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PROPAGATION OF SOUND WAVES THROUGH A LINEAR SHEAR LAYER - A CLOSED FORM SOLUTION

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PROPAGATION OF SOUND WAVES THROUGH A LINEAR
SHEAR LAYER - A CLOSED FORM SOLUTION

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Abstract

Closed form solutions are presented for sound propagation from a line source in or near a shear layer. The analysis is exact for all frequencies and is developed assuming a linear velocity profile in the shear layer. This assumption allows the solution to be expressed in terms of parabolic cylinder functions. The solution is presented for a line monopole source first embedded in the uniform flow and then in the shear layer. Solutions are also discussed for certain types of dipole and quadrupole sources. Asymptotic analysis of the exact solutions for small and large values of Strouhal number gives expressions that correspond to solutions previously obtained for these limiting cases.

Introduction

Sound waves passing through a shear layer are altered both in amplitude and direction. Hence, an observer situated in the far field on the opposite side of a shear layer from a sound source will, in general, hear sound which differs from that originally produced by the source. Any sound produced by an aircraft engine must pass through a shear layer or combination of shear layers. Consequently, a thorough understanding of this phenomenon is essential for determining the far field sound pressure amplitude, directivity, and directionality generated by an aircraft engine in flight. Such knowledge is also necessary for developing adequate correction procedures for data from open-jet anechoic wind tunnels used for simulating flight effects on noise sources.

There have been several studies of the effect of relative motion between two media on the propagation of sound waves. Ribner and Hine independently investigated plane waves impinging on a velocity interface between two moving streams. Gottlieb studied a sound source near a velocity discontinuity. He obtained far field solutions by evaluating the exact sound field integrals by the method of stationary phase. Slutsky, Tanagho and Horozic extended Gottlieb's analysis to include quadrupole sources embedded in the mean flow. Their analyses were developed for the source situated in an axisymmetric slug flow. Graham and Graham treated the propagation of plane waves through a finite shear layer. The same authors later investigated the effect of a finite shear layer on sound waves from a point source. They obtained velocity potential solutions by means of a series expansion about a singular point and presented results for very low frequencies, acoustic wavelengths much greater than the shear layer thickness.

Goldstein, obtained solutions for low frequency sound from multiple sources in axisymmetric shear flows with arbitrary velocity profiles. The results of his analysis show how the mean flow affects the radiation pattern from the sources.

Symbols

- \( A(k) \) arbitrary coefficient in equation (12)
- \( B(k) \) arbitrary coefficient in equation (14)
- \( b \) transformation parameter defined by equation (18)
- \( C(k) \) arbitrary coefficients in equation (20)
- \( c_0 \) speed of sound, \( 340.46 \text{ m/sec} \)
- \( f \) arbitrary unit source term in uniform flow region
- \( F_1 \) Fourier transform of \( f \)
- \( F \) externally applied force
- \( F_1 \) arbitrary unit source term in shear layer
- \( H_1 \) Fourier transform of \( E_1 \)
- \( k \) wave number in \( x \) direction
- \( k_0 \) wave number in \( x \) direction (region 1)
- \( k_1 \) wave number in \( x \) direction (region 1)
- \( k_0 \) wave number in \( y \) direction
- \( k_1 \) wave number in \( y \) direction
- \( L \) Mach number in shear flow
- \( M \) Mach number in uniform flow
- \( P(y,k) \) Fourier transform of \( p(y,x) \)
- \( p \) perturbation pressure, \( N/m^2 \)
- \( q \) external volume flow source, equation (52)
- \( r \) radial coordinate of observation point in cylindrical coordinate system, meters
- \( s \) Strohal number (reduced frequency)

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Figures 3 to 6 and 7(c). The dashed lines in the keys should be reversed; that is, the short-dash line denotes the high-frequency approximation, and the long-dash line denotes the low-frequency approximation.
The governing equations for sound propagation in each of these regions may be found in Ref. 14 (p. 10) and are given in terms of acoustic pressure as follows:

1. In the no flow region, \( U = 0 \) (region 1)
   \[
   \nu^2 p_1 = \frac{1}{c_0^2} \frac{\partial^2 p_1}{\partial t^2}
   \]

2. In the uniform flow region \( U = U_0 \) (region 2)
   \[
   \nu^2 p_2 - \frac{1}{c_0^2} \frac{\partial^2 p_2}{\partial t^2} = \sum_{j=1,2} f_j
   \]
   \[
   \text{where}
   \]
   \[
   f_j = \begin{cases} 
   6(x)\delta(y + h) \text{e}^{i\omega t} & \text{for } j = 1 \\
   0 & \text{for } j = 2
   \end{cases}
   \]
   \[
   \text{accounts for the presence } (j = 1) \text{ or absence } (j = 2) \text{ of the source of unit strength in the uniform flow.}
   \]

