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OPTIMAL SINGULAR CONTROL

WITH APPLICATIONS

TO TRAJECTORY OPTIMIZATION

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SUMMARY

This report presents a comprehensive discussion of the problem of singular control. Singular control enters an optimal trajectory when the so-called switching function vanishes identically over a finite time interval.

Using the concept of domain of maneuvrability, the problem of optimal switching is analyzed. Criteria for the optimal direction of switching are presented. The switching, or junction, between nonsingular and singular subarcs is examined in detail. It is shown that, in general, switching with singular arcs falls into one of two categories: a regular type where the control is discontinuous at the junction point, and a singular type where not only the control is discontinuous at the junction point, but is non-analytic. In this type of junction, entering or leaving a singular arc is effected by chattering control.

Junction between nonsingular and singular subarcs in which the control is continuous at the junction point is a rare phenomenon and usually is effected at some specified points in the phase space. This will require particular initial and final manifolds. Several theorems concerning the necessary, and also sufficient conditions for smooth junction are presented.

The concepts of quasi-linear control and linearized control are introduced. They are designed for the purpose of obtaining approximate

solution for the difficult Euler-Lagrange type of optimal control in the case where the control is nonlinear.

Some illustrative examples are presented as applications of the theorems formulated and of the concepts introduced.

I. INTRODUCTION

Optimal control problems in which the control variables appear only linearly admit the possibility of the existence of singular extremals. Along a singular optimal subarc the so-called switching function is identically zero and necessary condition for optimality is established by considering higher order variation of the Hamiltonian. In recent years, this problem of singular control has been studied by a number of authors [1-14]. The case is now no longer considered as just a mathematical singularity, as its name suggests, but because of frequent occurrence of optimal singular subarcs in trajectory optimization, singular control has become a reality and the inclusion of such subarcs in the overall optimal trajectory has to be considered. This, in turn, leads to the investigation of the problem of joining optimal singular and nonsingular subarcs [15-18].

On the other hand, the physical nature of the engineering problems encountered suggests that the linearity of the control variables in the majority of the cases is merely an approximation in mathematical modeling. Hence, although sometimes a singular solution is obtained through the use of linear control, the true solution to the physical problem is nonsingular since the control is nonlinear, or at most quasi-linear. It is then interesting to investigate the real physical problem that is quasi-linear in the control and analyze the solution which can be termed as quasi-singular.

Finally, one of the well established techniques for stability analysis is through the linearization of the equations of motion about a certain known solution, usually steady state solution, called the reference solution. The brilliant works of Poincaré [19] and followers [20] in establishing periodic solutions in the three and many-body problems, and also in space dynamics [21, 22], are testimonial of the usefulness of the approach. It is enlightening to use linearization in purely nonlinear control problems. One can then assess the behavior of the control about a certain solution. Furthermore, if this solution is near optimal, then linearization is a proven technique which allows one to obtain improved solution as long as the near optimality of the reference solution is valid.

The outline of this report is as follows. After this introductory section, optimal control problem using the notion of domain of maneuvrability is discussed in Section II. If certain components of the control vector enter the equations of motion linearly, the domain of maneuvrability, which is bounded by a hypersurface, has a portion of its boundary being a ruled surface on which the optimal control is singular. The problem of switching from one control to another one is investigated and the condition for a smooth junction between singular and nonsingular subarcs is established. In Section III, optimal control problems in which the control is quasi-linear is studied. The quasi-singular solution is obtained by the construction of a switching function and the fact that the junction between nonsingular and

quasi-singular subarcs is continuous is established. In Section IV, the technique of linearization is applied to investigate the behavior of the optimal control near a given solution. It is shown that through the linearization of the Hamiltonian near a suboptimal solution, a better solution can be obtained. In Section V, some applications of the theory to trajectory optimization are given and, finally, in the last section, Section VI, a summary of the new results and their usefulness in solving optimal control problems is presented.

II. SWITCHING THEORY

Consider a dynamical system defined by a n-vector $\vec{x}$ subject to the differential constraint

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}, t) \tag{1}$$

where $\vec{u}$ is an m-control vector belonging to a certain control space U

$$\vec{u} \in U(\vec{x}, t) \tag{2}$$

It is proposed to find the optimal control $\vec{u}^*$, as function of time, to bring the system from a certain initial manifold to a certain final manifold such that a certain final component of the state vector is minimized.

II. 1. Selection of The Optimal Control.

Following Contensou, we define the natural domain of maneuvrability $D(\vec{x}, t)$ at the time t, with state vector $\vec{x}$, as the reachable domain in the hodograph space [2-4]

$$\vec{V} = \dot{\vec{x}} = \vec{f}\ (\vec{x}, \vec{u}, t) \quad . \tag{3}$$

Introducing the n-adjoint vector $\vec{p}$, the optimal trajectory is obtained by selecting, at each instant t, the control vector $\vec{u}*$ in the control space U such that

$$\vec{u}* = \arg\sup_{\vec{u} \in U} H \quad , \text{ or } \quad H* = \sup_{\vec{u} \in U} H \tag{4}$$

where the Hamiltonian H is defined by

$$H = \vec{p} \cdot \vec{f} = \vec{p} \cdot \vec{V} \quad . \tag{5}$$

In the domain of maneuvrability (Fig. 1), the optimal condition (4) leads to the selection of the optimal operating point M* such that the projection of the vector $\vec{V}* = \vec{OM}*$ on the adjoint vector $\vec{p}$ is maximized. The point M* is necessarily on the boundary G of D. Only the convex portion of the boundary G can be used optimally. The concave part of the boundary has to be completed by the smallest convex ruled surface. In this case, the convex ruled surface is artificial. In the case where certain components of the control vector $\vec{u}$ enter the equations (3) linearly, there exists a natural, ruled part of the domain of maneuvrability.

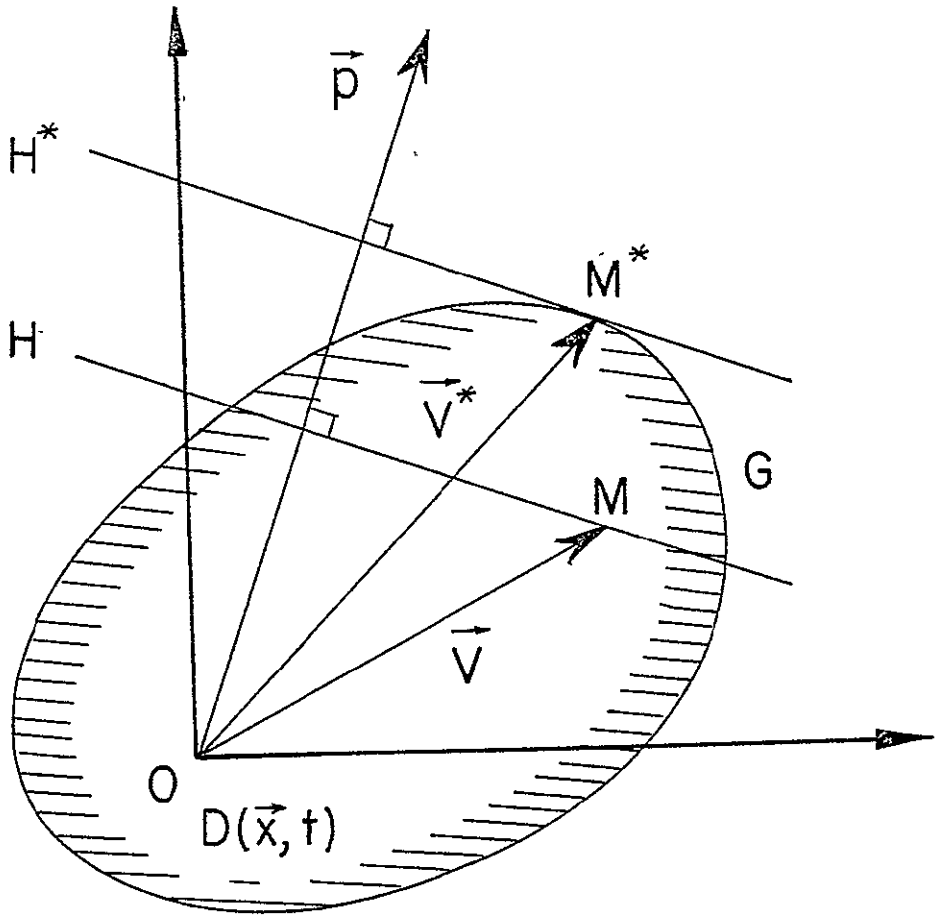

Fig. 1. Selection of The Optimal Velocity In The Domain of Maneuvrability.

II. 2. Switching of Optimal Control.

Consider a rectilinear part of the smallest convex domain $\overline{D}$ of the domain of maneuvrability D. This part can be natural R, or artificial $\overline{R}$ by convexizing. The convex domain $\overline{D}$ and the adjoint vector $\vec{p}$ vary with the time t. There may exist a time t_o such that, through the ruled

part R, or $\overline{R}$, the optimal operating point changes brusquely from M_{1_o} to M_{2_o} . At that point, the optimal control changes from $\vec{u}_1^*$ to $\vec{u}_2^*$. We have a switching of the optimal control.

The sequence in Fig. 2 shows a switching $M_{1_o} \rightarrow M_{2_o}$. If the sequence of the events is in the reverse direction, we have a switching $M_{2_o} \rightarrow M_{1_o}$.

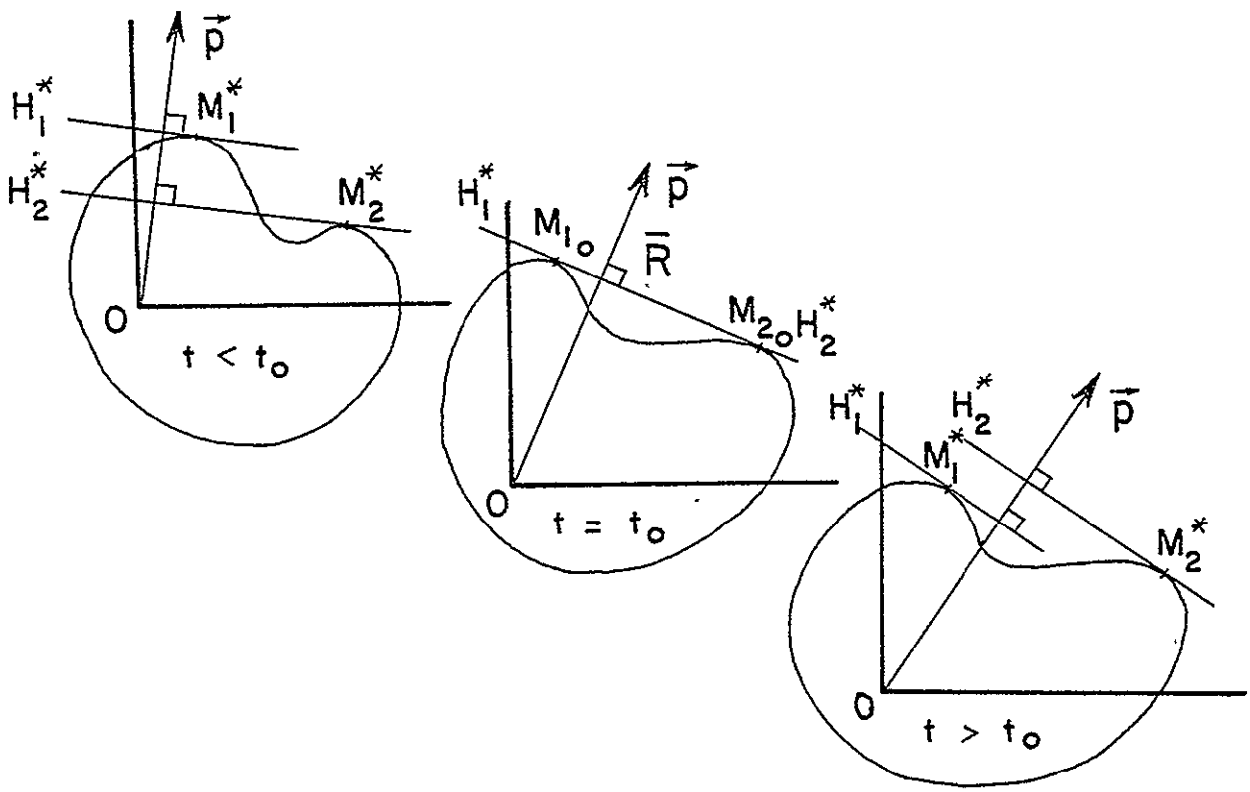

Fig. 2. Optimal Switching $M_{1_o} \rightarrow M_{2_o}$.

To study the direction of the switching, we consider the convex parts G_1 and G_2 of the boundary G of D near of the point M_{1_o} and M_{2_o}

respectively. Near the switching point, the optimality condition leads to the selection of the operating point, either on G_1, with the velocity $\vec{V}_1$ or on G_2, with the velocity $\vec{V}_2$ (Fig. 3).

Let

$$\vec{V} = (1-\lambda)\,\vec{V}_1 + \lambda\,\vec{V}_2 = (1-\lambda)\vec{f}(\vec{x},\vec{u}_1,t) + \lambda\vec{f}(\vec{x},\vec{u}_2,t) \quad (6)$$

where $\vec{u}_1$ and $\vec{u}_2$ are the values of $\vec{u}$ corresponding to the point $M_1 \in G_1$ and $M_2 \in G_2$ respectively. By varying λ in its interval $\lambda \in [0,1]$ we obtain all the points M on the segment $M_1\ M_2$ which is obviously within the convex domain of maneuvrability $\overline{D}$. The parameter λ, introduced artificially as defined in Eq. (6), constitutes a normalized linear control. We observe that, near the switching point, the optimal value of λ is either $\lambda = 0$, point M_1, or $\lambda = 1$, point M_2. Hence, it suffices to first select $\vec{u}_1^*$ and $\vec{u}_2^*$, and then the optimal value λ^* to have the optimal velocity $\vec{V}^*$. We have

$$\vec{u}_1^* = \vec{u}_1^*(\vec{p},\vec{x},t) = \arg\sup_{M_1 \in G_1} H_1$$

$$\vec{u}_2^* = \vec{u}_2^*(\vec{p},\vec{x},t) = \arg\sup_{M_2 \in G_2} H_2$$

with the corresponding maximized Hamiltonians

$$H_1^* = H_1^*(\vec{p},\vec{x},t) = \sup_{M_1 \in G_1} H_1$$

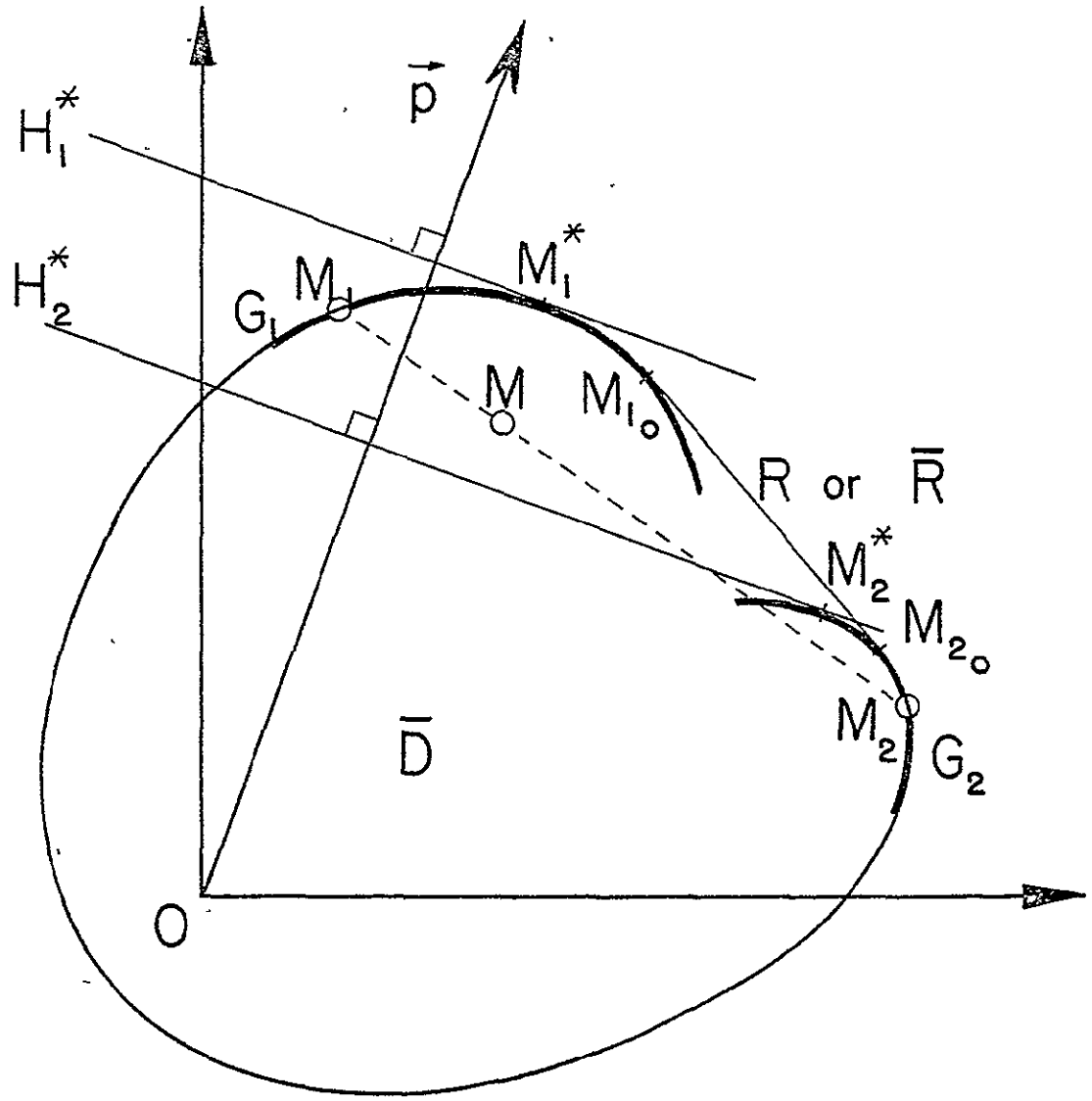

Fig. 3. Domain of Maneuvrability Near a Switching Discontinuity.

$$H_2^* = H_2^*(\vec{p}, \vec{x}, t) = \sup_{M_2 \in G_2} H_2$$

Since $H = \vec{p} \cdot \vec{V}$, we have the Hamiltonian by using Eq. (6)

$$\tilde{H} = (1-\lambda) H_1^* + \lambda H_2^* \quad . \tag{7}$$

To maximize $\tilde{H}$ with respect to λ, we have the following optimal solution

$$\lambda = \begin{cases} 0 & \text{if} \quad \dfrac{\partial \tilde{H}}{\partial \lambda} = H_2^* - H_1^* \lesseqgtr 0 \\ 1 & \end{cases} .$$

At the time of the switching, we have

$$H_2^* - H_1^* = 0 \tag{8}$$

Furthermore, the switching is from M_1 to M_2 if at that time

$$\frac{d}{dt} (H_2^* - H_1^*) > 0 \quad . \tag{9}$$

If the inequality is reversed, the switching is from M_2 to M_1.

