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DYNAMICAL THEORY OF STABILITY FOR ELASTIC RODS WITH NONLINEAR CURVATURE AND TWIST

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16. Abstract

Considering non-linear terms in the curvature as well as in the twist, the governing boundary value problem for lateral bending of elastic, transverse loaded rods is formulated by means of Hamilton's principle. Using the method of small vibrations, the associated linearized equations of stability are derived, which complete the currently accepted relations. The example of the simplest lateral bending problem illustrates the improved effect of the proposed equations.

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DYNAMICAL THEORY OF STABILITY FOR ELASTIC RODS WITH NONLINEAR CURVATURE AND TWIST

J. Wauer

SUMMARY /161*

Considering non-linear terms in the curvature as well as in the twist the governing boundary value problem for lateral bending of elastic, transverse loaded rods are formulated by means of Hamilton's principle. Using the method of small vibrations, the associated linearized equations of stability are derived, which complete the currently accepted relations. The example of the simplest lateral bending problem illustrate the improved effect of the proposed equations.

1. PROBLEM FORMULATION

The stability equations which describe the tipping of elastic rods have been derived several times recently [1, 2, 3, 4] with the assumption that the deviation between the undeformed and the deformed rod axis is so small that differentiations with respect to the two important position coordinates can be considered to be identical.

This restriction will be given up here. The Hamilton principle described in [4] will be the point of departure for our analysis. Therefore, it is the purpose of this paper to formulate this variational principle for describing the kinetic tipping with more extensive consideration of the rod curvature during deformation. Synthetic equilibrium analyses, which is the continuation

*Numbers in margin indicate pagination in original foreign text.

of the Federhofer analysis, were performed early by Chwalla [5]. However, these have been forgotten. It was only Ziegler [6] who considered this effect in the case of a compressed and twisted circular cross section. However, he did not pursue the most important aspects of this process. In our paper, it will also not be possible to derive the consequences for the most general tipping problem of elastic rods with longitudinal and transverse loading. We have to restrict ourselves to cases where the rod axis does not experience any longitudinal extension.

Using the Hamilton principle, we obtain the nonlinear boundary value problem from a variation. After this, one can transfer to the linearized stability equations using the method of small oscillations.

In order to focus the reader's attention to the important aspects of the problem, we will consider the Prandtl tipping rod with a pure moment load, without referring to a concrete special case.

2. VARIATIONAL PRINCIPLE

Let us consider an originally straight rod without preliminary /162 deformations of length l. We assume that it has the constant main bending stiffnesses EJ_1, EJ_2 (E modulus of elasticity, $J_{1,2}$ area moments of inertia), the constant twisting stiffness GJ_T (G shear modulus, J_T torsion area moment) and the constant mass per unit of length $\mu = \varrho A$ (ϱ Density, A cross-section area.)

We will assume all of the usual assumptions made in classical technical bending theory, and assume pure Saint-Venant torsion. We assume that the shear center and the center of gravity coincide. Also we will not consider the influences of damping.

The principle of Hamilton

$$\delta \int_{t_0}^{t_1} (T - V_i - V_a)\, dt + \int_{t_0}^{t_1} \delta W\, dt = 0 \qquad (1)$$

is conveniently used for the systematic analysis. In order to evaluate it, we therefore must first determine the kinetic energy T, the elastic potential V_i, the potential of the external forces V_a, and the virtual work δW of the potential-free forces.

The kinetic energy is given by

$$T = \frac{1}{2}\varrho \int_0^l \left[\iint_A v_P^2 \, dA\right] ds \,, \tag{2}$$

where v_P is the absolute velocity of a material rod particle P; S, points along the arc length of the rod which has a curvature in its general position.

The elastic potential V_1 is calculated from the sum of the shape change energy for bending

$$V_{iB} = \frac{1}{2} E J_1 \int_0^l p_1^2 \, ds + \frac{1}{2} E J_2 \int_0^l p_2^2 \, ds \tag{3}$$

and for twisting,

$$V_{iT} = \frac{1}{2} G J_T \int_0^l q^2 \, ds \tag{4}$$

where $p_{1,\ 2}$ are the curvatures and q is the torsion of the rod axis [7, 8].

