a flight situation the noise often consists of multiple pure tones and blade passage frequency. The dominant modes expected for the multiple pure tones (shock noise) can be estimated perhaps from rotor locked mode considerations. These modes will most likely be closed to cut-off. The modes for blade the passage frequency tones and their associated cut-off ratios can be estimated from rotor-stator interaction theory (Ref. 17).

References


Figure 1. - Geometry and steady-flow velocity profile.

Figure 2. - Constant attenuation and propagation contours for the least attenuated spinning mode using displacement boundary condition. Inlet Mach number, $M_0 = 0.4$; lobe number, $m = 7$; frequency parameter, $\eta = 1/k$; boundary layer thickness, $\delta^*/r_0 = 0$. 
OPEN SYMBOLS $\beta_m/n$
FILLED SYMBOLS $-\chi_m/n$
MACH NUMBER $M_o = 0$

LOBE NUMBER, $m$

$\bigcirc$ 0
$\bigtriangleup$ 1
$\square$ 20
$\triangle$ 50

$3.1 \frac{R_m \cos 2\varphi_m}{\cos 2\varphi_m}$

$8.3 \frac{R_m \cos 2\varphi_m}{R_m \cos 2\varphi_m}$

Figure 3. - Correlation of optimum resistance and reactance with mode eigenvalue.
Figure 4. - Comparison of correlating equation with exact calculations of optimum resistance for variations in boundary layer thickness.
Figure 5. - Comparison of correlating equation with exact calculations of optimum reactance for variations in boundary layer thickness.
Figure 6. Comparison of correlating equation with exact calculations of optimum resistance for variations of frequency parameter.
Figure 7. Comparison of correlating equation with exact calculations of optimum reactance for variations of frequency parameter.

Figure 8. Comparison of correlating equation with exact calculations of optimum resistance for variations in steady flow Mach number.
Figure 9. - Comparison of correlating equation with exact calculations of optimum reactance for variations in steady flow Mach number.
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<th>EXACT CALCULATION</th>
<th>APPROXIMATE EQUATION</th>
<th>FREQUENCY PARAMETER, $\eta$</th>
<th>MACH NUMBER, $M_0$</th>
<th>BOUNDARY LAYER THICKNESS, $\varepsilon$</th>
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Figure 10. - Comparison of exact calculations of optimum resistance with extrapolated correlating equation.
EXACT APPROXIMATE

CALCULATION EQUATION

FREQUENCY PARAMETER, \( \eta \)
MACH NUMBER, \( M_0 \)
BOUNDARY LAYER THICKNESS, \( \delta_0 \)

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<th>Approximate Equation</th>
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Figure 11. - Comparison of exact calculations of optimum reactance with extrapolated correlating equation.

Figure 12. - Deviation of high lobe number modes from the acoustic resistance correlation.
Figure 13. Deviation of high lobe number modes from the acoustic reactance correlation.