We define the switching function

$$\Phi = H_2^* - H_1^* = \Phi(\vec{p}, \vec{x}, t) \tag{10}$$

At the switching point, $\Phi = 0$. It suffices to analyze the sign of $d\Phi/dt$ at $\Phi = 0$ to determine the optimal direction for switching. In this respect we write the maximized Hamiltonian

$$H^* = (1-\lambda^*)H_1^* + \lambda^* H_2^* = H_1^* + \lambda^* \Phi \quad . \tag{11}$$

Now, consider an arbitrary function $F = F(\vec{p}, \vec{x}, t)$. Its total derivative is

$$\frac{dF}{dt} = \frac{\partial F}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial F}{\partial \vec{x}} \frac{d\vec{x}}{dt} + \frac{\partial F}{\partial t}$$

Since along an optimal trajectory

$$\frac{d\vec{x}}{dt} = \frac{\partial H^*}{\partial \vec{p}} , \qquad \frac{d\vec{p}}{dt} = - \frac{\partial H^*}{\partial \vec{x}} \tag{12}$$

we can use the Eq. (11) to write the derivative of F

$$\frac{dF}{dt} = - \frac{\partial F}{\partial \vec{p}} \frac{\partial H_1^*}{\partial \vec{x}} + \frac{\partial F}{\partial \vec{x}} \frac{\partial H_1^*}{\partial \vec{p}} + \frac{\partial F}{\partial t}$$

$$+ \lambda^* \left(\frac{\partial F}{\partial \vec{x}} \frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial \Phi}{\partial \vec{x}} \right) . \tag{13}$$

We define the temporary derivatives $D_1 F = dF/dt_1$, $D_2 F = dF/dt_2$ of a function F as the derivative of F using the Eq. (12) as generated by the Hamiltonian H_1^* and H_2^* respectively. Then the derivative $DF = dF/dt$ has the form

$$DF = D_1 F + \lambda^* \left(\frac{\partial F}{\partial \vec{x}} \frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial \Phi}{\partial \vec{x}} \right) . \tag{14}$$

Since $\partial F/\partial t_1 = \partial F/\partial t_2$, we can write the coefficient of λ^* in this equation

$$\frac{\partial F}{\partial \vec{x}} \frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial \Phi}{\partial \vec{x}} = \left(\frac{\partial F}{\partial \vec{x}} \frac{\partial H_2^*}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial H_2^*}{\partial \vec{x}} + \frac{\partial F}{\partial t_2} \right)$$

$$- \left(\frac{\partial F}{\partial \vec{x}} \frac{\partial H_1^*}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial H_1^*}{\partial \vec{x}} + \frac{\partial F}{\partial t_1} \right) .$$

$$= D_2 F - D_1 F \quad .$$

Hence, we can write Eq. (14) in the operational form

$$DF = D_1 F + \lambda^* (D_2 F - D_1 F) \tag{15}$$

valid for any arbitrary function F along an optimal trajectory.

When $F = \Phi$, the condition (9) coupled with Eq. (14), provides the condition for a switching from M_1 to M_2

$$D\Phi = D_1 \Phi = \frac{dH_2^*}{dt_1} - \frac{\partial H_1^*}{\partial t_1} > 0 \quad . \tag{16}$$

This condition, first derived in [18] , is a generalization to nonautonomous system of the condition given in [23] . (Application of this switching condition will be given in Section V.) By comparing the Eqs. (14) and (15), we see that, for $F = \Phi$, the coefficient of λ^* is zero and we have

$$D_1 \Phi = D_2 \Phi \quad . \tag{17}$$

Hence the equivalent condition for a switching from M_1 to M_2 is

$$\frac{dH_1^*}{dt_2} - \frac{\partial H_2^*}{\partial t_2} < 0 \ . \tag{18}$$

In the case where $D\Phi = D_1\Phi = D_2\Phi = 0$ at the time t_o of the switching, the direction of the switching is decided upon analyzing higher order derivatives of the switching function. The successive derivatives of Φ may contain the control which is discontinuous across the switching point so that in the neighborhood of the time $t = t_o$, the switching function is not analytic. To circumvent the difficulty, we consider separately the development Φ_1 and Φ_2 of the function Φ, respectively in the neighborhood of the points M_{1_o} and M_{2_o} as a series expansion. Then

$$\Phi_1(t) = B_1 \frac{(t-t_o)^{n_1}}{n_1!} + \cdots$$

$$\Phi_2(t) = B_2 \frac{(t-t_o)^{n_2}}{n_2!} + \cdots \tag{19}$$

where

$$B_1 = D_1^{n_1} \Phi \Big|_{t=t_o} \neq 0$$

$$B_2 = D_2^{n_2} \Phi \Big|_{t=t_o} \neq 0 \tag{20}$$

and n_1 and n_2 are the order of the first non zero successive derivatives at the time $t = t_o$. The direction of the switching depends on the orders, odd or even, of n_1 and n_2 and the signs of B_1 and B_2. In the neighborhood of $t = t_o$, the plots of the function Φ_1 and Φ_2 are one of the four types

shown in Fig. 4. These four types are denoted by I k and II k, $k = 1,2,3,4$ with the definition given in Table 1 for the types I k.

Table 1

Different type of curves representing Φ_1.

I 1	:	$B_1 > 0$	,	$(-1)^{n_1}$	$<$	0
I 2	:	$B_1 < 0$	,	$(-1)^{n_1}$	$>$	0
I 3	:	$B_1 < 0$	,	$(-1)^{n_1}$	$<$	0
I 4	:	$B_1 > 0$	,	$(-1)^{n_1}$	$>$	0

We have similar definition for the types II k.

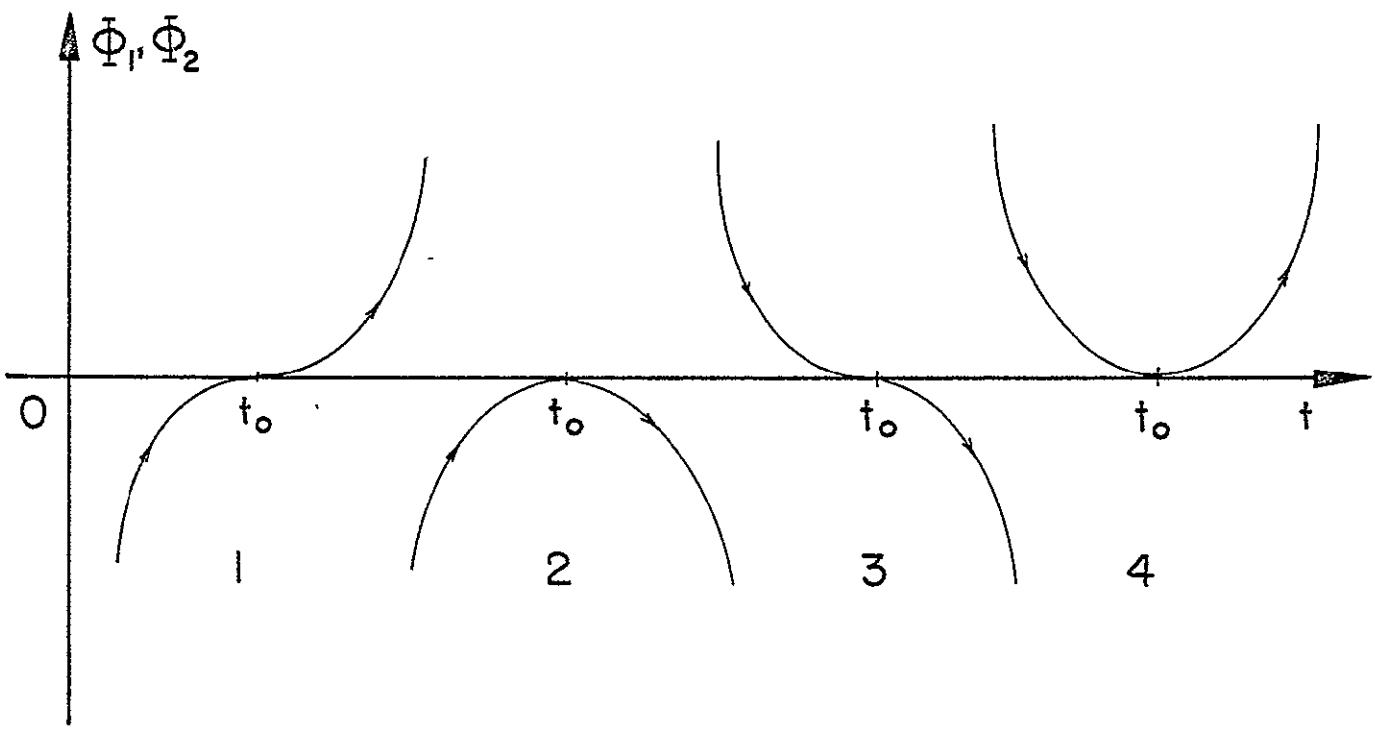

Fig. 4. Plots of Φ_1 and Φ_2 in the Neighborhood of t_o.

We distinguish two cases:

The Regular Case

This is the case where the first non zero successive derivative does not contain the control. The orders n_1 and n_2 on the one hand, and the coefficients B_1 and B_2 on the other hand are identical. The possible switchings are the following:

$$
\begin{array}{lcl}
(\mathrm{I}\,1\,,\ \mathrm{II}\,1) & : & M_1 \rightarrow M_2 \\
(\mathrm{I}\,2\,,\ \mathrm{II}\,2) & : & M_1 \rightarrow M_1 \\
(\mathrm{I}\,3\,,\ \mathrm{II}\,3) & : & M_2 \rightarrow M_1 \\
(\mathrm{I}\,4\,,\ \mathrm{II}\,4) & : & M_2 \rightarrow M_2 \quad .
\end{array}
$$

The proof is simple. For example, we consider the case (I 1, II 1) of Fig. 5. We see that before the time $t = t_o$ we must take $\Phi = \Phi_1$ since $\Phi_1 = H_2^* - H_1^* < 0$, that is $H_1^* > H_2^*$ in agreement with the maximum principle, and after the time $t = t_o$, we must take $\Phi = \Phi_2$ for $\Phi_2 = H_2^* - H_1^* > 0$, that is $H_2^* > H_1^*$. The switching is then from M_1 to M_2. In this regular case, the switching occurs at the junction between two nonsingular subarcs. The cases (I 1, II 1) and (I 3, II 3) are the ordinary switchings and the cases (I 2, II 2) and (I 4, II 4) are the false switchings.

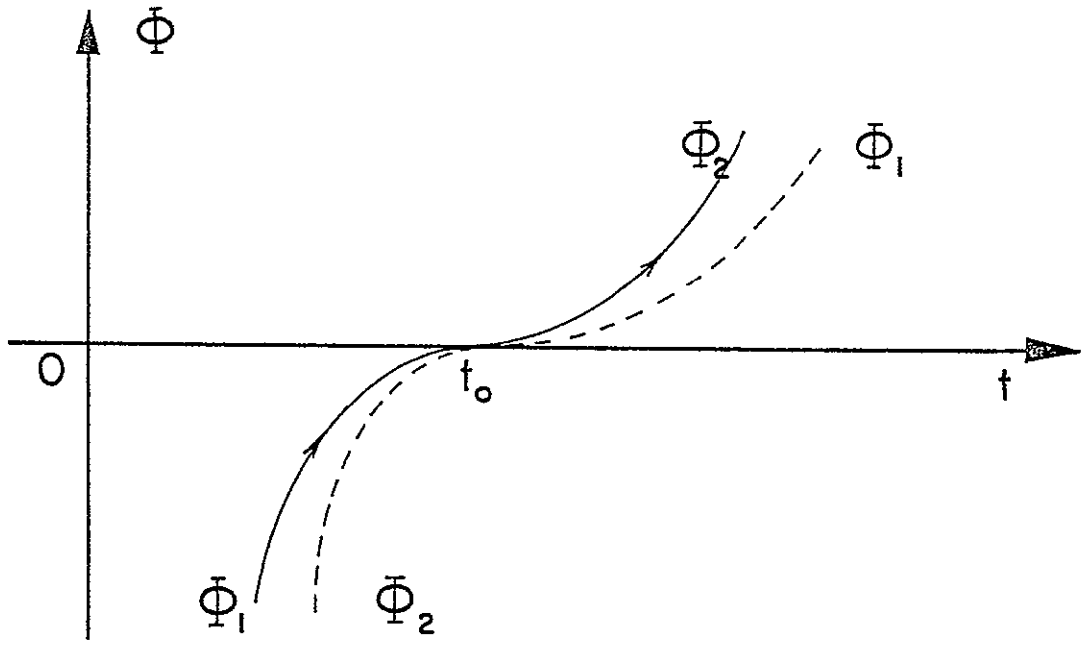

Fig. 5. Switching in the Case (I 1, II 1), $M_1 \rightarrow M_2$.

The Singular Case

This is the case where upon successive differentiation, the first non zero derivative contains the linear control λ^*. In general, this case corresponds to a junction with a singular arc as will be apparent from the discussion below.

We have seen that, at all time t

$$D\,\Phi = D_1\,\Phi = D_2\,\Phi \quad . \tag{21}$$

By taking the derivative of this equation, using the operational relation (15), we have

$$D^2\Phi = D_1^{\,2}\Phi + \lambda^* (D_2^{\,2}\Phi - D_1^{\,2}\Phi) \quad . \tag{22}$$

It may occur that the coefficient of λ^* vanishes identically. For this case, we have for all t in a closed interval containing t_o

$$D^2 \Phi = D_1^{\,2} \Phi = D_2^{\,2} \Phi \quad . \tag{23}$$

Continuing the operation until the coefficient of λ^* does not vanish identically, we have

$$D^k \Phi = D_1^{\,k} \Phi + \lambda^* (D_2^{\,k} \Phi - D_1^{\,k} \Phi) \quad . \tag{24}$$

In the case where $\vec{u}$ enters linearly the differential constraint (1), Kelley has shown in Ref. [5] that in taking the successive derivative of the switching function, the linear control appears for the first time only with an even derivative $k = 2q$, where q is the order of the singular arc. This is also true with respect to the artificial normalized linear control λ in the present formulation where $\vec{u}$ can be non-linear. A simple proof of this property can be found in Ref. [18] .

In summary in the singular case, we have $n_1 = n_2 = 2q$, $D_1^{\,2q} \Phi \not\equiv D_2^{\,2q} \Phi$. In the neighborhood of a singular arc, the derivative of the switching function Φ is

$$D^{2q} \Phi = D_1^{\,2q} \Phi + \lambda^* (D_2^{\,2q} \Phi - D_1^{\,2q} \Phi) \tag{25}$$

with all the $D_1^k \Phi \equiv D_2^k \Phi$ for $k \leq 2q - 1$. Furthermore, all these derivatives vanish at $t = t_o$. The integer q is called the order of the singular arc.

Let

$$B_1 = D_1^{2q} \Phi \Big|_{t = t_o}$$

$$B_2 = D_2^{2q} \Phi \Big|_{t = t_o} \tag{26}$$

and consider first the simplest case where $B_1 \neq 0$ and $B_2 \neq 0$. If B_1 and B_2 have the same sign that is if $B_1 B_2 > 0$, then since $n_1 = n_2 = 2q$, we have the false switching cases (I2, II2) if $B_1 < 0$ and (I4, II4) if $B_1 > 0$, with $M_1 \rightarrow M_1$ for $B_1 < 0$ and $M_2 \rightarrow M_2$ for $B_1 > 0$ respectively. As can be expected, these cases are rare.

If B_1 and B_2 have different signs, that is if $B_1 B_2 < 0$, we distinguish two cases. The first case is $B_1 < 0$, $B_2 > 0$, hence it is the case (I2, II4) as shown in Fig. 6. This case has some ambiguities. We can take either Φ_1 or Φ_2 before t_o and also either Φ_1 or Φ_2 after t_o. Furthermore, by writing

$$\Phi = (1 - \lambda^*) \Phi_1 + \lambda^* \Phi_2 \tag{27}$$

since Φ_1 and Φ_2 have different signs, we can select $\lambda^* \in [0, 1]$ to make Φ identically zero before or after t_o. If the rectilinear part of the domain of maneuvrability is natural, R, λ^* has an intermediary value between 0 and 1; the arc is a singular arc. If the rectilinear part is artificial, $\overline{R}$, obtained by convexizing, we can render Φ identially zero by switching rapidly λ^* between 0 and 1; the arc is a chattering arc.

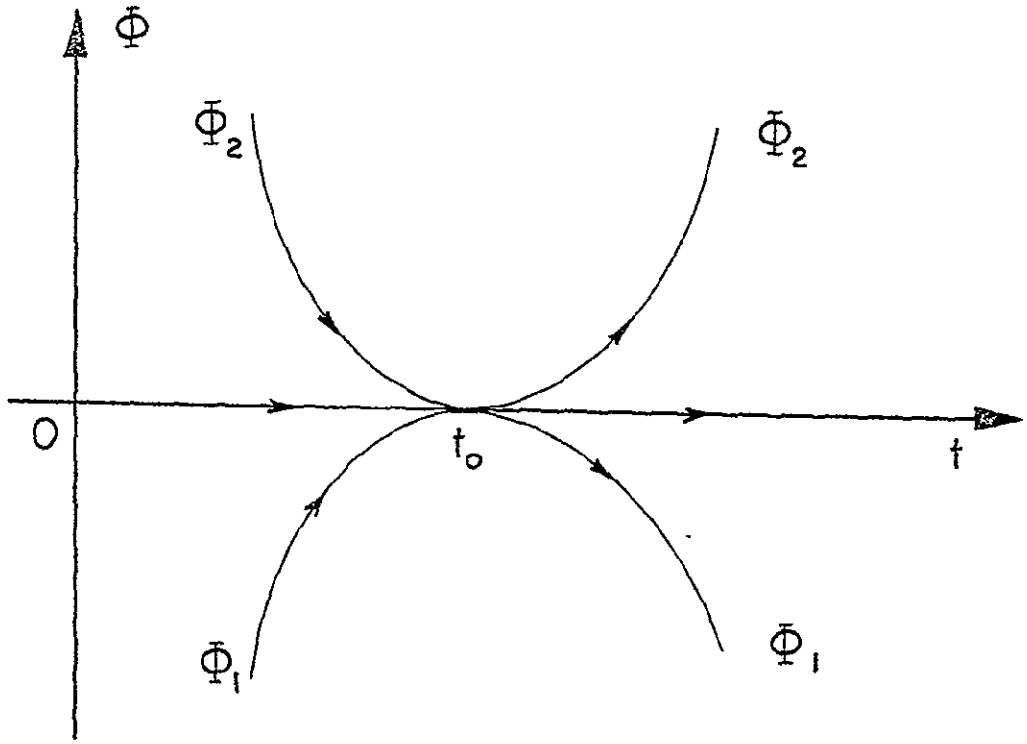

Fig. 6. Switching in the Case (I2, II4).