These relationships can be generalized for the more general tipping process for both transverse and axial loading, by adding the following longitudinal extension energy

$$V_{iL} = \frac{1}{2} E A \int_0^l \varepsilon^2 \, ds \,, \tag{5}$$

within the elastic potential V_1, where ε is the longitudinal extension of the central line of the rod.

We will discuss the calculation of V_a and δW later.

3. ANALYSIS OF THE STATE OF DEFORMATION

Two coordinate systems shown in Figure 1 are used to describe the rod deformations. A first, space-fixed x, y, z reference system (unit vectors e_x, e_y, e_z) is attached to one of the rod ends, where the z-axis coincides with the undeformed rod axis. A second body-fixed ξ, η, ζ system (unit vectors e_ξ, e_η, e_ζ) is established at an arbitrary point z. It points along the main moments of inertia of a cross-section and is rotated with respect to the space-fixed system, and it is also tangential to the central line of the rod.

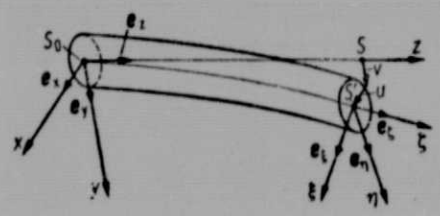

Figure 1: Deformation state of a rod, cross section.

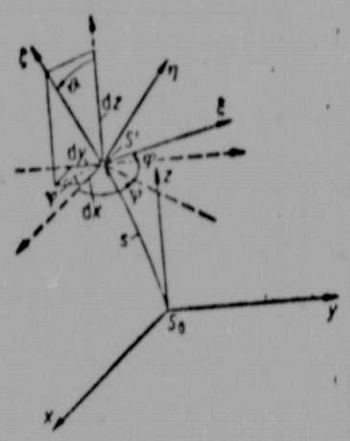

Figure 2: Description of deformation using Euler angles.

Consequently, the coordinate origin S' of the body-fixed system /163
is therefore the geometric locus of all deformed cross-sectional centers of gravity. In the general case, it is displaced by the coordinates u, v, with respect to the initial point S. These quantities are measured in the space-fixed x, y, z system. The second coordinate system, whose rotation with respect to the (x, y, z) system, is described using the Euler angles ψ, ϑ, φ according to figure 2, describes the stiffnesses $EJ_{1,\,2}$ and GJ_T.

Primes will refer to derivatives with respect to the arc length S. From the Kirchhoff gyroscope analogy [9], one finds the kinematic relationships

$$p_1 = \psi' \sin\vartheta \sin\varphi + \vartheta' \cos\varphi, \quad p_2 = \psi' \sin\vartheta \cos\varphi - \vartheta' \sin\varphi, \quad q = \psi' \cos\vartheta + \varphi' \tag{6}$$

between the characteristics $p_{1,\,2}$, q of the rod axis and the Euler angles ψ, ϑ, φ .

As can be seen from Figure 2, the differential quotients

$$x' = \sin\vartheta \sin\psi, \quad y' = -\sin\vartheta \cos\psi, \quad z' = \cos\vartheta \tag{7}$$

relate the space-fixed coordinates x, y, z and the Euler angles ψ, ϑ, φ.

This means we are now in the position of formulating the Hamilton principle (1) as a function of the deformation quantities ψ, ϑ, φ . In order to specify the boundary conditions, however, it is necessary to transfer from the Euler angles ψ, ϑ, φ to the center of gravity displacements u, v and the effective rotation of a cross-section τ.