3. In the shear layer
   \[
   \nu^2 \left( \frac{\partial^2 p}{\partial y^2} - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \right) + 2U' \frac{\partial^2 p}{\partial x \partial y} = \delta_j
   \]
   \[
   \text{where the prime denotes differentiation with respect to } y,
   \]
   \[
   \frac{\partial \delta_j}{\partial y} = \frac{\partial \delta_j}{\partial x}
   \]
   \[
   \delta_j = \begin{cases} 
   0 & \text{for } j = 1 \\
   \frac{D}{c_0} \left( \delta(x) \delta(y + h) \text{e}^{i\omega t} \right) & \text{for } j = 2
   \end{cases}
   \]
   \[
   \text{accounts for the absence } (j = 1) \text{ or presence } (j = 2) \text{ of the source in the shear layer.}
   \]

The radiation relationship between the transverse particle velocity and the acoustic pressure is given by

\[
\nu'_0 \frac{\partial p}{\partial y} = \frac{\partial \delta_j}{\partial y}
\]

The linear velocity profile in the shear layer is given by

\[
U(y) = \frac{U_0}{h} y
\]

For the boundary conditions we have the acoustic radiation condition which requires outward propagating waves at \( y = \pm t \). At the edges of the shear layer we require continuity of pressure

\[
p = p_1 \text{ at } y = 0 \text{ and } p = p_2 \text{ at } y = h
\]
and continuity of particle displacement which can be expressed in terms of the transverse perturbation velocity (since there is no slip)

\[ v = v_1 \text{ at } y = 0 \text{ and } v = v_2 \text{ at } y = b \]  

(3)

Assuming a simple harmonic time dependence and applying the Fourier transforms

\[ P(y, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(x, y) e^{ikx} \, dx \]

\[ V(y, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x, y) e^{ikx} \, dx \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x) e^{ikx} \, dx = \frac{1}{2\pi} \]

gives the following system of second order ordinary differential equations and boundary conditions.

In region 1

\[ p''_1 + \left[ \frac{c^2}{c_o^2} - k^2 \right] p_1 = 0 \]  

(6)

In region 2

\[ p''_2 + \left[ \frac{(c^2 - Nk)^2}{c_o^2} - k^2 \right] p_2 = \delta_{Tj} \quad j = 1, 2 \]

(7)

where

\[ \delta_{Tj} = \begin{cases} \frac{1}{2\pi} \delta(y + h) & j = 1 \\ 0 & j = 2 \end{cases} \]

and

\[ N_0 = U_0/c_o \]

In the shear layer

\[ p'' = -\frac{2M^k}{(kN - c_o^2)} p' + \left( kN - \frac{c^2}{c_o^2} \right) p = \delta_{Tj} \]

(8)

where

\[ \delta_{Tj} = \begin{cases} \frac{1}{2\pi} \delta(y + h) & j = 2 \\ 0 & j = 1 \end{cases} \]

and

\[ p' = -\ln c_o \left( \frac{c^2}{c_o^2} + kN \right) \]  

(9)

with the boundary conditions

\[ p = p_1 \quad v = v_1 \text{ at } y = 0 \]

(10)

\[ p = p_2 \quad v = v_2 \text{ at } y = b \]

(11)

The solution of this system will depend on whether the source is located in the uniform flow or in the shear layer. However, the procedure is the same in either case.

The solution of equation (6) in the no flow region which satisfies the radiation condition has the form

\[ P_1 = A(k)c_0^{-\gamma_1 y} \]  

(12)

where

\[ \gamma_1 = \sqrt{k^2 - (c/\rho c_0)^2} \]

(13)

When the source is in the uniform flow region, the solution to equation (7) which satisfies the radiation condition has the form

\[ P_2 = B(k)c_0^{-\gamma_2 y} + \frac{1}{\gamma_2} \gamma_2 \gamma_1 c_0^{-\gamma_1 y} \]  

(14)

where

\[ \gamma_2 = \sqrt{k^2 - (c/\rho c_0 - kN)^2} \]

(15)

The assumption of a linear velocity profile in the shear permits use of the solution obtained by Goldstein and Rice which expresses the acoustic pressure in the shear layer in terms of the parabolic cylinder functions of Weber. Hence, following Goldstein and Rice, the new independent variable \( s \) and dependent variable \( U \) are defined by

\[ s = \sqrt{\frac{1}{2N^k}} \left( \frac{c}{c_o} - kN \right) \]  

