NASA Contractor Report 2947

Insensitive Control Technology Development

C. A. Harvey and R. E. Pope

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C. A. Harvey and R. E. Pope
Honeywell Systems & Research Center
Minneapolis, Minnesota

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SECTION I

INTRODUCTION AND SUMMARY

The work described in this report represents a continuation of the effort initiated under NASA contract NAS 1-13680, Study of Synthesis Techniques for Insensitive Aircraft Control Systems. The by-products of that contract were two new advanced theoretical concepts for insensitive controller design that had been developed by contract consultants Professor William A. Porter of the University of Michigan and Professor David L. Kleinman of the University of Connecticut. The concept developed by Professor Porter has been designated the Finite Dimensional Inverse method whereas that developed by Professor Kleinman has been termed the Maximum Difficulty concept. At the conclusion of the initial effort, neither of the concepts had been developed to a point where the resultant insensitive controller designs could be evaluated on a realistic flight control example. The objective of this contract effort, NAS 1-14476, Insensitive Control Technology Development, was to extend the theoretical base of the two concepts to workable insensitive controller synthesis techniques and to evaluate the resultant insensitive controller designs on a realistic flight control example. As in the first study, the C-5A longitudinal dynamics model was used as the test bed for evaluation.

The results of the study of the two concepts are summarized below

FINITE DIMENSIONAL INVERSE CONCEPT

The present formulation of this concept is much more suited to trajectory-type sensitivity problems than to stationary flight control problems. The controller designed for the C-5A example using this concept involves time-varying gains. The controller's performance for this example fulfilled the theoretical predictions, but the present formulation of the concept limited the reduction in sensitivity to certain selected outputs. For the C-5A example design, the control surface displacements were suitable as selected outputs. Unfortunately, reductions in sensitivity of these outputs did not yield reductions in sensitivity of other important responses, such as bending and torsion moments.

Despite these deficiencies with respect to the C-5A example, the concept exhibits certain promising aspects such as on-line parameter identification and sensitivity reduction of minimum phase input-output systems. The study also indicates that although the computational requirements are severe, they are not beyond current capabilities.

MAXIMUM DIFFICULTY CONCEPT

The objective of the Maximum Difficulty concept was extended to exploration of a technique devised by Professor Kleinman which utilizes the Information Matrix element of parameter identification theory. In its complete form, this concept requires insensitive controller design for a flight condition with minimal controllability index and, at the same time, desensitizing system responses to variations in uncertain parameters. The latter function, involving the Information Matrix approach, received a majority of the attention in the contract and will be the main subject of the Maximum Difficulty concept discussion.

In addition to the development of the Information Matrix concept, Professor Kleinman participated heavily in the actual design and preliminary evaluations. Formal evaluation of the concept was performed on the 15-state Case 4R residualized C-5A model using the criteria defined in the initial phase. The evaluation revealed that the insensitive controller designed with the Information Matrix approach performed as well as the top-ranked controllers of the previous study.

SECTION II

SYMBOLS

UPPER CASE

B	Bending moment
D, $D(\alpha)$	Control input coefficient matrix in response equation
$F(\alpha)$	Plant coefficient matrix
$G(\alpha)$, G_1	Control input coefficient matrix
$H(\alpha)$	Coefficient matrix of state vector in output or response equation
I	Identity matrix
J	Performance index
K	Gain matrix
K_o	Nominal gain matrix
K(t)	A Riccati matrix
M	Finite dimensional inverse of T_o, Information Matrix
M_w	Stability derivative (pitching moment due to vertical velocity)
N(a, b)	Normal distribution (a = mean, b = standard deviation)
N(t)	Inverse of K(t)
Q	Weighting matrix

S	Weighting matrix
$S_m(\beta)$	Sensitivity index
T	Torsion moment
$T(\alpha)$	Input-output transformation
T_o	Nominal plant input-output transformation
W	Weighting matrix
$X(t)$	Coefficient matrix in the differential equation representation of M

LOWER CASE

e_i	Basis vectors
$p(t)$	State vector for M
$\bar{q}$	Dynamic pressure
$q(t)$	Alternate state vector for M
u	Control input vector
v	Input to M
w	Output vector
x	State vector
y_i	Input vectors to the nominal plant giving outputs ξ_i
z	Output of M

GREEK SYMBOLS

Upper Case

$\Lambda(t)$	Scaling matrix in the differential equation representation of M
$\Pi(t)$	Symmetric matrix function of the $\xi_i(t)$

Lower Case

α	Vector of parameters
β	Scalar design parameter in insensitive compensator implementations
γ_i	Scaling factors
δT	Perturbation transformation
δa	Aileron displacement
δe_i	Inboard elevator displacement
ζ	Damping ratio
η	Scalar white noise input
ξ_i	Outputs corresponding to specific inputs and plant variations
ρ	Noise/signal ratio
σ	Standard deviation
ω	Frequency

SECTION III

FINITE DIMENSIONAL INVERSE COMPENSATOR

One of the major objectives of this study was the development and quantitative evaluation of the finite dimensional inverse technique for compensation for parameter uncertainty. The finite dimensional inverse concept, conceived in the previous insensitive controller study, is based on the concept of the a priori construction of a set of inverse functions which are derived from a finite number of input-output pair relationships. The input-output pairs are specified by type of input, selected output, and a combination of uncertainties that represent variations in plant behavior from a nominal or no-uncertainty condition.

In operation, the measured outputs of the plant are used to determine the degree of mapping of plant outputs on the prestored outputs at off-normal conditions. The degree of mapping then dictates the formation of the feedback signals using the inverse functions that are used to compensate for the plant operating at off-nominal conditions. In essence, the inverse functions represent the change in control which is necessary to negate the effect of parameter uncertainties.

In investigating the finite dimensional inverse concept, the specific goal of this part of the study was to implement the concept for a simple illustrative example and for the C-5A example[1] and to examine its performance and limitations via simulation.

The illustrative example is a first order system with two parameters. This example served two purposes. First, it provided a simple problem for purposes of debugging software and examining numerical aspects of implementation. Second, it permitted extensive analysis of the effects of nonlinearity with respect to the plant parameters; variation in the design parameter which governs the attainable degree of insensitivity; and sensor noise.

The C-5A example provided a more realistic test for the concept. Additional effects which could be examined with this example were those associated with authentic disturbances, unmodeled dynamics, authentic types of parameter uncertainty, and the choice of outputs of interest.

Two forms of the compensator were examined. One included a simulation of the nominal as a model; the other excluded this model. In general, the performance of the compensators was consistent with theoretical predictions: both configurations yielded reductions in sensitivity. The second configuration exhibited a tendency toward an initial high gain instability for a constant sensitivity gain. This could be alleviated with a time-varying sensitivity gain. The concept as implemented is based on the assumption that the system outputs are linearly dependent on the parameter variations. The effect of actual nonlinear dependence was found to be significant in the sense that the reduction in sensitivity for large parameter variations differed significantly from the linear theoretical predictions. But the sensitivity was reduced even for large parameter variations.

In the C-5A example it was found that the compensator provides reduced sensitivity in terms of the outputs of the system, but not in terms of the system states or other system responses of interest. To be of real benefit for such an example, this deficiency would need to be remedied. It was also found that the compensators did provide reduced sensitivity to gust disturbances and were not seriously affected by unmodeled dynamics.

Details of the mathematical formulation and the experimental results are described below.

MATHEMATICAL FORMULATION

Consider a linear system represented by an input-output transformation $T(\alpha)$ with α denoting an r-dimensional vector of parameters. The system may be represented in state variable form as

$$T(\alpha): \quad \dot{x} = F(\alpha)x + G(\alpha)u, \quad x(0) = 0 \qquad (1)$$
$$w = H(\alpha)x$$

where $T(\alpha)$ maps the input u into the output w. Let us assume without loss of generality that the nominal value of the parameter vector is zero and let T_o denote the nominal system. We assume that the dependence of $T(\alpha)$ on α is sufficiently smooth so that linearization about zero is an adequate model. If $\{e_1, e_2, \ldots, e_r\}$ is a basis for R^r, then the linearization of $\delta T(u)$ becomes

$$\delta T(u) = \sum_{i=1}^{r} \alpha_i \xi_i \qquad (2)$$

where

$$\delta T = T(\alpha) - T_o \tag{3}$$

$$\alpha = \sum_{i=1}^{r} \alpha_i e_i \tag{4}$$

$$\xi_i = [T(e_i) - T_o]u, \quad i = 1, 2, \ldots, r \tag{5}$$

We note that ξ_i generally depends on u.

For the moment, suppose that u is known a priori so that the ξ_i may be computed. Let

$$X = \text{Linear Span } \{\xi_1, \xi_2, \ldots, \xi_r\} = \text{Range } \delta T(u) \tag{6}$$

For simplicity of discussion, we will assume a single input u. The extension to more than one input is straightforward. Assuming that X is contained in the range of T_o, there exist functions $\{y_1, y_2, \ldots, y_r\}$ such that

$$\xi_i = T_o y_i, \quad i = 1, 2, \ldots, r \tag{7}$$

We call a map M a finite-dimensional inverse of T_o if M is linear, bounded, and satisfies

$$M\xi_i = y_i, i=1, \ldots, r \tag{8}$$

Assuming the set of functions $\{\xi_i(t)\}$ are linearly independent on every finite interval $[0, T]$, the map M may be realized in the following state variable or differential equation form.[2,3] Let M map v into z. Then M is given by

$$z(t) = y(t)\,\Lambda(t)\,K(t)\,p(t) \tag{9}$$

$$\dot{p}(t) = -\tfrac{1}{2}X(t)\,p(t) + \Lambda(t)\,\xi(t)\,v(t), \quad p(0) = 0 \tag{10}$$

where

$$y(t) = \text{row}\,(y_1(t), y_2(t), \ldots, y_r(t)) \tag{11}$$

$$\xi(t) = \text{col}\,(\xi_1(t), \xi_2(t), \ldots, \xi_r(t)) \tag{12}$$

$$\Lambda(t) = \text{diag}\,(\lambda_{ii}), \quad \lambda_{ii} = \left[\int_0^t \xi_i^2(s)ds\right]^{-\frac{1}{2}} \tag{13}$$

$$X(t) = \text{diag}(x_{ii}), \quad x_{ii} = \xi_i^2(t)/\int_0^t \xi_i^2(s)ds \tag{14}$$

and K(t) is a symmetric r x r matrix. The matrix K(t) is the inverse of a symmetric matrix N(t) and satisfies the Riccati differential equation.

$$\dot{K}(t) = \tfrac{1}{2}\{X(t)\,K(t) + K(t)\,X(t)\} - K(t)\,\Pi(t)K(t) \tag{15}$$

where the elements of the matrices N(t) and $\Pi(t)$ are defined as

$$N_{ij}(t) = \int_0^t \xi_i(t)\cdot\xi_j(t)/\sqrt{\int_0^t \xi_i^2(s)ds \int_0^t \xi_j^2(s)ds} \tag{16}$$

$$\Pi_{ij}(t) = \xi_i(t)\cdot\xi_j(t)/\sqrt{\int_0^t \xi_i^2(s)ds \int_0^t \xi_j^2(s)ds} \tag{17}$$

The vector p(t) may be computed in the following manner as an alternative to Equation (10). Let us define the vector q(t) as the solution of the following differential equation:

$$\dot{q}(t) = \xi(t)\cdot v(t), \quad q(0) = 0 \tag{18}$$

Then p(t) is given by the equation

$$p(t) = \Lambda(t)q(t) \tag{19}$$

since

$$\frac{d}{dt}\left[\Lambda(t)q(t)\right] = \dot{\Lambda}(t)\, q(t) + \Lambda(t)\, \dot{q}(t)$$
$$= -\tfrac{1}{2}X(t)\left[\Lambda(t)q(t)\right] + \Lambda(t)\, \xi(t)v(t) \tag{20}$$

The numerical solution for p(t) via Equations (18) and (19) was found to be much better behaved than the solution for p(t) via Equation (10) for t near zero.

The finite dimensional inverse may be used as a compensator to reduce sensitivity. It may also be used in a parameter identifier mode. Two compensator configurations are shown schematically in Figures 1 and 2. The on-line identifier configuration is shown schematically in Figure 3.

Notation in these figures will be used in the remaining discussion. In particular, we note that

w is the output of the uncompensated system

w_o is the output of the nominal system

$\hat{w}$ is the output of the compensated system with model

$\bar{w}$ is the output of the compensated system without the model

Similar notation is used for the state vectors of these systems. For example, x is the state vector of the uncompensated system.

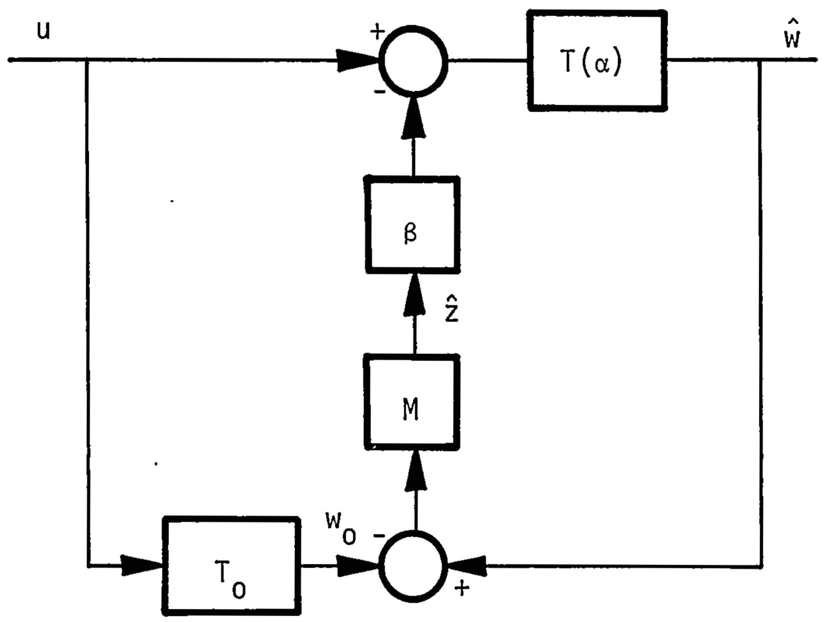


Figure 1. Compensated System with Model

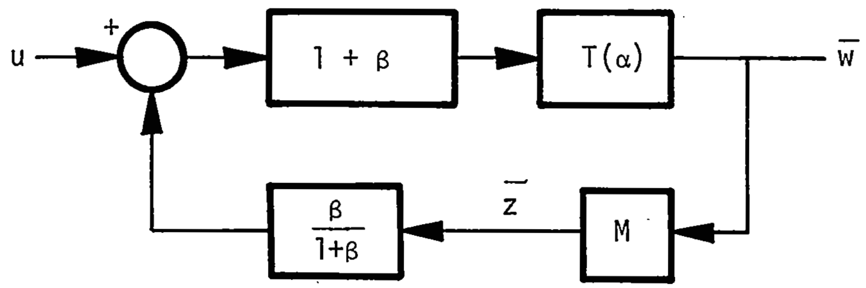


Figure 2. Compensated System without Model

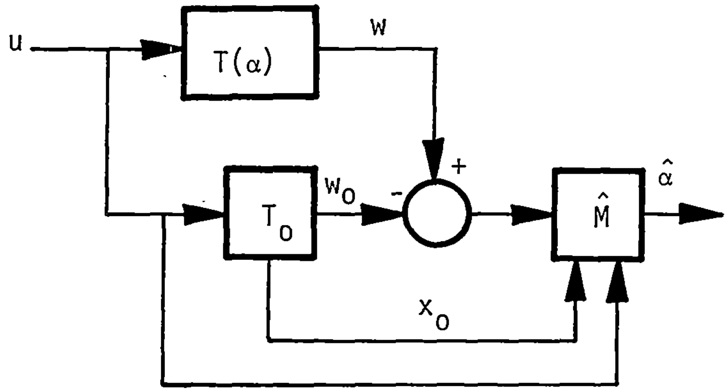


Figure 3. On-Line Identifier

The map M for the compensated systems was computed according to the equations given above with the $\{\xi_i\}$ precomputed. The only difference in M between Figures 1 and 2 is in the inputs and outputs. In Figure 1, the input v is taken to be $\hat{w} - w_o$ and the output is called $\hat{z}$. In Figure 2 the input to M is $\overline{w}$ and the output is called $\overline{z}$. The map $\hat{M}$ for the on-line identifier was computed in a similar manner, except that y(t) is called $\hat{y}$, the r x r identity matrix. The $\{\xi_i\}$ were replaced by $\{\hat{\xi}_i\}$ computed on line from the equations

$$\hat{\xi}_i = \left[\frac{\partial T}{\alpha_i}\right]_{\alpha=0} u, \quad i = 1, 2, \ldots, r \tag{21}$$

The output, z, is called $\hat{\alpha}$.

