COUPLING OF ELECTROMAGNETIC WAVES AND SPACE CHARGE WAVES IN TYPE "O" TRAVELING WAVE TUBES

P. Ricci


NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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**Author(s):** P. Ricci

**Performing Organization Name and Address:** Leo Kanner Associates, Redwood City, California 94063

**Sponsoring Agency Name and Address:** National Aeronautics and Space Administration, Washington, DC 20546

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**Abstract:** H. Derfler has observed that a parameter defined by Pierce's perturbation method does not have the same physical significance as an analogous parameter described by a differently derived equation of W. Kleen. A modification of Pierce's method is proposed, which yields an equation of Derfler's type, and also allows quicker and easier calculation of a given traveling wave tube's parameters.
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Summary

The first section of this paper gives a brief presentation of the electromagnetic problem connected with traveling wave tubes, with special reference to J.R. Pierce's classic works on the subject, and discussing the fourth-degree equation this author developed to calculate gain in these tubes. Then an observation made by H. Derfler at the International Congress on Microwave Tubes at Paris, 1956, is noted, according to this observation, a parameter defined by Pierce's equation does not have the same physical significance as an analogous parameter defined by an equation developed in a different manner by W. Kleen, even though this latter may be written in such a way as to be formally identical to that of Pierce. To verify this fact, Derfler calculated the electromagnetic field in a radially symmetrical traveling wave tube (one of the few cases which can be approached analytically by the direct route). The author got a resulting equation for calculating gain in the tube which, while still of the fourth degree, could be considered a combination of Pierce's and Kleen's two equations. In subsequent sections a modification of Pierce's perturbation method is proposed. This new form yields an equation of the same type as Derfler's, yet also (more importantly) allows quicker and easier calculation of the parameters of a given traveling wave tube. Then the physical significance of our calculation is discussed, and finally, to confirm our theory, the formulas developed are specialized for the radially symmetrical case studied by Derfler, obtaining in this case the

* Numbers in the margin indicate pagination in the foreign text.
same results as he did.

1. The Electromagnetic Problem of a Traveling Wave Tube

Type "O" traveling wave tubes consist basically (cf. Fig. 1) of a periodic structure and an electron beam, suitably accelerated and focused by means of an electron gun. The beam is under the influence of a very intense static magnetic field which prevents scattering in the transverse direction. The theory of interaction between the kinetic energy of the particles and the electromagnetic energy of the wave supported by the periodic structure may be developed under the following simplifying hypotheses

1) High frequency signals are quite small.

2) They possess a dependency on $z$ (axial direction of the beam) of the type $e^{-\sqrt{z}}$.

3) There are no positive ions.

4) The beam does not achieve relativistic speeds.

Fig. 1. Example of periodic electron guide

Key: a. Electron beam
5) The axial magnetic field mentioned above is intense enough to prevent transverse motion (with respect to $z$) of the particles. One may also assume that at an initial time the electrons all leave the electron gun at the same speed $u_0$. Now, if the plasma pulsation is defined as

$$\omega_p = \sqrt{\frac{e^2}{m_0 e_0}}$$

where

- $\rho$ = density of charge of the beam at exit from the electron gun
- $\frac{e}{m}$ = ratio between charge and mass of the electron
- $e_0$ = dielectric constant of vacuum

then, by calculating the current density as a function of the longitudinal component $E_z$ of the electric field (cf. [7] p. 183-84), one easily finds that the beam can be considered as a medium with a tensorial dielectric constant given by

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\omega_p^2}{(j\omega - \epsilon_0)} \end{bmatrix}$$

From Eq. 2 one also finds that the TM modes are the only ones capable of feeling the effects of the beam. In fact, given that the particles can vibrate only in the longitudinal direction, only the longitudinal component of the electric field can modulate the beam and draw from it the necessary energy to amplify signals.
By introducing Eq. 2 into Maxwell's equations, one finds that the TM modes mentioned above (cf. for instance [7] p. 166) can be derived from a vector potential:

\[ A = \mathcal{I}(r) e^{-r Z_0} \]

where \( Z_0 \) is the versor of the z-axis solution of the homogeneous equation

\[ \frac{\partial^2 \phi}{\partial x^2} + \left( k^2 - l^2 \right) \left( f(\sqrt{\omega^2 - k^2}) \right) \phi = 0 \]