We will assume small deformations, and therefore in (6) and (7) we will assume a small inclination of the central line of the rod with respect to the space-fixed z-axis. However, we will not linearize with respect to ϑ as was done in [2], but we will expand up to quadratic terms. Then, from (6), we obtain

$$p_1 = \psi'\vartheta \sin\varphi + \vartheta' \cos\varphi, \quad p_2 = \psi'\vartheta \cos\varphi - \vartheta' \sin\varphi, \quad q = \psi'\left(1 - \frac{\vartheta^2}{2}\right) + \varphi' \tag{8}$$

and from (7) we find

$$x' = \vartheta \sin\psi, \quad y' = -\vartheta \cos\psi, \quad z' = 1 - \frac{\vartheta^2}{2}. \tag{9}$$

The two first equations in (9) also apply for the derivatives of the dependent variables u, v in the x and y directions [10]. If we differentiate with respect to the arc length s, we then find

$$u'' = \vartheta' \sin\psi + \vartheta\psi' \cos\psi, \quad v'' = -\vartheta' \cos\psi + \vartheta\psi' \sin\psi. \tag{10}$$

If we formulate the expressions

$$\left.\begin{array}{l} u'' \sin(\psi + \varphi) - v'' \cos(\psi + \varphi), \quad u'' \cos(\psi + \varphi) + v'' \sin(\psi + \varphi), \\ (\psi + \varphi)' - \frac{1}{2}(u'v'' - v'u''), \end{array}\right\} \tag{11}$$

we can see that they coincide with $p_{1,\,2}$, q according to (6) if we use an addition theorem, where we take into account the "Quadratic approximation". According to figure 2, the effective rotation angle of a rod cross-section angle is /164

$$\tau = \psi + \varphi \tag{12}$$

for small inclinations. Therefore, the curvatures $p_{1,\,2}$ and the torsion q can be written in the form

$$p_1 = u'' \sin\tau - v'' \cos\tau, \quad p_2 = v'' \cos\tau + u'' \sin\tau, \tag{13}$$

$$q = \tau' - \tfrac{1}{2}(u'v'' - v'u'') \tag{14}$$

which only contain the usual quantities u, v, τ and their derivatives.

If we also assume a small twist angle τ, then in addition to the torsion q (14) we can write the curvatures $p_{1,\,2}$ in the following form

$$p_1 = -v'' + \tau u'', \quad p_2 = u'' + \tau v'' \tag{15}$$

up to and including quadratic coupling terms. Therefore, we have a consistent approximation for the quantities $p_{1,\,2}$, q which are based on the theory of elasticity, and which is not found in the literature

according to research by the author. There, one usually finds

$$p_1 = -v'' + \tau u'', \quad p_2 = u'' + \tau v'', \quad q = \tau' \tag{16}$$

[2, 4]. This is a consequence of the initially formulated assumption that derivatives with respect to s and z can be considered to be identical. The coupling terms in (14), (15) have the same order of magnitude, and therefore only the approximation given here will be completely satisfactory, even though in practice the original relationship (16) will remain useful.

When calculating the absolute velocity, there are no special features compared with [4]. We find

$$\boldsymbol{v}_P = (\dot{u} - \eta\dot{\tau})\,\boldsymbol{e}_x + (\dot{v} + \xi\dot{\tau})\,\boldsymbol{e}_y, \tag{17}$$

where ξ, η are the coordinates of P' in the local systems and dots are derivatives with respect to time.

This means that the potential V_1 and the kinetic energy T for the tipping support can be formulated without the longitudinal extension of the rod axis, and without even considering a specific problem. If we introduce the radius of inertia k_s

$$T = \frac{\mu}{2}\int_0^l (\dot{u}^2 + \dot{v}^2 + k_s^2\dot{\tau}^2)\,ds \tag{18}$$

which is known from the literature [4], as well as

$$V_1 = \frac{EJ_1}{2}\int_0^l (-v'' + \tau u'')^2\,ds + \frac{EJ_2}{2}\int_0^l (u'' + \tau v'')^2\,ds + \frac{GJ_T}{2}\int_0^l \left[\tau' - \frac{1}{2}(u'v'' - v'u'')\right]^2 ds, \tag{19}$$

if (14), (15) are substituted in (3), (4).