The sensitivity to variations in α of the system shown in Figure 1 may be represented by the index:

$$S_m(\beta) = \lim_{\|\alpha\| \to 0} \|(T_m(\alpha, \beta) - T_o)(T(\alpha) - T_o)^{-1}\| \tag{22}$$

where $T_m(\alpha, \beta)$ is the closed-loop map from u to $\hat{w}$. For a single-input/single-output system, this index is the limit as $\|\alpha\|$ tends to zero of the ratio of the error between the compensated system and the nominal system outputs to the error between the uncompensated system and the nominal system outputs, i.e.,

$$S_m(\beta) = \lim_{\|\alpha\| \to 0} \frac{\hat{w} - w_o}{w - w_o} \tag{23}$$

In the general case of an arbitrary M,

$$T_m(\alpha, \beta) = (I + \beta T(\alpha) M)^{-1} T(\alpha) (I + \beta M T_o) \tag{24}$$

and it follows that

$$S_m(\beta) = \| (I + \beta T_o M)^{-1} \| \tag{25}$$

Thus, with $M = T_o^{-1}$, we would have

$$S_m(\beta) = (1 + \beta)^{-1} \tag{26}$$

The M that was constructed above is an approximation to T_o^{-1} and in fact is equal to T_o^{-1} on a finite dimensional subspace. Thus, we can expect that the compensator will exhibit reduced sensitivity.

For the system of Figure 2, defining a similar sensitivity index,

$$S(\beta) = \lim_{\|\alpha\| \to 0} \|(T(\alpha, \beta) - T_o)(T(\alpha) - T_o)^{-1}\| \tag{27}$$

where $T(\alpha, \beta)$ is the closed-loop map from u to $\overline{w}$. This leads to the same result as obtained for $S_m(\beta)$, i.e.,

$$S(\beta) = \|(I + \beta T_o M)^{-1}\|$$

Our major interest was to assess the utility of this concept to reduce sensitivity.

As a by-product of our computational analysis, we also examined the utility of the concept for on-line parameter identification for the simple illustrative example. The configuration is shown in Figure 3. In this case, $\hat{M}$ is a finite dimensional inverse to the map $\hat{T}(u)$ from α to $(T(\alpha) - T_o) u = w - w_o$.

THE ILLUSTRATIVE EXAMPLE

Several experiments were performed with the following scalar system assuming two uncertain parameters:

$$T(\alpha_1, \alpha_2): \; \dot{x}(t) = -(1 + \alpha_1) x(t) + (1 + \alpha_2) u(t), \; x(0) = 0$$

$$w(t) = x(t)$$

where the nominal values of α_1 and α_2 are zero. In addition to examining the performance of the two compensators (with and without an explicit model), the performance of an on-line identifier was analyzed.

The partial inverse for the compensators was constructed assuming for simplicity that the input u was a unit step. As such, the response of the nominal system denoted by x_0 (t) and the input-output pairs, $[y_i(t), \ \xi_i(t)]$, corresponding to perturbations in α_1 are:

$$x_0(t) = 1 - e^{-t} \tag{28}$$

$$y_1(t) = -(1 - e^{-t}) \tag{29}$$

$$\xi_1(t) = te^{-t} - (1 - e^{-t}) \tag{30}$$

$$y_2(t) = 1 \tag{31}$$

$$\xi_2(t) = 1 - e^{-t} \tag{32}$$

The pairs $[y_i(t), \ \xi_i(t)]$ were computed from the equations

$$\dot{\xi}_1 = -\xi_1 - x_0\alpha_1, \ \xi_1(0) = 0 \tag{33}$$

$$\dot{\xi}_2 = -\xi_2 + u\,\alpha_2, \ \xi_2(0) = 0 \tag{34}$$

$$y_i = \dot{\xi}_i - \xi_i, \ i = 1, 2 \tag{35}$$

with $\alpha_1 = \alpha_2 = u = 1$. Equations (33) and (34) which define ξ_1 and ξ_2 are the variational equations associated with the parameters α_1 and α_2. Equations (35) which define y_1 and y_2 are the nominal input-output relations.

The on-line identifier was constructed by choosing the output to be

$$\hat{\alpha} = \begin{bmatrix} \text{Estimate of } \alpha_1 \\ \text{Estimate of } \alpha_2 \end{bmatrix} \tag{36}$$

In this case the input-output pairs $[\hat{y}_i, \xi_i]$ are given by

$$\hat{y}_1 = [1 \;\; 0], \; \hat{y}_2 = [0 \;\; 1] \tag{37}$$

$$\dot{\xi}_1 = -\xi_1 - x_0, \; \dot{\xi}_2 = -\xi_2 + u, \; \xi_i(0) = 0 \tag{38}$$

with ξ_i being computed on line for arbitrary inputs u(t).

Experiments were conducted to examine the effects on the compensator performance of

- Magnitude and direction of the parameter vector α,
- Magnitude of the design parameter β, and
- Sensor noise.

Transient responses for a five-second interval were computed for various combinations of α and β with and without sensor noise. Noise-to-signal ratios less than or equal to one caused no significant changes in performance. Quantitative results for the variations in α and β with no sensor noise are summarized below.

A typical response plot of the output w(t) is shown in Figure 4 for a step input.

The responses of the compensated systems are closer to the nominal response than is the uncompensated response. The effect of the parameter β is also evident in Figure 4. The larger β yields less deviation from the nominal. The error ratios corresponding to the responses of Figure 4 are shown in Figure 5. Both these figures display a significant initial transient

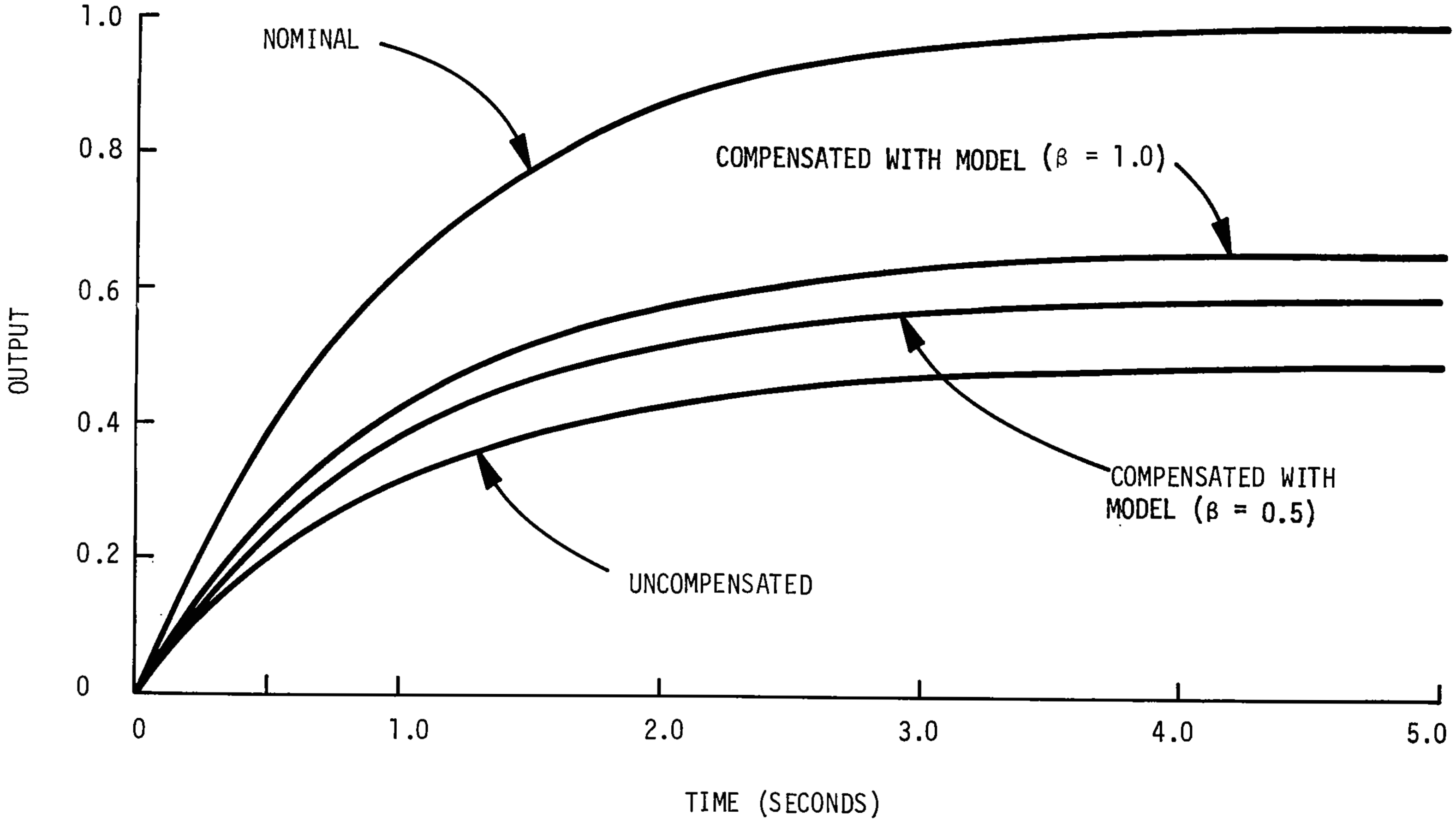


Figure 4. Responses to Unit Step ($\alpha_1 = 0$, $\alpha_2 = -0.5$)

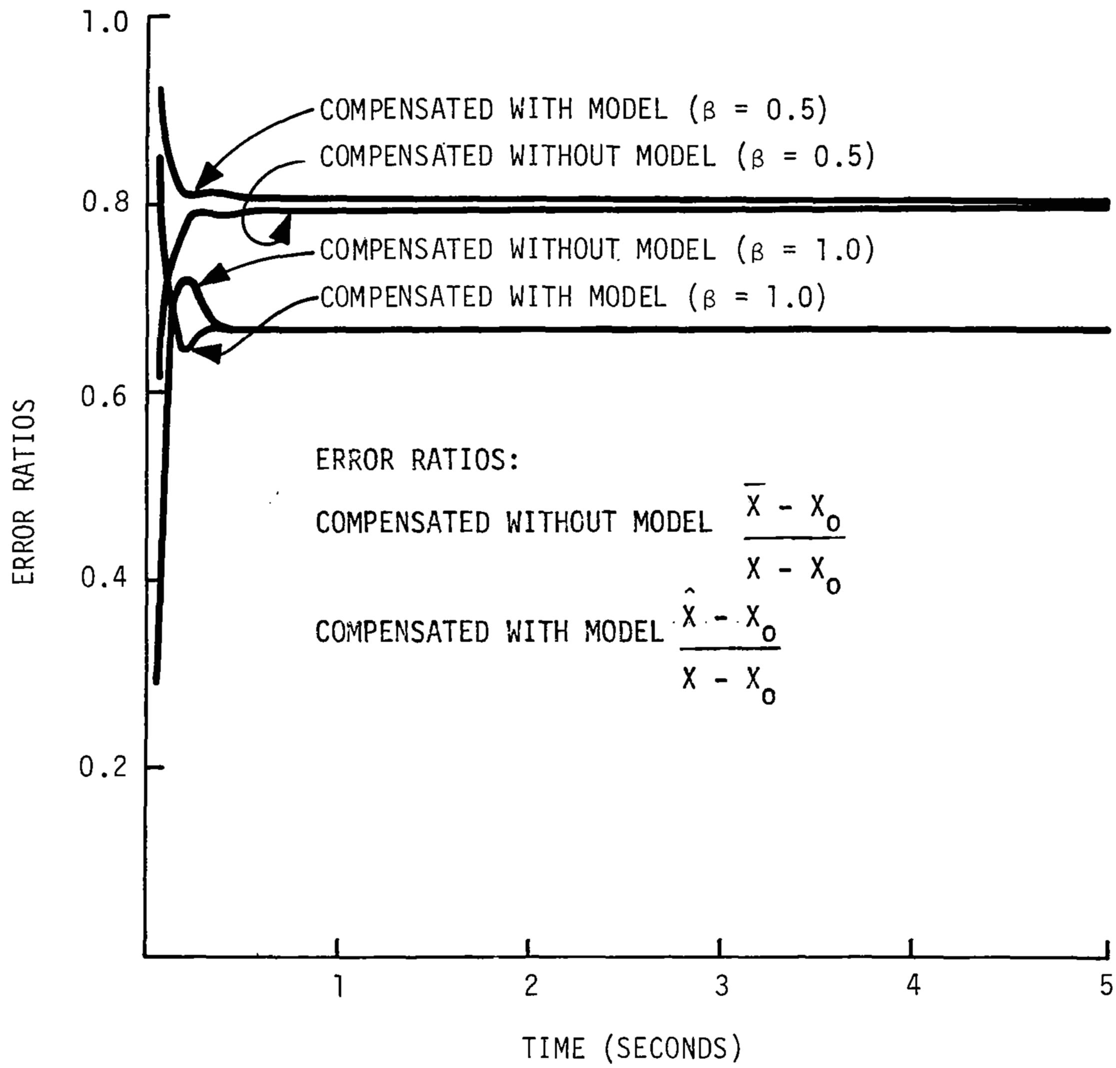


Figure 5. Error Ratios for Step Responses ($\alpha_1 = 0$, $\alpha_2 = -0.5$)

for the compensator without the model. This transient is caused by the initial high gain on the system output. In the compensator with the model, this high gain is multiplied by the error rather than the output, and the initial transient is greatly subdued. A time-varying β could be introduced to alleviate the initial transient of the compensator without the model.