Thus the problem of determining the field in a structure containing an electron beam becomes one of the beam's contours. This problem is not too complex in the case in which the structure supporting the electromagnetic waves is a continuous structure of the type of a normal wave guide. The problem is also easily solved using Birdsall and Whinnery's charts [2]. In fact, the constant \( \Gamma \) may be found, with the continuity of the tangential fields \( E_z \) and \( H_t \) imposed on the beam contour, by the equation

\[ \left( \frac{H_t}{E_z} \right)_i = Y_c(r) = \left( \frac{H_t}{E_z} \right)_e \]

where the indices i and e indicate, respectively, internal and external quantities of the beam, and in the case of continuous structures one may set [2]
Having this quantity at our disposal, the problem can be immediately solved with the aforementioned charts. The constant thus determined corresponds to the propagation constant of so-called space charge waves (waves which exist only in the presence of the space charge of the beam). In continuous-structure tubes these waves are the only waves with a velocity lower than that of light. But the case is different for traveling wave tubes in which the structure surrounding the beam is of a periodic type. This structure propagates waves with phase velocities slower than that of light even in the absence of the electron beam ("cold" case). Furthermore, to obtain maximum signal amplification, the beam is given a velocity $u_0$ close to that of one of the slow supported waves in the cold case (synchronous wave). Thus $Y^e$ has a pole [2] for $\Gamma = \Gamma_m^o = J\beta e$, such that Eq. 6 cannot be set up in this case, furthermore, it is not easy to impose the contour conditions on the periodic structure. Because of this, a direct solution of the problem generally turns out to be very complicated. An appropriate solution, however, can be obtained in this case by using the perturbation method.

We owe this method to the works of Pierce [1] and Wainstein [5]. It consists essentially of the perturbation of the cold solution ($\omega_p = 0$ throughout the structure) in such a way as to obtain the serial development of the transcendental equation (approximating Eq. 5) for the $Y^e$ poles. Thus one can remove from the sum the term $\Gamma = \Gamma_m^o$ relative to the pole, by writing it separately, and replace $\Gamma$ in the other terms with the quantity $J\beta e$, which does not differ greatly from $T$ in analogy to what was done in writing Eq. 6. Thus Eq. 5 is approximated with a fourth-degree equation. In this equation there appears a variable term $\omega$ to $\Gamma^2$ in which the factor $1$ is a parameter generally indicated by $Q$ and called the space charge parameter. This parameter, however, despite its name, does not take account of the phenomena due expressly to the modulation of the space charge. This has been
noted elsewhere by H. Derfler [4], who calculated, in order to test such an assertion, the field in a circular guide periodically loaded with thin discs and crossed by an electron beam concentric with it. He found that to Pierce's parameter Q there is added a term due to the space charge waves. In fact, the structure in the presence of the beam is conceptually different from the cold structure; suffice it to consider that (Eq. 4) for each eigenvalue \( k_T \) of the problem there are 4 waves, instead of the two that generally result in homogeneous problems (two of these waves correspond to the direct and inverse ones in the absence of the beam, and the other two are space charge waves introduced by the beam, having phase velocities close to \( u_0 \)).

2. The Modified Perturbation Method

We thus want to reconsider the perturbation method to include space charge phenomena as well. To this end we will first perturb the cold solution, then we will consider the effect of modulation of the space charge by perturbing the space charge waves in a guide with perfectly conductive walls (in which there are no slow waves in the absence of the beam). By combining the two cases we will obtain a fourth-degree equation analogous to those of Pierce and Wainstein, in which there furthermore appears a parameter R which may be considered the generalization of the parameter calculated by Derfler. First, therefore, we will assume that if the field in the absence of the beam is derived from an eigenfunction \( T_m^0 \), the current inside the beam equals \( J^0 T_m^0 \) where \( J_m^0 \) can be found from Eqs. 2 and 4, and equals (writing \( T_m \) in place of \( T \))

\[
J_m^0 = j \omega e \frac{\omega^2}{(\omega - \gamma m)} E_m.
\]

This does not introduce any very gross error, since, as we
will see, the equation for the calculation of $\Gamma_m$ may be written in a variational form with respect to $T_m$. Then, to find the field throughout the guide section, we rewrite Eq. 4 to include Eq. 7, in the form:

$$I(r-r_0) = \sum_j J_n \Gamma_n,$$

where $I(r-r_0)$ is a zero function outside the beam and equal to $I$ on the inside. Now the second member quantity of Eq. 8 may be developed serially for the cold waves $T_n$, in the following manner.