In summary, for the case (I2, II4) of Fig. 6, we have the following possible switchings

$$\begin{matrix} M_1 \\ M_2 \end{matrix} \rightarrow M_1 \qquad \begin{matrix} M_1 \\ M_2 \end{matrix} \rightarrow M_2$$

$$\begin{matrix} M_1 \\ M_2 \end{matrix} \rightarrow S \qquad S \rightarrow \begin{matrix} M_1 \\ M_2 \end{matrix}$$

$$S \rightarrow S$$

where S denotes the singular arc, either natural or artificial by chattering. In practical applications, the ambiguity is removed by considering the initial and the final conditions.

There remains the case where $B_1 > 0$, $B_2 < 0$, that is the case (I 4, II 2) as shown in Fig. 7.

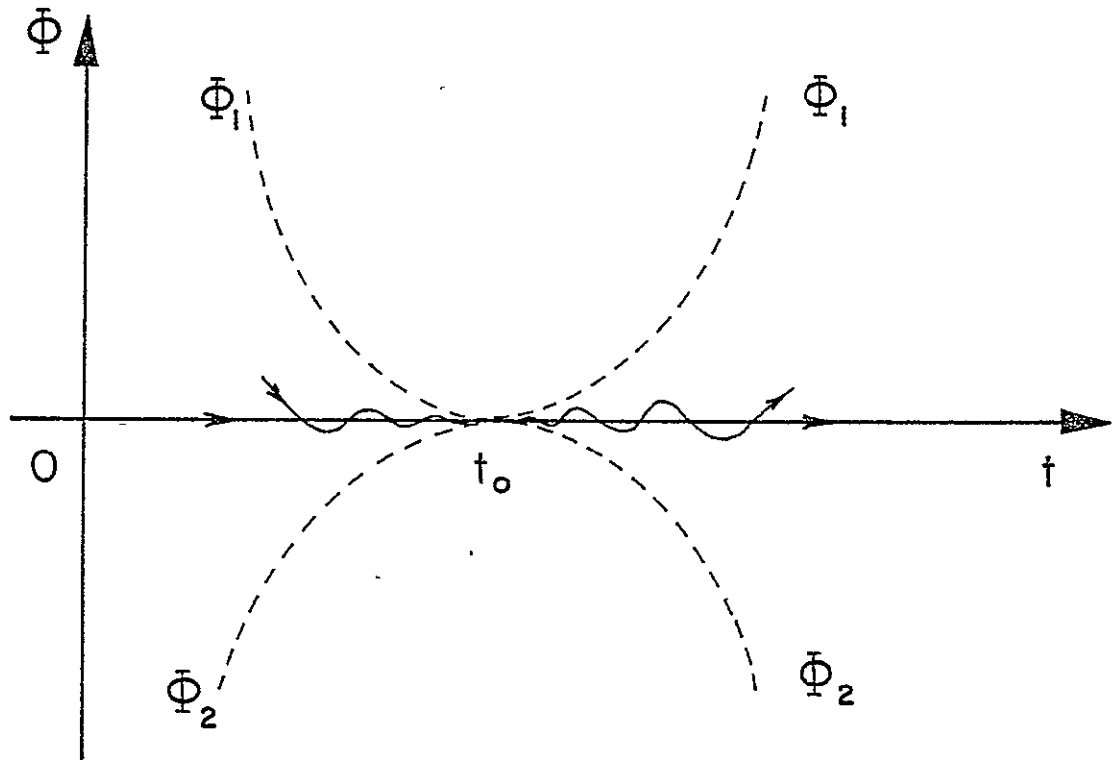

Fig. 7. Switching in the Case (I 4, II 2).

This is the case of singular switching. The junction between subarcs is connected in a singular manner. In this case, immediately before and after the time t_o, we cannot take $\Phi = \Phi_1$ for any finite time

interval for $\Phi_1 > 0$, that is $H_2^* > H_1^*$, in violation of the maximum principle. On the other hand, for the same reason, we cannot take $\Phi = \Phi_2$. The only natural possibility is to combine Φ_1 and Φ_2 to make $\Phi = 0$. By Eq. (27), since Φ_1 and Φ_2 have different signs, if the rectilinear part of the domain of maneuvrability is natural, λ^* has an intermediary value in its interval $\lambda^* \in [0, 1]$. The arc before or after, or both before and after the time t_o is a singular arc. If the rectilinear part is artificial, obtained by convexizing, λ^* can only have the value 0 or 1, we can render $\Phi = 0$ by switching λ^* rapidly between its extreme values. The arc is a chattering arc.

The chattering arc can occur with a linear control, that is for a natural rectilinear part of the domain of maneuvrability. Referring to Fig. 7, we consider the case where the trajectory is not totally singular in a time interval $t \in [t_1, t_2]$ containing t_o. Let us assume that the singular arc occurs after the time t_o. By assumption, before t_o, the arc is not singular, obtained by using an intermediate value of λ^*. Also since neither Φ_1 nor Φ_2 can be used for any finite time interval, the entering of a singular arc at $t = t_o$ can only be obtained, with Φ neither positive, $\Phi = \Phi_1$, nor Φ negative, $\Phi = \Phi_2$, that is with $\Phi = 0$, by rapid switching of λ^* between its limits 0 and 1. The same arguments apply when the trajectory leaves a singular arc. In this case, although the control is linear, with the possibility of using an intermediate value

of λ^*, entering or leaving a singular arc is effected by chattering. For this reason, we refer to the case of Fig. 7 as the case of singular switching. The results of the discussion are summarized in Table 2, with $\mathscr{S}$ denoting singular switching.

	I1	I2	I3	I4
II 1	$M_1 \to M_2$	*	*	*
II 2	*	$M_1 \to M_1$	*	singular switching $\mathscr{S}$
II 3	*	*	$M_2 \to M_1$	*
II 4	*	$M_1 \to M_1$, $M_2 \to M_2$, $M_1 \to S$, $M_2 \to S$, $S \to M_1$, $S \to M_2$, $S \to S$	*	$M_2 \to M_2$

Table 2. Optimal Switchings.

$(B_1 \neq 0, \ B_2 \neq 0, \ n_1 = n_2 \leq 2q)$

Remark

Table 2 is not complete in the sense that it does not present all the possible cases of optimal switching. The reason is that Table 2 is restricted to the case

$$B_1 \neq 0 \ , \ B_2 \neq 0 \tag{28}$$

where B_1 and B_2 are the first non vanishing values of the derivatives evaluated at $t = t_o$ for an order $n_1 = n_2 \leq 2q$ where q is the order of the singular arc. Nevertheless, it will be shown in the following that the condition (28) is generally satisfied. Under this condition, junction with a singular arc only occurs in two cases. In case (I4, II2), entering or leaving a singular arc is effected by chattering control with increasing frequency as t approaches t_o.

To clarify the meaning in subsequent analysis we use the following definitions as given in [16] :

Definition 1. A real-valued function g is said to be piecewise analytic on an interval (a, b) if for each $t_o \in (a, b)$, there exist $t_1 \in (a, t_o)$ and $t_2 \in (t_o, b)$ such that g is analytic on the open subintervals (t_1, t_o) and (t_o, t_2).

Definition 2. A junction between singular and nonsingular subarcs of the control is said to be a nonanalytic junction if the control is not piecewise analytic in any neighborhood of the junction.

From the Definition 1, chattering control is not piecewise analytic in the neighborhood of t_o and the junction in case (I4, II2) is a nonanalytic junction.

On the other hand, in case (I2, II4) where junction between singular and nonsingular arcs is also possible, the normalized control λ^* is either $\lambda^* = 0$ or $\lambda^* = 1$ on the nonsingular arc, and from Eq. (30), is given by

$$D_1^{2q}\Phi + \lambda^*(D_2^{2q}\Phi - D_1^{2q}\Phi) = 0 \tag{29}$$

on the singular arc. At the junction point, we have

$$B_1 + \lambda^*(B_2 - B_1) = 0 . \tag{30}$$

Since this is the case where $B_1 < 0$ and $B_2 > 0$, it is clear that at the junction point, on the singular side, λ^* is specified by $0 < \lambda^* < 1$. Hence the control is piecewise analytic but is discontinuous at the junction.

II. 3. Junction With Singular Arc.

By a systematic discussion of optimal switching, we have seen that junction with singular arc is usually through chattering control or if the control is piecewise analytic, it is discontinuous at the junction. It remains to investigate the cases where the control is continuous, or even smooth at a junction between nonsingular and singular arcs. In this respect several interesting theorems have been formulated by McDanell and Powers [16]. The objective of the present analysis is to complement their results for non symmetric control and to supply additional rules with practical applications. From now on, we shall restrict ourselves to the case where certain components of the control vector $\vec{u}$ enter the equations of motion linearly. Also, at any given interval, on the singular arc, there is only one linear component of the control that is singular. If $u(t)$ is that component, then $u \in U(\vec{x}, t)$, or explicitly

$$K_1(\vec{x},t) \leq u \leq K_2(\vec{x},t) \quad . \tag{31}$$

We shall rule out the trivial case where at the switching point $t = t_o$

$$K_1(\vec{x}(t_o),t_o) = K_2(\vec{x}(t_o),t_o) \quad . \tag{32}$$

Then obviously $B_1 = B_2$ and we either have a false switching or a regular switching between non singular arcs with the control being continuous at the junction $t = t_o$, $u(t_o) = K_1 = K_2$.

We continue to use our normalized linear control λ^* and the temporary differential operators D_1 and D_2 which prove to be very effective in formulating practical rules for continuous control across a junction.

Along a singular arc, the switching function vanishes identically. Hence, from Eq. (25) we constantly have

$$D^{2q}\Phi = A\lambda^* + C = 0 \tag{33}$$

where

$$A = D_2^{2q}\Phi - D_1^{2q}\Phi$$

$$C = D_1^{2q}\Phi \quad . \tag{34}$$

Equation (33) can be solved for the singular control λ^*. The necessary condition for optimality of a singular subarc derived by Kelley and Contensou [3-5] , also called the generalized Legendre-Clebsch condition states that

Theorem (Generalized Legendre-Clebsch condition). On an optimal singular subarc of order q, it is necessary that

$$(-1)^q \frac{\partial}{\partial \lambda^*} \left[\frac{d^{2q}}{dt^{2q}} \left(\frac{\partial H^*}{\partial \lambda^*} \right) \right] \leq 0 \quad . \tag{35}$$

In the present formulation, it is expressed by the condition

$$(-1)^q A \leq 0 \quad . \tag{36}$$

In the following, we shall refer to the condition as the GLC condition and by the strengthened GLC condition we mean that strict inequality holds in (36).

First, we have seen that, under the condition in Table 2, the control at a junction between singular and nonsingular subarcs is discontinuous.

Hence, we have

Lemma

Let

$$\bar{B}_1 = D_1^{2q+r_1} \Phi \Big|_{t=t_o} \neq 0$$

$$B_2 = D_2^{2q+r_2} \Phi \Big|_{t=t_o} \neq 0 \tag{37}$$

be the first non vanishing derivatives evaluated at a junction point between a nonsingular subarc and a singular subarc of order q. Then, a necessary

condition for the control to be continuous at the junction point is that either $r_1 > 0$ or $r_2 > 0$ or both $r_1 > 0$ and $r_2 > 0$.

This simple rule is in fact very useful. In an optimal control problem in which optimal singular subarcs are suspected one can immediately single out the region where a continuous junction is possible by writing the necessary condition at the junction point

$$\Phi(t_o) = 0$$

$$D_i^n \Phi \Big|_{t=t_o} = 0 \tag{38}$$

$$i = 1 \text{ or } 2$$

$$n = 1, 2, \ldots, 2q \ .$$

This condition, together with other necessary conditions given below can restrict further the region where junction is continuous. Hence in general, continuous control at a junction point only occurs in very special cases.

We can now prove the following main theorem.

Theorem 1.

Suppose the strengthened GLC condition is satisfied at a point t_o, on an optimal trajectory, where a nonsingular control u_i is joined with a singular control u_s. Let q be the order of the singular arc. Then, for the control to be continuous at t_o, it is necessary that

$$D_i^{n-1}\,\Phi\Big|_{t=t_o} = 0, \quad D_i^{n}\,\Phi\Big|_{t=t_o} \neq 0 \tag{39}$$

$$i = 1 \text{ or } 2$$

$$n = 1, 2, \ldots, 2q + r$$

where $r > 0$ and $q + r$ is an odd integer.

Proof: From the lemma, for the control to be continuous at the junction with a singular subarc, it is necessary that $D^{n-1}\,\Phi\,|_{t=t_o} = 0$ for $n = 1, 2, \ldots, 2q$, and $B_1 = D_1^{2q}\,\Phi\,|_{t=t_o} = 0$ or $B_2 = D_2^{2q}\,\Phi\,|_{t=t_o} = 0$. The strengthened GLC condition prevents the case where both B_1 and B_2 are zero. On the singular side of the junction, the normalized singular control λ^* is given by

$$B_1 + \lambda^*(B_2 - B_1) = 0 \quad . \tag{40}$$

If $B_1 = 0$, $\lambda^*(t_o) = 0$ and since, on the nonsingular side, $\lambda^* = 0$ corresponds to $u_1(t)$, continuous junction is made with u_1. Similarly, if $B_2 = 0$, $\lambda^* = 1$ on the singular side and since on the nonsingular side $\lambda^* = 1$ corresponds to $u_2(t)$, continuous junction is made with u_2. This question of junction settled we next consider:

a. Case of q even. If $B_1 = 0$, junction is made with u_1 and $\Phi_1 < 0$ on the nonsingular side. Since q is even, by the strengthened GLC condition $B_2 < 0$ and and the representative curve for Φ_2 is of the

type II 2 (Fig. 8). For singular arc to exist Φ_1 and Φ_2 must have different signs on the singular side. Therefore $\Phi_1 > 0$ on the singular side and the representative curve for Φ_1 has an inflection point at $t = t_o$. The function Φ_1 in the neighborhood of $t = t_o$ is of the form

$$\Phi_1(t) = \overline{B}_1 \frac{(t-t_o)^n}{n!} + \dots \tag{41}$$

where

$$\overline{B}_1 = D_1^n \Phi \Big|_{t=t_o} \neq 0 \tag{42}$$

and $n = 2q + r$ is an odd integer. Hence, r is an odd integer and $q + r$ is an odd integer.

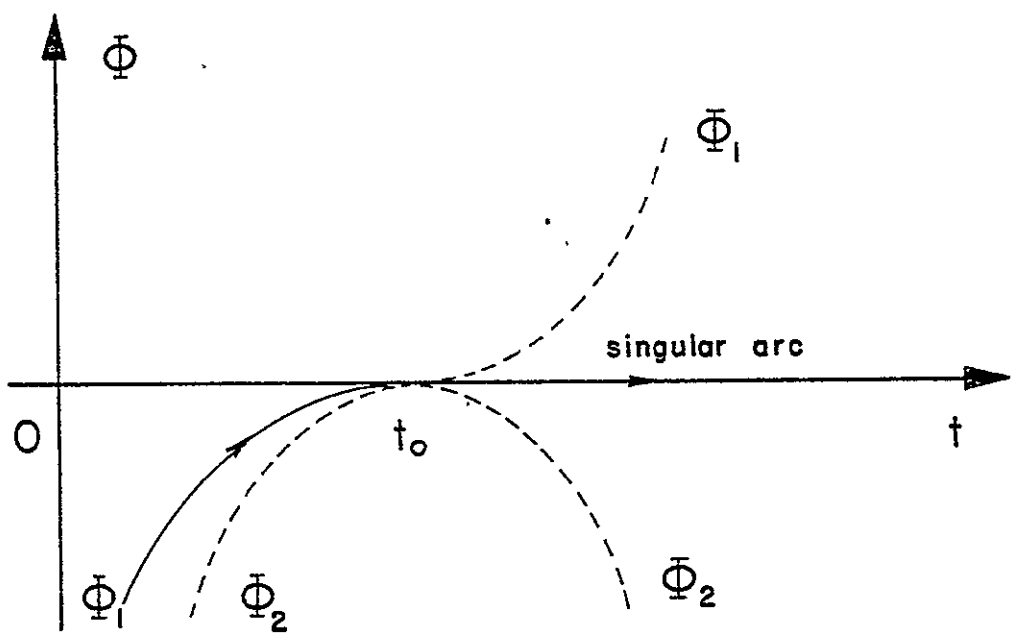

Fig. 8. Switching in the Case (I1, II2).

We notice that the curve Φ_1 is of the type I1 or I3. If $\overline{B}_1 > 0$, it is of the type I1 as shown in Fig. 8 and the switching is $M_1 \rightarrow S$. If $\overline{B}_1 < 0$ the curve Φ_1 is of the type I3 and the switching is $S \rightarrow M_1$.

Similarly, if $B_2 = 0$, junction is made with u_2 and $\Phi_2 > 0$ on the nonsingular side. Since q is even, by the strengthened GLC condition, $B_1 > 0$ and the representative curve for Φ_1 is of the type I4 (Fig. 9). For singular arc to exist, Φ_1 and Φ_2 must have different signs on the singular side. Therefore $\Phi_2 < 0$ on the singular side and the representative curve for Φ_2 has an inflection point at $t = t_o$. In the neighborhood of $t = t_o$, the series expansion of Φ_2 is of the form

$$\Phi_2(t) = \overline{B}_2 \frac{(t-t_o)^n}{n!} + \ldots \tag{43}$$

where

$$\overline{B}_2 = D_2^{\,n} \Phi \Big|_{t=t_o} \neq 0 \; . \tag{44}$$

and $n = 2q + r$ is an odd integer. Hence $q + r$ is an odd integer.

We notice that the curve Φ_2 is of the type II1 or II3. If $\overline{B}_2 < 0$, it is of the type II3 as shown in Fig. 9 and the switching is $M_2 \rightarrow S$. If $\overline{B}_2 > 0$, the curve is of the type II1 and the switching is $S \rightarrow M_2$.

b. Case of q odd. If $B_1 = 0$, junction is made with u_1 and $\Phi_1 < 0$ on the nonsingular side. By similar arguments, as carried above, we can deduce that $B_2 > 0$ and $\Phi_1 < 0$ on the singular side. Hence the curve

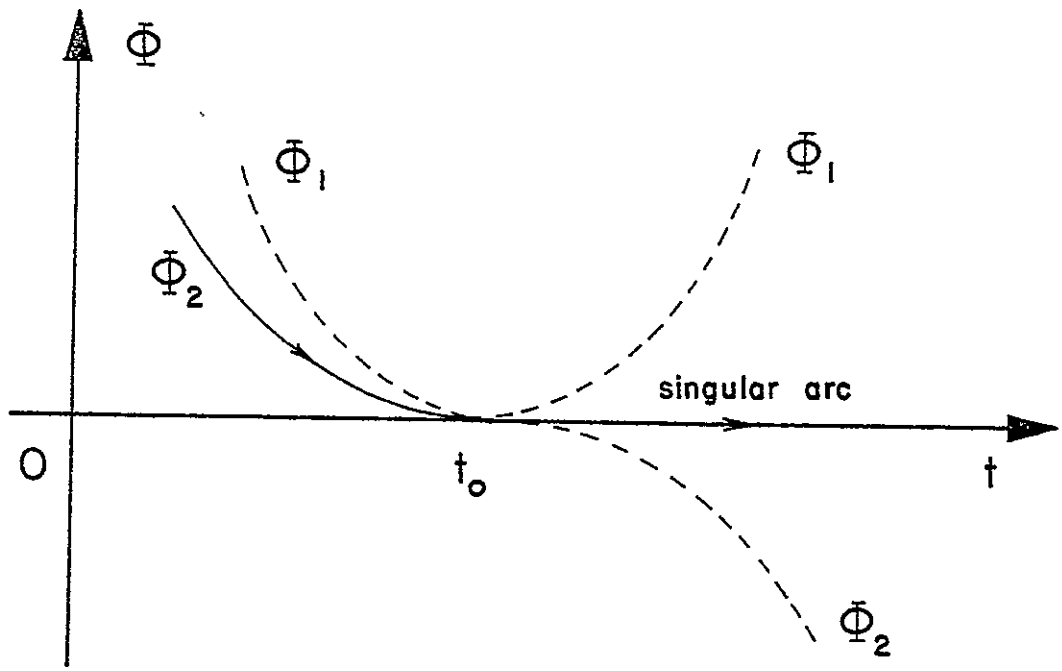

Fig. 9. Switching in the Case(I4, II3) .