The theoretically predicted values for the error ratio is $(\beta + 1)^{-1}$ which gives 1/2 for $\beta = 1$ and 2/3 for $\beta = 1/2$. These values are somewhat less than the steady state values shown in Figure 5. The parameters for this figure are $\alpha_1 = 0$ and $\alpha_2 = -1/2$. The error ratios for a case with $\alpha_1 = 0$ and $\alpha_2 = -1/10$ are shown in Figure 6 for $\beta = 1$. In this case of smaller parameter magnitudes, the steady state ratios very closely approximate the theoretically predicted value of 1/2. The direction of the vector (α_1, α_2) also influences the degree of compensation. The error ratios after five seconds for 12 different values of (α_1, α_2) and three values of β are given in Table 1. The data generally confirm the theory for small values of α_1 and α_2 and the predicted trend in β. The major deviation occurs when the parameters are equal and are of the same sign. In this case the steady state values of the outputs for the compensated and uncompensated systems are equal. Thus, the denominator of the error ratio is approaching zero, and the fact that the ratios have the magnitudes shown is an indication that the compensators behave well even in this case.

The on-line identifier was also evaluated by computing five-second transient responses. Responses to a unit step input with zero initial condition for several values of (α_1, α_2) are shown in Figure 7. Four general characteristics are evident in this figure. First, if α_1 is zero, the estimate of α_2 is exact to within the computer word-length accuracy. Second, for

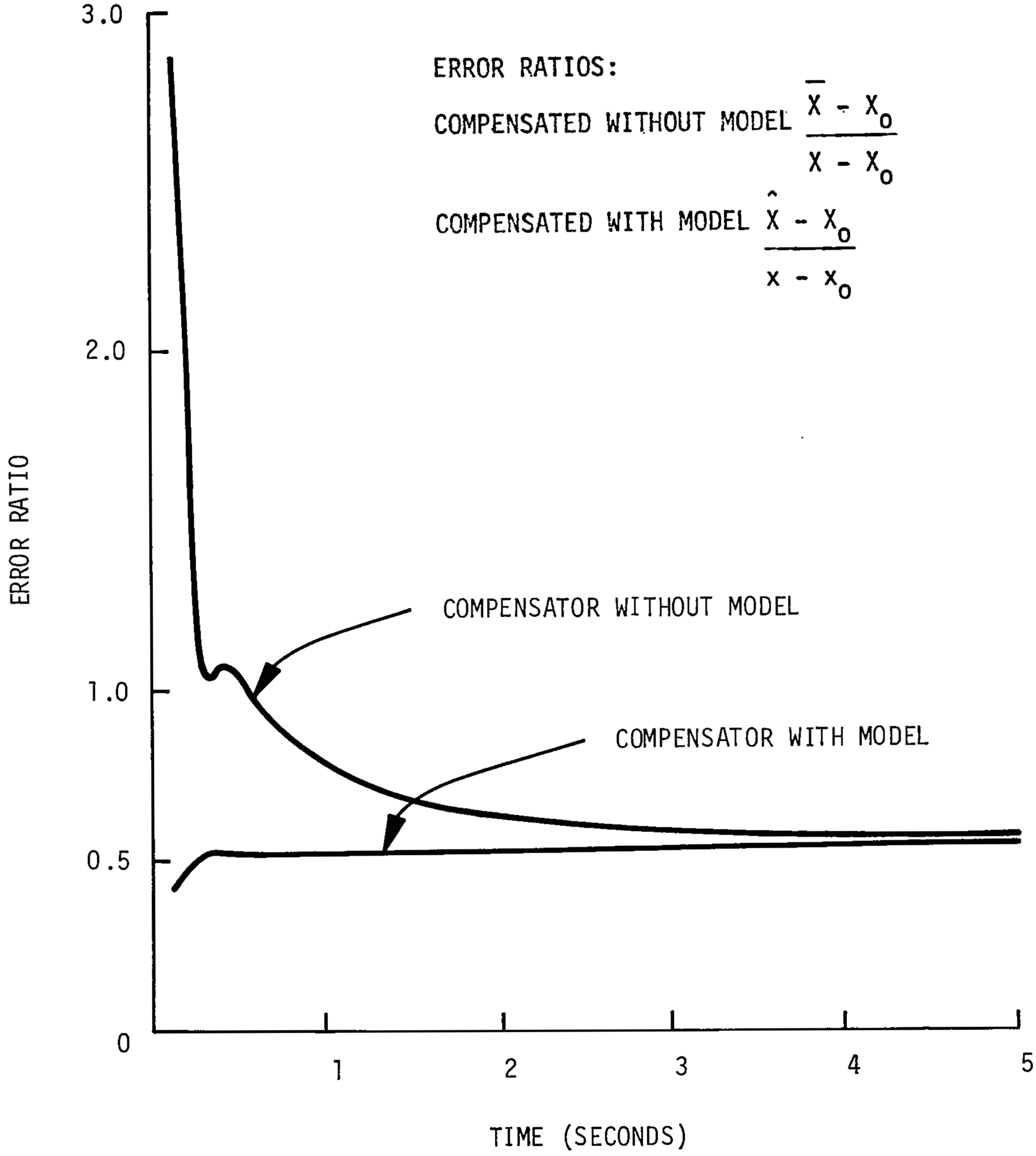


Figure 6. Error Ratios for Step Responses ($\alpha_1 = 0.0$, $\alpha_2 = -0.1$) with $\beta = 1$ and Sensor Noise

TABLE 1. ERROR RATIOS FOR VARIOUS VALUES OF α_1, α_2, β

		$\beta = 0.5$ Theoretical Ratio .67		$\beta = .75$ Theoretical Ratio .57		$\beta = 1.0$ Theoretical Ratio .50	
α_1	α_2	$\frac{\hat{x}-x_o}{x-x_o}$	$\frac{\bar{x}-x_o}{x-x_o}$	$\frac{\hat{x}-x_o}{x-x_o}$	$\frac{\bar{x}-x_o}{x-x_o}$	$\frac{\hat{x}-x_o}{x-x_o}$	$\frac{\bar{x}-x_o}{x-x_o}$
0.1	0	0.685	0.686	0.592	0.606	0.521	0.559
0	0.1	0.645	0.642	0.547	0.528	0.474	0.428
0.1	0.1	0.738	0.630	0.606	-0.0784	0.485	-1.153
-0.1	0.1	0.624	0.622	0.524	0.515	0.452	0.430
0.5	0	0.742	0.742	0.658	0.661	0.591	0.599
0	0.5	0.571	0.568	0.468	0.459	0.396	0.376
0.4	0.4	1.024	0.885	0.827	0.267	0.624	-0.593
-0.4	0.4	0.485	0.484	0.384	0.380	0.316	0.308
1.0	0	0.792	0.792	0.718	0.719	0.656	0.661
0	1.0	0.498	0.495	0.396	0.386	0.327	0.309
0.7	0.7	1.292	1.075	0.997	0.248	0.687	-0.852
-0.7	0.7	0.342	0.340	0.251	0.249	0.198	0.193

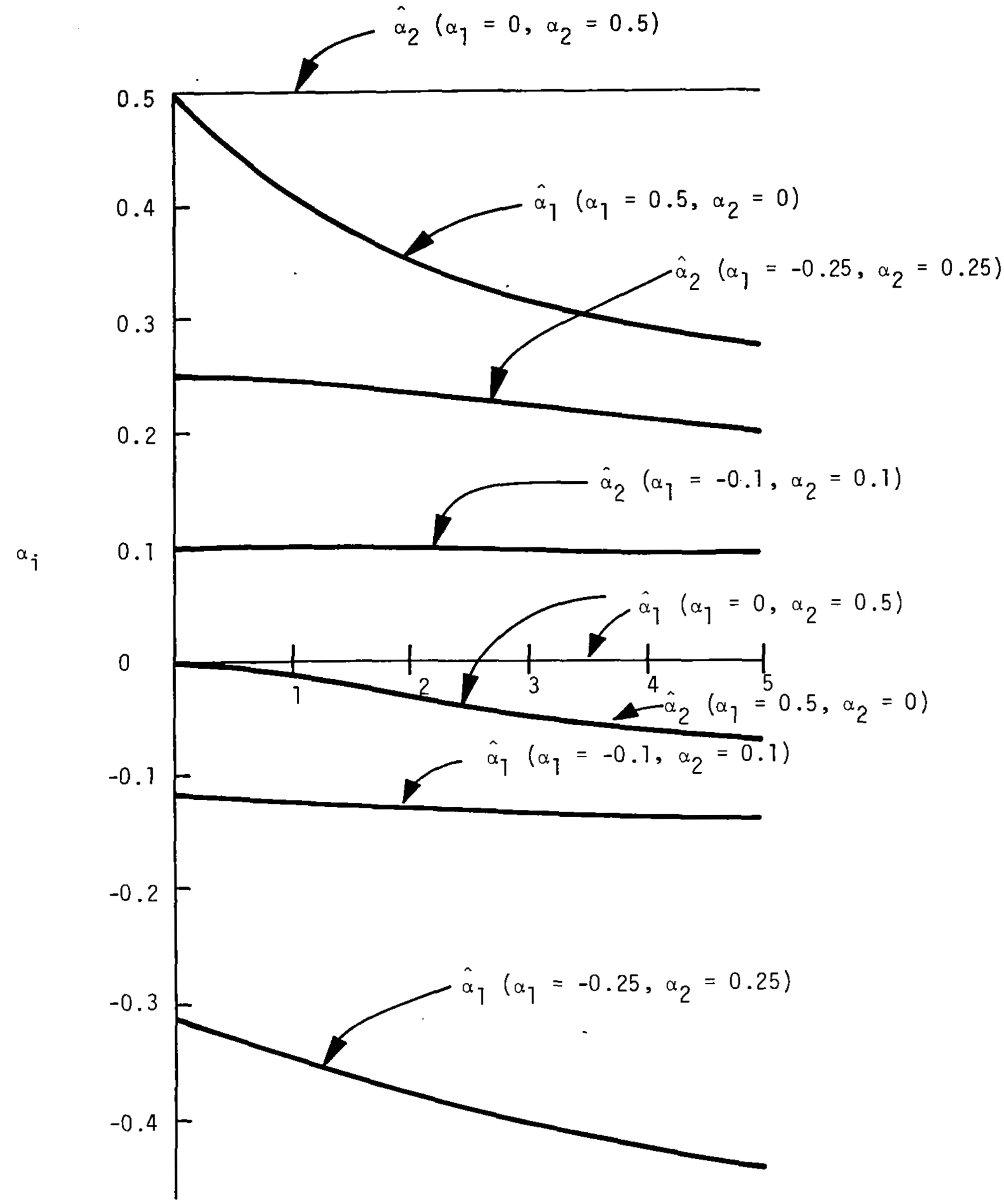


Figure 7. On-Line Identifier Responses to Step Input, u = 1

nonzero α_1, the estimate of α_2 is more accurate than the estimate of α_1. Third, for nonzero α_1, the accuracy of the estimate is greatest in the initial phase of the response. Fourth, accuracy is better for small magnitudes than for large magnitudes of $\alpha_1^2 + \alpha_2^2$ when α_1 is nonzero.

The first characteristic is a consequence of the fact that the system output is linear in α_2 for zero initial conditions. This fact also contributes to the estimates of α_2 being more accurate than the estimates of α_1. The third characteristic is a consequence of the fact that as the system approaches steady state, the estimate is given by

$$\begin{bmatrix} \hat{\xi}_1^2 & \hat{\xi}_1 \hat{\xi}_2 \\ \hat{\xi}_1 \hat{\xi}_2 & \hat{\xi}_2^2 \end{bmatrix}_{ss} \hat{\alpha}_{ss} = 1/2 \begin{bmatrix} \hat{\xi}_1 (x-x_o) \\ \hat{\xi}_2 (x-x_o) \end{bmatrix}_{ss} \tag{39}$$

which gives the linear relation

$$(\hat{\alpha}_2 - \hat{\alpha}_1)ss = (x-x_o)_{ss} \tag{40}$$

For this example, the only correct estimate satisfying Equation (40) occurs when $\alpha_1 = 0$ and α_2 is arbitrary. The fourth characteristic is a manifestation of the nonlinearity with respect to α_1 of the system output.

Responses of the on-line identifier to a sinusoidal input are shown in Figure 8. The same general characteristics as for the step input occur.

Additional responses were computed for other values of (α_1, α_2) and for cases with "sensor noise" added in the simulation. The noise caused no serious degradation in performance.