4. TIPPING ROD WITH MOMENT LOAD

In order to calculate the external potential V_a in the virtual work δW, one must specify the problem. We will consider a bending rod loaded by discrete end moments. We will not consider continuous bending moments, or continuously distributed forces and torsion moments.

In addition to the constant direction moment pairs

$$\boldsymbol{M}_{Br}(s = 0) = M_r\boldsymbol{e}_x, \quad \boldsymbol{M}_{Br}(s = l) = -M_r\boldsymbol{e}_x \tag{20}$$

we will assume the associated loads

$$M_{Bf}(s=0) = M_f e_\xi(l), \qquad M_{Bf}(s=l) = -M_f e_\xi(l) \tag{21}$$

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Together they bend the rod in the basic state around the principal axis which is stiffer (bending stiffness EJ_1), before the stability limit is reached.

According to the Kirchhoff analogy, we can then use the relationship

$$dW = (M_\xi p_1 + M_\eta p_2 + M_\zeta q)\, ds \tag{22}$$

for calculating the work [9]. The curvatures $p_{1,2}$ and the torsion q are known as a function of u, v, τ from (14), (15). M_ξ, M_η, M_ζ are the components of the moment vectors (20), (21) in the ξ, η, ζ system. Therefore, we must only decompose the constant direction moments (20) into body-fixed coordinates. If we carry out this decomposition, then M_B can be written in the form

$$M_{Br}(0) = M_r(e_\xi - \tau e_\eta + u' e_\zeta)|_0, \qquad M_{Br}(l) = M_r(-e_\xi + \tau e_\eta - u' e_\zeta)|_l \tag{23}$$

and according to (22), we finally get

$$\delta W = M_r(\delta v' - u'\,\delta\tau)|_0^l + M_f(\delta v' - \tau\,\delta u')|_0^l. \tag{24}$$

It is found that a moment load which follows the rod as well as a constant direction moment load does not have a potential. Therefore, we have

$$V_a = 0 \tag{25}$$

and the virtual work is finally given by (24).

The variations specified in (1), can then be evaluated using (18), (19), (24), and (25). The results are the coupled and nonlinear motion equations for tipping in space of an elastic rod, with constant direction and following moment loads and nonlinear boundary conditions. We find the following for the bending deformations

$$\left.\begin{aligned} EJ_2 u'''' + (EJ_2 - EJ_1)(\tau v'')'' + \frac{GJ_T}{2}[(\tau' v'')' + (\tau' v')''] + \mu \ddot{u} = 0, \\ EJ_1 v'''' + (EJ_2 - EJ_1)(\tau u'')'' - \frac{GJ_T}{2}[(\tau' u'')' + (\tau' u')''] + \mu \ddot{v} = 0 \end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{aligned}
EJ_2u'' + (EJ_2 - EJ_1)\,\tau v'' + \frac{GJ_T}{2}\tau' v' + M_f\tau = 0 \quad &\text{or} \quad u' = 0,\\
EJ_2u''' + (EJ_2 - EJ_1)\,(\tau v'')' + \frac{GJ_T}{2}(\tau' v')' = 0 \quad &\text{or} \quad u = 0,\\
EJ_1v'' + (EJ_2 - EJ_1)\tau u'' - \frac{GJ_T}{2}\tau' u' - (M_r + M_f) = 0 \quad &\text{or} \quad v' = 0,\\
EJ_1v''' + (EJ_2 - EJ_1)\,(\tau u'')' - \frac{GJ_T}{2}(\tau' u')' = 0 \quad &\text{or} \quad v = 0,
\end{aligned}\right\} \quad s = 0 \text{ and } 1 \tag{27}$$

For the twist, we find

$$GJ_T[\tau' - \tfrac{1}{2}(u'v'' - v'u'')]' - (EJ_2 - EJ_1)\,u''v'' - \mu k_s^2\ddot{\tau} = 0 \tag{28}$$

where

$$GJ_T[\tau' - \tfrac{1}{2}(u'v'' - v'u'')] + M_r u' = 0 \quad \text{or} \quad \tau = 0, \quad s = 0 \text{ and } l \tag{29}$$