(16)

and

\[ p = \frac{b^2}{2 + b} \frac{d}{ds} \left( e^{-b^2/2s} \right) \]

(17)

where

\[ b = \frac{k}{2M^k} \]

(18)
These transformations are introduced into equation (8) which is then integrated once as in Ref. 13 to obtain Weber’s equation.

$$U'' = (1/4 \varepsilon^2 + b)U = 0 \quad (19)$$

where the primes now denote differentiation with respect to $\varepsilon$. This equation has as its solutions an arbitrary linear combination of parabolic cylinder functions $U(\varepsilon, t)$ (Abramowitz). Thus the solution to equation (8) is written as an arbitrary linear combination

$$P = c_1(k)P_b(t) + c_2(k)P_w(t) \quad (20)$$

or

$$P(t) = U(b + 1/2) + (b - 1/2)U(b - 1/2) \quad (22)$$

Having obtained these solutions (Eqs. 12), (14) and (20)) the boundary conditions (Eqs. 10) and (12) are applied at the edges of the shear layer to evaluate the following expression for the coefficient $A(k)$ in the far field pressure expression (Eq. 12),

$$A(k) = \frac{\gamma_2^2(\varepsilon^2 + b)}{2} \left[ p^b(\varepsilon, t_1) p^w(\varepsilon, t_2) - p^b(\varepsilon, t_2) p^w(\varepsilon, t_1) \right] \quad (23)$$

where

$$\gamma_1 = p^b(\varepsilon, t_2) - \gamma_2 p^b(\varepsilon, t_2) \quad (24)$$

$$\gamma_1 = p^b(\varepsilon, t_1) + \gamma_2 p^b(\varepsilon, t_1) \quad (25)$$

and

$$p^b(\varepsilon, t) = \frac{dp^b(\varepsilon, t)}{d\varepsilon} \bigg|_{\varepsilon=\varepsilon_1} \quad (26)$$

where

$$\gamma = \sqrt{\frac{1 - U_0 \cos \omega \varepsilon}{\varepsilon_0}} \quad (27)$$

and

$$\gamma = \sqrt{\frac{1 - U_0 \cos \omega \varepsilon}{\varepsilon_0}} \quad \text{at } \varepsilon = 0 \quad (28)$$

and

$$\gamma = \sqrt{\frac{1 - U_0 \cos \omega \varepsilon}{\varepsilon_0}} \quad \text{at } \varepsilon = 0 \quad (29)$$

Putting the appropriate expressions for $\gamma_2$ and for $P^b(t_1)$ and $P^w(t_1)$ in the numerator of equation (22) yields,

$$A(k) = \frac{\gamma_2^2(\varepsilon^2 + b)}{2} \left[ p^b(\varepsilon, t_1) p^w(\varepsilon, t_2) - p^b(\varepsilon, t_2) p^w(\varepsilon, t_1) \right] \quad (23)$$

This expression for $A(k)$ can be put into equation (12) for $P_1$ which is the Fourier Transform solution for the far field pressure pattern in region 1. The actual pressure is then obtained from the Fourier Inversion integral

$$P_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) e^{\gamma k x + \gamma y} dk \quad (27)$$

with $A(k)$ given by equation (26). The above integral is of the form

$$\int_{-\infty}^{\infty} f(k) e^{-\gamma k} dk$$

where

$$f(k) = \gamma(k \cos \theta + \gamma y) \quad (28)$$

with $\gamma$ defined by equation (12). This type of integral can be evaluated by the method of stationary phase or saddle point method 17, 18 (for large positive $x = \sqrt{x^2 + y^2}$).

The details of the evaluation of equation (27) are presented in Ref. 12. The result, given here in cylindrical coordinates, in the following expression for the far field pressure directivity for a line monopole source located in the uniform flow

$$P_{1m} = \frac{\gamma_2^2}{2 \pi} \frac{\min \omega}{(b + 1/2)\gamma(b - 1/2)} \quad (29)$$

where

$$\omega = \sqrt{(1 - U_0 \cos \omega \varepsilon)^2 - \cos^2 \omega \varepsilon \gamma \omega}$$

and

$$\gamma = \sqrt{\frac{1 - U_0 \cos \omega \varepsilon}{\varepsilon_0}} \quad \text{at } \varepsilon = 0 \quad (29)$$

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$$A(k) = \frac{\gamma_2^2(\varepsilon^2 + b)}{2} \left[ p^b(\varepsilon, t_1) p^w(\varepsilon, t_2) - p^b(\varepsilon, t_2) p^w(\varepsilon, t_1) \right] \quad (23)$$