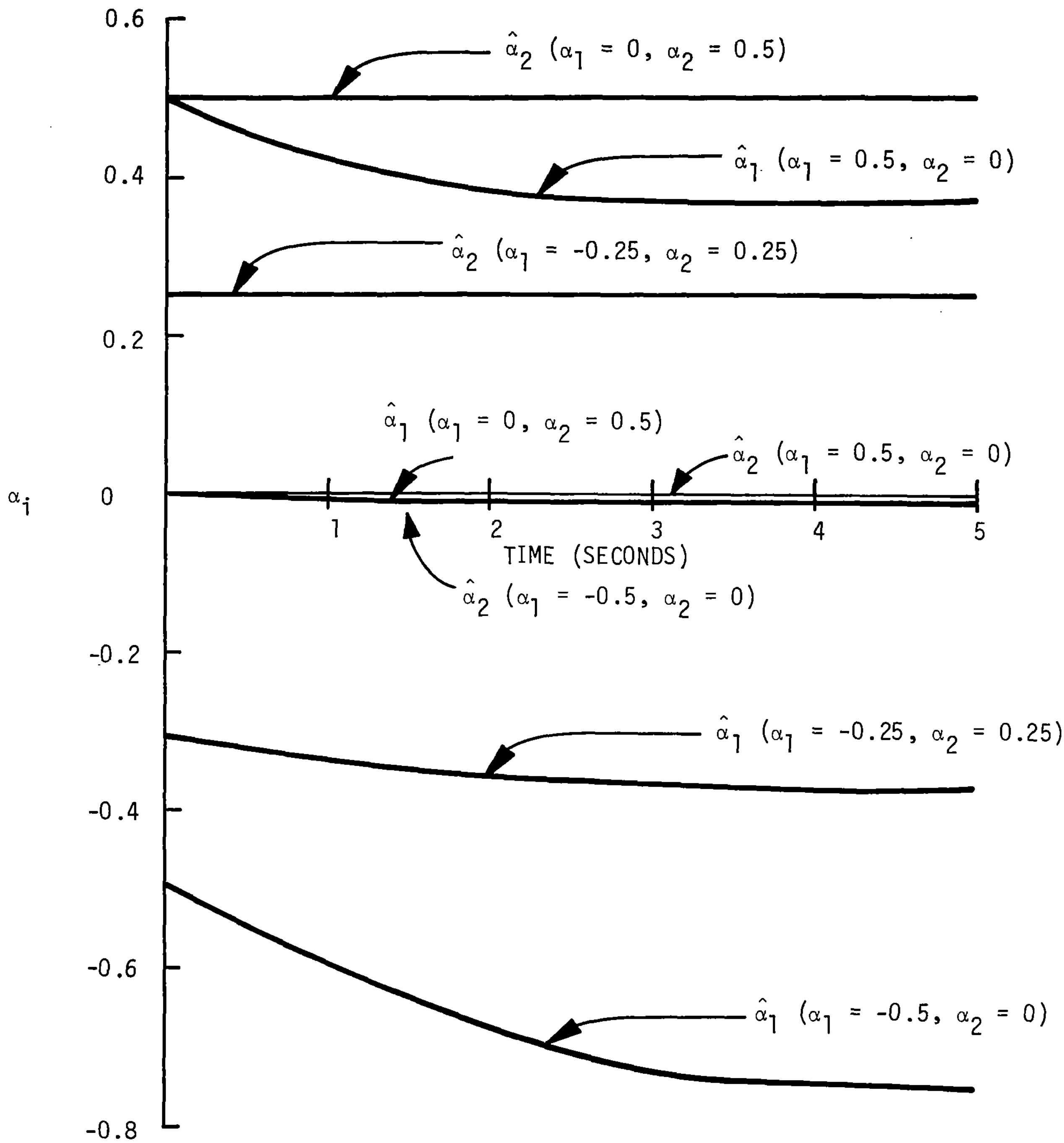


Figure 8. On-Line Identifier Responses to u = sin(10t)

In summary, the compensator and the on-line identifier fulfilled the theoretical expectations.

THE C-5A EXAMPLE

A seventh order model of the C-5A was used as the design model. This model was generated from the 79th order model described in Reference 1 as Case 1. The unsteady aerodynamics were truncated as in Reference 1 to arrive at a 42nd order model called Case 2. Then a 13th order model called Case 5 was computed by the process of residualization. This model retained only one flexure mode. The seventh order gust model was then approximated by a first order gust model yielding the seventh order reduced model called Case 5R'. The data for this model are given in the Appendix A. The parameter variations considered were the same as those in Reference 1.

In Reference 1, a 15th order model called Case 4R which retained two flexure modes was used as a design model. Comparison of the open-loop performance of Case 4R, Case 5R, and Case 5R' models is given in Table 2. The results indicate a high degree of consistency.

A nominal controller for Case 5R' was computed using quadratic optional theory with the same response vector and weights as used in the nominal controller design for Case 4R. The response vector and quadratic weights are given in Table 3. The closed-loop performance for the nominal parameter setting for Case 4R and Case 5R' is shown in Table 4. Again, there is a high degree of consistency.

TABLE 2. OPEN-LOOP PERFORMANCE COMPARISON

	Case 4R	Case 5R	Case 5R'
Maneuver load bending (N-m)	0.0427×10^6	0.0427×10^6	0.0427×10^6
Gust load bending (N-m)	0.12×10^6	0.12×10^6	0.12×10^6
Gust load torsion (N-m)	0.179×10^5	0.179×10^5	0.174×10^5
Short period frequency (rad/sec)	1.62	1.55	1.55
Short period damping (sec^{-1})	0.56	0.57	0.57

TABLE 3. RESPONSE VECTOR AND QUADRATIC WEIGHTS

Response Vector	Physical Quantity	Weight
r_{d_1}	B_1 = bending moment at wing root	1×10^{-10}
r_{d_2}	T_1 = torsion moment at wing root	1×10^{-9}
r_{d_3}	$\dot{B}_1$ = rate of change of bending moment at wing root	5.5×10^{-13}
r_{d_4}	T_1 = rate of change of torsion moment at wing root	1×10^{-11}
r_{d_5}	δ_a = aileron displacement	0.32×10^{8}
r_{d_6}	δ_{e_i} = inboard elevator displacement	0
r_{d_7}	δ_{a_m} = function of aileron displacement and aileron command	1×10^{6}
r_{d_8}	$\dot{\delta}_{e_i}$ = inboard elevator rate	1×10^{4}
r_{d_9}	r_{CF} = control follower response	2×10^{5}

TABLE 4. CLOSED-LOOP PERFORMANCE COMPARISON

	Criterion	Case 4R	Case 5R'
Maneuver load control bending % change	< -30%	-40%	-41%
Gust load--bending alleviation	< -30%	-35%	-34%
Percent change--torsion	< + 5%	-31%	-27%
Handling ω_{sp}(rad/sec)	> 1.6	2.13	2.16
Qualities ζ(sec^{-1})	0.7 - 0.8	72	0.73
Stability Gain δa	> 6db	∞	∞
δe		29db	∞
Phase δa	> 45°	∞	∞
δe		∞	∞

The equations used to implement the compensator for this example are given below. In this case the input is two-dimensional and the inverse is based on independent step inputs in each channel. The equations are given for a general two-dimensional output.

The equations for the nominal states and outputs are

$$\dot{x}_i = (F_o + G_1 K_o)\, x_i + G_1\, u_i \tag{41}$$

$$w_i = (H_o + D_o K_o)\, x_i + D_o\, u_i \tag{42}$$

where i = 1, 2 represents step inputs on the aileron and inboard elevators, respectively, and the subscript o indicates nominal value.

The equations for the variations in states and outputs are

$$\dot{x}\,(i,j) = (F_o + G_1 K_o)\, x\,(i,j) + (F_j - F_o)\, x_i \tag{43}$$

$$\xi\,(i,j) = (H_o + D_o K_o)\, x\,(i,j) + (H_j - H_o)\, x_i + (D_j - D_o)\, u_i \tag{44}$$

where F_j, H_j, and D_j are computed from the general expressions for F, H, and D in terms of q_f, ω_f, and M_{w_f} with j = 1, 2, 3 denoting the following specific variations:

$j = 1 : q_f = 1.0,\ \omega_f = 1.0,\ M_{w_f} = 0.8$

$j = 2 : q_f = 1.0,\ \omega_f = 0.75,\ M_{w_f} = 1.0$

$j = 3 : q_f = 1.25,\ \omega_f = 1.0,\ M_{w_f} = 1.0$

The equations relating the variational outputs that are to be outputs of the nominal system are

$$\dot{\hat{x}}(i,j) = (F_o + G_1 K_o)\,\hat{x}(i,j) + G_1\, y(i,j) \tag{45}$$

$$\hat{\xi}(i,j) = (H_o + D_o K_o)\,\hat{x}(i,j) \tag{46}$$

where $y(i,j)$ is the inverse input defined such that $\xi(i,j) = \hat{\xi}(i,j)$. An expression for $y(i,j)$ will now be derived. This derivation will assume that the number of outputs equals the number of controls.

Let us introduce

$$e(i,j) = x(i,j) - \hat{x}(i,j) \tag{47}$$

and

$$\tilde{e}(i,j) = \xi(i,j) - \hat{\xi}(i,j) \tag{48}$$

Then

$$\begin{aligned} \dot{e}(i,j) &= \overline{F}_o\, x(i,j) + (F_j - F_o)\, x_i - \overline{F}_o\, \hat{x}(i,j) - G_1 y(i,j) \\ &= \overline{F}_o\, e(i,j) + (F_j - F_o)\, x_i - G_1\, y(i,j) \end{aligned} \tag{49}$$

where

$$\overline{F}_o = (F_o + G_1 K_o)$$

Defining $\overline{H}_o = (H_o + DK_o)$, Equation (48) may be written as

$$\tilde{e}(i,j) = \overline{H}_o\, e(i,j) + (H_j - H_o)\, x_i + (D_j - D_o)\, u_i \tag{50}$$

Assuming $u_i = 0$ and differentiating Equation (50), we have

$$\begin{aligned} \dot{\tilde{e}}(i,j) &= \overline{H}_o\, \dot{e}(i,j) + (H_j - H_o)\, \dot{x}_i \\ &= \overline{H}_o\, [\overline{F}_o\, e(i,j) + (F_j - F_o)\, x_i - G_1\, y(i,j)] + (H_j - H_o)\, \dot{x}_i \end{aligned} \tag{51}$$

Now $y(i,j)$ is to be such that $\tilde{e}(i,j)$ is identically zero, which is equivalent to $\dot{\tilde{e}}(i,j)$ being zero since $x(i,j)$, $\hat{x}(i,j)$, x_i, and u_i are initially zero. Thus, setting the left-hand side of Equation (51) to zero, we may solve for $y(i,j)$ obtaining

$$y(i,j) = [\overline{H}_o G_1]^{-1} \{[\overline{H}_o \overline{F}_o e(i,j) + \overline{H}_o (F_j - F_o) x_i] + (H_j - H_o)\dot{x}_i\} \tag{52}$$

Using this expression for $y(i,j)$, Equation (49) may be rewritten as

$$\dot{e}(i,j) = [I - G_1(\overline{H}_o G_1)^{-1} \overline{H}_o] [\overline{F}_o e(i,j) + (F_j - F_o) x_i]$$
$$- G_1(\overline{H}_o G_1)^{-1} (H_j - H_o) \dot{x}_i \tag{53}$$

Now the dot products (for $i, j = 1, 2, 3$) are defined as

$$\dot{v}_{ij} = \xi(1,i) \cdot \xi(1,j) \tag{54}$$

$$\dot{v}_{i+3,\, j+3} = \xi(2,i) \cdot \xi(2,j) \tag{55}$$

$$\dot{v}_{i,\, j+3} = \dot{v}_{j+3,\, i} = \xi(1,i) \cdot \xi(2,j) \tag{56}$$

$$\dot{q}_j = \xi(1,j) \cdot w_{measured} \tag{57}$$

$$\dot{q}_j = \xi(2,j) \cdot w_{measured} \tag{58}$$

Then defining

$$p_k = \begin{cases} q_k / \sqrt{v_{kk}} & \text{if } v_{kk} \neq 0 \\ 0 & \text{if } v_{kk} = 0 \end{cases}, \quad k = 1, 2, \ldots, 6 \tag{59}$$

$$\hat{y}_j = \begin{cases} y(1,j) / \sqrt{v_{jj}} & \text{if } v_{jj} \neq 0 \\ 0 & \text{if } v_{jj} = 0 \end{cases}, \quad j = 1, 2, 3 \tag{60}$$

$$\hat{y}_{j+3} = \begin{cases} y(2,j) \,/\sqrt{v_{j+3,j+3}} & \text{if } v_{j+3,j+3} \neq 0 \\ 0 & \text{if } v_{j+3,j+3} = 0 \end{cases}, \quad j = 1,2,3 \tag{61}$$

$$K = N^{-1} \tag{62}$$

$$N_{kk} = 1, \; k = 1,2,\ldots,6 \tag{63}$$

$$N_{k\ell} = v_{k\ell} \,/\sqrt{v_{kk} v_{\ell\ell}}, \; k,\ell = 1,2,\ldots,6 \tag{64}$$

the output of the compensator is

$$z = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_6] \, K \, p \tag{65}$$

where $\hat{y}$ is $y\Lambda$ and $\Lambda = \text{diag}\,(\ldots v_{ii}^{-\frac{1}{2}}\ldots)$.

In the first attempt to implement the compensator, the first two responses, bending moment and torsion moment, were chosen as the outputs. Unfortunately, the transfer matrix from the inputs to these outputs has a zero in the right half-plane. This caused the coefficient matrix in Equation (53), namely $[I - G_1(\bar{H}_o G_1)^{-1} \bar{H}_o]\bar{F}_o$, to have eigenvalues in the right half-plane leading to divergent functions $y(i,j)$. Rather than implement this compensator with internal instability, it was decided to choose the aerodynamic surface positions, δa and δe_i, as the outputs. This choice eliminated the internal instability.

Evaluation

Experiments were conducted to examine the quantitative performance of the compensator. The parameter variations considered were the three major parameter variations of Reference 1, i.e., dynamic pressure, $\bar{q}$;

structural frequency, ω; and the stability derivative, M_w. Dynamic pressure variations cause changes in all the aerodynamic terms and, hence, most of the elements of the coefficient matrices. Structural frequency variations induce variations in a significant subset of coefficients in the complete model. The variation in M_w permits examination of the effect of variation in a single coefficient. The physical motivation for treating these variations is discussed in Reference 1. This example also has permitted examination of the effect of authentic gust disturbances. A single sample on a five-second interval was used to examine the gust effect.

To examine the effect of unmodeled dynamics, the compensator designed for the 7th order model was used in conjunction with the 15th order model. Experiments were also conducted to test the effect of the design parameter, β. The model was incorporated in the compensator for all the C-5A experiments. Another experiment conducted tested the effect of recycling the time-varying gains of the compensator. These gains were computed off line for a five-second interval.