Φ_1 is of the type I2 and the curve Φ_2 is of the type II4 as shown previously in Fig. 6. In this case, in the neighborhood of $t = t_o$, the series expansion of Φ_1 is given by Eq. (41) with $\bar{B}_1 < 0$ and $n = 2q + r$ is an even integer. Therefore r is even and again $q + r$ is an odd integer.

Similarly, if $B_2 = 0$, continuous junction is made with u_2 and we have the same case (I2, II4) with $B_1 < 0$, $\bar{B}_2 > 0$. In the neighborhood of $t = t_o$, the series expansion of Φ_2 is given by Eq. (43) with $n = 2q + r$ being an even integer. Hence $q + r$ is an odd integer.

The results of discussion, in terms of the direction of switching, are summarized in Table 3 below.

	I 1	I 2	I 3	I 4
II 1	*	*	*	$S \to M_2$ q even
II 2	$M_1 \to S$ q even	*	$S \to M_1$ q even	*
II 3	*	*	*	$M_2 \to S$ q even
II 4	*	$M_1 \to M_1$ $M_1 \to M_2$ M_2 M_2 $M_1 \to S$ $S \to M_1$ M_2 M_2 $S \to S$	*	*

q odd

Table 3. Optimal Switching, $(-1)^q (B_2 - B_1) < 0$.

Remarks.

As compared with Table 2, we have 4 new cases of junction with singular arc for q even. For the case of q odd we have the same ambiguities as in Table 2. The difference here is that since the first non-vanishing derivative, either for Φ_1 or Φ_2, occurs for an order $n > 2q$, there is a possibility of continuous control at the junction with the singular arc. In other words, if $B_1 = 0$, the switchings $M_1 \to S$ and $S \to M_1$ are continuous, while the other two possible switchings $M_2 \to S$ and $S \to M_2$ are discontinuous. Similarly, if $B_2 = 0$, the switchings $M_2 \to S$ and $S \to M_2$ are continuous, while the two other possible switchings $M_1 \to S$

and $S \to M_1$ are discontinuous. Theorem 1, for q odd only provides the necessary condition for continuous junction. On the other hand, for the case of q even, as shown in Table 3, not only that continuous junction between nonsingular and singular subarcs is established but the direction of the switching is uniquely determined. Hence, we have

Theorem 2.

For a continuous junction between nonsingular subarc and singular subarc of an even order, the condition in Theorem 1 is also sufficient. Furthermore, the conditions for entering/leaving the singular subarc are

$$\text{(i)} \quad B_1 = 0 \ , \quad \overline{B}_1 \gtrless 0 \tag{45}$$

$$\text{(ii)} \quad B_2 = 0 \ , \quad \overline{B}_2 \lessgtr 0 \ . \tag{46}$$

Of course, in the theorem, condition (i) applies to junction with u_1 and condition (ii) applies to junction with u_2 while the upper inequality sign is for entering and the lower inequality sign for leaving the singular subarc.

We now can prove the following important theorem, first discovered by McDanell and Powers for symmetric control [16].

Theorem 3.

Under the condition in Theorem 1 and, in addition, if the control is piecewise analytic in a neighborhood of t_o, then r is the lowest order derivative of u which is discontinuous at t_o.

Proof: The normalized linear control λ^* is related to u by the relation

$$u = (1 - \lambda^*)u_1 + \lambda^* u_2 \tag{47}$$

where

$$u_1 = K_1(\vec{x}, t)$$

$$u_2 = K_2(\vec{x}, t) \quad . \tag{48}$$

The analyticity of u implies the analyticity of K_1 and K_2, which in turn, from Eq. (47) implies the analyticity of λ^*. In fact, for the proof we only require that λ^* possesses successive derivatives up to a certain order r. We define

$$\alpha(t) = D_1^{2q}\,\Phi = F(\vec{p}, \vec{x}, t)$$

$$\beta(t) = D_2^{2q}\,\Phi - D_1^{2q}\,\Phi = G(\vec{p}, \vec{x}, t) \tag{49}$$

and write the equation (29)

$$\alpha(t) = -\lambda^*\,\beta(t) \quad . \tag{50}$$

This equation is constantly satisfied along a singular arc, and since λ^* possesses successive derivatives up to a certain order r, we have

$$\alpha^{(r)} = -\sum_{i=0}^{r} \binom{r}{i} \beta^{(r-i)} \lambda^{*(i)} \tag{51}$$

where $\binom{r}{i}$ are the binomial expansion coefficients. Equation (51) is in the form for the proof in the case of junction with u_1, $\lambda^* = 0$. By writing the condition (39) in Theorem 1, for r = 1, 2, . . . , and noticing

that $D(\ ) = D_1(\ )$ for junction with u_1, and $\beta(t_o) \neq 0$ because of the strengthened GLC condition, we have at $t = t_o$ on the singular side

$$\lambda^{*(i)} = 0 \quad , \ i = 1, 2, \ldots, r-1$$

$$\lambda^{*(r)} \neq 0 \quad . \tag{52}$$

Since on the nonsingular side $\lambda^{*(i)} = 0$ for all $i = 0, 1, \ldots,$ the control is discontinuous at the rth derivative.

For junction with u_2, we have the same proof, using the change of variables

$$\begin{aligned} \delta^* &= 1 - \lambda^* \\ \alpha(t) &= D_2^{2q}\,\Phi \\ \beta(t) &= D_1^{2q}\,\Phi - D_2^{2q}\,\Phi \ . \end{aligned} \tag{53}$$

Of course, when $r = 0$ we have the condition of the lemma, that is, of Table 2, and the control is discontinuous.

Theorem 3 generalizes McDanell and Powers main theorem to nonsymmetric control. Furthermore, when used in conjunction with Theorem 1, not only that it predicts that the two-order q and r are of different types; that is, if q is odd, then r is even, and if q is even, r is odd, but also, in many cases, by using Theorem 1 we can actually compute the integer r without evaluating the singular control itself. This assessment will be illustrated by examples given in Section V.

To complete the anlysis, we shall prove the following theorem:

Theorem 4.

At the junction point between nonsingular and singular subarcs, the jump in the discontinuous r^{th} derivative of the control is given by

$$u_{s_1}^{(r)} - u_1^{(r)} = (u_1 - u_2) \frac{D_1^{2q+r} \Phi}{D_2^{2q} \Phi} \tag{54}$$

for a junction with $u_1 = K_1(\vec{x}, t)$, and

$$u_{s_2}^{(r)} - u_2^{(r)} = (u_2 - u_1) \frac{D_2^{2q+r} \Phi}{D_1^{2q} \Phi} \tag{55}$$

for a junction with $u_2 = K_2(\vec{x}, t)$.

Proof: For a junction with u_1 , evaluating the rth derivative of Eq. (47) at $t = t_o$, using the relations (52) we have

$$u_{s_1}^{(r)} = u_1^{(r)} + \lambda^{*(r)} (u_2 - u_1) \ . \tag{56}$$

On the other hand, from Eq. (51), at $t = t_o$

$$\lambda^{*(r)} = - \frac{D_1^{2q+r} \Phi}{D_2^{2q} \Phi} \ . \tag{57}$$

Using (57) into (56), we have the relation (54). For junction with u_2 we can either use the transformation (53) or simply permute the indices in Eq. (54) to have Eq. (55).

III. QUASI - SINGULAR CONTROL

In most engineering problems, the control is hardly linear. Most often, it enters linearly the equation of motion of a dynamical system through approximating in mathematical modeling. With such an approximation, the optimal control, if it is not of the bang-bang type, is singular and hence at the junction point it is subject to discontinuity as discussed in the previous section. Although the singular solution obtained may be satisfactory, the true solution is nonsingular. It is then of interest to consider the real physical problem that is quasi-linear in the control and analyze the approximation necessary to obtain the intermediate solution which can be termed as quasi-singular.

In this exploratory work, we shall restrict ourselves to the case of one single component for the control.

The hodograph is given by

$$\vec{V} = \dot{\vec{x}} = \vec{f}(\vec{x}, u, t) \tag{58}$$

where

$$u_1 \leq u \leq u_2 \tag{59}$$

and u_1 and u_2 are two arbitrary functions of $\vec{x}$ and t as defined in Eq. (48). If the control u is linear

$$\vec{V} = \vec{f}_1(\vec{x}, t) + u\,\vec{f}_2(\vec{x}, t) \tag{60}$$

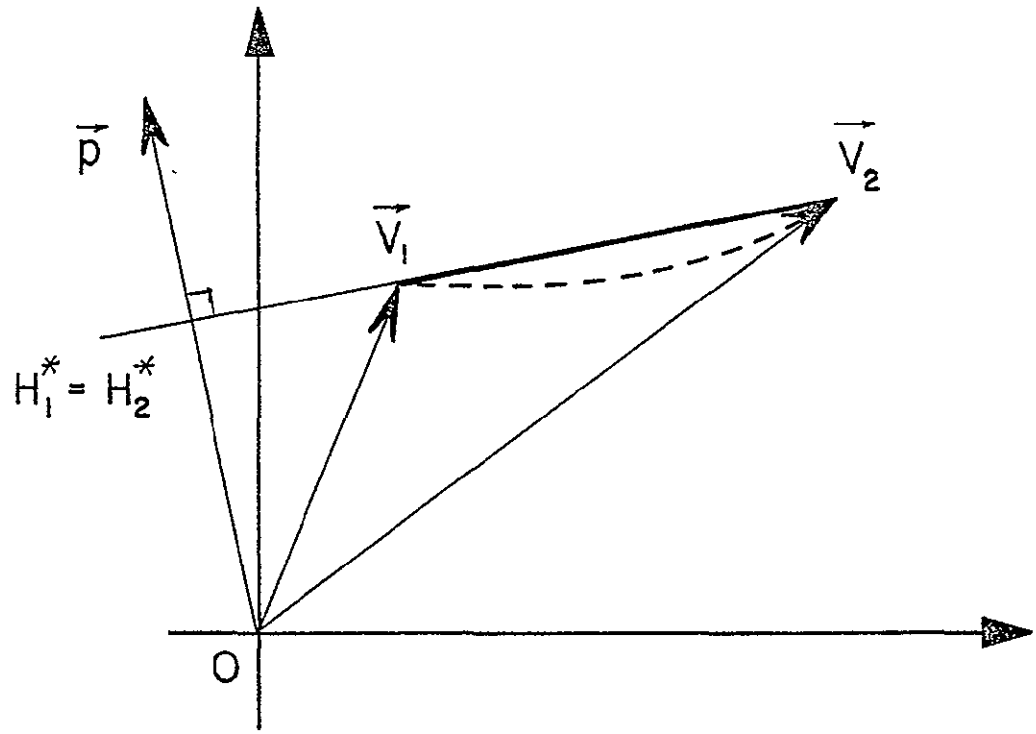

Fig. 10. Chattering Control For Nonlinear Control.

and the domain of maneuvrability is a segment of a straight line (Fig. 10). If u is quasi-linear, the segment is quasi-rectilinear. We have two cases.

If the domain of maneuvrability is concave as shown in a dashed line in Fig. 10, the control is either $u^* = u_1$ or $u^* = u_2$ depending on whether $H_1^* > H_2^*$ or $H_2^* > H_1^*$ where

$$H_1^* = \vec{p} \cdot \vec{f}(\vec{x}, u_1, t)$$

$$H_2^* = \vec{p} \cdot \vec{f}(\vec{x}, u_2, t) \quad . \tag{61}$$

In the case where we have

$$\Phi = H_2^* - H_1^* \tag{62}$$

identically zero for a finite time interval, the resulting arc is a chattering arc. This type of control, also called sliding control, is studied extensively in [24,27]. The condition for chattering control is obviously, for a finite time interval

$$H_1^* = H_2^*$$

$$\left(\frac{\partial H}{\partial u}\right)_{u=u_1} < 0 \quad . \tag{63}$$

It is more frequent that the domain of maneuvrability is convex (Fig. 11). The optimal control is either $u^* = u_1$, or $u^* = u_2$, of the boundary type, or $u_1 < u^* < u_2$. In the latter case, it is said to be of the Euler-Lagrange type. In this case, u^* is obtained by solving

$$\frac{\partial H}{\partial u} = 0 \quad . \tag{64}$$

In general, we have

$$u^* = u^*(\vec{p}, \vec{x}, t) \quad . \tag{65}$$

If the equations for the adjoint vector $\vec{p}$ cannot be integrated analytically, the optimal control cannot be expressed in terms of the state vector $\vec{x}$ and possibly the time t, and usually numerical solution has to be sought. In the case of linear control, because of additional relations from the equation $D^n \Phi = 0$, $n = 0, 1, \ldots, 2q$, we may obtain more information about the control, and even about the trajectory itself. Hence, in problems in which the control is quasi-linear, simplification is made by linearizing the control. In doing so, singular control may be obtained explicitly, but the equations of motion suffer in accuracy since they no longer describe the actual trajectory. Hence, it is better to maintain the exact equations with quasi-linear control and use approximate method to obtain near optimal solution for the control.

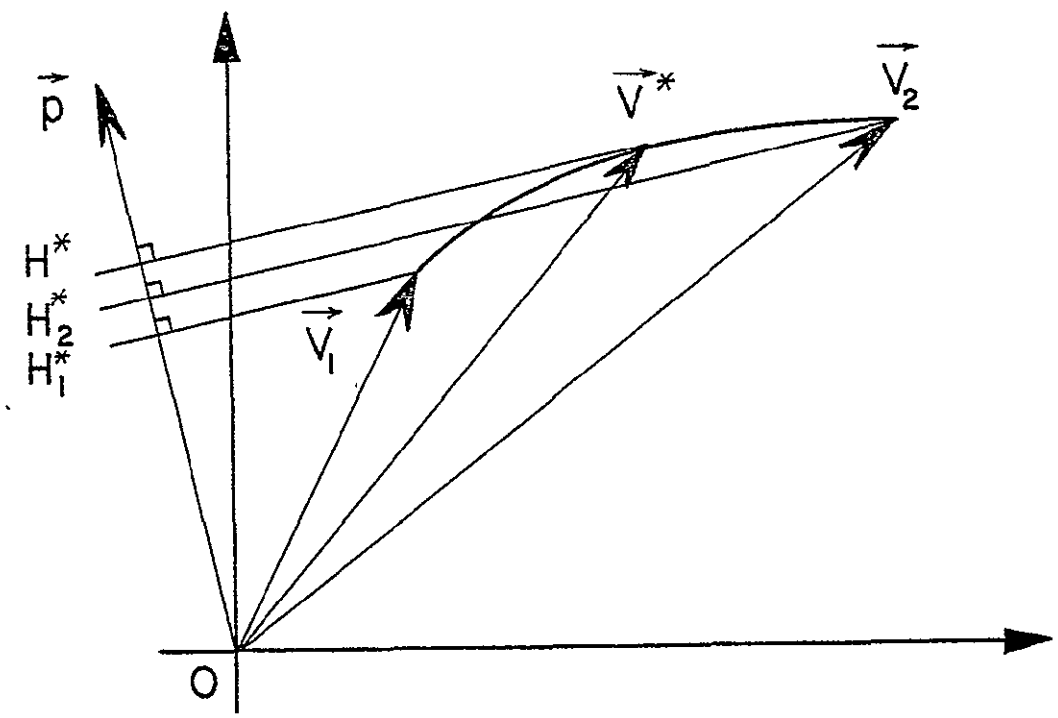

Fig. 11. Euler-Lagrange Type of Optimal Control

Referring to Fig. 11, if the control is quasi-linear, then the domain of maneuvrability is near rectilinear. This means that during the time interval where the control is of the Euler-Lagrange type

$$H^* \approx H_1^* \approx H_2^* \quad . \tag{66}$$

The optimal velocity $\vec{V}^*$ can be approximated by

$$\vec{V}^* = (1-\lambda^*)\ \vec{V}_1 + \lambda^*\ \vec{V}_2 \tag{67}$$

where λ^* is an intermediary value between 0 and 1. This in turn leads to the approximation for the maximized Hamiltonian H^*

$$\overline{H}^* = (1-\lambda^*)\ H_1^* + \lambda^*\ H_2^* \quad . \tag{68}$$

The control is now linear and since λ^* has an intermediary value, it is singular. In other words, the Euler-Lagrange type of optimal control has been approximated by a singular type control. While the physical equations of motion are retained in their exact form for the purpose of evaluating the actual performance, the near optimal control, with the assumption of quasi-linear control, is sought using the approximate maximized Hamiltonian as given in Eq. (68). Using this Hamiltonian, we have for the state and adjoint equations, the canonical system

$$\frac{d\vec{x}}{dt} = \frac{\partial \overline{H}^*}{\partial \vec{p}} \quad , \quad \frac{d\vec{p}}{dt} = -\frac{\partial \overline{H}^*}{\partial \vec{x}} \quad . \tag{69}$$

The normalized linear control λ^* is a handy device in developing

the switching theory. In practical application, we can return to the physical control u through Eq. (47). Then, the approximate Hamiltonian $\overline{H}^*$ is given by

$$\overline{H}^* = \frac{(u_2 H_1^* - u_1 H_2^*)}{(u_2 - u_1)} + \frac{(H_2^* - H_1^*)}{(u_2 - u_1)} u \quad . \tag{70}$$

If singular control to this transformed problem is sought, then we have the condition for singular arc

$$\Phi = \frac{(H_2^* - H_1^*)}{(u_2 - u_1)} = 0 \tag{71}$$

and along the singular arc

$$\overline{H}^* = \frac{(u_2 H_1^* - u_1 H_2^*)}{(u_2 - u_1)} \quad . \tag{72}$$

Once the problem has been solved, and in the case where the suboptimal intermediary control u^* has been obtained either explicitly as function of the time, or in terms of the state variable $\vec{x}$, and the time, actual performance can be evaluated using the original state equations which, as previously stated, are quasi-linear in the control. The error committed, using this approach, is of the order of $\epsilon = (H_2^* - H_1^*)/(u_2 - u_1)$. Its analysis requires further investigation.