$$\nabla^2 T - (k^2 - \Gamma_m^2) T = -\frac{J_m}{\gamma_m} \sum_j J_n \Gamma_n \Gamma_m I(\Gamma_m - \Gamma_n),$$

by using these waves' property of orthogonality [5]

$$\int \frac{E_m \times H_m - E_m \times H_m}{\gamma_m} \cdot dS = N \delta_{m,n},$$

where $\delta_{m,n}$ is Kroenecker's symbol. Then, also considering the function $T$ which can be developed serially for these waves, one gets [[1], chap. 6],

$$\nabla^2 \left( \sum_n A_n \Gamma_n \Gamma_m \right) - (k^2 - \Gamma_m^2) \sum_n A_n \Gamma_n \Gamma_m = -\sum_n J_n \Gamma_n^2,$$

where the coefficients $J_n$ can be determined using the aforementioned condition of orthogonality. Then, breaking up Eq. 10 into an infinite number of equations for each value of $n$, one can calculate $A_n$ as a function of $J_n$, and find, from the vector potential thus obtained, the field $E'_2 = E'_f$ corresponding to it. Thus
one gets (cf. also [5]).

\[ E_f = \sum_{m} T_m^\infty (\frac{1}{j\omega e} - \frac{1}{j\omega e} - \frac{1}{j\omega e}) T_m^\infty Z_m^\infty Z_m^\infty \]

where:

\[ Z_m^\infty = \frac{2(E_m^\infty)^2}{\left(\frac{E_m^\infty E_n^\infty - E_m^\infty E_n^\infty T_m^\infty T_m^\infty \right)} \]

\[ \psi_m = \int T_m^\infty T_m^\infty dS \]

where \( S_1 \) is the section of the beam and \( S \) is the section of the guide in which the waves are propagated. The field \( E_f \) thus calculated results from the perturbation of cold waves, to it we add a field \( E_{sp} \) due to the modulation of the beam in such a way that the longitudinal field equals

\[ E_L = E_f - E_{sp} \]

To calculate \( E_{sp} \) the reasoning is completely parallel to that above. Let us consider a sum of space charge waves uncoupled from the synchronous mode. These waves may be precisely the space charge waves of a guide with perfectly conductive walls of a section \( S \) geometrically identical to that of the periodic structure. These waves, due to the Coulomb interaction of the electrons, may be considered as the eigensolution of Poisson's equation ([7] p 169 and 200 ff.) Our problem is to find \( E_{sp} \) on the inside of the beam. To do this we observe that on the contour of the latter, \( S \) is written (indicating the normal versor on the contour \( s \) of the beam) as:

\[ \frac{\delta \rho_s}{\delta n} \left( \frac{\delta \rho_s}{\delta n} \right) - \left( \frac{\delta \rho_s}{\delta n} \right) = \frac{\delta \rho_s}{\delta n} \left( \frac{\delta \rho_s}{\delta n} \right) - \left( \frac{\delta \rho_s}{\delta n} \right) = \left( \frac{\delta \rho_s}{\delta n} \right) - \left( \frac{\delta \rho_s}{\delta n} \right) \]

Actually a guide with perfectly conductive walls does not have poles for \( \Gamma = J\beta e \). Then, applying Green's lemma to two functions \( T_n \) and \( T_m \) relative to two space charge waves, one gets:
That is, for the functions found in the approximative manner using Eq. 13, the property of orthogonality equals:

\[ \int T_m T_n \, dS = N_n \delta_{mn} \]

Using this relationship we can repeat the line of reasoning used already with the perturbation of colds ayes, writing:

\[ J_{2m} = \beta_m (T_m + T_{pm}) \]

The significance of the symbols is obvious (in the preceding treatment we set \( T_{fm} \approx T_m \)).

The current is zero outside the beam, thus we can write:

\[ J_{2pm} = J_m T_{pm} \cos (\theta - \phi) \]

In this manner we get:

\[ \nabla^2 T_{pm} - (K_m^2 - \Delta_m^2) T_{pm} = -J_m \sum \frac{\int T_{pm} T_n \, dS}{\int T_n^2 \, dS} T_m \]

Analogously to what was done for Eq. 10, remembering:

\[ \beta_e^2 = \left| \Gamma_m \right|^2 \approx K_m^2 \]
one finds:

$$E_{spm} = E_{spm}^0 T_{spm} = -\frac{P_m^2}{j\omega e_0} \sum R_n^2 \int T_{spm}(t) T_n dS / \int T_n^2 dS \quad (16)$$

where

Then setting

$$\frac{E_{spm}^0}{j\omega e_0} = -R_m^2 \quad (16b)$$

one gets the integral equation in $T_{spm}$:

$$R_m^2 T_{spm}(t) = \sum R_n^2 \frac{T_{spm}(t) T_n(t)}{\int T_n^2 dS} T_n(t) dS \quad (17)$$