In the recycle experiment, the gains for the first fourth of this interval were used repeatedly in each succeeding fourth of the interval. This experiment was motivated by two considerations. The first is that the example is essentially stationary and that infinite data lengths are impractical. The second is that the assumption of linearity of the outputs with respect to parameter variations is less valid for longer time intervals than for short ones. This phenomenon is shown in Figure 9. The two components of $\xi(2, 1)$, the output of the variational equation associated with M_w for an inboard elevator step input, are shown along with the actual increments

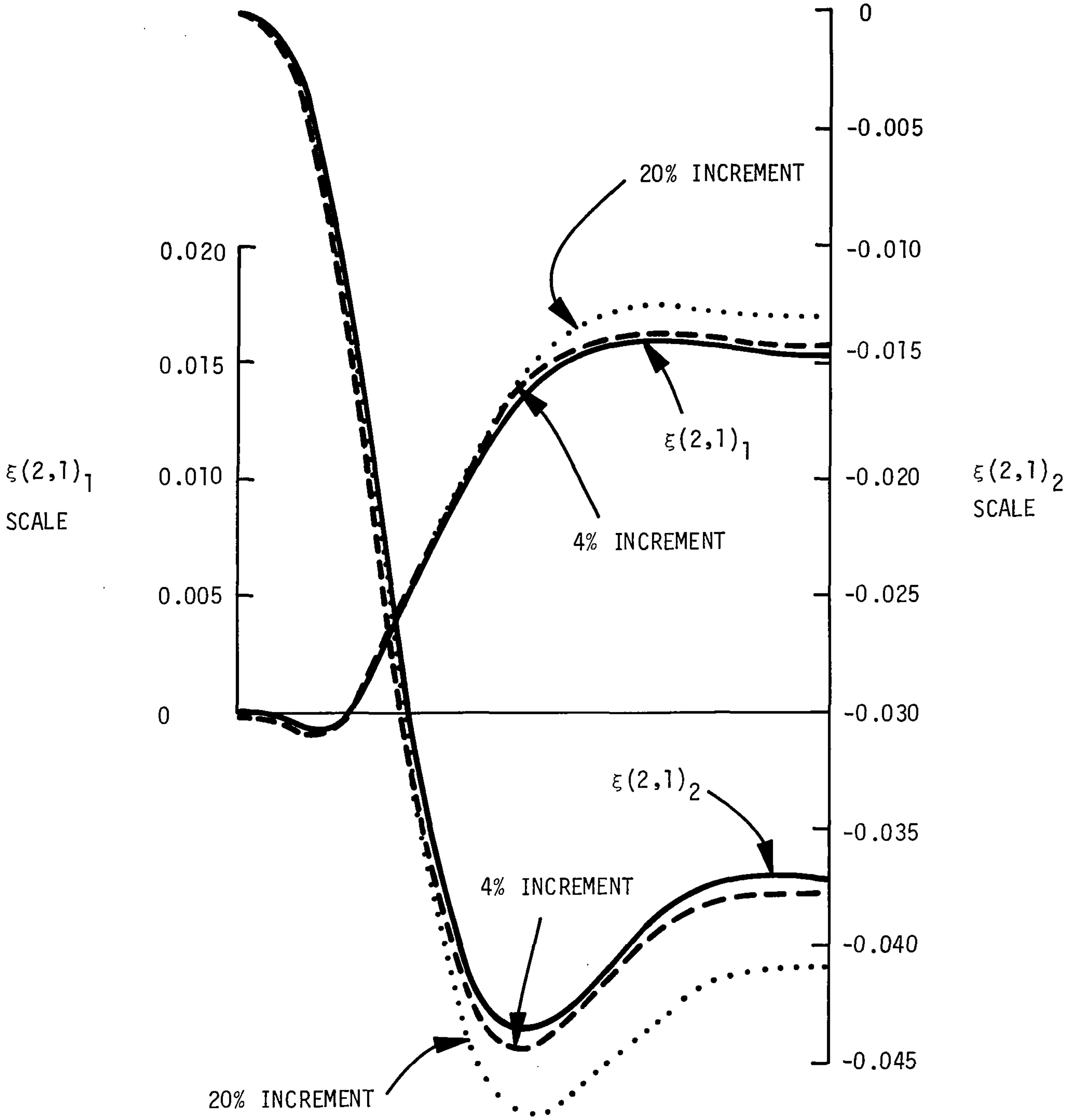


Figure 9. Variational Output and Incremental Outputs for Unit Step δe_i, M_w Variation

in the outputs corresponding to 4 and 20 percent variations in M_w for the same step input. The nonlinear effect is clearly more pronounced in the later part of the five-second interval. This phenomenon is common to all the components of the ξ's.

To determine the effects of the design parameter, β, responses to gusts and step inputs of 0.02 radian magnitude in u_1 and u_2 were computed. To determine the effects of different parameter variations, such responses were computed for independent variations of individual parameters and two cases of variations in all three parameters. These latter two cases were found to be "worst case" variations in Reference 1.

Figures 10 and 11 show the deviations from nominal of control surface deflections in response to step inputs for $\beta = 0$, 0.5, 1.0, and 3.0 with the model in the loop and a 20 percent variation in M_w. The reduction in these deviations for increasing β is consistent with the theoretical prediction.

The effect of recycling is shown with $\beta = 1$ in Figure 12 for the same parameter variation and a step in u_1. The recycling induces a severe transient following the start of each recycle. This is due in part to the fact that the compensator output is zero for an initial interval, and in part to the effective high gain in the system following this zero output interval. Although the recycling tended to deteriorate the δa response, it improved the δe_i response.

Figures 13 and 14 show the effect of changing the input. The inputs for these responses were chosen to be a positive step for 1 sec, followed by

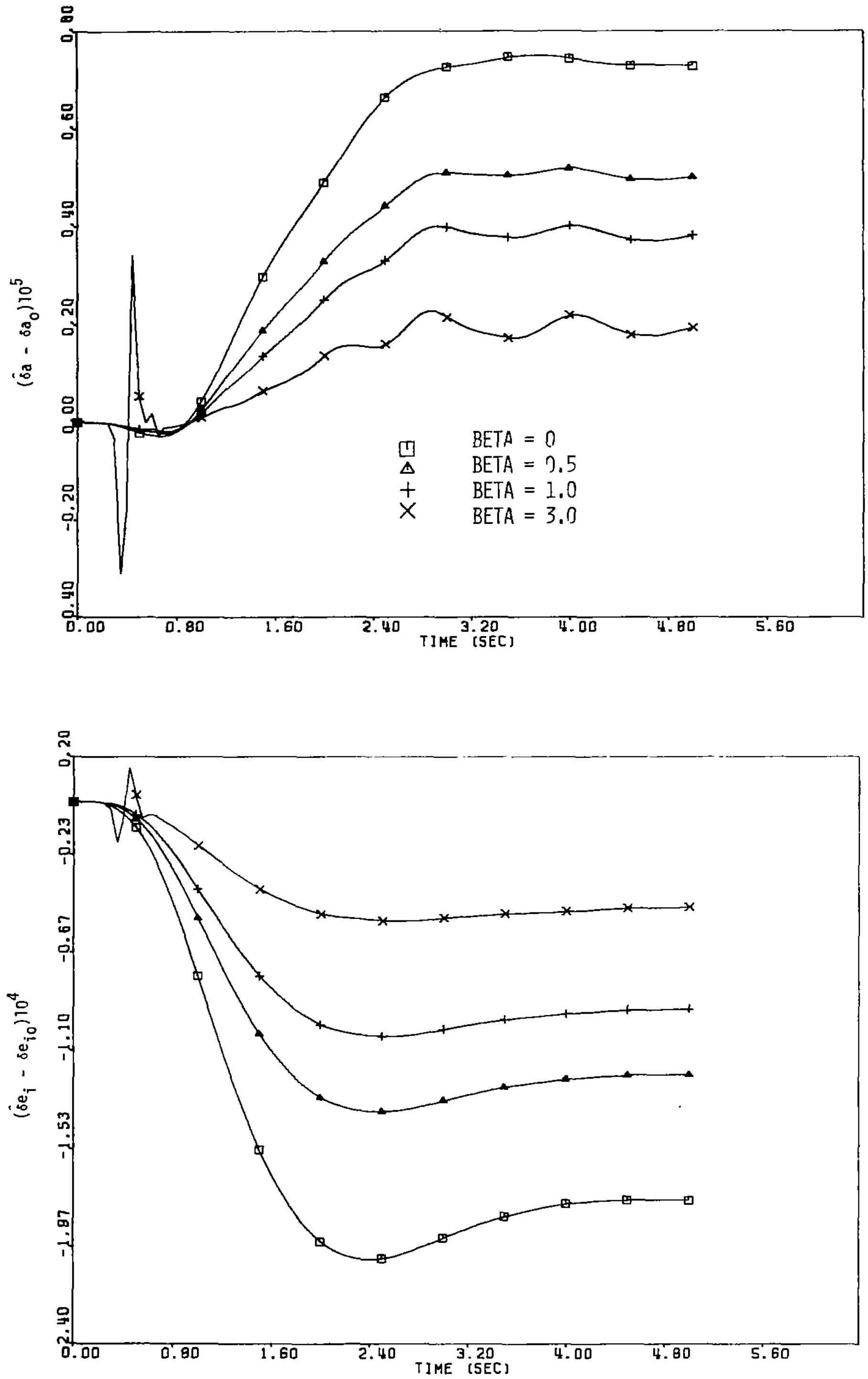


Figure 10. Response to $u_1 = 0.02$ for $\overline{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

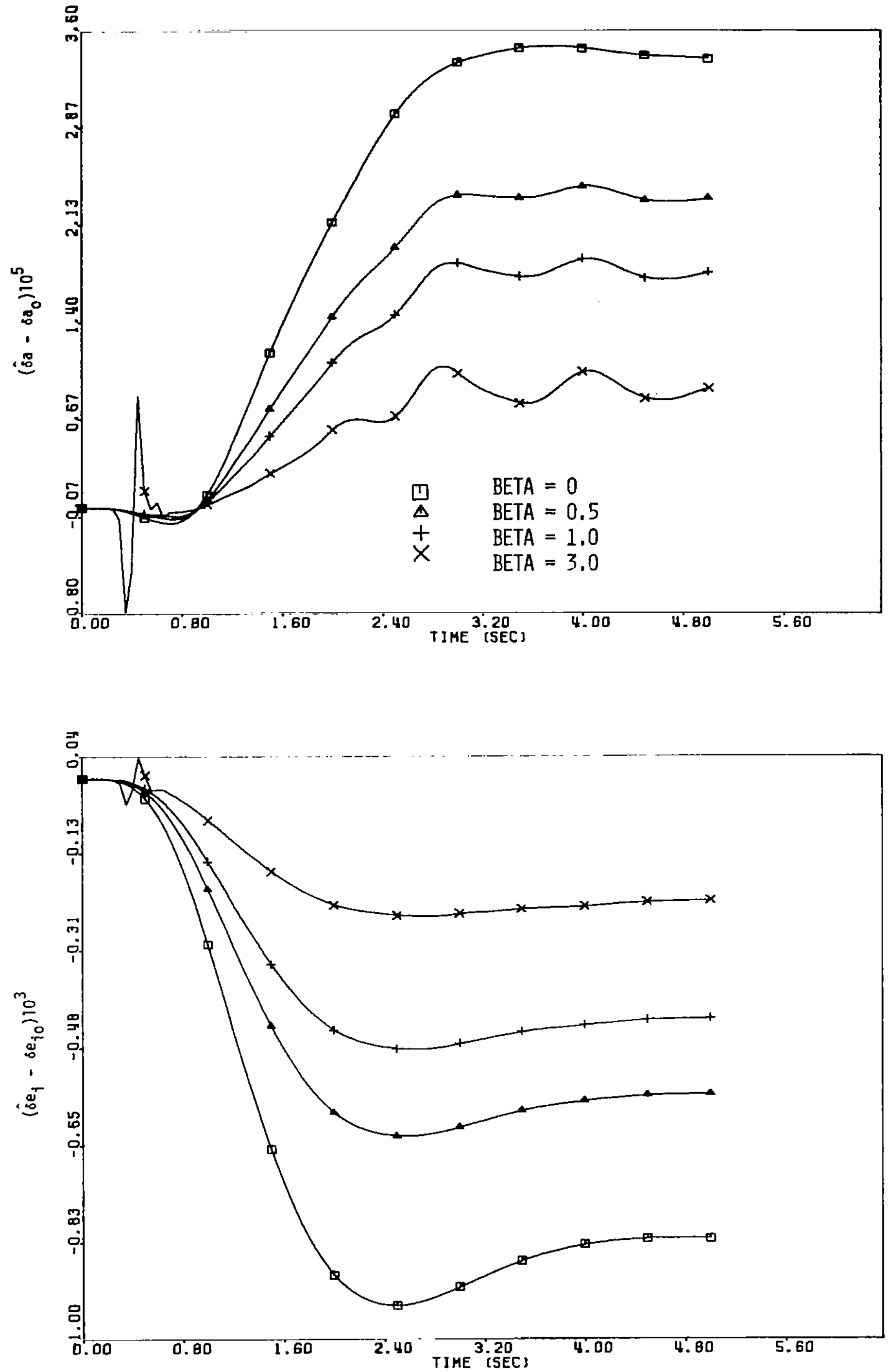


Figure 11. Response to $u_2 = 0.02$ for $\bar{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

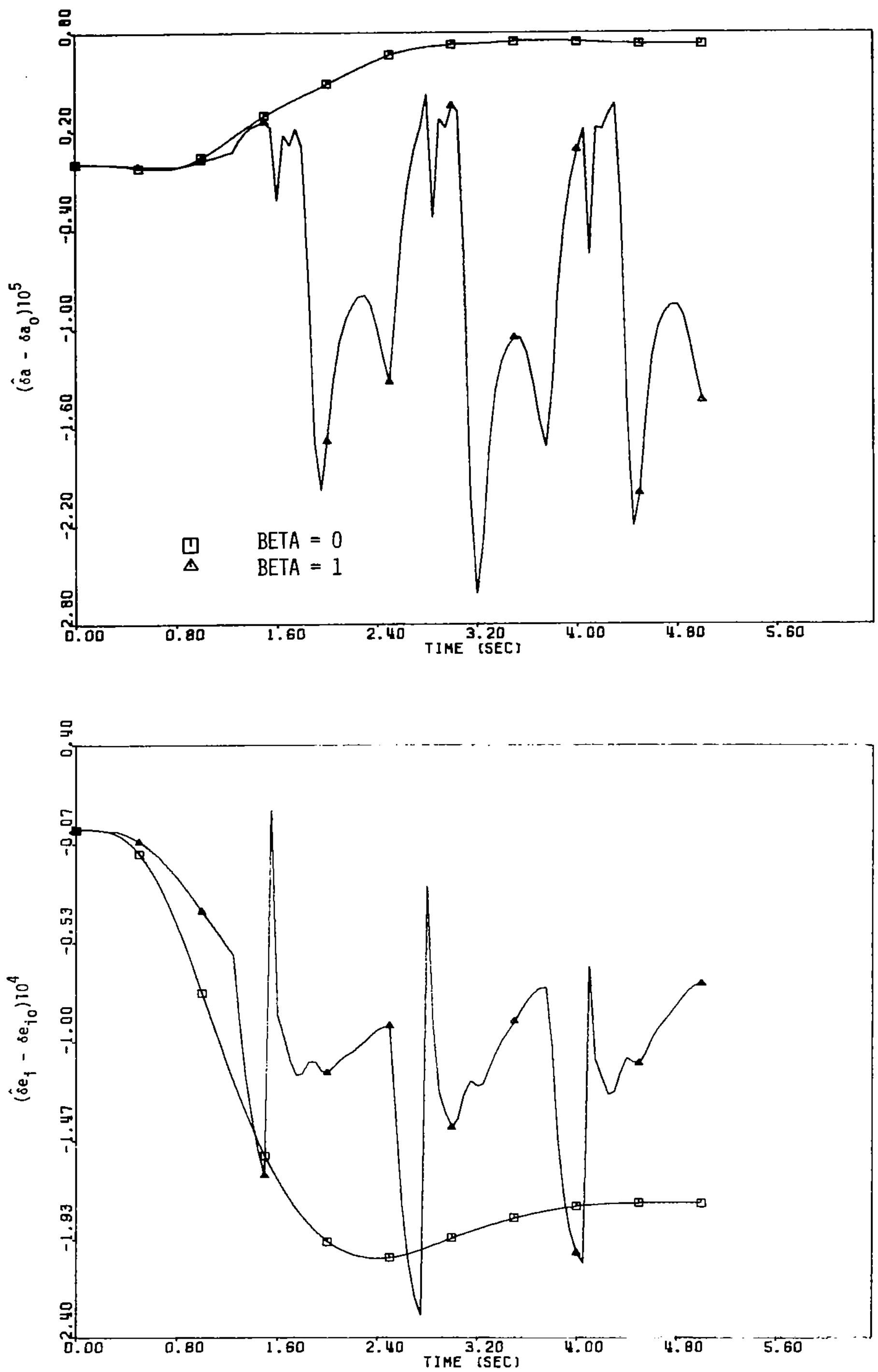


Figure 12. Response to $u_1 = 0.02$ for $\bar{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$ with recycle every 1.25 second