Here we will investigate the stability basic state which here is a pure bending deformation v_o around the stiffest bending principle axes. On top of this, we superimpose the infinitesimal additional displacements

$$u = 0 + \bar{u}, \quad v = v_0 + \bar{v}, \quad \tau = 0 + \bar{\tau} \tag{30}$$

which are the components of the basic motion. These trial solutions (30) are substituted into the nonlinear boundary value problem (26) to (29). For the zero-order terms in the $\bar{u}$, $\bar{v}$, $\bar{\tau}$ we find a boundary value problem whose solution can be given in the form

$$v_o'' = \frac{M_r + M_f}{EJ_1} = \text{const.}, \quad v_o' = \frac{M_r + M_f}{EJ_1}\left(s - \frac{l}{2}\right) = v_o'(s) \tag{31}$$

The first order terms in the dashed quantities (we leave the dash off) results in the differential equations /166

$$\left.\begin{aligned}
EJ_2u'''' + (EJ_2 - EJ_1)\,(v_o''\tau)'' + \frac{GJ_T}{2}[(v_o''\tau')' + (v_o'\tau')''] + \mu\ddot{u} = 0,\\
GJ_T\left[\tau' - \frac{1}{2}(v_o''u' - v_o'u'')\right]' - (EJ_2 - EJ_1)\,v_o''u'' - \mu k_s^2\ddot{\tau} = 0
\end{aligned}\right\} \tag{32}$$

and the boundary conditions

$$\left.\begin{aligned}
EJ_2u'' + [(EJ_2 - EJ_1)\,v_o'' + M_f]\,\tau + \frac{GJ_T}{2}v_o'\tau' = 0 \quad &\text{or} \quad u' = 0,\\
EJ_2u''' + (EJ_2 - EJ_1)\,(v_o''\tau)' + \frac{GJ_T}{2}(v_o'\tau')' = 0 \quad &\text{or} \quad u = 0,\\
GJ_T\left[\tau' - \frac{1}{2}(v_o''u' - v_o'u'')\right] + M_r u' = 0 \quad &\text{or} \quad \tau = 0,
\end{aligned}\right\} \quad s = 0 \text{ and } 1 \tag{33}$$

which are the linear stability equations. A comparison shows that the nonlinear initial problem (26) - (29) and the stability

equations (32), (33) are a further continuation of the relationships given in [4], caused by the torsion q which has been expanded by the nonlinear terms q (14).

As a specific example, we will first consider a rod which is supported by forks. Just like a joint, such an attachment cannot take up any bending moment; however, it does prevent the transverse displacement and the twisting of the end cross-section. The stability equations are substantially simplified and the interesting difference between the following and the constant direction end moments is lost. Both are conservative and we will simply write M for their sum ($M_r + M_f$) in the future. For the theory, this deficiency is not very serious, because the more complete consideration of rod torsion (14) is still in effect. There only exist a number of experimental results for a tipping support supported by forks.

This means that the inertia terms in (32), (33) do not have to be carried along for this special case. The tipping moment of interest is the smallest load in which the non-trivial equilibrium position exists. This means that it is the lowest branching point of the eigen value problem (32), (33) which does not involve time.