From this, multiplying both sides by $T_{spm}$ and integrating with $S_1$, one gets the variational expression of $R_m^2$ which may be considered the mean reduction factor for the plasma frequency:

$$R_m^2 = \frac{\sum \left( \int T_{spm}(t) T_n(t) dS \right)^2 / \int T_n^2 dS}{\int (T_{spm}(t))^2 dS} \quad (18)$$

However, we have seen that for $T_{spm}$ in the domain $S_1$, the development appearing in the second member of Eq. 15 applies. One can thus write

$$R_m^2 \sum \left( \frac{\int T_{spm}(t) T_n(t) dS / \int T_n^2 dS}{\int T_{spm}(t) T_n(t) dS / \int T_n^2 dS} R_n^2 \right) = \frac{\sum R_n^2 R_m^2}{\sum (\int T_{spm}(t) T_n(t) dS / \int T_n^2 dS)} \quad (19)$$
To calculate this expression one may consider the field due to modulation of the space charge as being proportional point by point to \( T_{fm} \sim T_{spm} \).

Under this hypothesis one can write

\[
E_{zm} = (E_{zm}^* + E_{spm}^*) T_{fm} = (E_{zm}^* - R_{m}^* \frac{j}{\omega \epsilon_0} T_{fm}) \tag{21a}
\]

Introducing Eq. 20 into Eq. 7 one gets:

\[
J_{zm} = j \omega \epsilon_0 \frac{\beta^2_p}{(j \beta - \Gamma_m)^2 + \beta^2 p} E_{zm} \tag{21b}
\]

where

\[
\beta_p = \frac{\omega_p}{\omega_0} \tag{21c}
\]

Finally, by comparing Eq. 21 with Eq. 11 we get,

\[
\frac{(j \beta - \Gamma_m)^2 + \beta^2 p}{j \omega \epsilon_0} T_m = \frac{1}{j \omega \epsilon_0} T_m^* + \sum_n \frac{T_n^*}{\Gamma_m - \Gamma_n} Z_{mn} \eta_n T_n \tag{21d}
\]

Then, multiplying the preceding by \( T_m^0 \) and integrating with \( S_i \), one gets an equation for the variational propagation constants in the last eigenfunction. We can now make the usual equations for this type of problem (cf. [7] chap. 10, and [5,6])

\[
R^2 = \frac{\omega \epsilon_0}{j \omega} \int_{T_m^0} ds \sum_n \frac{T_n^*}{\Gamma_m - \Gamma_n} Z_{mn} (\eta_n)^2 \tag{21e}
\]

\[
Q_i = \frac{R^2 (\frac{i}{\omega} \omega_0)}{C} \tag{21f}
\]

\[
C = 4 \sqrt{Z_m T_m^0} \tag{21g}
\]

\[
Z_m = Z_m^* \int_{T_m^0} ds \tag{21h}
\]
where \( I_0 \) and \( V_0 \) are the continual current and tension of the beam. One thus obtains the fourth-degree equation:

\[
\left( \frac{I_0}{I_m} - 1 \right)^2 + \frac{F_m^2}{F_m^2} \beta_0^2 + \frac{C_0}{C_m} \beta_0 \left( \frac{I_0}{I_m} \right)^2 = 2 \gamma \frac{F_m^2}{F_m^2} \beta_0 \left( \frac{I_0}{I_m} \right)^2 
\]

In this, in addition to the parameter \( Q \) calculated by Pierce and Wainstein, there appears the term due to the space charge waves.

It is now opportune, however, by analogy to Werfler, to define a parameter \( Q_{\text{tot}} \) as:

\[
Q_{\text{tot}} = Q + \frac{F_m^2}{4C} \left( \omega_d \right)^2 
\]

In this way the equation takes on the form of Pierce's equation. This makes it possible to calculate the projections and verifications, utilizing the studies already performed by various authors [1,7,8] for a quick solution to Pierce's equation. In the second place, if it is impossible to assume that the electrons all come out of the electron gun at the same velocity, it is necessary to derive a differential equation from Eq. 22 for the function \( f(z_1u_2) \) of the electron density of the phase plane (setting the symbol \( \frac{\delta}{\delta z} \) in place of the quantity \( -T_m \)) so as to resolve at the same time Boltzmann's equation, which \( f(z_1u_2) \) must satisfy. This study was performed by D.A. Watkins and H. Rynn [3] for Pierce's equation. The result was that they found a further term (due to the velocity distribution function) to be added to Pierce's parameter \( Q \). Thus one finds that one can define a comprehensive parameter for all three effects examined.