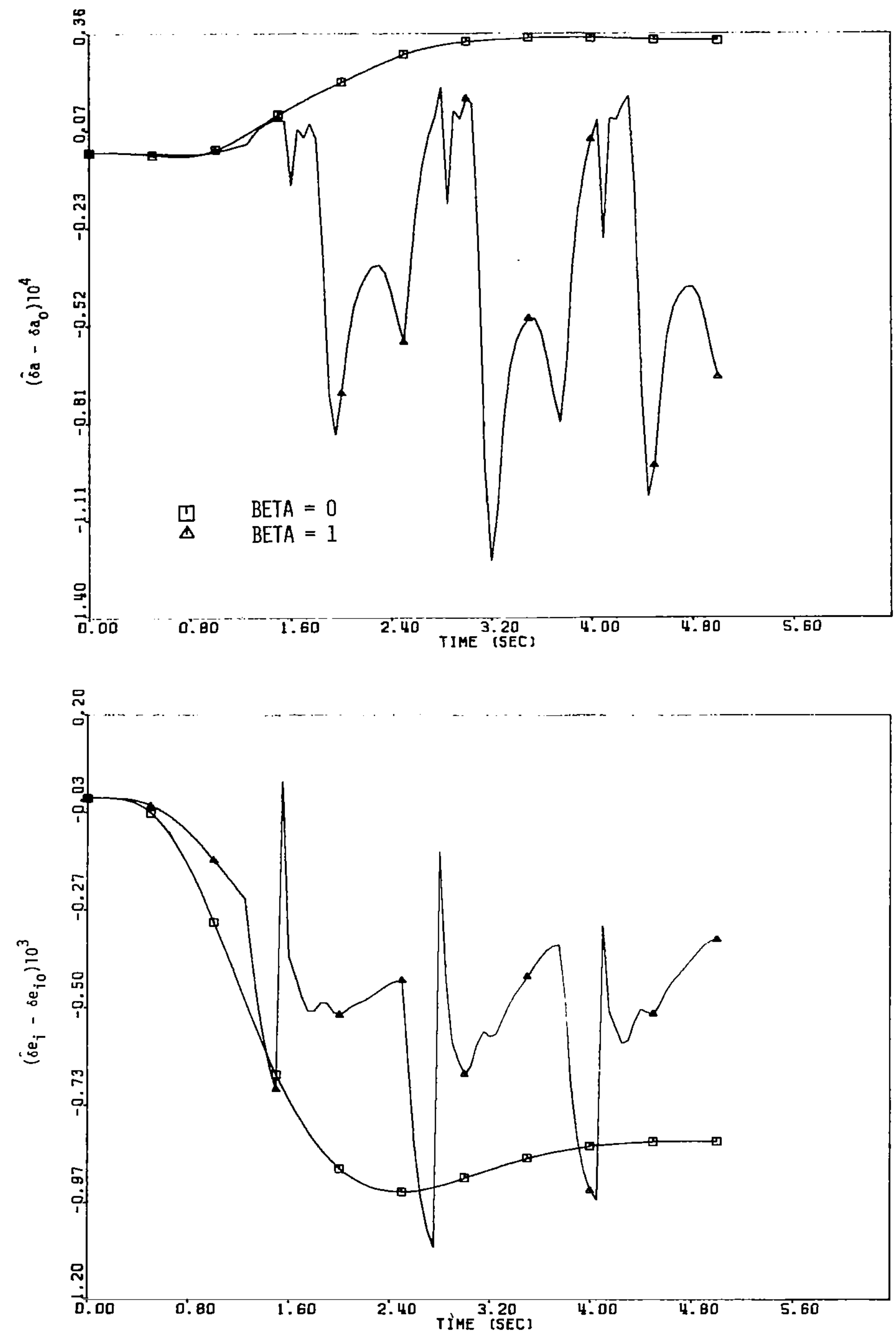


Figure 12. Response to $u_1 = 0.02$ for $\overline{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$ with recycle every 1.25 second (concluded)

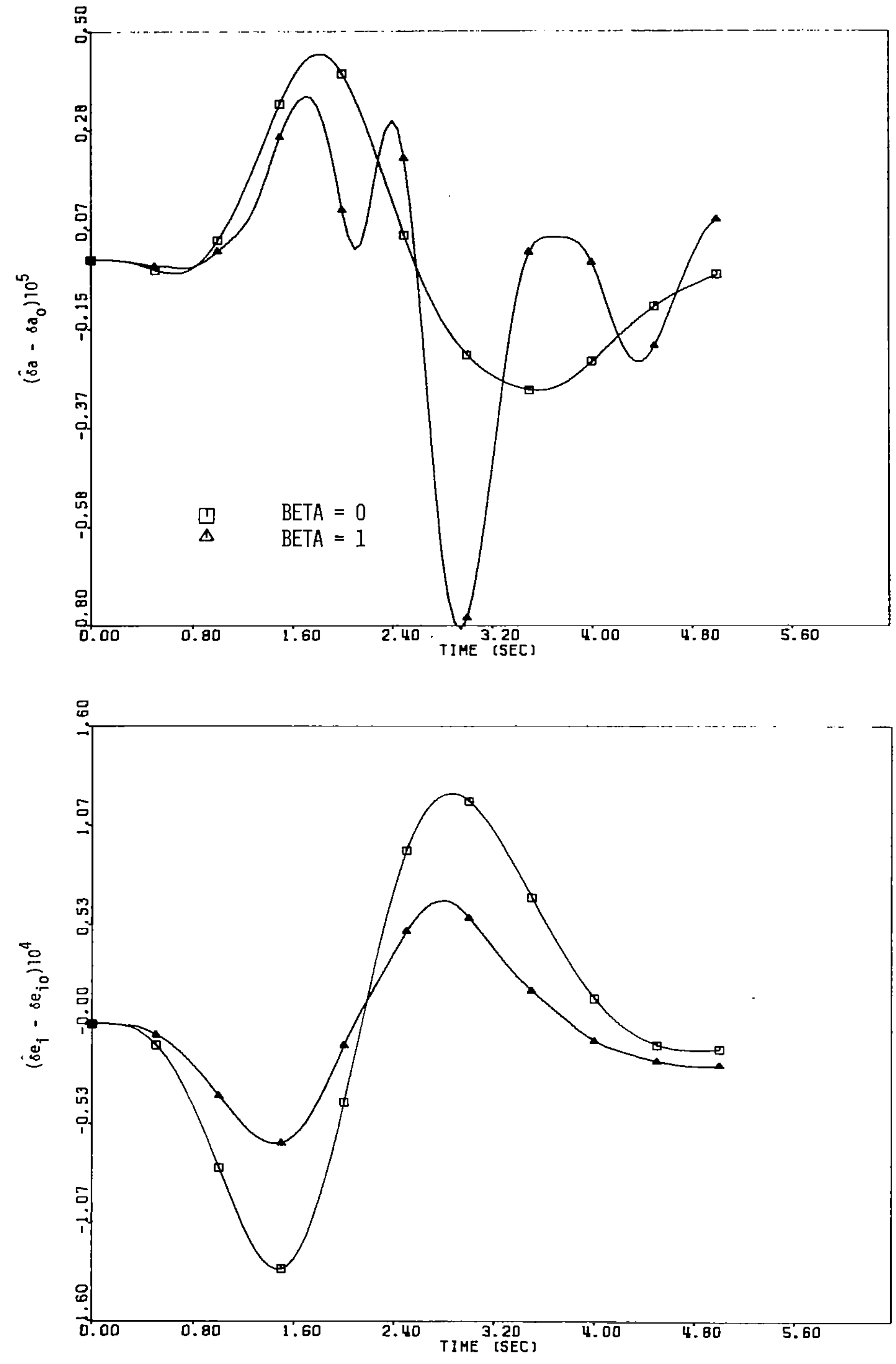


Figure 13. Response to u_1 Doublet for
$\overline{q}_f = \omega_f = 1.0$, $M_{W_f} = 0.8$

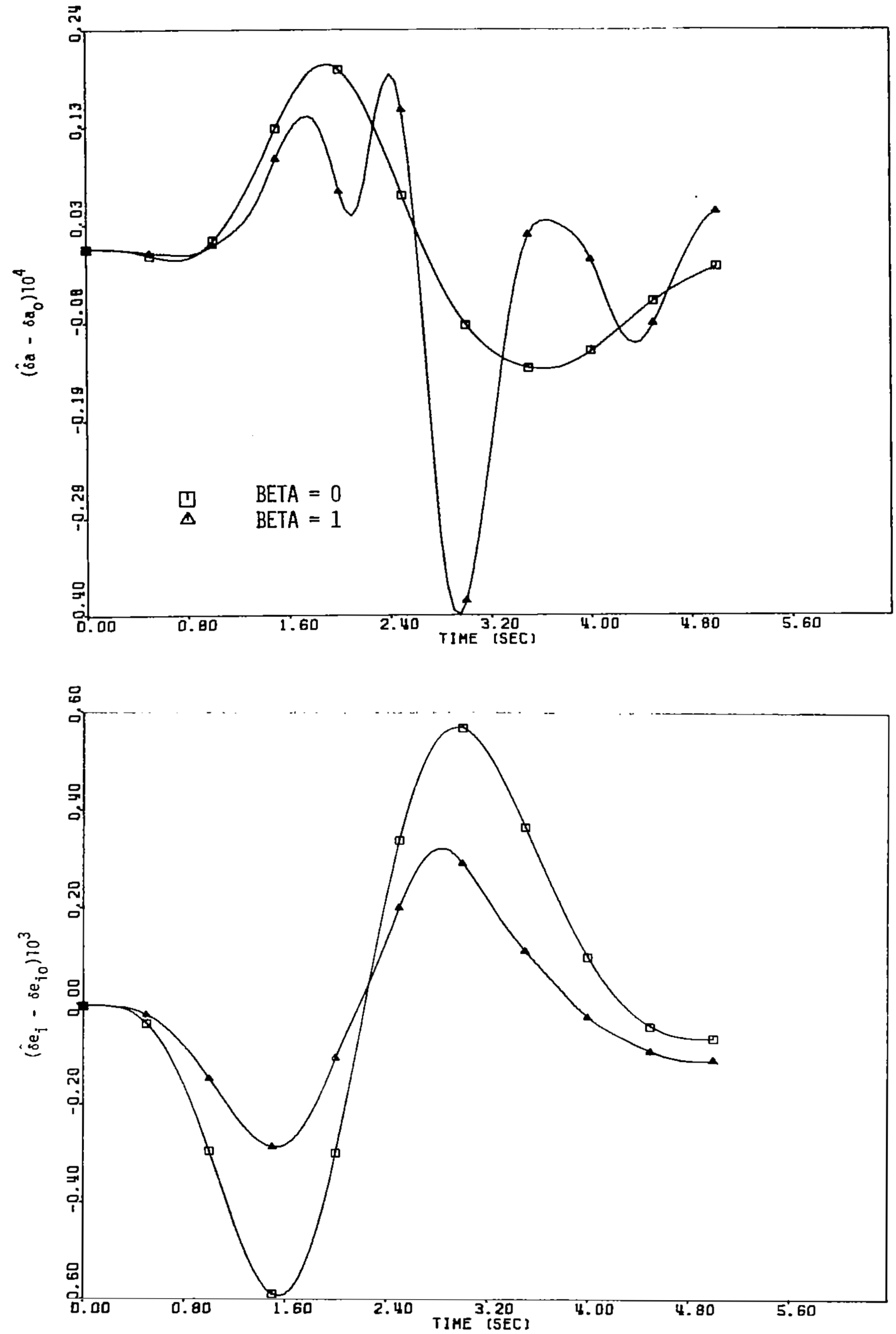


Figure 14. Response to u_2 Doublet for $\overline{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

a negative step for 1 sec, and then zero for 3 sec. This type of input is more realistic for an aircraft.

The gust response for this parameter variation is shown in Figures 15 and 16. The effect of β is shown in Figure 15, and the effect of recycling is shown in Figure 16. In Figure 15, the nonlinear effect is evident with better performance in the earlier portion of the interval than in the later portion. In this case, recycling seems to improve performance generally in spite of the induced transients.

Figures 17 and 18 display the effect of β for step inputs for the so-called "worst" Case 1 condition. Again, the trend is consistent with the theory.

The steady state relative errors for the parameter variations considered are summarized in Table 5 for step inputs. The trend is generally in accordance with the theory. Exceptions do occur in cases where one of the ratios is negative.

Table 6 presents the gust response statistics for the same parameter variations. Here the mean and standard deviations are computed for the time series of numerical integration on the five-second interval for a single gust sample. Again, the results are generally in accord with the theory.