IV. LINEARIZED SINGULAR CONTROL

In the case where the control is strongly nonlinear, it may be impossible to obtain analytical solution for the Euler-Lagrange type of optimal control (Fig. 12). Yet, in the numerical search for the optimal trajectory, it is helpful to know an approximate optimal control for this type of subarc. In some favorable case, using some physical properties of the trajectory, say a certain equilibrium condition, or steady state condition which occurs when a certain number of state variables vary slowly, one may readily obtain an approximate control $u_o(\vec{x}, t)$ called the reference solution. The objective is to improve this solution to obtain a better control.

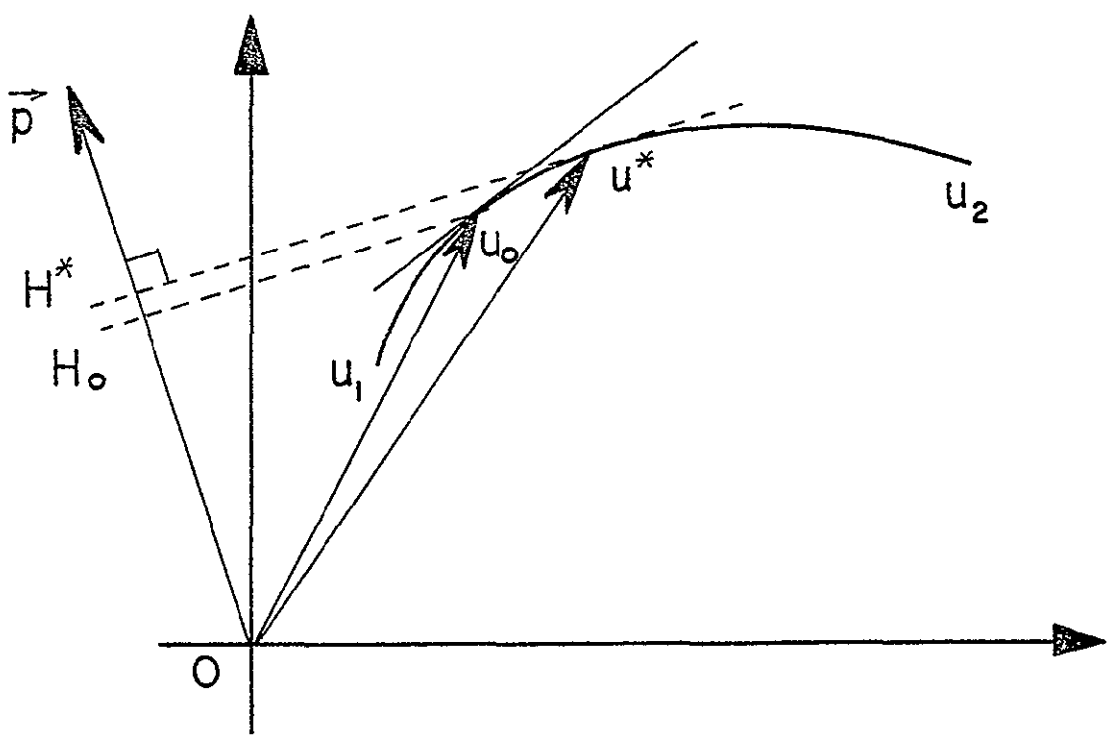

Fig. 12. Linearization of the Domain of Maneuvrability.

If the reference solution is near optimal, then

$$H^* \approx H_o(\vec{p}, \vec{x}, u_o, t) \quad . \tag{73}$$

Therefore, as first-order approximation we can use the linearized Hamiltonian

$$\overline{H}^* = H_o + \left(\frac{\partial H}{\partial u}\right)_o (u - u_o) \quad . \tag{74}$$

Geometrically, this is the same as replacing the domain of maneuvrability, near the point $u = u_o$, by the tangent at that point. The transformed problem is linear in the control u and since the optimal control is not of the boundary type, it is singular. We have the condition for the singular control

$$\left(\frac{\partial H}{\partial u}\right)_o = 0 \quad . \tag{75}$$

By solving the transformed problem, the pertinent state and adjoint equations are

$$\frac{d\vec{x}}{dt} = \frac{\partial \overline{H}^*}{\partial \vec{p}} \quad , \quad \frac{d\vec{p}}{dt} = - \frac{\partial \overline{H}^*}{\partial \vec{x}} \quad . \tag{76}$$

Once the problem has been solved, and in the case where the suboptimal intermediary control u^* has been obtained either explicitly as function of the time, or in terms of the state variable $\vec{x}$, and the time t, actual performance can be evaluated using the original state equations. The error committed, using this approach is of the order of $\epsilon = (\partial H/\partial u)_o$. Its analysis requires further investigation.

V. APPLICATIONS

In this section we shall give some applications of the theory developed in the previous sections. The first two examples are applications of the switching theory developed in Section II. The last two examples are illustrations of the theory of quasi-linear control given in Section III, and linearized control given in Section IV.

V. 1. Smooth Junction With q Odd.

The dynamical system is governed by

$$\begin{aligned} \dot{x}_i &= x_{i+1}\ , \quad i = 1, \ldots, q\ ; \quad q \text{ odd}\ ; \quad q \geq 1 \\ \dot{x}_{q+1} &= u \\ \dot{x}_{q+2} &= \frac{1}{2}(x_2^{\ 2} - x_1^{\ 2}) \end{aligned} \tag{77}$$

with the linear control u subject to the constraint

$$-1 \leq u \leq 1 \quad . \tag{78}$$

It is assumed that the vector $\vec{x} = (x_1, \ldots, x_{q+1})$ belongs to a certain initial manifold G_i at the initial time t_i and a certain final manifold G_f at the final time t_f while

$$x_{q+2}(t_i) = 0, \quad x_{q+2}(t_f) = \text{minimum} \quad . \tag{79}$$

This example is given in [16] with $q = 1$.

The Hamiltonian of the system is

$$H = \sum_{i=1}^{q} p_i x_{i+1} + p_{q+1} u + \frac{1}{2} p_{q+2} (x_2^2 - x_1^2) \quad . \tag{80}$$

We notice that p_{q+2} = constant = -1. The other adjoint components are governed by

$$\begin{aligned} \dot{p}_1 &= -x_1 \\ \dot{p}_2 &= x_2 - p_1 \\ \dot{p}_i &= -p_{i-1} \ , \quad i = 3, \ldots, q+1 \quad . \end{aligned} \tag{81}$$

We take

$$\begin{aligned} H_1^* &= (H)_{u^*} = -1 \\ H_2^* &= (H)_{u^*} = 1 \quad . \end{aligned} \tag{82}$$

Hence, the switching function is

$$\Phi = H_2^* - H_1^* = 2 p_{q+1} \quad . \tag{83}$$

In evaluating the derivatives $D_1()$ and $D_2()$, the equations for the adjoints are the same, while in the equations for the states, we simply have for the components x_{q+1}

$$\frac{d}{dt_1}(x_{q+1}) = -1 \ , \quad \frac{d}{dt_2}(x_{q+1}) = 1 \ . \tag{84}$$

Taking the derivative of Φ q times we have

$$
\begin{aligned}
D_1 \Phi &= -2p_q = D_2 \Phi \\
D_1^2 \Phi &= 2\, p_{q-1} = D_2^2 \Phi \\
&\vdots \\
D_1^q \Phi &= (-1)^{q-1}\, 2(x_2 - p_1) = D_2^q \Phi
\end{aligned}
\tag{85}
$$

where it should be noted that signs alternate. We notice that q is odd, hence $(-1)^{q-1} = 1$. Taking the derivative of the last equation in (85) q more times and applying (77) to eliminate time derivatives,

$$
\begin{aligned}
D_1^{q+1} \Phi &= 2(x_3 + x_1) = D_2^{q+1} \Phi \\
D_1^{q+2} \Phi &= 2(x_4 + x_2) = D_2^{q+2} \Phi \\
&\vdots \\
D_1^{2q-1} \Phi &= 2(x_{q+1} + x_{q-1}) = D_2^{2q-1} \Phi \\
D_1^{2q} \Phi &= 2(x_q - 1)\,, \quad D_2^{2q} \Phi = 2(x_q + 1)\ .
\end{aligned}
\tag{86}
$$

Hence, the order when q is odd is the order of the singular arc. Also, we notice that

$$
A = D_2^{2q} \Phi - D_1^{2q} \Phi = 4 \quad . \tag{87}
$$

Hence, the strengthened GLC condition $(-1)^q\ A < 0$ is satisfied, not only at the junction point but everywhere along the singular arc.

On the singular arc

$$
\lambda^* = - \frac{D_1^{2q} \Phi}{D_2^{2q} \Phi - D_1^{2q} \Phi} = \frac{1}{2}\,(1 - x_q) \quad . \tag{88}
$$

Since

$$u = (1 - \lambda^*) u_1 + \lambda^* u_2 = 2\lambda^* - 1 \tag{89}$$

we have

$$u = -x_q \quad . \tag{90}$$

Therefore

$$x_q \dot{x}_q + x_{q+1} \dot{x}_{q+1} = 0 \quad . \tag{91}$$

In the (x_q, x_{q+1}) plane, singular arcs are the circles

$$x_q^2 + x_{q+1}^2 = R^2 \quad . \tag{92}$$

If we are interested in continuous junction, then either $B_1 = 0$, or $B_2 = 0$. Therefore we have the possibilities

$$x_q = 1 , \quad \text{junction with } u_1 = -1$$

or

$$x_q = -1 , \quad \text{junction with } u_2 = +1 \quad . \tag{93}$$

Furthermore, since q is odd, r is even. Hence, for junction with u_1

$$D_1^{2q+1} \Phi = 2 x_{q+1} = 0$$

$$D_1^{2q+2} \Phi = -2 < 0 \quad . \tag{94}$$

Without knowing the singular control, from our Theorems 1 and 3 we have found that its derivative is discontinuous at r = 2.

For junction with u_2 , we have similarly

$$D_2^{2q+1}\,\Phi = 2\,x_{q+1} = 0$$

$$D_2^{2q+2}\,\Phi = 2 > 0 \tag{95}$$

For the case of q odd, the theorems only give the necessary conditions. This means that, in the (x_q, x_{q+1}) plane the only points with possible continuous junction for the control, with discontinuity for the second derivative, are the points

$$x_q = 1\,,\ x_{q+1} = 0\,,\ \text{junction with } u_1 = -1$$

$$x_q = -1,\ x_{q+1} = 0\,,\ \text{junction with } u_2 = +1\ . \tag{96}$$

Furthermore, since all the derivatives in Eq. (86) have to be zero, up to 2q-1, we have

$$x_1 = -x_3 = x_5 = \ldots\ldots = (-1)^{\frac{1}{2}(q-1)}\,x_q$$

$$x_2 = -x_4 = x_6 = \ldots\ldots = (-1)^{\frac{1}{2}(q-1)}\,x_{q+1}\ . \tag{97}$$

These equations are valid along the singular arc. Hence, the projection of the singular arc into the plane with odd ordered, or even ordered coordinates are straight lines. For a pair of coordinates with different orders, we notice that

$$\dot{x}_i = x_{i+1}$$

$$\dot{x}_{i+1} = x_{i+2} = -x_i\ .$$

Hence,

$$x_i \dot{x}_i + x_{i+1} \dot{x}_{i+1} = 0 \quad .$$

The projections of the singular arc are concentric circles.

$$x_i^2 + x_{i+1}^2 = R_i^2$$

$$i = 1, \ldots, q \quad . \tag{98}$$

Returning to the points with possible continuous junction for the control, since (96) have to be satisfied at the junction point, in addition to (97) these relations uniquely determine 2 points for the position vector $\vec{x} = (x_1, x_2, \ldots, x_{q+1})$ where continuous junction is possible. The component x_{q+2} represents the cost and is additive. It can be taken arbitrarily at the junction point if one proposes to construct artificially optimal trajectory with continuous junction from that point.

For example, let us take the point $x_q = 1$. Then

$$x_1 = -x_3 = x_5 = \ldots\ldots = (-1)^{\frac{1}{2}(q-1)}$$

$$x_2 = x_4 = \ldots\ldots\ldots\ldots\ldots = x_{q+1} = 0.$$

This junction point is uniquely defined. For the case of q odd the conditions are only necessary. We must impose the condition that junction between nonsingular and singular subarcs exists, and the junction is made with $u_1 = -1$. As seen in Table 3, the singular arc can be before or after the junction point. If it is before we can integrate backward with $u = u_s$ and

forward with $u = u_1 = -1$ to a certain initial manifold G_i and a certain final manifold G_f. Hence this type of continuous junction is indeed rare and depends on the boundary condition. The equations are simple enough so that the integration can be performed easily and the solution obtained in closed form. We have the same discussion for junction with $u_2 = 1$ at the point $x_q = -1$.

Before obtaining explicitly the singular control, we would like to use our Theorem 4 of Section II to predict that the jump in the discontinuous second derivative of the control is

$$\ddot{u}_{s_1} - \ddot{u}_1 = (-2)\ \frac{(-2)}{4} = 1$$

at the junction with $u_1 = -1$, and

$$\ddot{u}_s - \ddot{u}_2 = (2)\ \frac{2}{(-4)} = -1$$

at the junction with $u_2 = 1$.

Now, for the singular control, since

$$\dot{x}_q = x_{q+1}\ , \quad \dot{x}_{q+1} = u \quad , \quad u = -x_q \tag{99}$$

the equation for x_q is

$$\ddot{x}_q + x_q = 0 \quad . \tag{100}$$

Hence, integrating

$$-x_q = u = C_1 \sin t + C_2 \cos t \tag{101}$$

where C_1 and C_2 are constants of integration. By translation of the time to $t = 0$ at the junction point, we see that the singular controls starting at

u_1 and u_2 are respectively

$$u_{s_1} = -\cos t$$

$$u_{s_2} = \cos t \quad . \tag{102}$$

We easily verify the condition of continuity and the jump in the second derivative as predicted by the theory.

V. 2. Smooth Junction With q Even.

The following example of smooth junction with q even was given by Maurer [17] for a special case of the boundary conditions. In light of the new information supplied by our theorems the present treatment of the problem is general in the sense that we can predict the location of the junction points for continuous control and also the boundary manifolds that can lead to such junction.

The dynamical system is governed by

$$\dot{x}_i = x_{i+1} \ , \quad i = 1, \ldots, q; \ q \text{ even}; \quad q \geq 2$$

$$\dot{x}_{q+1} = u$$

$$\dot{x}_{q+2} = \frac{1}{2}(x_1^2 + x_2^2) \tag{103}$$

with the linear control u subject to the constraint

$$-1 \leq u \leq 1 \quad . \tag{104}$$

It is assumed that the vector $\vec{x} = (x_1, \ldots, x_{q+1})$ belongs to a certain initial manifold G_i at the initial time t_i and a certain final manifold G_f at the final time t_f while

$$x_{q+2}(t_i) = 0 \quad , \quad x_{q+2}(t_f) = \text{minimum} \quad . \tag{105}$$

The treatment is identical to the case of q odd. The difference here is that for q even, the conditions stated in Theorems 1 and 3 of Section II are also sufficient for a continuous junction for the control. As before, the switching function is

$$\Phi = H_2^* - H_1^* = 2p_{q+1} \quad . \tag{106}$$

The q th derivative of this function is found to be

$$D_1^{\,q}\,\Phi = (-1)^{q-1}\,2(x_2 - p_1) = D_2^{\,q}\,\Phi \tag{107}$$

with $(-1)^{q-1} = -1$ since q is even. Taking the derivative of this equation q more times and noticing that we now have $\dot{p}_1 = x_1$, we have

$$\begin{aligned}
D_1^{\,q+1}\,\Phi &= 2(x_1 - x_3) = D_2^{\,q+1}\,\Phi \\
D_1^{\,q+2}\,\Phi &= 2\,(x_2 - x_4) = D_2^{\,q+2}\,\Phi \\
&\vdots \\
D_1^{\,2q-1}\,\Phi &= 2\,(x_{q-1} - x_{q+1}) = D_2^{\,2q-1}\,\Phi \\
D_1^{\,2q}\,\Phi &= 2(x_q + 1) \;,\; D_2^{\,2q}\,\Phi = 2(x_q - 1)
\end{aligned} \tag{108}$$

Hence, q even is the order of the singular arc. Also,

$$A = D_2^{\,2q}\,\Phi - D_1^{\,2q}\,\Phi = -4 \tag{109}$$

so that for q even, the strengthened GLC condition $(-1)^q A < 0$ is satisfied, not only at the junction point but also everywhere along the singular arc. For the singular control, we have the relation

$$u = x_q \quad . \tag{110}$$

The continuous junction can only be made at $B_1 = 0$ or $B_2 = 0$. Hence at the point of continuous junction

$$x_q = -1 \ , \quad \text{junction with } u_1 = -1$$

$$x_q = 1 \ , \quad \text{junction with } u_2 = 1 \ . \tag{111}$$

Since q is even, by Theorem 1, r is odd and therefore, at the continuous junction with u_1

$$D_1^{2q+1} \Phi = 2 x_{q+1}$$

$$D_1^{2q+2} \Phi = -2 < 0 \ . \tag{112}$$

To satisfy the theorem, for the first non-vanishing derivative, r is odd. Therefore $x_{q+1} \neq 0$ and $r = 1$. The control at the junction point is continuous but its derivative is discontinuous. We have the same conclusion for the case of junction with $u_2 = 1$ at $x_q = 1$.

Along the singular arc, since all the derivatives of Φ up to 2q-1 vanish identically, we have

$$x_1 = x_3 = \ldots = x_{q+1}$$

$$x_2 = x_4 = \ldots = x_q \quad . \tag{113}$$

The projections of the singular arc, in the plane of both odd, or both even ordered coordinates are first bisectors of the axes.

On the other hand, for a pair of coordinates of different orders

$$\dot{x}_i = x_{i+1}$$

$$\dot{x}_{i+1} = x_{i+2} = x_i \quad . \tag{114}$$

Therefore

$$\dot{x}_i \dot{x}_i = x_{i+1} \dot{x}_{i+1} \quad . \tag{115}$$

Upon integrating

$$x_i^2 - x_{i+1}^2 = \pm R_i^2 \quad . \tag{116}$$

The projections of the singular arc into these planes are equilateral hyperbolas, or in the degenerate case, $R_i = 0$, are bisectors in the (x_i, x_{i+1}) planes. For compatibility with Eq. (113), the R_i^2 are the same and we have the equations of the singular arc

$$x_i^2 - x_q^2 = \pm R^2 \quad , \quad i = 1, 3, \ldots, q+1$$

$$x_2 = x_4 = \ldots = x_q \quad . \tag{117}$$

Maurer considered the case $R^2 = 0$, but in general, the condition imposed

is only that the junction point must be at

$$x_2 = x_4 = \dots = x_q = \pm 1$$

$$x_1 = x_3 = \dots = x_{q+1} = k \tag{118}$$

where k is an arbitrary constant. From these points we can construct the optimal trajectory with continuous junction by integrating backward and forward. Again we see that boundary conditions are very special and continuous junction is indeed rare. Before evaluating the singular control, we would like to show that, for this case of q even, we also have sufficient condition.