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and

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Putting the appropriate expressions for $\gamma_2$ and for $P^b(t_1)$ and $P^w(t_1)$ in the numerator of equation (22) yields,
\[ x(b - 1/2)u(b + 1/2) \]  

(30)

and

\[ R^+ = \left( \frac{\alpha}{c_0} \right)^{-1} R^- (1 - \sin \theta)u(b - 1/2) \]

\[ + (1 + \sin \theta)(b - 1/2)u(b + 1/2) \]  

(31)

The frequency is defined in terms of Strouhal number as follows:

Recall that

\[ b = \frac{k}{2N} = \frac{1}{2N} \frac{\omega \cos \theta}{c_0} \]

and that

\[ \xi = \sqrt{\frac{2\pi}{N}} \frac{\omega}{c_0} (1 - N \cos \theta) \]

where \( k = k_1 = k_2 \cos \theta \) was used.

Note that \( b/\omega c_0 / N_0 \) is a nondimensional group of parameters that occurs several places in the pressure expression. Hence we define

\[ \text{Strouhal number} = S = \frac{b \omega}{N_0} \]  

(32)

Then

\[ b = 1/2 \frac{1}{S} \cos \theta \]

and

\[ \xi = \sqrt{\frac{2\pi}{N}} \frac{\omega}{c_0} (1 - N \cos \theta) \]

This Strouhal number is essentially a comparison of the shear layer thickness to the acoustic wavelength, and is referred to as a nondimensional frequency.

Now placing the source in the shear layer,

\[ c_1 = 2, c_2 = 0, c_{12} = \frac{1}{2} \xi (\psi(b)) \]  

(33) and following exactly the same procedure yields the far field expression for acoustic pressure

\[ P_{\text{Im}} = \left[ \frac{1}{2\pi} \frac{\xi}{c_0} \right]^{1/2} \left( \frac{-\xi \sin \theta}{1 - \frac{1}{2} N \cos \theta} \right)^2 \]

\[ \times \left\{ 1 + \left( \frac{c_{12} \xi}{c^2 - R^2} \right) \right\} \left( \frac{c_{12} \xi}{c^2 - R^2} \right) \]

\[ \times \left[ \frac{f^+(t_0) N^0}{c^2 - R^2} \right] \left[ \frac{f^+(t_0) N^0}{c^2 - R^2} \right] \]

\[ \times e^{i(\omega - \xi t) + \xi t} \]  

(33)

where

\[ \xi = \sqrt{\frac{2\pi}{N}} \frac{\omega}{c_0} (1 - k \frac{b}{N_0}) \]  

(34)

It is noted that when the source is placed at the uniform flow edge of the shear layer, in which case \( b = 0 \) and \( \xi = 0 \), equation (33) becomes equal to equation (29). Both equations (29) and (33) reduce to the same spatial result regardless of source location when the line between the source and observer is oriented at 90° to the flow direction (i.e., \( \theta = 90^\circ \) in Eqs. (29) and (33)). This result is

\[ P_{\text{Im}} = \frac{1}{2\pi} \frac{\xi}{c_0} \left( \frac{-\xi \sin \theta}{1 - \frac{1}{2} N \cos \theta} \right)^2 \]

(35)

This also agrees with Gottlieb's result\(^2\) at \( \theta = 90^\circ \) when the shear thickness \( b \) is set equal to zero.

Approximate Solutions for Limiting Cases

of Low and High Frequency

When considering the far field pressure for limiting values of frequency it is actually limiting values of nondimensional frequency or Strouhal number (defined by Eq. (32)) which are of interest. Careful examination of the far field pressure equations (Eqs. (29) and (33)) along with the definitions of \( b \) and \( \xi \) reveals the strong dependence of the pressure upon Strouhal number. This dependence occurs through the parabolic cylinder functions which can be evaluated for small or large arguments by means of appropriate asymptotic expansions found in Refs. 16 and 19. The parabolic cylinder functions appear only through \( \lambda(k) \) all evaluated for \( k = k_1 = \omega c_0 \cos \theta \) By evaluating these expressions for the limiting cases of very small and very large values of the Strouhal number the approximate expressions for the far field pressure will be obtained. These approximate expressions for the far field pressure can also be obtained by solving the differential equations asymptotically for arbitrary velocity profiles using methods discussed in sec 6.7.1 of Ref. 14.