The final experiment consisted of testing the effects of unmodeled dynamics. Figures 19, 20, and 21 present comparisons of step responses for the 7th and 15th order systems with the three individual parameter variations. Gust responses for these cases are shown in Figures 22, 23, and 24. The

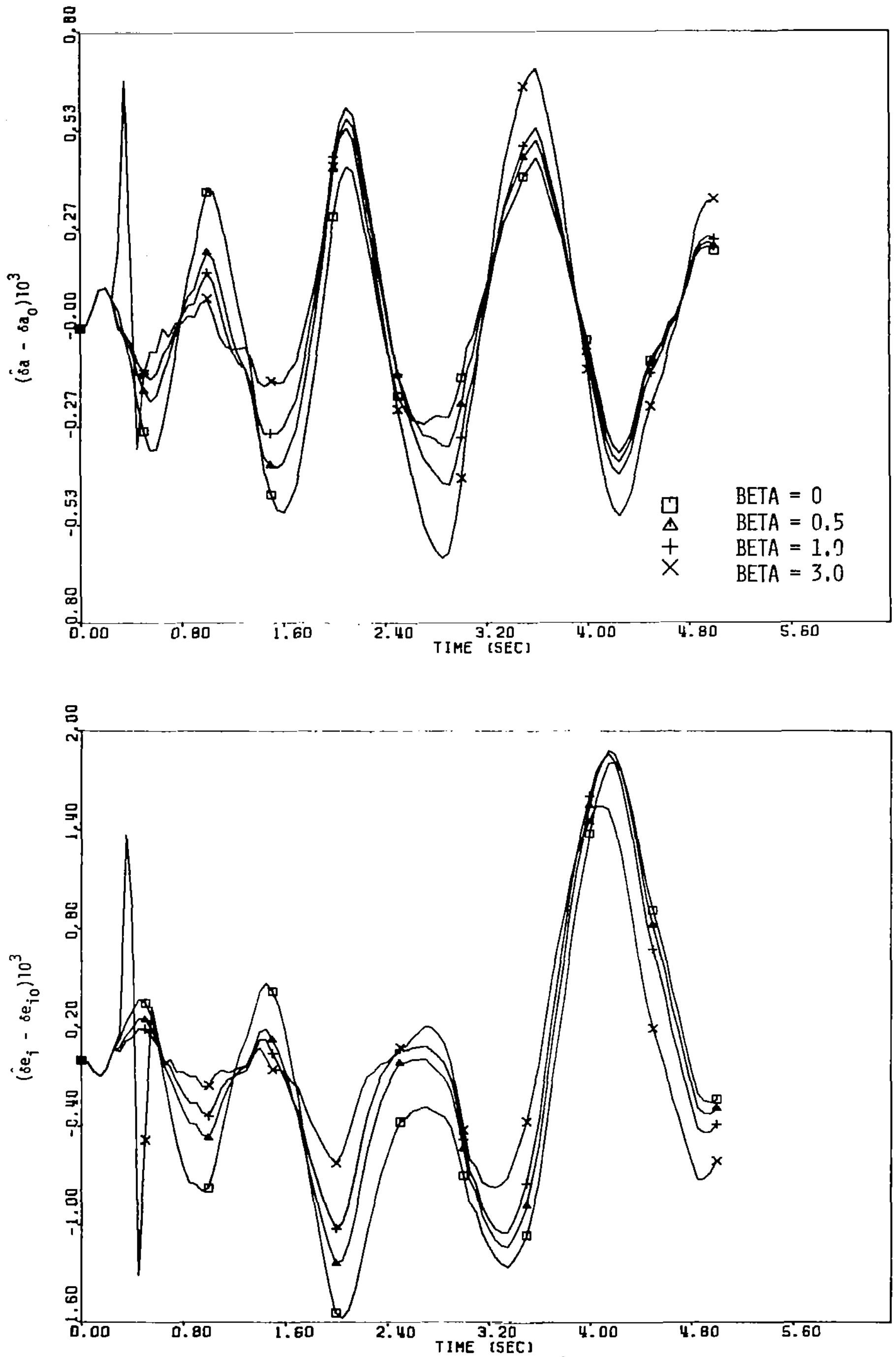


Figure 15. Gust Response for $\bar{q}_f = \omega_f = 1.0$, $M_{W_f} = 0.8$

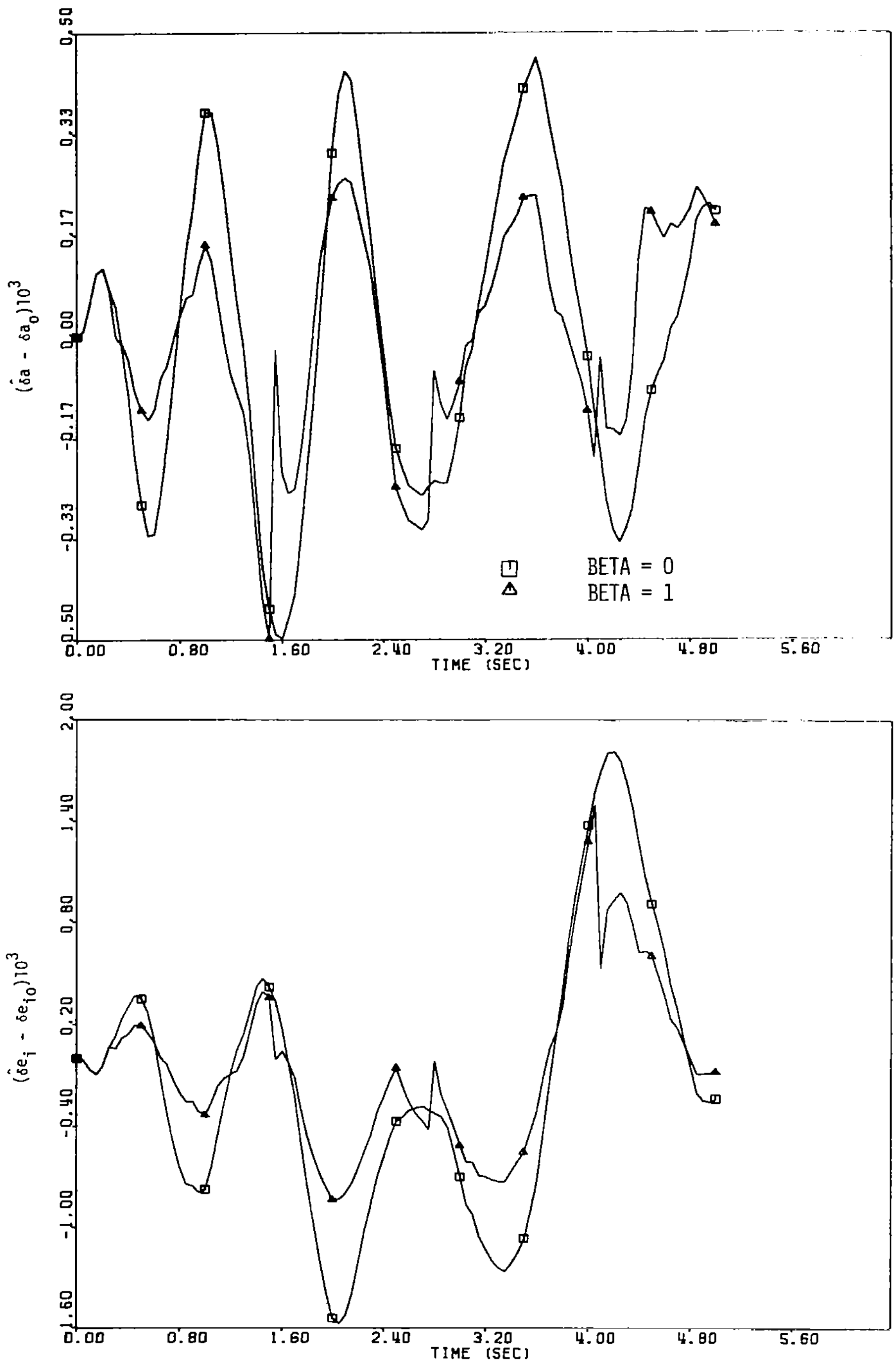


Figure 16. Gust Response with Recycle for $\bar{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

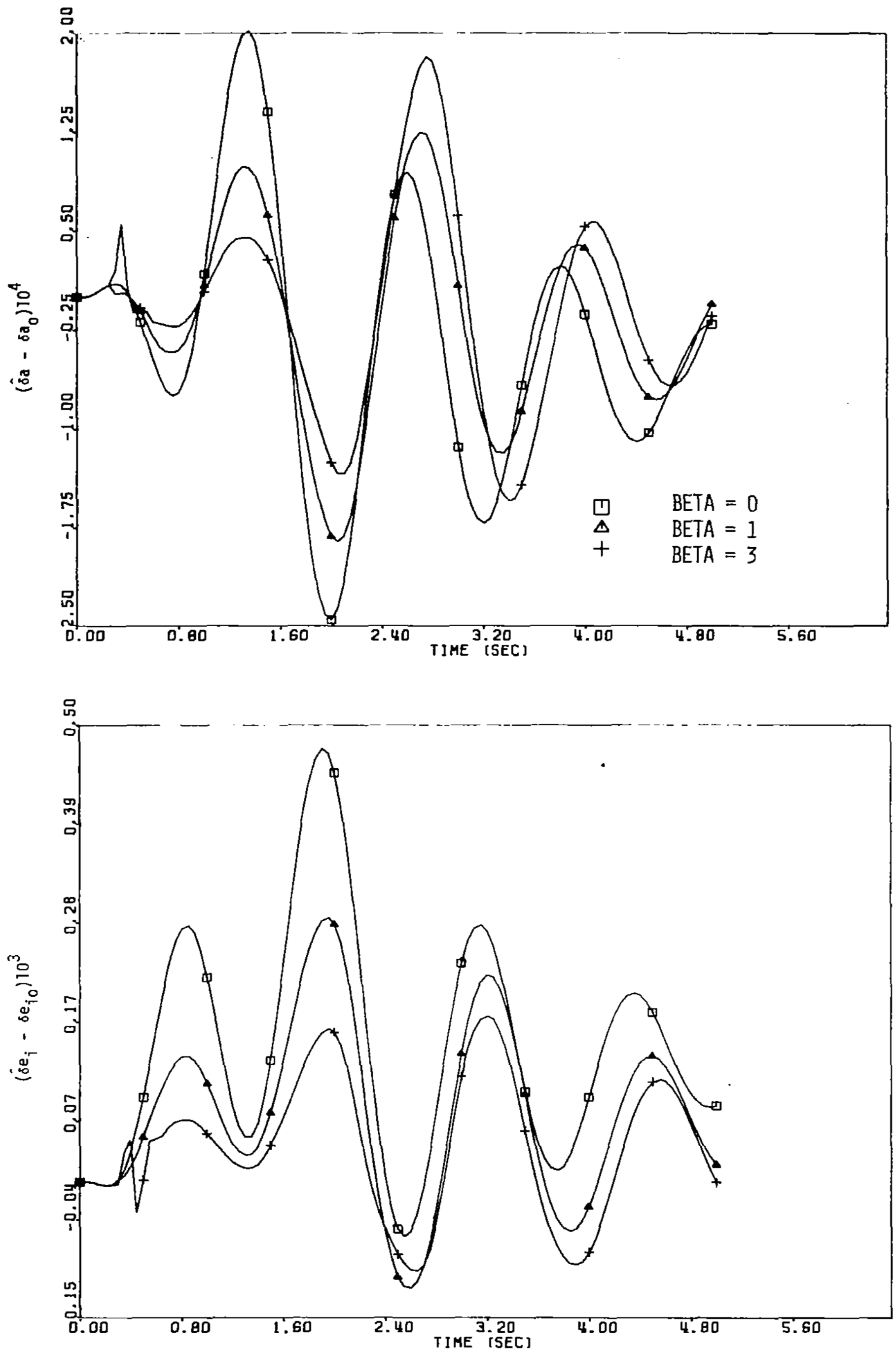


Figure 17. Response to $u_1 = 0.02$ for $\overline{q}_f = 1.25$, $\omega_f = 0.75$, $M_{W_f} = 0.80$

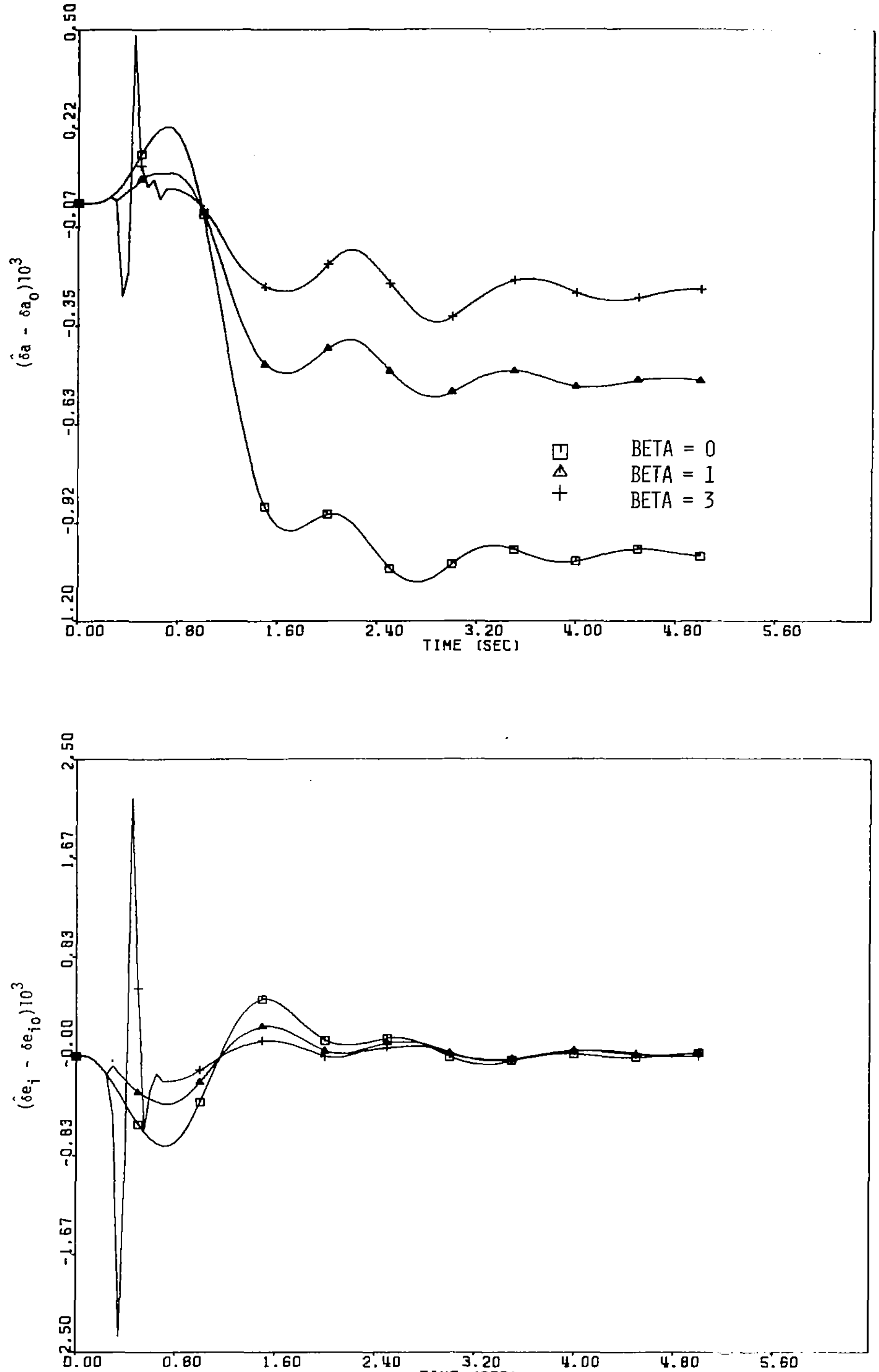


Figure 18. Response to $u_2 = 0.02$ for $\overline{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.80$