Because of the fact that the coefficients partially depend on time according to (31), it is not possible to rigorously calculate the corresponding eigen values. An approximate calculation must be used. In order to avoid special rod dimensions, it is appropriate to make the system (32) and the corresponding boundary conditions (33) dimensionless. In addition to the position coordinate

$$\sigma = \frac{s}{l} \tag{34}$$

we will introduce the nondimensional variables

$$U = \frac{u}{l}, \quad T \equiv \tau, \tag{35}$$

the stiffness ratios

$$\gamma_1 = \frac{EJ_2}{EJ_1}, \quad \gamma_2 = \frac{GJ_T}{EJ_1} \tag{36}$$

and the eigenvalue

$$\lambda = \frac{Ml}{EJ_2} \tag{37}$$

If we use the new coordinates and parameters, we find that

$$\left.\begin{aligned} U'''' - (1-\gamma_1)\lambda T'' + \tfrac{1}{2}\gamma_2\lambda\left[T + \left(\sigma - \tfrac{1}{2}\right)T'\right]'' = 0, \\ \frac{\gamma_2}{\gamma_1}T'' + (1-\gamma_1)\lambda U'' - \tfrac{1}{2}\gamma_2\lambda\left[U' - \left(\sigma - \tfrac{1}{2}\right)U''\right]' = 0, \end{aligned}\right\} \tag{38}$$

with the boundary conditions /167

$$U = 0, \quad U'' + \tfrac{1}{2}\left(\sigma - \tfrac{1}{2}\right)\gamma_2\lambda T' = 0, \quad T = 0, \quad \sigma = 0 \text{ and } 1. \tag{39}$$

This dimensionless eigenvalue problem can then be associated with the variation problem, so that the connection with one of the direct methods of variational calculus has been established.

$$\delta\left\{\frac{1}{2}\int_0^1\left(U''^2 + \frac{\gamma_2}{\gamma_1}T'^2\right)d\sigma - \frac{\lambda}{2}\int_0^1\left[2(1-\gamma_1)TU'' + \gamma_2\left[T'U' - \left(\sigma - \tfrac{1}{2}\right)T'U''\right]\right]d\sigma\right\} = 0 \tag{40}$$

In order to approximately calculate the eigenvalues using the Ritz method, the variational problem (40) is stated in algebraic terms. For this purpose, we will use a single term trial solution for the calculation discussed here

$$U(\sigma) = a_1 U_1(\sigma), \qquad T(\sigma) = b_1 T_1(\sigma). \tag{41}$$

We will use the eigen functions

$$U_1 = T_1 = \sin \pi\sigma \tag{42}$$

as position functions, which are the eigen functions of the tipping support for which the nonlinear parts in the torsion q (14) can be ignored [4]. These satisfy all of the geometric boundary conditions in (39) as required.

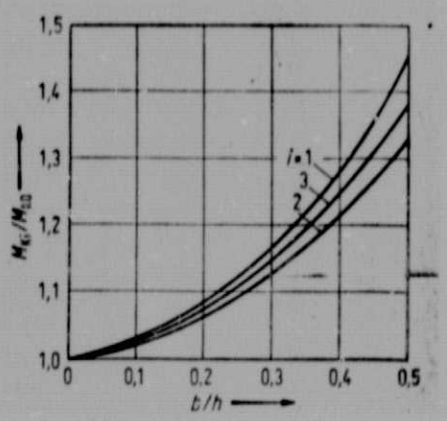

Figure 3: Critical tipping moment of a rectangular cross-section profile support.

After carrying out the variations given in (40), we obtain a

homogeneous equation system for the free coefficients a_1, b_1 in (41). If the associated determinant is set equal to 0, then with (36) and (37) we find the corresponding critical tipping moment with the lowest order to be

$$M_{K1} = \frac{\pi}{l} \frac{\sqrt{GJ_T EJ_2}}{1 - \frac{EJ_2}{EJ_1} - \frac{1}{4}\frac{GJ_T}{EJ_1}}. \tag{43}$$

If, as in [4], we had ignored the rod inclination during the differentiation, we would have obtained

$$M_{K2} = \frac{\pi}{l} \frac{\sqrt{GJ_T EJ_2}}{1 - \frac{EJ_2}{EJ_1}} \tag{44}$$

On the other hand, from rigorous nonlinear equations we would have obtained

$$M_{K3} = \frac{\pi}{l} \sqrt{\frac{GJ_T EJ_2}{\left(1 - \frac{EJ_2}{EJ_1}\right)\left(1 - \frac{GJ_T}{EJ_1}\right)}} \tag{45}$$