By replacing the parabolic cylinder functions in equations (29) and (33) with their asymptotic expansions\(^1\) for small arguments the low frequency approximations for far field pressure are obtained. For the source in the uniform flow

\[ P_{\text{Im}} = \frac{1}{2\pi} \frac{\xi}{c_0} \left( \frac{-\xi \sin \theta}{1 - \frac{1}{2} N \cos \theta} \right)^2 \]

\[ \times \left\{ 1 - \frac{c_{12} \xi}{c^2 - R^2} \right\} \left\{ 1 - \frac{c_{12} \xi}{c^2 - R^2} \right\} \]

(36)

\[ \text{ORIGINA} \text{L PAGE IS OF POOR QUALITY} \]
For the source in the shear layer region

\[
\frac{1}{2} \left[ \frac{L}{2 \pi \omega c_0^2} \right]^{1/2} \frac{1}{(1 - \frac{b}{h} \cos \theta)^2} \frac{(1 - M_e \cos \theta)^2}{(1 - M_e \cos \theta)^2}
\]

\[
\left[ \frac{(1 - \frac{b}{h} \cos \theta)^2 \sin \theta + \sqrt{(1 - \frac{b}{h} \cos \theta)^2 - \cos \theta}}{c_0} \right]
\]

(37)

It should be pointed out that when the source is located at the edge of the ahar layer joining the uniform flow (\(b = h\)) equations (36) and (37) reduce to the same result.

Replacing the parabolic cylinder functions with their asymptotic expansions\(^{16,19}\) for large arguments gives the approximate high frequency expressions for the far field pressure. The results are as follows.

For the source in the uniform flow

\[
P_{\text{f}} = \left[ \frac{1}{2 \pi \omega c_0^2} \right]^{1/2} \frac{1}{(1 - \frac{b}{h} \cos \theta)^2} \frac{(1 - M_e \cos \theta)^2}{(1 - M_e \cos \theta)^2}
\]

\[
\times \left[ \frac{(1 - \frac{b}{h} \cos \theta)^2 \sin \theta + \sqrt{(1 - \frac{b}{h} \cos \theta)^2 - \cos \theta}}{c_0} \right]
\]

(38)

For the source in the shear layer

\[
P_{\text{f}} = \left[ \frac{1}{2 \pi \omega c_0^2} \right]^{1/2} \frac{1}{(1 - \frac{b}{h} \cos \theta)^2} \frac{(1 - M_e \cos \theta)^2}{(1 - M_e \cos \theta)^2}
\]

\[
\times \left[ \frac{(1 - \frac{b}{h} \cos \theta)^2 \sin \theta + \sqrt{(1 - \frac{b}{h} \cos \theta)^2 - \cos \theta}}{c_0} \right]
\]

(39)

Note that again these results reduce to the same expression when the source is located at the edge of the shear layer joining the uniform flow; that is, \(b = h\).

### Multipole Sources

This analysis can be extended to multipole sources located either in the uniform flow or in the shear layer. For a multipole line source located in the uniform flow region, the far field pressure may be obtained by differentiating the monopole solution with respect to source location (Ref. 12). Since the source location is only dependent upon the transverse coordinate direction (\(y - direction\)) only lateral multipole sources will be of interest here. Thus far field pressure due to a lateral line dipole source in the uniform flow is the first derivative with respect to transverse source location (\(b\)) of the monopole expression (Eq. (29)).

\[
P_{\text{f}} = \frac{\partial P_{\text{f}}}{\partial b} = \frac{1}{c_0^2} \frac{\partial}{\partial b} \left[ \frac{1}{2 \pi \omega c_0^2} \right]^{1/2} \frac{1}{(1 - \frac{b}{h} \cos \theta)^2} \frac{(1 - M_e \cos \theta)^2}{(1 - M_e \cos \theta)^2}
\]

\[
\times \left[ \frac{(1 - \frac{b}{h} \cos \theta)^2 \sin \theta + \sqrt{(1 - \frac{b}{h} \cos \theta)^2 - \cos \theta}}{c_0} \right]
\]

(40)

The lateral line quadrupole expression is obtained by differentiating the monopole expression (Eq. (29)) twice with respect to transverse source location (\(b\)).

\[
P_{\text{f}} = \frac{2}{3 \pi \omega c_0^2} \frac{\partial^2 P_{\text{f}}}{\partial b^2} = \frac{1}{c_0^2} \frac{\partial^2}{\partial b^2} \left[ \frac{1}{2 \pi \omega c_0^2} \right]^{1/2} \frac{1}{(1 - \frac{b}{h} \cos \theta)^2} \frac{(1 - M_e \cos \theta)^2}{(1 - M_e \cos \theta)^2}
\]

\[
\times \left[ \frac{(1 - \frac{b}{h} \cos \theta)^2 \sin \theta + \sqrt{(1 - \frac{b}{h} \cos \theta)^2 - \cos \theta}}{c_0} \right]
\]

(41)