For junction point with $u_1 = -1$, $x_q = -1$, $x_{q+1} = k$. Then at the junction point, froms Eqs. (109) and (112) with $D_1^{2q}\Phi = 0$

$$D_1^{2q+1}\Phi = 2k \quad , \qquad D_2^{2q}\Phi = -4 \quad . \tag{119}$$

Then the switching is either the type (I 1, II 2) or (I 3, II 2) depending on whether the value k selected is $k > 0$ or $k < 0$. As seen in Table 3, the only switching is between nonsingular arc, with u_1, and singular arc. The condition is sufficient, and also the direction of switching is determined. We have $u_1 \to u_s$ for $k > 0$ and $u_s \to u_1$ for $k < 0$.

Similarly, for junction point with $u_2 = 1$, $x_q = 1$, $x_{q+1} = k$, we have

$$D_2^{2q+1}\Phi = 2k \quad , \qquad D_1^{2q}\Phi = 4 \quad . \tag{120}$$

Then the switching is either the type (I4, II1) or (I4, II3), depending on whether the value of k selected is $k > 0$, or $k < 0$. As seen in Table 3, the only switching is between nonsingular arc, with u_2, and singular arc. The direction of switching is given by Theorem 2. We have $u_s \rightarrow u_2$ if $k > 0$ and $u_2 \rightarrow u_s$ if $k < 0$. The construction of the optimal trajectory by integrating forward and backward must be based on this theorem.

Finally, from Theorem 3, we have the jump in the discontinuous derivative of the control

$$\dot{u}_{s_1} - \dot{u}_1 = (-2) \frac{2k}{(-4)} = k$$

at the junction with $u_1 = -1$, and

$$\dot{u}_{s_2} - \dot{u}_2 = (2) \frac{2k}{(4)} = k$$

at the junction with $u_2 = 1$.

We can easily verify all the results by obtaining the equation for the singular control from

$$\dot{x}_q = x_{q+1} \quad , \quad \dot{x}_{q+1} = u \; , \quad u = x_q \quad . \tag{121}$$

Hence

$$\ddot{u} = u \tag{122}$$

and the solution is

$$u = C_1 e^{t} + C_2 e^{-t} \; . \tag{123}$$

For junction with u_1 , the control is

$$u = \frac{(k-1)}{2} e^{t} - \frac{(k+1)}{2} e^{-t} \tag{124}$$

by taking $t = 0$ at the time of switching.

For junction with u_2, the control is

$$u = \frac{(1+k)}{2} e^{t} + \frac{(1-k)}{2} e^{-t} \quad . \tag{125}$$

The derivative of the singular control has the jump as predicted. The example given by Maurer [17] is the case of junction with u_2 and $k = 1$, and also for special case of the initial and final manifold G_i and G_f.

As a concluding remark, it is interesting to notice the following. Assume that we are on a singular arc with say, x_q increasing. We also recall that

$$\begin{aligned} D_1^{2q} \Phi &= 2(x_q+1) \\ D_2^{2q} \Phi &= 2(x_q - 1) \quad . \end{aligned} \tag{126}$$

Suppose that we plan to leave the singular arc at a point $-1 < x_q < 1$ to enter a nonsingular arc leading to the final manifold. Then $B_1 > 0$, $B_2 < 0$ and the switching is ot the type (I4, II2). As seen in Table 2, the junction is nonanalytic, and Maurer's conjecture that entering the nonsingular arc by chattering is correct. When $x_q = 1$, $D_2^{2q} \Phi = 0$, so that junction is made with $u_2 = 1$. If the singular arc is not left at this point,

then for $x_q > 1$ both $D_1^{2q}\Phi$ and $D_2^{2q}\Phi$ are positive and we do not have the conditions for a singular arc since they must be of different signs. Therefore, the singular arc terminates at $x_q = 1$.

V. 3. Example of Quasi-Singular Control.

The present study is not merely an academic exercise to display a certain peculiarity in optimal control theory. It has been motivated by an urgent need of mathematical tool in solving a number of engineering problems of interest. We shall give two examples in flight mechanics, namely the problems of finding the maximum range in thrusting flight and in coasting flight of an aerospace vehicle. The flight is to take place in the dense layer of the atmosphere. Hence a model of non-rotating Earth with constant gravitational acceleration is adequate.

For flight in a vertical plane, the equations of motion are

$$\begin{aligned}
\frac{dX}{dt} &= V \cos \gamma \\
\frac{dZ}{dt} &= V \sin \gamma \\
m\frac{dV}{dt} &= T \cos \alpha - \frac{1}{2} \rho S C_D V^2 - mg \sin \gamma \\
m V \frac{d\gamma}{dt} &= T \sin \alpha + \frac{1}{2} \rho S C_L V^2 - mg \cos \gamma \\
\frac{dm}{dt} &= - \frac{c}{g} T .
\end{aligned} \tag{127}$$

Standard notation has been used. In particular, α is the angle of attack,

measured from the thrust line, and c is the specific fuel consumption, assumed constant. The drag polar is assumed parabolic and is of the form

$$C_D = C_{D_o} + K C_L^2 \tag{128}$$

where C_{D_o} and K are constant. For most vehicles, parabolic drag polar is a good approximation in the range of angle of attack of interest. The controls are the angle of attack α, or equivalently the lift coefficient C_L, and the thrust magnitude T, subject to the constraint

$$0 \leq T \leq T_{max} . \tag{129}$$

In the first problem, we shall consider the problem of maximizing the range in the case of constant altitude, thrusting flight. Hence the flight path angle $\gamma = 0$, and we have the equations, with the assumption of small angle of attack, $T \cos \alpha \approx T$, $T \sin \alpha \approx T\alpha$

$$\begin{aligned} \frac{dX}{dt} &= V \\ m \frac{dV}{dt} &= T - \frac{1}{2} \rho S C_D V^2 \\ \frac{dm}{dt} &= - \frac{c}{g} T . \end{aligned} \tag{130}$$

The equation for γ becomes a constraining relation

$$T \alpha + \frac{1}{2} \rho S C_L V^2 = mg . \tag{131}$$

Because of this relation, the angle of attack can be expressed in terms of

T, and the thrust is the unique control of the problem.

To simplify the problem, several authors [28, 29] have ignored the small term $T\alpha$ in Eq. (131). Then the equations (130) are linear in T, and the variable thrust arc is a singular arc. As has been said in the introductory section, that the physical equations are nonlinear in T. So it is more rational to use the exact equations and find the suboptimal solution rather than using the approximate equations to obtain optimal solution. It should be emphasized that neglecting the term $T\alpha$, or even using the approximation $T\cos\alpha \approx T$, can introduce serious errors in the analysis of such vehicles as the delta wing type flying at high angle of attack in the low speed regime.

To compare the two approaches, we first solve the linear problem by neglecting $T\alpha$ in Eq. (131). Then

$$C_L = \frac{2mg}{\rho S V^2} \quad . \tag{132}$$

Using in Eq. (128) for C_D and then in Eq. (130) for the equation in V we have the state equations

$$\begin{aligned} \frac{dX}{dt} &= V \\ \frac{dV}{dt} &= \frac{1}{m}\left[T - \frac{1}{2}\rho S C_{D_o} V^2 - \frac{2Km^2g^2}{\rho S V^2} \right] \\ \frac{dm}{dt} &= -\frac{c}{g} T \quad . \end{aligned} \tag{133}$$

The Hamiltonian of the system is

$$H = P_x V - \frac{p_V}{m} \left[\frac{1}{2} \rho S C_{D_o} V^2 + \frac{2 K m^2 g^2}{\rho S V^2} \right]$$

$$+ \frac{T}{m} \left(p_V - \frac{c}{g} m p_m \right) . \tag{134}$$

We shall take

$$H_1^* = (H)_{T=0}$$

$$H_2^* = (H)_{T=T_{max}} . \tag{135}$$

Hence, the switching function is

$$\Phi = H_2^* - H_1^* = \frac{T_{max}}{m} \left(p_V - \frac{c}{g} m p_m \right) . \tag{136}$$

According to the general theory, if

$p_V - \frac{c}{g} m p_m > 0$, we use H_2^* , $T = T_{max}$

$p_V - \frac{c}{g} m p_m < 0$, we use H_1^* , $T = 0$

$p_V - \frac{c}{g} m p_m = 0$ for a finite time interval, we use T = variable.

To obtain the direction of switching, we evaluate the derivative $D_1 \Phi$.

Noticing that in general

$$\frac{dp_x}{dt} = 0$$

$$\frac{dp_V}{dt} = -p_x + \frac{p_V}{m}\left[\rho S C_{D_o} V - \frac{4Km^2g^2}{\rho S V^3}\right]$$

$$\frac{dp_m}{dt} = -\frac{p_V}{m^2}\left[\frac{1}{2}\rho S C_{D_o} V^2 - \frac{2Km^2g^2}{\rho S V^2}\right] + \frac{p_V}{m^2}T \tag{137}$$

we have the derivative $D_1(\ )$ or $D_2(\ )$ by simply using $T = 0$, or $T = T_{max}$ in the state equations (133) and adjoint equations (137).

We first notice that

$$p_x = C_1 \tag{138}$$

where C_1 is a constant of integration. Furthermore, we have the Hamiltonian integral

$$H = 0 \ . \tag{139}$$

The derivative $D_1\Phi$ is easily found with the help of Eq. (139). Also we notice here the usefulness of the relation $D\Phi = D_1\Phi$, Eq. (16), given in Section II because the D_1 derivative is simpler due to the fact that $T = 0$ in all the equations. We have

$$D_1\Phi = \frac{p_V T_{max}}{m^2 V}\left[\frac{1}{2}\rho S C_{D_o} V^2 \left(1 + \frac{cV}{g}\right) - \frac{2Km^2g^2}{\rho S V^2}\left(3 + \frac{cV}{g}\right)\right] \tag{140}$$

Since we maximize the range, $C_1 > 0$, it is seen from the Hamiltonian integral that $p_V > 0$. Then, when a coasting arc, $T = 0$, is joined with a maximum thrust arc, $T = T_{max}$, according to our theory developed in Section II, the direction of switching is from $T = 0$ to $T = T_{max}$ if $D_1 \Phi > 0$, that is if

$$mg < \frac{1}{2} \rho S V^2 \sqrt{\frac{C_{D_o}}{K}} \sqrt{\frac{1 + \frac{V}{g/c}}{3 + \frac{V}{g/c}}} \tag{141}$$

If the inequality reverses, the switching is from $T = T_{max}$ to $T = 0$. Defining the dimensionless quantities

$$u = \frac{V}{g/c} \quad , \qquad w = \frac{2mc^2}{g \rho S} \sqrt{\frac{K}{C_{D_o}}} \tag{142}$$

we can plot the curve

$$w = u^2 \sqrt{\frac{1+u}{3+u}} \tag{143}$$

in the mass-velocity space (w, u) (Fig. 13). Below this curve, the switching is from $T = 0$ to $T = T_{max}$, while above the curve, the switching is from $T = T_{max}$ to $T = 0$. On the curve, we are entering or leaving a singular arc. Hence, Eq. (141) with equality sign is the equation for the singular arc. By taking the derivative of this equation we have a relation for evaluating the singular thrust control. This singular curve has been found by Hibbs [28] for the case of constant C_{D_o} and K and by

Miele [29] in a generalized version, when these characteristics depend on the Mach number. The novelty here is our rigorous treatment of the direction of the switching. Furthermore, we now consider the case where the control is quasi-linear, that is we shall retain the term $T\alpha$ in the constraining relation (131).

The lift coefficient C_L, as function of the angle of attack α is given by

$$C_L = C_{L_o} + C_{L_\alpha} \alpha \tag{144}$$

where C_{L_o} and C_{L_α} are two characteristic constant coefficients. Upon substituting into Eq. (131) and solving for C_L, we have

$$C_L = \frac{mg + \dfrac{C_{L_o}}{C_{L_\alpha}} T}{\dfrac{1}{2}\rho S V^2 + \dfrac{T}{C_{L_\alpha}}} . \tag{145}$$

The equations of motion now become

$$\begin{aligned} \frac{dX}{dt} &= V \\ \frac{dV}{dt} &= \frac{1}{m}\left[T - \frac{1}{2}\rho S C_{D_o} V^2 - \frac{2Km^2g^2}{\rho S V^2} \frac{\left(1+\dfrac{C_{L_o} T}{mgC_{L_\alpha}}\right)^2}{\left(1+\dfrac{2T}{\rho S C_{L_\alpha} V^2}\right)^2}\right] \\ \frac{dm}{dt} &= -\frac{c}{g} T . \end{aligned} \tag{146}$$

As compared to the Eqs. (133), the thrust control is nonlinear, but since as has been assumed that the perturbing quantity $T\alpha$ is small as compared to the weight mg, the thrust is characterized as quasi-linear. It is possible to linearize the thrust T in Eq. (146), but then we shall deal with approximate equations of motion, a situation we sought to avoid. We seek to obtain approximate optimal variable thrust control to Eq. (146) by constructing the approximate Hamiltonian as given by Eq. (72)

$$\overline{H}^* = p_x V - \frac{p_V}{m}\left[\frac{1}{2}\rho\, S\, C_{D_o} V^2 + \frac{2Km^2g^2}{\rho\, S V^2}\right] + \Phi\, T \tag{147}$$

with the switching function being

$$\Phi = \frac{1}{m}\left(p_V - \frac{c}{g}\, m\, p_m\right) + \frac{2Kmg^2 p_V}{T_{max}\,\rho\, SV^2}\left[1 - \frac{\left(1+\dfrac{C_{L_o} T_{max}}{m g C_{L_\alpha}}\right)^2}{\left(1+\dfrac{2T_{max}}{\rho S C_{L_\alpha} V^2}\right)^2}\right]. \tag{148}$$

The objective here is to show the correction to the singular curve (143), so that to simplify the calculation, we linearize the square bracket in Eq.(148) to obtain

$$\Phi = \frac{1}{m}\left[\left(1 + \frac{2\epsilon\, m g}{\rho\, SV^2}\cdot\sqrt{\frac{K}{C_{D_o}}}\right) p_V - \frac{c}{g}\, m\, p_m\right] \tag{149}$$

where

$$\epsilon = \frac{2\sqrt{K C_{D_o}}}{C_{L_\alpha}} \left(\frac{2mg}{\rho S V^2} - C_{L_o} \right) > 0 \qquad (150)$$

ϵ is a small quantity and we shall take it as constant being equal to its average value.

The problem is solved with the approximate Hamiltonian $\overline{H}*$, with Φ as given by Eq. (149). The state and adjoint equations in this variational problem are generated by the approximate Hamiltonian $\overline{H}*$. In particular for the derivative $D_1()$, we have

$$\frac{dV}{dt_1} = -\frac{1}{m}\left[\frac{1}{2}\rho S C_{D_o} V^2 + \frac{2Km^2g^2}{\rho S V^2}\right]$$

$$\frac{dm}{dt_1} = 0$$

$$\frac{dp_V}{dt_1} = -p_x + \frac{p_V}{m}\left[\rho S C_{D_o} V - \frac{4Km^2g^2}{\rho S V^3}\right]$$

$$\frac{dp_m}{dt_1} = -\frac{p_V}{m^2}\left[\frac{1}{2}\rho S C_{D_o} V^2 - \frac{2Km^2g^2}{\rho S V^2}\right] \quad . \qquad (151)$$

The Hamiltonian integral exists, and along the singular arc we have

$$p_x V = \frac{p_V}{m}\left[\frac{1}{2}\rho S C_{D_o} V^2 + \frac{2Km^2g^2}{\rho S V^2}\right] \qquad (152)$$

which shows that $p_V > 0$ since $p_x > 0$. Using the Eqs. (151) and (152)

to evaluate $D_1 \Phi$, with Φ given by Eq. (149), we have

$$D_1 \Phi = \frac{p_V}{m^2 V} \left[\frac{1}{2} \rho S C_{D_o} V^2 \left(1 + \frac{cV}{g} + \frac{6 \epsilon m g}{\rho S V^2} \sqrt{\frac{K}{C_{D_o}}} \right) \right.$$

$$\left. - \frac{2 K m^2 g^2}{\rho S V^2} \left(3 + \frac{cV}{g} + \frac{2 \epsilon m g}{\rho S V^2} \sqrt{\frac{K}{C_{D_o}}} \right) \right] \quad (153)$$

At a junction point between a coasting arc, $T = 0$, and a maximum thrust arc, $T = T_{max}$, if $D_1 \Phi > 0$, the direction of switching is from $T = 0$ to $T = T_{max}$. If $D_1 \Phi < 0$, the optimal switching is from $T = T_{max}$ to $T = 0$. If $D_1 \Phi = 0$, the junction is with a singular arc. By taking the derivative of the equation $D_1 \Phi = 0$, we have the equation for evaluating the approximate optimal variable thrust control. Setting $D_1 \Phi = 0$, and using the dimensionless variables (142), we have the equation for the singular arc

$$\left(1 + u + 3\epsilon \frac{w}{u^2}\right) = \frac{w^2}{u^4} \left(3 + u + \epsilon \frac{w}{u^2}\right) . \quad (154)$$

From Eq. (151), we notice that we can write

$$\epsilon = \frac{1}{E^* C_{L_\alpha}} \left(\frac{2 m g}{\rho S V^2} - C_{L_o} \right) \quad (155)$$

and $E^* = 1/2 \sqrt{K C_{D_o}}$ is the maximum lift-to-drag ratio. It is seen

that neglecting the component $T\alpha$, that is to take $\epsilon = 0$, is a good approximation when the vehicle has high maximum lift-to-drag ratio. Using series expansion we obtain the approximate solution for the equation (154).

$$w = u^2\left[\sqrt{\frac{1+u}{3+u}} + \frac{\epsilon(4+u)}{(3+u)^2} + O(\epsilon^2)\right] . \tag{156}$$

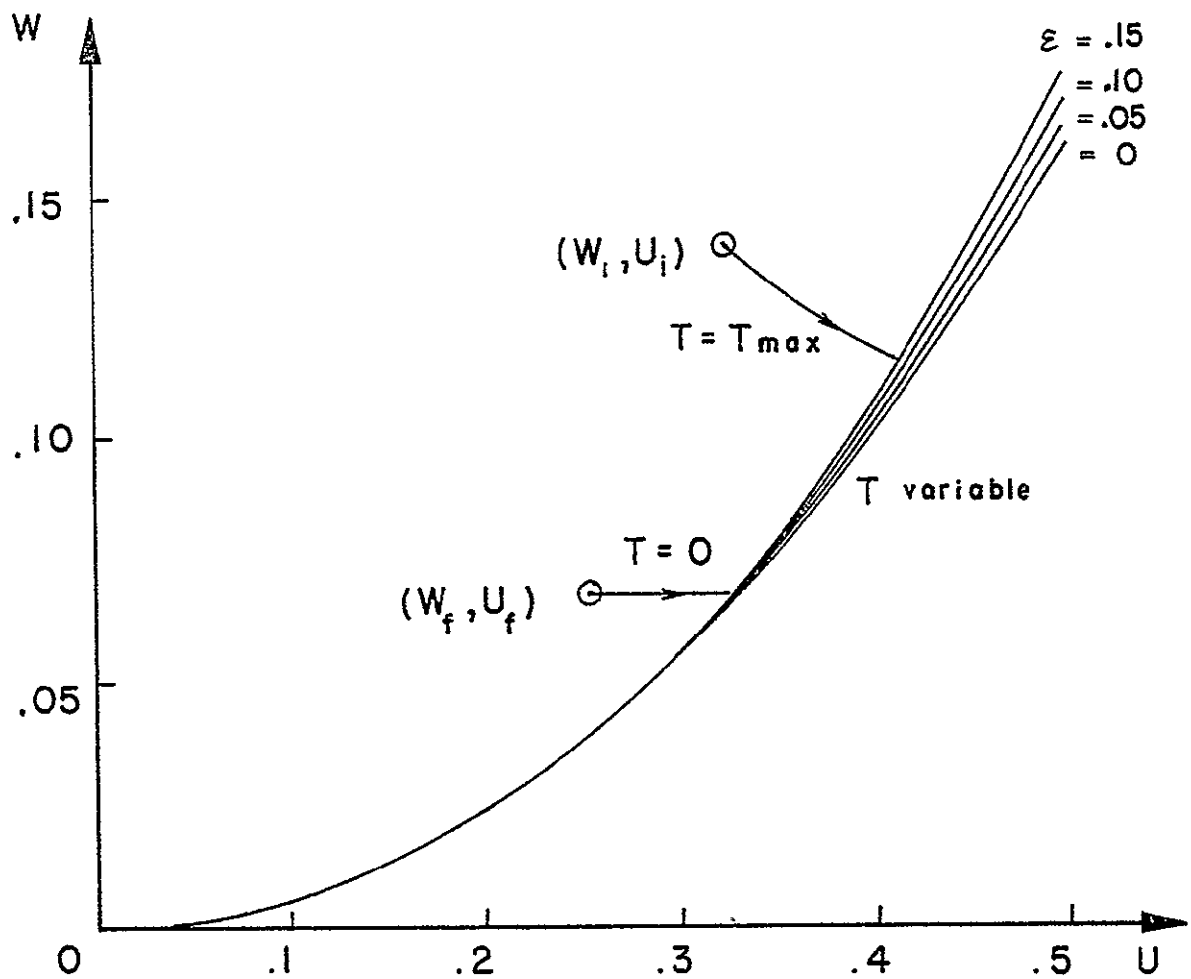

Fig. 13. The Singular Arc in the (w, u) Space.