[9]. For $EJ_2 \ll EJ_1$ and $GJ_T \ll EJ_1$, all of the formulas (43) to (45) lead to the Prandtl tipping moment

$$M_{K0} = \frac{\pi}{l} \sqrt{GJ_T EJ_2}. \tag{46}$$

We will qualify our result using the example of a rectangular steel strip (Height h, width b, G = 0.385E). The nondimensional tipping moments M_{ki}/M_{ko} (i = 1, 2, 3) are plotted as a function of the width-height ratio b/h in the range $0 \leq b/h \leq 1/2$, (see Figure 3). It is known that the critical moment can increase considerably with increasing width. Comparison of the result shows the usefulness of the approximation (43) and (44). It seems that (44) is even more suitable in practice, because in this way we have a certain safety margin. By using a multiple-term trial solution (41) it would be possible to improve the result (43). From the theoretical point of view, (43) is better than (44). /168

5. SUMMARY AND OVERVIEW

Tipping problems of elastic rods are of theoretical and practical interest. Usually, one has to deal with complicated boundary value problems. This means that it is advantageous to derive the

differential equations and boundary conditions from a variational principle.

One of the main assumptions in this process up to the present was a very small deviation between the undeformed and deformed rod axis, so that differentiation with respect to the position coordinates could be considered to be identical. Here this restriction is disregarded. If the central line does not experience any longitudinal extension, then from the analogy between the motions of a gyroscope and the elastic behavior of a rod, one can derive nonlinear relationships for the curvatures and for the torsion. These compliment previously known relationships because nonlinearities occur in the torsion which are of the same order of magnitude as in the other characteristics of the rod axis.

Using the method of small oscillations of small expanded boundary value problem, we derive the corresponding stability equations. Using the simple example of a bending-tipping problem we demonstrate the improved performance of the suggested equations. Among other things, we find that the constant direction moment and the following moments are not conservative [10], a fact which is known from twisted rods.

If it is desired to extend our results to the compressed tipping rod, then a nonlinear longitudinal extension will be added to the previously mentioned nonlinear quantities. We may assume that the expressions (14) and (15) for $p_{1,\ 2}$ and q will remain unchanged and that the expression

$$\varepsilon = w' + \frac{1}{2}\left(u'^2 + v'^2 + k_2^2 \tau'^2\right)$$

will be used for the extension of the rod longitudinal axis, just like in [4].

REFERENCES

1. Nishino, F.; Kasemet, C.: Lee, S. L: Variational Formulation of Stability Problems of Thin-Walled Members. Ing. Arch. 43 (1974) 58-68.

2. Vielsack, P: Influence of Pre-stresses on the Stability of Slender Elastic Rods. Ing.-Arch 43 (1974) 985-995.

3. Vielsack, P: Linear Stability Theory of Elastic Rods According to the Second Approximation. Ing.-Arch 44 (1975), 143-152.

4. Wauer, J. Derivation of the Stability Equation for Tipping of the Elastic Rods with Following Load. Z. Angew. Math. Mech. in print.

5. Chwalla, E: Tipping Stability of Straight Supports with Double-Symmetric Cross Section. Forsch-H, Stalbau, Heft 2, 1939.

6. Ziegler, H. Buckling of Straight Rod Under Torsion. Z. Angew. Math. Phys. 3, (1952), 96-119.

7. Leipholz, H.: Festigkeitslehre für den Konstrukteur (Strength of Materials for Designers). Berlin/Heidelberg/New York, 1969, 109 ff.

8. Love, A. E. H.: Mathematical Theory of Elasticity, 4. Ed., Cambridge 1934, 388 ff.

9. Kovari, K.: Spatial Branching Problems of the Thin Elastic Rod with Finite Deformations. Ing.-Arch. 37 (1968) 393-416.

10. Ziegler, H.: Stability Problems of Straight Rods and Shafts Z. Angew Math. Phys. 2 (1951), 265-289.