To obtain far field pressure solutions for multipole sources located in the shear layer the equation for wave propagation in a transversely sheared mean flow must be considered. This is given in Chapter 1 of Goldstein\(^{14}\) in a general form as:

\[
\frac{D}{D\xi} \left( \frac{2}{c_0^2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} P \right) - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial y} P \cdot \frac{\partial}{\partial y} = q
\]

(42)

where \(q\) is an externally applied volume force arising from the momentum equation and \(q\) is an external volume flow source within the fluid which arises from the continuity equation. If the vol-
ume flow source is put equal to zero (i.e., no mass is added to the flow) the two remaining terms on the right hand side of equation (42) are due to the externally applied force \( F \). Goldstein\(^4\) shows that in flows with no mean velocity an external force behaves as a volume dipole source of strength \( \mathbf{F} \).

Now a dipole source is not uniquely defined in a transversely sheared mean flow. If we want to extend its definition so that its strength still coincides with an external force we can define it so that \( \mathbf{T} \) on the right hand side of equation (42) is the strength of a general dipole.

Similarly a quadrupole can be defined so that it corresponds to an external stress. This is done by replacing \( \mathbf{T} \) with \( \mathbf{S} \) where \( \mathbf{S} \) is now a tensor. Hence in particular the source term representing a lateral quadrupole of strength \( \mathbf{S} \) in the transverse direction is

\[
\frac{D_0}{c} \frac{\partial}{\partial y} \left( \frac{1}{\rho} \mathbf{v} \cdot \mathbf{S} \right) = 2 \mathbf{v} \cdot \mathbf{S}.
\]

Hence, the solution to equation (42) with \( \mathbf{q} = \mathbf{0} \) can be obtained by differentiating the monopole solution with respect to source location \( y \) once for the dipole (associated with the force vector) and twice for the quadrupole (associated with the stress tensor).

Only the more interesting quadrupole result will be given here.

\[
P_{1q} - P_{1m} \left( \frac{1}{\rho} \mathbf{v} \cdot \mathbf{S} \right)
\]

where \( P_{1m} \) is given by equation (32). This expression gives the far field pressure for a point lateral quadrupole source located in the shear layer.

Inspection of equation (43) indicates that the first term inside the curly brackets will dominate at low frequencies while the second term (containing \( \mathbf{v} \) ) will dominate at high frequencies. Thus the product of these respective terms with the low and high frequency approximations for \( P_{1m} \) will give the low and high frequency approximations for the far field quadrupole pressure pattern, respectively.

The high frequency approximation is

\[
P_{1q} - P_{1m} \left( \frac{1}{\rho} \mathbf{v} \cdot \mathbf{S} \right)
\]

where \( P_{1m} \) is given by equation (37).

\[
\cos^2 (1 - \nu/3) \mathbf{H} \mathbf{J} \mathbf{O} \mathbf{V} \mathbf{G} = 0
\]

where \( P_{1m} \) is given by equation (39).

The details of the proceeding analysis may be found in Ref. 12.

Results and Discussion

The far field pressure pattern for sound radiating from a given type of point source in a uniform flow or a linear shear layer may be controlled by a number of different factors. Among these are Mach number of the uniform flow, frequency of the sound, shear layer thickness and source location with respect to the shear layer. The effects of varying these parameters are determined from the appropriate pressure solutions given in the proceeding section. It should be noted that three dimensional and turbulent scattering were ignored in obtaining the pressure expressions.

The sound pressure field was calculated by programming the various pressure expressions using standard Fortran techniques. The parameter \( \nu \) has been normalized to 0.305 motor. The parabolic cylinder functions occurring in the exact pressure expressions were calculated using a computer program which was originally developed in connection with other research being performed at NASA Lewis Research Center. This program was capable of calculating up to 30 terms in the series for parabolic cylinder functions with complex arguments at higher frequencies this proved to be inadequate. However, at the frequencies where computational difficulties were encountered with the exact solutions the approximate expressions had already shown good agreement with the exact solutions. These calculations were made using single precision arithmetic which gave nine significant decimal digits. The directivity calculations were made at 5 degree intervals.

Figures 3 to 7 show pressure level referenced to \( \mathbf{v} \) (i.e., Pressure Level - 10 \log \mathbf{P} / \mathbf{P}_0 \) plotted against the angle from the positive x-axis (downstream flow direction). These curves are for a source of unit strength, a uniform flow Mach number of 0.8 and a shear layer thickness of 0.15. The curves compare the exact and approximate calculations derived in the previous sections as indicated in the various figures.