This curve, for different values of ϵ is plotted in Fig. 13. A typical trajectory is also plotted. The initial point (w_i, u_i) is above the singular curve so that the trajectory starts with a maximum thrust arc. When the mass w and speed u satisfy relation (156), the trajectory enters a singular arc until the final mass w_f. The trajectory terminates with a coasting arc, $T = 0$, until the speed reaches the final value u_f. It is seen that, by retaining the component $T\alpha$ of the thrust, singular arc begins and ends with a speed slower than the corresponding one for the case where that component is neglected. Also, it should be noted that singular arc is obtained by approximation. In practice, the control is nonsingular so that both the angle of attack and the thrust vary continuously. In the (w, u) plane, the exact optimal trajectory, not only is continuous but also has continuous derivative. This information is useful for a numerical calculation of the true optimal trajectory based on the approximate solution obtained by using quasi-singular control.

V. 4. Example of Linearized Singular Control.

In this last example, we shall consider the glide of an aerospace vehicle. With $T = 0$, the equations of motion (127) become

$$\frac{dX}{dt} = V \cos \gamma$$

$$\frac{dZ}{dt} = V \sin \gamma$$

$$\frac{dV}{dt} = - \frac{\rho S C_D V^2}{2m} - g \sin \gamma$$

$$\frac{d\gamma}{dt} = \frac{\rho S C_L V}{2m} - \frac{g}{V} \cos \gamma \quad . \tag{157}$$

It is proposed to find the angle of attack modulation, or equivalently the variation of the lift coefficient, to maximize the range for a given altitude drop from Z_i to Z_f. The glide of a shuttle vehicle is an illustrative example.

We shall use an exponential atmosphere of the form

$$\rho = \rho_o \; e^{-\beta Z} \tag{158}$$

where β is the inverse of the scale height, assumed constant, and ρ_o is the density of the atmosphere at sea level. We shall use a normalized lift coefficient defined as

$$\lambda = \frac{C_L}{C_L^*} = C_L \sqrt{\frac{K}{C_{D_o}}} \tag{159}$$

where $C_L^* = \sqrt{C_{D_o}/K}$ is the lift coefficient corresponding to the maximum lift-to-drag ratio. If $C_D^* = 2\,C_{D_o}$ is the drag coefficient for maximum lift-to-drag ratio, we have the normalized drag coefficient

$$\frac{C_D}{C_D^*} = \frac{1 + \lambda^2}{2} \quad . \tag{160}$$

The use of the normalized lift coefficient is suggested by the fact that when $\lambda = 1$, the flight is at maximum lift-to-drag ratio.

It is convenient for the analysis to use the following dimensionless variables

$$w = \frac{2\beta m}{\rho S C_L^*} \quad , \quad u = \frac{V^2}{g/\beta} \quad , \quad x = \beta X \ . \tag{161}$$

The dimensionless kinetic energy u is used to replace the speed while the apparent wing loading w is used to replace the altitude. When ρ varies w varies in the same direction as the altitude. Of course x is the dimensionless longitudinal distance. With these dimensionless variables and using x as the independent variable, we have the dimensionless equations of motion

$$\begin{aligned}
\frac{dw}{dx} &= w \tan\gamma \\
\frac{du}{dx} &= -\frac{u(1+\lambda^2)}{E^* w \cos\gamma} - 2\tan\gamma \\
\frac{d\gamma}{dx} &= \frac{\lambda}{w\cos\gamma} - \frac{1}{u} \\
\frac{dx}{dx} &= 1
\end{aligned} \tag{162}$$

where E^* is the maximum lift-to-drag ratio. In this formulation, the only characteristic that enters the equations is the maximum lift-to-drag ratio E^* and the results obtained are valid for any vehicle, whether it is

a sail plane with high E^*, a fighter aircraft with moderate E^*, or a shuttle vehicle with low maximum lift-to-drag ratio. The only restriction, besides the flat Earth model, and exponential atmosphere, is that the speed range is such that the aerodynamic characteristic coefficients are independent of the Mach number.

It is assumed that the lift coefficient is bounded by

$$\lambda_{min} \leq \lambda \leq \lambda_{max} \quad . \tag{163}$$

The Hamiltonian of the system is

$$H = p_w \, w \tan\gamma - p_u \left[\frac{u(1+\lambda^2)}{E^* \, w \cos\gamma} + 2 \tan\gamma \right] + p_\gamma \left[\frac{\lambda}{w \cos\gamma} - \frac{1}{u} \right] + p_x \quad . \tag{164}$$

H is maximized for $\lambda = \lambda_{min}$, or $\lambda = \lambda_{max}$, or a variable λ obtained from the equation $\partial H / \partial \lambda = 0$. This gives the optimum relation for lift modulation

$$\lambda = \frac{E^* \, p_\gamma}{2 \, u \, p_u} \quad . \tag{165}$$

Although we have the case of coasting flight, $T = 0$, this problem is more difficult to solve than the previous one because the present control λ is not linear or quasi-linear but parabolic. It is expressed in terms of p_γ

and p_u by Eq. (165) but in general the equations for p_γ and p_u cannot be integrated in closed form. Hence, the exact solution for λ is not known.

We can use linearized theory to obtain a better solution for λ if some good approximate solution λ_o is known. This can be done in the case of steady state glide which occurs for large altitude drop.

The optimal control depends on the boundary condition. In general, for large altitude drop, after some initial maneuver the trajectory stabilizes along a variable λ, of the Euler-Lagrange type solution, where the variations in the speed and the flight path angle are both small. To find the solution for steady state glide, we use the assumption $du/dx \approx 0$, $d\gamma/dx \approx 0$ in Eq. (162) to have

$$- \tan \gamma = \frac{u(1+\lambda^2)}{2E^* w \cos \gamma}$$

$$\lambda = \frac{w \cos \gamma}{u} \quad . \tag{166}$$

Hence,

$$- \tan \gamma_i = \frac{(1+\lambda^2)}{2 E^* \lambda} \tag{167}$$

where $\tan \gamma_i$ has a nearly constant value. On the other hand, from the first two equations of system (157), we have

$$\frac{dX}{dZ} = \operatorname{cotg} \gamma_i \quad . \tag{168}$$

Hence, to maximize the range X_f for a given altitude drop, $Z_1 - Z_f$, we

must use the smallest value of $-\gamma_i$, that is, from Eq. (167), we must select

$$\lambda_o = 1 \quad . \tag{169}$$

The value γ_i is given by

$$-\tan \gamma_i = \frac{1}{E^*} \quad . \tag{170}$$

We obtain the classical solution which states that for glide with maximum range we must use the flattest glide with maximum lift-to-drag ratio. We call this solution the zeroth order solution because it is definitely not the optimal solution. It is expected that the optimal solution, when the trajectory has stabilized in the variable λ arc, is near this zeroth order solution. Linearized theory based on this solution, as developed in Section IV, can be applied in this case.

Using Eq. (74), we can construct the linearized Hamiltonian using the reference solution $\lambda_o = 1$.

$$\overline{H}^* = p_x + p_w w \tan \gamma - 2 p_u \tan \gamma - \frac{p_\gamma}{u} + \frac{1}{w \cos \gamma} \left(p_\gamma - \frac{2 u p_u}{E^*}\right) \lambda \, . \tag{171}$$

The state and adjoint equations, generated from this Hamiltonian are

$$\frac{dw}{dx} = w \tan \gamma$$

$$\frac{du}{dx} = - \frac{2u\lambda}{E^* w \cos \gamma} - 2 \tan \gamma$$

$$\frac{d\gamma}{dx} = \frac{\lambda}{w \cos \gamma} - \frac{1}{u} \tag{172}$$

and

$$\frac{dp_w}{dx} = - p_w \tan \gamma + \frac{1}{w^2 \cos \gamma} \left(p_\gamma - \frac{2up_u}{E^*}\right)\lambda$$

$$\frac{dp_u}{dx} = - \frac{p_\gamma}{u^2} + \frac{2p_u \lambda}{E^* w \cos \gamma}$$

$$\frac{dp_\gamma}{dx} = - \frac{p_w w}{\cos^2 \gamma} + \frac{2p_u}{\cos^2 \gamma} - \frac{\sin \gamma}{w \cos^2 \gamma} \left(p_\gamma - \frac{2up_u}{E^*}\right)\lambda \tag{173}$$

The control is variable, so that in the transformed problem, it is singular. The switching function is constantly zero and we have

$$p_\gamma = \frac{2up_u}{E^*} \quad . \tag{174}$$

This equation is seen to be derived from the optimal relation (165) when the approximation $\lambda = 1$ is used. By taking the derivative of Eq. (174), using the Eqs. (172) and (173), we have

$$2p_u \left[1 + \frac{2 \cos^2 \gamma}{E^{*2}} (1 + E^* \tan \gamma)\right] = w p_w \quad . \tag{175}$$

Again, by taking the derivative of this equation, we have the relation for the lift control

$$\frac{4(1+E^{*2})\cos^2\gamma}{E^{*3}}\left(\frac{\lambda}{w\cos\gamma}-\frac{1}{u}\right)=0\ . \tag{176}$$

Therefore, we have the first-order solution

$$\lambda_1=\frac{w\cos\gamma}{u}\ . \tag{177}$$

Referring to the exact equations (162) this relation shows that to maximize the range we must keep constant flight path angle, a result in agreement with standard flight technique. The difference with the maximum lift-to-drag ratio glide is that if we use $\lambda = 1$, and integrate the equations of motion, the resulting flight path angle is not constant, as has been assumed in the steady state solution, but presents an oscillatory behavior with relatively large amplitude. On the other hand, the exact numerical solution obtained also displays an oscillation in the flight path angle near a certain reference value, but with smaller amplitude (Fig. 14). This reference value is the one that must be used for the first-order solution (177). In practice, the initial flight path angle γ_i, speed u_i and apparent wing loading w_i are prescribed. If the average glide angle γ_s can be evaluated from the exact numerical solution, one can use a constant value λ_i, to be selected such that the integration leads to

$$\frac{d\gamma}{dx}=\frac{\lambda_i}{w\cos\gamma}-\frac{1}{u}=0$$

when $\gamma = \gamma_s$. After that the flight path angle is kept at this constant

value by using the first-order solution (177). The convergence of the solution by repeated application of the linearized theory requires further study. Nevertheless, for the problem of maximum range as considered here, it appears that both the zeroth order solution (169) and first-order solution (177) provide good results as compared to the exact numerical solution, with the first-order solution giving a significant improvement over the zeroth order solution.

This suggests that, again, we can use the first-order solution (177) as a reference solution to linearize the Hamiltonian. We have

$$\overline{H}^* = p_x + p_w w \tan\gamma + p_u \left[\frac{w \cos\gamma}{E^* u} - \frac{u}{E^* w \cos\gamma} - 2 \tan\gamma \right] - \frac{p_\gamma}{u} + \left(\frac{p_\gamma}{w \cos\gamma} - \frac{2 p_u}{E^*} \right) \lambda \ . \tag{178}$$

The state and adjoint equations generated from this Hamiltonian are

$$\begin{aligned} \frac{dw}{dx} &= w \tan\gamma \\ \frac{du}{dx} &= \frac{w \cos\gamma}{E^* u} - \frac{u}{E^* w \cos\gamma} - 2 \tan\gamma - \frac{2\lambda}{E^*} \\ \frac{d\gamma}{dx} &= \frac{\lambda}{w \cos\gamma} - \frac{1}{u} \end{aligned} \tag{179}$$

and

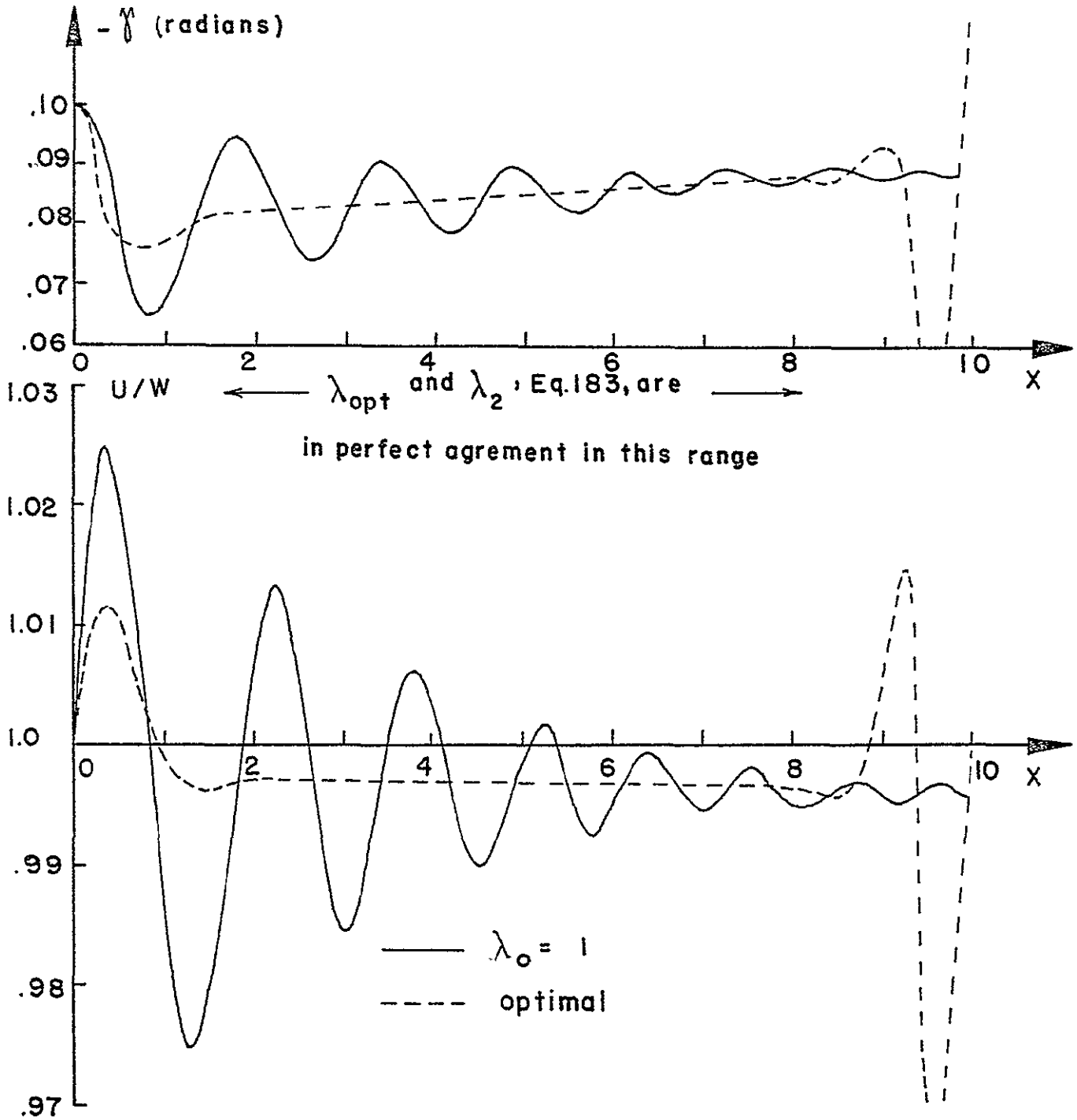

Fig. 14. Variations of the Flight Path Angle and the Dimensionless Dynamic Pressure for Glide with Maximum Range.

$$\frac{dp_w}{dx} = -p_w \tan\gamma - p_u\left[\frac{\cos\gamma}{E^* u} + \frac{u}{E^* w^2 \cos\gamma}\right] + \frac{p_\gamma \lambda}{w^2 \cos\gamma}$$

$$\frac{dp_u}{dx} = p_u\left[\frac{w\cos\gamma}{E^* u^2} + \frac{1}{E^* w\cos\gamma}\right] - \frac{p_\gamma}{u^2}$$

$$\frac{dp_\gamma}{dx} = -\frac{p_w w}{\cos^2\gamma} + p_u\left[\frac{w\sin\gamma}{E^* u} + \frac{u\sin\gamma}{E^* w\cos^2\gamma} + \frac{2}{\cos^2\gamma}\right]$$

$$- \frac{p_\gamma \lambda \sin\gamma}{w\cos^2\gamma} \quad . \tag{180}$$

The control is variable, so that in the linearized problem, it is singular. We constantly have the switching function vanishing, that is,

$$E^* p_\gamma = 2 p_u w \cos\gamma \quad . \tag{181}$$

This equation is seen to be derived from the optimal relation (165) when the approximation $\lambda = \frac{w}{u}\cos\gamma$ is used. By taking the derivative of Eq. (181), using the Eqs. (179) and (180), we have

$$E^* w p_w = p_u\left[2E^* - \frac{2\cos^2\gamma}{E^*} + \frac{2w^2\cos^4\gamma}{E^* u^2} - \frac{(1+2u)}{u}\right.$$

$$\left. w\cos^2\gamma\sin\gamma + \frac{u\sin\gamma}{w}\right] \quad . \tag{182}$$

By taking the derivative of this equation we have the relation for the lift control

$$\lambda_2 = \frac{w \cos\gamma}{u} \quad \frac{P}{Q} \tag{183}$$

where

$$P = E^{*2} u^2 [\, 6 \tan^2\gamma - 4u(1-\tan^2\gamma) + 2u^2 \tan^2\gamma + (1+u)\, C]$$
$$- 2E^* u(3+u)(2+C)\, w \sin\gamma + 2C(2+C)\, w^2 \cos^2\gamma$$
$$Q = E^{*2} u^2 [\, 2(1+u)\tan^2\gamma - 4u - C] - 2E^* u(4+C) w \sin\gamma$$
$$+ 8 w^2 \cos^2\gamma \tag{184}$$

and

$$C = 1 - \frac{u^2}{w^2 \cos^2\gamma} \quad . \tag{185}$$

From the zeroth and first-order solution, Eqs. (169) and (177), it is seen that C is a small quantity.