\[
Q_{\text{tot}} = Q + \frac{F_m^2}{4C} \left( \frac{\nu_d}{\omega_d} \right)^2 + \frac{S}{\nu_C} 
\]
where for a semi-Maxwell distribution function [3], S equals

\[
\frac{3}{4} \left( \frac{K T_c}{C_1^2} \right)^{\frac{3}{2}}
\]

where

\[ K = \text{Boltzmann's constant} \]
\[ T_c = \text{temperature of cathode}. \]

3. Physical significance of the Results

Marcuvits [6] demonstrated that where the modes of the cold guide are orthogonal with respect to those of a guide suitably conjugate to it, the modes of the electron guide are orthogonal with respect to the modes of the electron guide conjugate to it and are obtained from the cold conjugate guide and the same electron beam. In the case of a periodic structure not containing dielectrics with tensorial constants, the conjugate guide coincides with the same periodic structure and the property of orthogonality takes on the form

\[
\int (E_n \times H_m - E_m \times H_n) \, Z_0 \, dS + \int (V_n \cdot Z_m + V_m \cdot Z) \, dS = N_n S_m
\]

where

\[ V = \frac{\mu_0 \mu_i}{\eta} \]

with \( \mu_1 \) being the alternative component of the electron velocity.

From this property one can see that if one perturbs the cold solution consisting of fields for which the orthogonality relationships applies

\[
\int (E_n \times H_m - E_m \times H_n) \, Z_0 \, dS = N_n S_m
\]

13
then these fields are coupled by way of the current they excite in the beam. The parameter \( Q \) defined by Pierce and Wainstein's equations in fact contains a sum of terms which can be interpreted as the mutual coupling impedances of the various modes across the beam. However, it has already been observed that the presence of the beam introduces an ensemble of space charge waves. These also are coupled to the synchronous wave. The parameter \( R_m \) we defined actually also contains a sum of terms which may be considered as being due to mutual coupling impedances between the synchronous modes and the space charge waves. For this very reason, the field due to serial beam modulation for the space charge waves in a guide with perfectly conductive walls was developed. This also explains why the parameter \( Q_{tot} \) was obtained as the sum of two terms. The first results from the slow wave corresponding to imaginary eigenvalues. In other words, the parameter \( Q \) expresses the coupling of the synchronous wave with the other waves supported by the periodic structure, and the parameter \( R_m \) expresses the coupling of this same wave with those waves supported by the beam.

Note that by substituting the value \( T_{sp} = I_o(\beta e r) \) in the expression for \( R_m \) (in the case of cylindrical symmetry) one obtains for \( f_n \) (Eq. 19)

\[
\begin{align*}
  f_n & = \frac{C_n / \beta e b}{1 + (-q_n)^2} \sqrt{1 + \left(\frac{q_n}{H}\right)^2} \\
  & = \frac{I_o(\beta e b) \left[ 1 + \frac{\beta e b}{H} \right] I_i(\beta e b)}{I_o(\beta e b) \left[ 1 + \frac{\beta e b}{H} \right] I_i(\beta e b)},
\end{align*}
\]

where

\[
\begin{align*}
  C_n & = |K_n|^2 b^2, \\
  D_n & = \frac{q_n}{I_o(\beta e b)} \left( \frac{I_i(\beta e b)}{I_o(\beta e b) \left[ 1 + \frac{\beta e b}{H} \right] I_i(\beta e b)} \right),
\end{align*}
\]

and

\[
H = \frac{C_n}{D_n (q_n)} = -\beta e b \frac{I_i(\beta e b) K_o(\beta e b) + K_i(\beta e b) I_o(\beta e b)}{I_o(\beta e b) K_o(\beta e Q) + K_i(\beta e Q) I_o(\beta e Q)}.
\]
The expression for $f_n$ coincides except for one independent constant with the formula (of the same type as Eq 19) calculated by Derfler.

\[ f_n^D = \frac{\mu f_{eb}}{1 + \left(\frac{\sigma_n}{\beta e_b}\right)^2} \frac{I_n(\beta e_b)}{I_e(\beta e_n)} \left[ 1 + \beta e_b \frac{I_\infty(\beta e_\infty)}{I_e(\beta e_\infty)} \right] \]

N.B. In these formulas $b$ represents the radius of the circumference which defines the beam, and $a$ the radius which defines (in the case of cylindrical symmetry) the circle corresponding to the section $S$. 