Source in the Uniform Flow

Figure 3 shows results for a source in the uniform flow. When treating a point source located in the uniform flow, the source is considered to be within a few wavelengths of the shear layer. If the source is many wavelengths from the shear layer the problem degenerates to the plane wave case already treated by Goldstein and Graham\(^7\).

One of the most significant characteristics of the pressure pattern in Fig. 3 is the designated...
"zone of silence." This region is determined by the quantity \( \sqrt{(1 - N_0 \cos \theta)^2 - \cos^2 \phi} \) which appears in \( G^0 \) (Eq. (29)) as well as in the exponent of the far field pressure expression (Eq. (29)). The angle at which this square root becomes zero is referred to as the critical angle and is defined by

\[
\phi_c = \cos^{-1} \left( \frac{1}{1 + N_0} \right)
\]

This is the smallest angle, \( \phi \), that the transmitted wave may make with the \( x \) axis. Note that this angle is completely controlled by the Mach number of the uniform flow. The "zone of silence" is defined as the region where \( \phi < \phi_c \). In this region the square root term acts as an exponential damping factor. Hence the "zone of silence" is actually a zone of attenuated sound propagation.

For the region in which \( \phi > \phi_c \), the square root term contributes only to the phase of the sound wave and no longer affects the amplitude.

The far field pressure patterns for dipole and quadrupole sources in the uniform flow are shown in Figs. 3 and 5. These far field pressure patterns also exhibit a well defined zone of silence. However, rather than peaking at the critical angle the pressure level goes to zero due to the fact that the above term is a multiplicative factor in the pressure expressions (Eqs. (60) and (61), respectively) which therefore go to zero at \( \phi = \phi_c \) in each case.

The curves shown in Figs. 3, 4, and 5 are all for a Strouhal number of 1 which means that the acoustic wavelength is approximately equal to the shear layer thickness. Additional calculations were made for Strouhal numbers ranging from 0.25 up to about 1.0. For Strouhal numbers above 3.0, the computer program for calculating the parabolic cylinder functions encountered computational difficulties for angles near 90°. Hence, the exact expression for the sound field is reliable only for certain regions of the sound field at high frequency. Outside of the zone of silence both of the approximate calculations agree with the exact result over the range of Strouhal numbers considered. This behavior indicates that neither the radiation pattern nor the method of calculation are particularly sensitive to frequency outside of the zone of silence. This corresponds to the assertion of previous researchers\(^1\) that the shear layer thickness has essentially no effect on the sound radiation pattern except inside the zone of silence \( (\cdot \cdot \cdot \cdot) \). These characteristics carry over directly to the dipole and quadrupole calculations.

Finally note that the pressure level tends to \( 0^0 \) and 180°. This corresponds to zero pressure at these angles. This is attributed to the cancellation of incident waves by reflected waves at small incidence angles. This is what Gottlieb\(^1\) refers to as the "Lloyd's Mirror Effect."

Source in the Shear Layer

The sound radiation field due to a volume flow source embedded in the shear layer is similar in nature to that of a monopole source in the uniform flow. A typical radiation pattern for a monopole source in the shear layer is shown in Fig. 6 for a Strouhal number of 1.

For the source in the shear layer there will appear to be a zone of silence again determined by \( \sqrt{(1 - N_0 \cos \theta)^2 - \cos^2 \phi} \) for both the exact and low frequency results; however, it enters the pressure expression (Eq. (38)) only through the coefficients of the parabolic cylinder functions given by \( G^0 \) (Eq. (30)) and does not enter directly into the exponent of the exact pressure equation or the low frequency result. Hence, there is no exponential damping factor in either of these equations. However, the combination of algebraic terms:

\[
(1 - N_0 \cos \phi)^2 \sin \phi + \sqrt{(1 - N_0 \cos \theta)^2 - \cos^2 \phi}
\]

appearing in the low frequency pressure expression (Eq. (37)) results in a rapid dropoff of the sound levels at angles less than \( \phi_c \cos^{-1} \frac{1}{1 + N_0} \). This behavior is much like the exponential "damping in the zone of silence at high frequencies or when the source is in the uniform flow. It should be noted that this rather remarkable feature is independent of frequency while the exponential damping factors found in the high frequency approximation for the source in the shear layer and for all results for the source in the uniform flow do depend on frequency.

The peak in the low frequency approximation curve occurs at \( \phi = \phi_c \) (i.e., when the square root in the denominator of the second factor is zero). As \( \theta \) decreases below \( \phi_c \), the denominator of the second factor approaches \( 1 \) (i.e., the square root) while \( \sin \phi \) approaches zero thus causing the decrease in sound level as \( \phi \) approaches zero. At angles near 90°, \( \sin \phi \) is near 1 causing the sound level to "flatten out" in this region.