Equation (183) gives the second-order solution for the lift control. As compared to the first-order solution, Eq. (177), the ratio P/Q is the correctional factor. The oscillation of this factor near the value 1 provides the small oscillation in the flight path angle.

When $\lambda = w \cos\gamma/u$, the flight path angle passes through an extremum. Hence by writing P = Q, we have the equation of a surface in the (γ, u, w) space

$$R = E^{*2} u^2 [\, 2(u^2+u+2)\tan^2\gamma + (2+u)C\,] - 2E^* u[\, 2(1+u)+(u+2)C]$$
$$w \sin\gamma + 2(C^2+2C-4)\, w^2 \cos^2\gamma = 0 \quad . \tag{186}$$

It is convenient to use a cylindrical coordinates system (u, γ, w) to represent the surface R = 0 (Fig. 15).

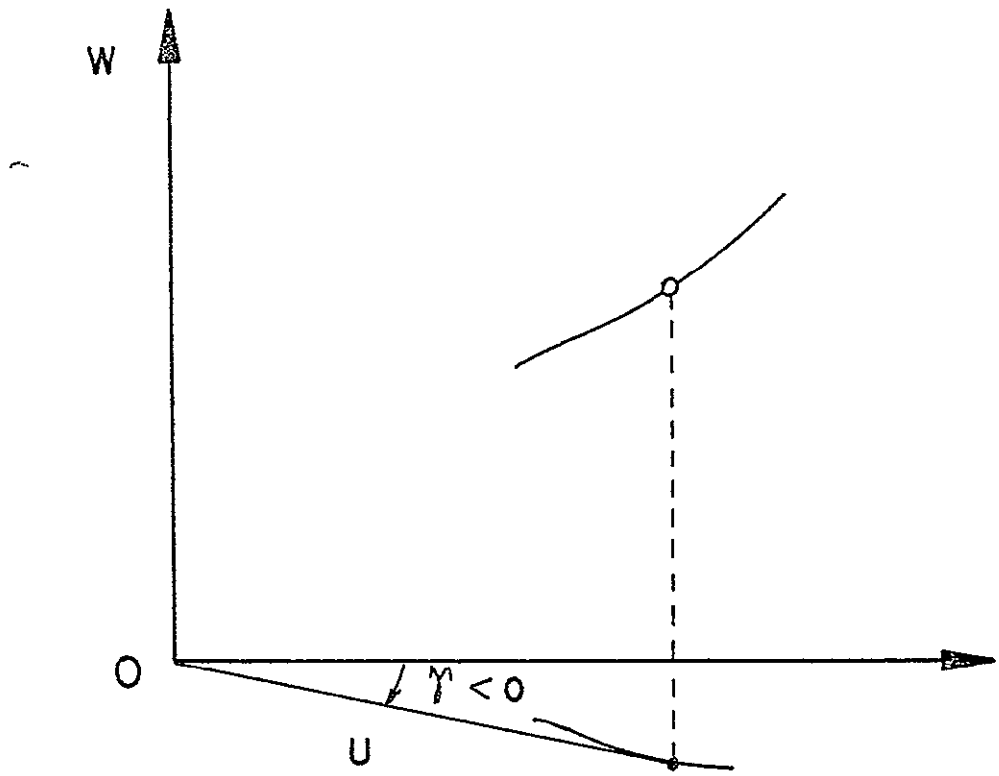

Fig. 15. Trajectory in the Cylindrical Coordinates System.

We notice that the equation for the flight path angle, using the variable lift control (183), can be written as

$$\frac{d\gamma}{dx} = \frac{R}{uQ} \quad . \tag{187}$$

Along the variable lift arc of the trajectory, γ remains small and changes its direction of variation each time the trajectory intersects the surface

$R = 0$. Hence the variable lift arc presents oscillation in γ. The intersection of the surface $R = 0$ and the plane $\gamma = 0$ is given by the equation

$$[\, E^{*2}(2+u) - 2\,]\; C^2 - [\, E^{*2}(2+u) + 4]\;\; C + 8 \;=\; 0 \tag{188}$$

where

$$C \;=\; 1 \;-\; \frac{u^2}{w^2} \quad . \tag{189}$$

Since in general $E^* > 1$, Eq. (188) has positive roots if real roots exist. Therefore, the trace of the surface $R = 0$ in the (u, w) plane is in the region $w > u$. The condition for real roots is

$$E^{*4}(2+u)^2 - 24\, E^{*2}(2+u) \;+ 80 \;\geq\; 0 \quad . \tag{190}$$

Explicitly, we have the conditions

$$u < \frac{4}{E^{*2}} - 2 \quad , \quad \text{or} \quad u > \frac{20}{E^{*2}} - 2 \quad . \tag{191}$$

They are generally satisfied for moderate E^* and positive u. It is expected that the trajectory has a slow variation in γ, considering that this variation is zero for the first-order solution. Therefore, the variable lift arc stays close to the surface $R = 0$. Using the approximation $E^* \tan\gamma \approx E^* \sin\gamma \approx -1$, $\cos\gamma \approx 1$ in Eq. (186) we have the following approximate equation for the variable lift arc (or linearized singular arc) in the (u, w) plane

$$[\,2(u^2+u+2)+E^{*2}(2+u)\,C]\;\frac{u^2}{w^2}+2[\,2(1+u)+(u+2)C\,]\;\frac{u}{w}+2(C^2+2C-4)\;=\;0 \tag{192}$$

where C is given by Eq. (189).

Using a parameter k defined as

$$\frac{u}{w}\;=\;k \tag{193}$$

we have a quadratic equation in u

$$2k^2u^2+[\,E^{*2}k^2(1-k^2)+2k^2+2k(3-k^2)\,]\;u$$
$$+\,2(1-k)[\,E^{*2}k^2(1+k)-k^3+k^2+3k-1\,]\;=\;0\;. \tag{194}$$

The two equations (193) and (194) can be considered as parametric equations for u and w with parameter k. The linearized singular curves for different values of E* are plotted in Fig. 16. It is seen that for high maximum lift-to-drag ratio the curves are close to the curve u = w. Furthermore, by the definition (161) of w and u, we see that the dimensionless quantity η defined as

$$\eta\;=\;\frac{u}{w}\;=\;\frac{\rho\,SC_L{}^*V^2}{2\,m\,g} \tag{195}$$

is the dimensionless dynamic pressure which is a measure of the indicated speed. Therefore if $u \approx w$, the indicated speed is nearly constant during the glide.

To support the present linearized theory, exact numerical solution has been generated for a vehicle with E* = 10, a typical value for a modern

fighter aircraft. The initial conditions used are

$$w_i = 0.5, \quad u_i = 0.5, \quad \gamma_i = -\frac{1}{E^*} = -0.1 \tag{196}$$

while the final conditions on w and u are

$$w_f = 0.215 \quad u_f = 0.213 \quad . \tag{197}$$

The exact numerical solution is plotted in Fig. 16 as a dashed line. It is seen that the trajectory nearly follows the singular curve as predicted by the theory. In general, for a given u_i and w_i the optimal trajectory, for maximum range glide, quickly joins a path near the approximate singular curve $u \approx w$ and stays in its vicinity until near the end when again it deviates to match the prescribed final conditions. For a large altitude drop, the initial and final arcs are short and, during the main portion of the glide, linearized singular control provides the analytical solution to the maximum range glide problem.

The near constancy of the flight path angle γ and the dimensionless dynamic pressure $\eta = u/w$ along the glide path for maximum range is explicitly displayed in Fig. 14.

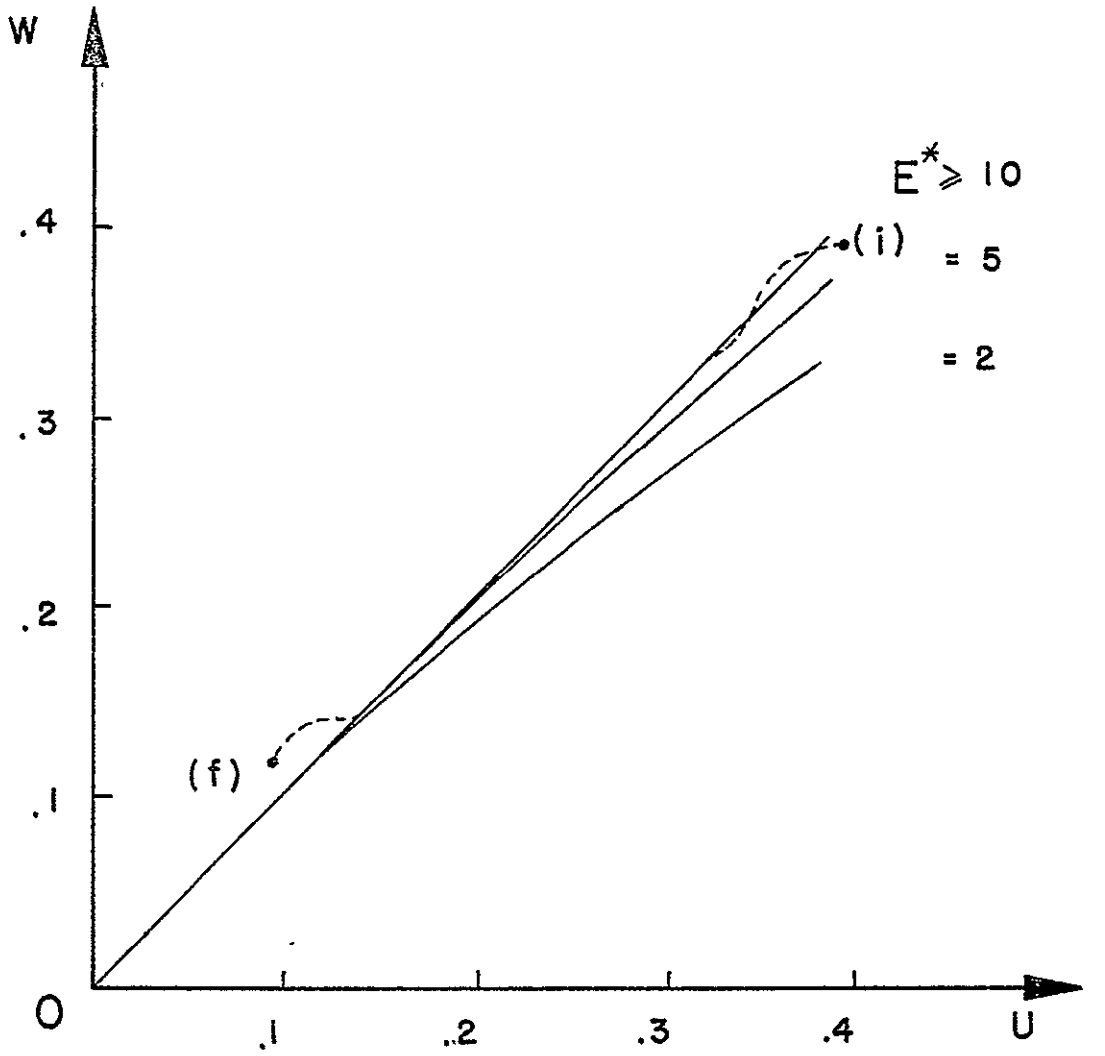

Fig. 16. The Approximate Linearized Singular Arc in the (u. w) Plane.

VI. CONCLUSION

In this report, we have presented a comprehensive discussion of the problem of singular control. Singular control enters an optimal trajectory when the so-called switching function vanishes identically over a finite time interval.

Using the concept of domain of maneuvrability, the problem of optimal switching is analyzed. Criteria for the optimal direction of switching are presented. The switching, or junction, between nonsingular and singular subarcs is examined in detail. It is shown that, in general, switching with singular arc is one of two categories: A regular type where the control is discontinuous at the junction point, and a singular type where not only that the control is discontinuous at the junction point, but is non-analytic. In this type of junction, entering or leaving a singular arc is effected by chattering control which requires a rapid switching of the control between its extreme limits.

Junction between nonsingular and singular subarcs in which the control is continuous at the junction point is a rare phenomenon and usually is effected at some specified points in the phase space. Hence, it requires particular initial and final manifolds leading to and coming from these points. Conditions for smooth junction are derived. The discussion of singular arc and junction with singular arc is carried out with the

mathematical rigor in optimal control theory. From the more practical aspect for solving engineering problems, the concepts of quasi-linear control and linearized control are introduced. They are designed for the purpose of obtaining approximate solution for the difficult Euler-Lagrange type of optimal control when either the dynamical system considered is quasi-linear in the control, or in the case of strongly nonlinear control, that a certain reference solution for the control, usually steady state case control, is known as function of the state variables and the time.

Some illustrative examples are presented as applications of the theorems formulated and of the concepts introduced.

A logical continuation of this work is the analysis of the error committed when quasi-linear control or linearized control theory is employed in solving nonlinear control problems. In this respect, a rigorous proof that the approximate solution indeed tends to the optimal solution is in order. Numerical applications of the linearized theory in some engineering problems tend to support this conjecture.

REFERENCES

1. CONTENSOU, P., "Note sur la Cinématique Générale du Mobile Dirigé," Communication presented to the Association Technique et Aéronautique, Paris, Vol. 45, p. 836, 1946.

2. CONTENSOU, P., "Etude Théorique des Trajectoires Optimales Dans Un Champ de Gravitation. Application au Cas d'un Centre d' Attraction Unique," Astronautica Acta, Vol. 8, Nos. 2-3, 1962.

3. CONTENSOU, P., "Quelques Propriétés Générales des Domaines de Manoeuvrabilité à Frontière Semi-linéaire. Application aux Transferts d' Orbites," Advanced problems and methods for space flight optimization, B. Fraeijs de Veubeke, Ed., Pergamon Press, 1969.

4. CONTENSOU, P., "Conditions d' Optimalité pour les Domaines de Manoevrabilité à Frontière Semi-Affine." Lectures Notes in Mathematics 112, Springer Verlag, 1970.

5. KELLEY, H., KOPP, R. E., and MOYER, H. G., "Singular Extremals," Topics in Optimization, G. Leitmann, Ed., Academic Press, 1967.

6. FULLER, A. T., "An Optimum Non-Linear Control System," paper presented at the IFAC Congress, Moscow, USSR, 1961.

7. FULLER, A. T., "Further Study of an Optimum Non-Linear Control System," Journal of Electronics and Control, Vol. 17, No. 3, 1964.

8. ROBBINS, H. M., "Optimality of Intermediate Thrust Arcs of Rocket Trajectories," AIAA Journal, Vol. 3, No. 6, 1965.

9. MARCHAL, C., "Généralisation Tridimensionnelle et Etude de l' Optimalité des Arcs à Poussée Intermediaire de Lawden (dans un champ Newtonien), LaRecherche Aérospatiale, No. 123, 1968.

10. LAWDEN, D. F., "Intermediate Thrust Arcs in a Gravitational Field," Astronautica Acta, Vol. 8, Nos. 2-3, 1962.

11. BREAKWELL, J. V., "Minimum Impulse Transfer," Progress in Astronautics and Aeronautics: Celestial Mechanics and Astrodynamics, V. Szebehely Ed., Academic Press, 1964.

12. ARCHENTI, A., and MAREC, J. P., "Transferts Orbitaux: Intégration Numérique des Arcs Singuliers Alternatifs de Contensou," La Recherche Aérospatiale, No. 6, 1970.

13. ARCHENTI, A., and MAREC, J. P., "On the Optimality of Contensou's Singular Alternating Arcs in the Optimal Transfer Problem," Journal of Optimization Theory and Applications, Vol. 8, No., 1, 1971.

14. ARCHENTI, A., and VINH, N. X. "The Intermediate- Thrust Arcs and Their Optimality in a Central, Time Invariant Force Field," Journal of Optimization Theory and Applications, Vol. 11, No. 3, 1973.

15. MARCHAL, C., "Theoretical Research in Deterministic Optimization," ONERA Publication No. 139, 1971.

16. McDANELL, J. P., and POWERS, W. F., "Necessary Conditions for Joining Optimal Singular and Nonsingular Subarcs," SIAM Journal of Control, Vol. 9, No. 2, 1971.

17. MAURER, H., "An Example of a Continuous Junction for a Singular Control Problem of Even Order," SIAM Journal of Control, Vol. 13, No. 4, 1975.

18. VINH, N. X., "Sur La Théorie de Commutation de Contensou," La Recherche Aérospatiale, No. 1977-2, 1977.

19. POINCARÉ, H., "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. 2, Dover Reprint, 1957.

20. MOULTON, F. R., "Periodic Orbits," Carnegie Institute of Washington, Washington, D. C., 1920.

21. VINH, N. X., "Sur Les Solutions Périodiques du Mouvement Plan de Libration des Satellites et des Planètes," Celestial Mechanics, Vol. 8, 1973.

22. BELETSKII, V. V., and LAVROVSKII, E. K., "Satellite Periodic Motions in the Vicinity of Mercurian Type Resonance," paper presented at the 27th Congress of the International Astronautical Federation, Anaheim, California, 1976.

23. MAREC, J. P., "Trajectoires Spatiales Optimales," Lectures Notes, ENSAE, 1973.

24. GURMAN, V. I., "Optimal Process of Singular Control," Avtomatika i Telemekhanika, No. 5, 1965.

25. GURMAN, V. I., "Investigation of a Certain Class of Optimal Sliding Controls," Avtomatika i Telemekhanika, No. 7, 1965.

26. GURMAN, V. I., "The Method of Multiple Maxima and the Conditions of the Relative Optimum of Degenerate Controls," Avtomatika i Telemekhanika, No. 12, 1967.

27. KROTOV, V. F., BUKREEV, V. Z., and Gurman, V. I., "New Variational Methods in Flight Dynamics," NASA Translation TTF-657, 1971.

28. HIBBS, A., "Optimum Burning Program for Horizontal Flight," Journal of the American Rocket Society, Vol. 22, No. 4, 1952.

29. MIELE, A., "Lagrange Multipliers and Quasi-Steady Flight Mechanics," Journal of the Aerospace Sciences, Vol 26, No. 9, 1959.