The square root \( \sqrt{(1 - N_0 \cos \theta)^2 - \cos^2 \phi} \) does not appear in the high frequency approximation (Eq. (39)). However, this approximation does contain the square root \( \sqrt{(1 - 0.6N_0 \cos \theta)^2 - \cos^2 \phi} \) both in the denominator and in the exponent. Hence, this square root determines a "critical angle" \( \phi_c \) for the high frequency expression given by

\[
\phi_c = \cos^{-1} \left( \frac{1}{1 + 0.6N_0} \right)
\]

Note that \( 0.6N_0 \) is the local Mach number of the flow at the source location in the shear layer. This accounts for the fact that the peaks for the high frequency approximation occur at slightly different angles than the peaks of the exact and low frequency curves.

It should be noted that both of the square root expressions arise originally from the expression

\[
\sqrt{k^2 - (\nu_0 - \nu_0)^2 - \cos^2 \phi}
\]

Hence the square root is zero at branch points of
the complex k-plane and as shown above these points determine "critical angles." It is at these points that the high frequency expansion breaks down.

The idea of a dipole source or a quadrupole source (or for that matter any other order multipole source) in a transversely sheared mean flow is rather vague since there is no unique way of defining such a source. In the previous section these sources are defined so that they correspond to an externally applied force and an externally applied stress, respectively. The curves shown in Fig. 7 are for a quadrupole source located at the center of the shear layer, a mean flow Mach number of 0.8 and Strouhal numbers of 0.5, 1.0, and 6.0, respectively.

The exponential damping is most easily seen in the high frequency approximation given by equation (39). However, this feature is not as pronounced as might be expected. This is because of a peaking resulting from the coefficient in equation (45).

One further point of significance which applies in general to this type of quadrupole is that the high frequency approximation equation (44) resembles the exact expression for a quadrupole in the uniform flow equation (45) multiplied by a doppler factor \( 1 - \frac{h}{n} \frac{1}{\mu_0 c} \), where the Mach number in each case is that of the flow at the source location.

Summary and Conclusion

An analysis has been developed for calculating the far field acoustic pressure for sound waves propagating through a linear shear layer. The sound waves are produced by a line source either located near the shear layer in a uniform flow or located in the shear layer itself. Closed form analytic expressions for the far field pressure have been obtained for several different types of sources (i.e., monopole, dipole, etc.). The definitions of these sources depend upon whether the source is located in the uniform flow or in the shear layer. Approximate expressions are given for limiting cases of high and low frequencies for each type of source. These approximate expressions give very good agreement with the exact results for their respective limiting cases.

The exact pressure expression for a monopole source reduces to previously known results for such special cases as zero shear layer thickness, propagation of the sound waves perpendicular to the flow direction, and plane waves propagating through a velocity discontinuity. The exact expressions for multipole source in the uniform flow are simple extensions of the monopole result. The expressions for the monopole or volume flow source in the shear layer become identical with the monopole expression in the uniform flow when the source is placed at the edge of the shear layer. The quadrupole source solution in the shear layer is an extension of the monopole solution in the shear layer. Consequently, all of the pressure expressions developed here can be reduced to known results for special limiting cases thus providing a check for the analysis. Hence analytic expressions have been developed for calculating the sound field due to various types of line sources located in a uniform flow or in a shear layer for the entire acoustic frequency spectrum. The asymptotic expansions of these exact expressions for the limiting cases of low and high frequency correspond to results presented elsewhere for shear layers of arbitrary velocity profiles.

References


Figure 1. - Propagation of aircraft noise through shear layers.

Figure 2. - Schematic of simplified shear layer used in analysis.
Figure 3. - Comparison of exact and approximate pressure expressions for a monopole source in the uniform flow.

Figure 4. - Comparison of exact and approximate pressure expressions for a transverse dipole source in the uniform flow.
Figure 5. - Comparison of exact and approximate pressure expressions for a transverse quadrupole source in the uniform flow.

Figure 6. - Comparison of exact and approximate pressure expressions for a monopole source in the shear layer.
Figure 7. Comparison of exact and approximate pressure expressions for a transverse quadrupole source in the shear layer.
Figure 7. - Concluded.

EXACT CALCULATION (EQ. (43))
--- HIGH FREQUENCY APPROXIMATION (EQ. (44))
--- LOW FREQUENCY APPROXIMATION (EQ. (45))

FLOW DIRECTION

\[
\theta = 6.0
\]
\[
M_0 = 0.8
\]
\[
\delta = 0.153 \text{ m}
\]
\[
h = 0.5 \delta
\]