TABLE 5. STEADY STATE RELATIVE ERRORS

Plant	β	Step Input	$\left[\dfrac{\hat{x}(5) - x_o(5)}{x(5) - x_o(5)}\right]_{t=5}$	$\left[\dfrac{\hat{x}(6) - x_o(6)}{x(6) - x_o(6)}\right]_{t=5}$
F1	1	$u_1 = 0.02$	0.52	0.52
	3	$u_1 = 0.02$	0.27	0.26
	1	$u_2 = 0.02$	0.52	0.52
	3	$u_2 = 0.02$	0.27	0.26
F2	1	$u_1 = 0.02$	0.20	0.19
	3	$u_1 = 0.02$	-0.99	0.30
	1	$u_2 = 0.02$	0.48	0.50
	3	$u_2 = 0.02$	0.21	0.23
F3	1	$u_1 = 0.02$	0.56	0.28
	3	$u_1 = 0.02$	0.56	-0.03
	1	$u_2 = 0.02$	0.49	0.65
	3	$u_2 = 0.02$	0.25	0.34
WC1	1	$u_1 = 0.02$	0.27	0.22
	3	$u_1 = 0.02$	0.70	-0.01
	1	$u_2 = 0.02$	0.50	0.87
	3	$u_2 = 0.02$	0.24	-0.04
WC2	1	$u_1 = 0.02$	-0.01	0.34
	3	$u_1 = 0.02$	-0.01	0.34
	1	$u_2 = 0.02$	0.57	0.78
	3	$u_2 = 0.02$	0.26	0.71

TABLE 6. GUST RESPONSE STATISTICS

Plant	β	$(\overline{\hat{x}_5})$	$\sigma_{\hat{x}_5}$	$(\overline{\hat{x}_6})$	$\sigma_{\hat{x}_6}$	$[\lvert\overline{\hat{x}_5}\rvert+\lvert\overline{\hat{x}_6}\rvert]$	$\sqrt{\left(\sigma_{\hat{x}_5}^2+\sigma_{\hat{x}_6}^2\right)/2}$
F1	0	0.447-6	0.248-3	-0.148-3	0.818-3	0.740-4	0.604-3
F1	0.5	0.146-4	0.249-3	-0.426-4	0.739-3	0.286-4	0.551-3
F1	1	0.138-4	0.262-3	-0.292-5	0.692-3	0.083-4	0.523-3
F1	2	0.100-4	0.290-3	0.186-4	0.614-3	0.143-4	0.480-3
F1	3	0.825-5	0.324-3	0.174-4	0.602-3	0.128-4	0.483-3
F2	0	-0.245-4	0.394-3	-0.895-4	0.923-3	0.570-4	0.710-3
F2	1	-0.183-4	0.384-3	0.392-4	0.780-3	0.288-4	0.615-3
F2	3	-0.206-4	0.432-3	0.376-4	0.672-3	0.291-4	0.565-3
F3	0	-0.403-5	0.280-3	-0.109-3	0.851-3	0.565-4	0.633-3
F3	1	0.825-5	0.298-3	-0.139-5	0.740-3	0.482-5	0.564-3
F3	3	0.609-5	0.370-3	0.647-6	0.689-3	0.337-5	0.553-3
WC1	0	-0.212-4	0.463-3	-0.144-3	0.106-2	0.826-4	0.818-3
WC1	1	-0.171-4	0.470-3	0.119-6	0.914-3	0.086-4	0.727-3
WC2	0	0.353-5	0.197-3	-0.152-3	0.571-3	0.780-4	0.427-3
WC2	1	0.250-4	0.184-3	-0.939-5	0.463-3	0.172-4	0.352-3

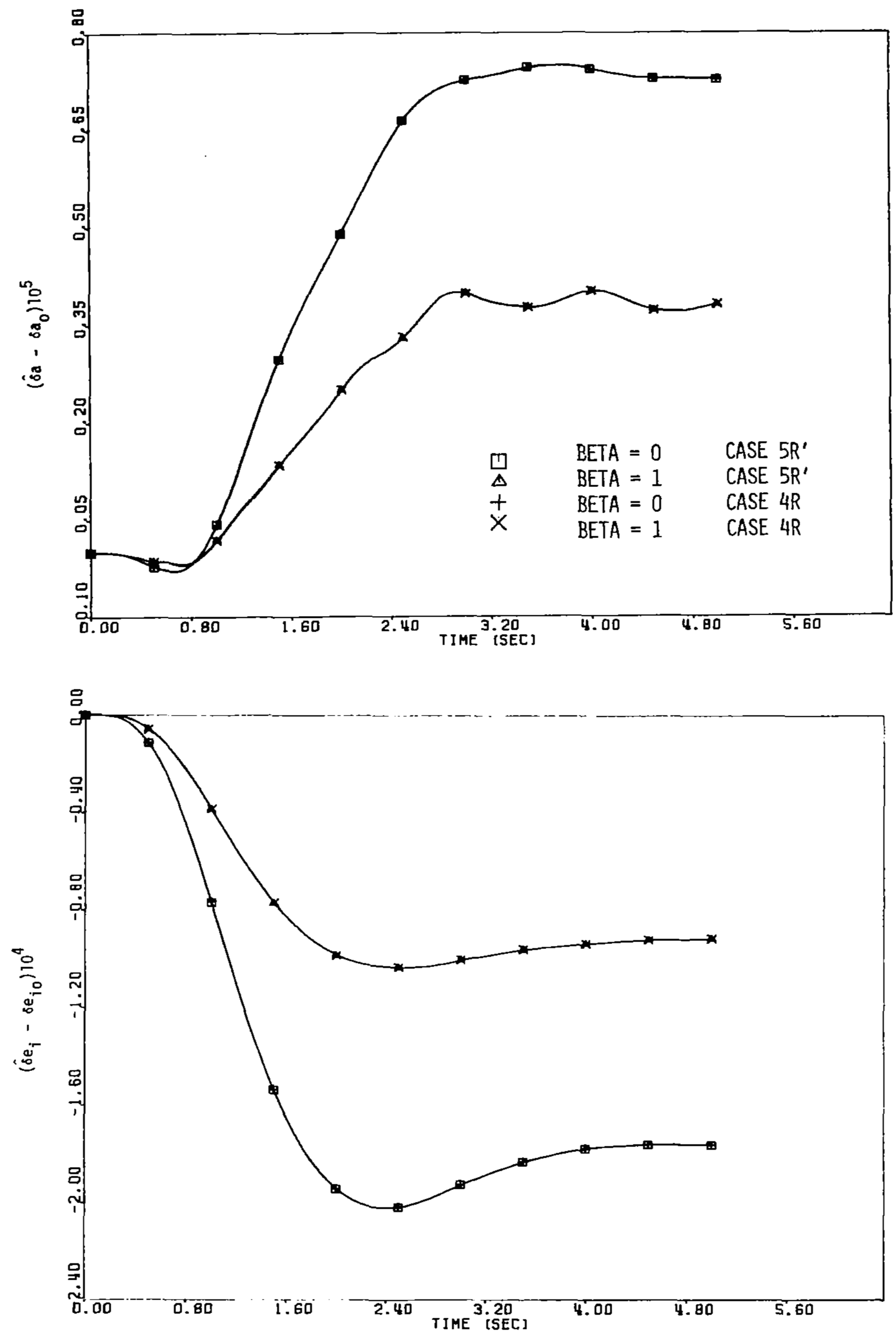


Figure 19a. Case 4R and Case 5R' Responses to $u_1 = 0.02$ for $\overline{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

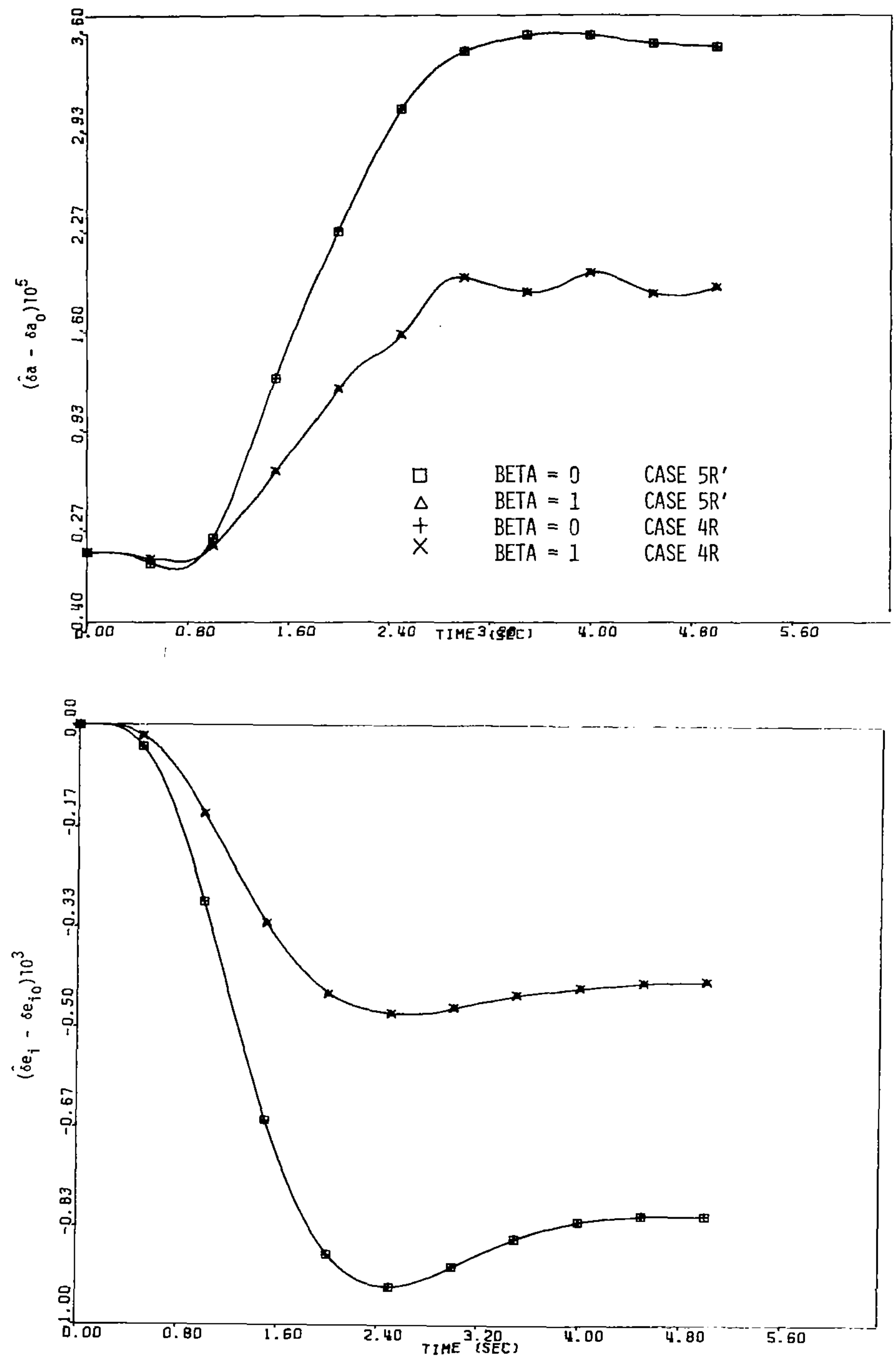
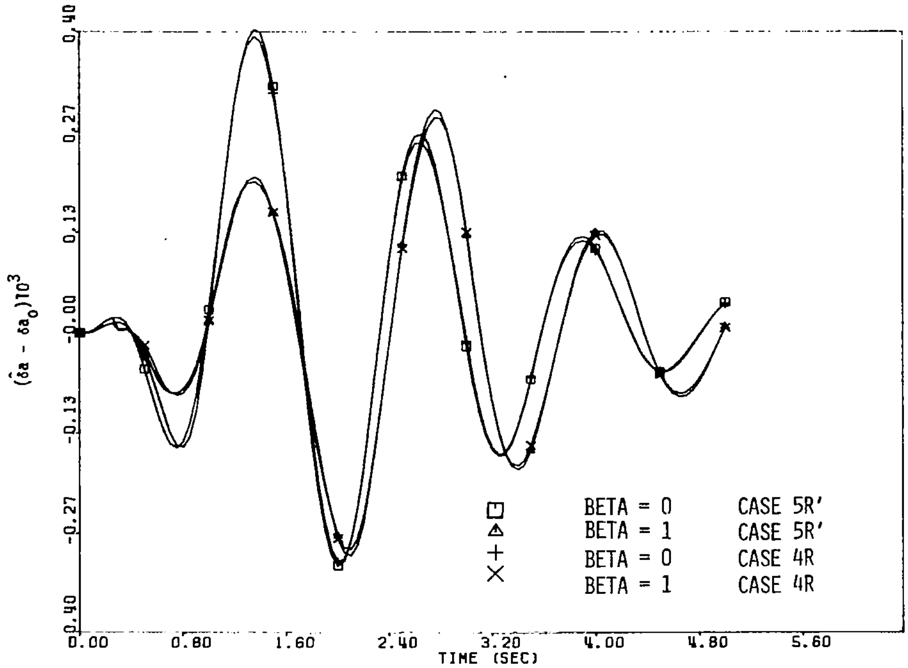


Figure 19b. Case 4R and Case 5R' Responses to $u_2 = 0.02$ for $\overline{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$



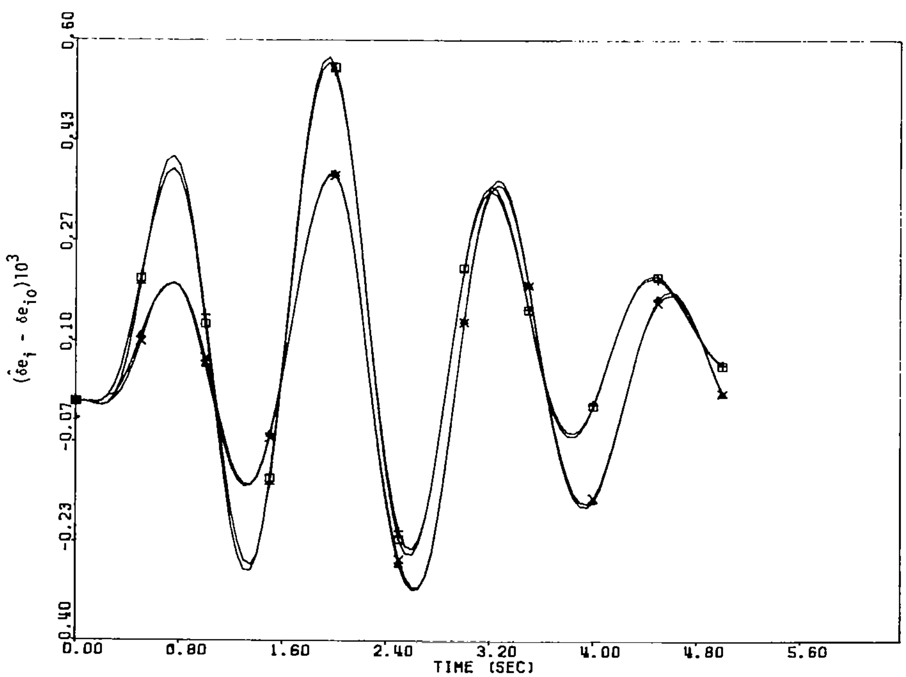


Figure 20a. Case 5R' and Case 4R Responses to $u_1 = 0.02$ for $\overline{q}_f = M_{w_f} = 1.0$, $\omega_f = 0.75$

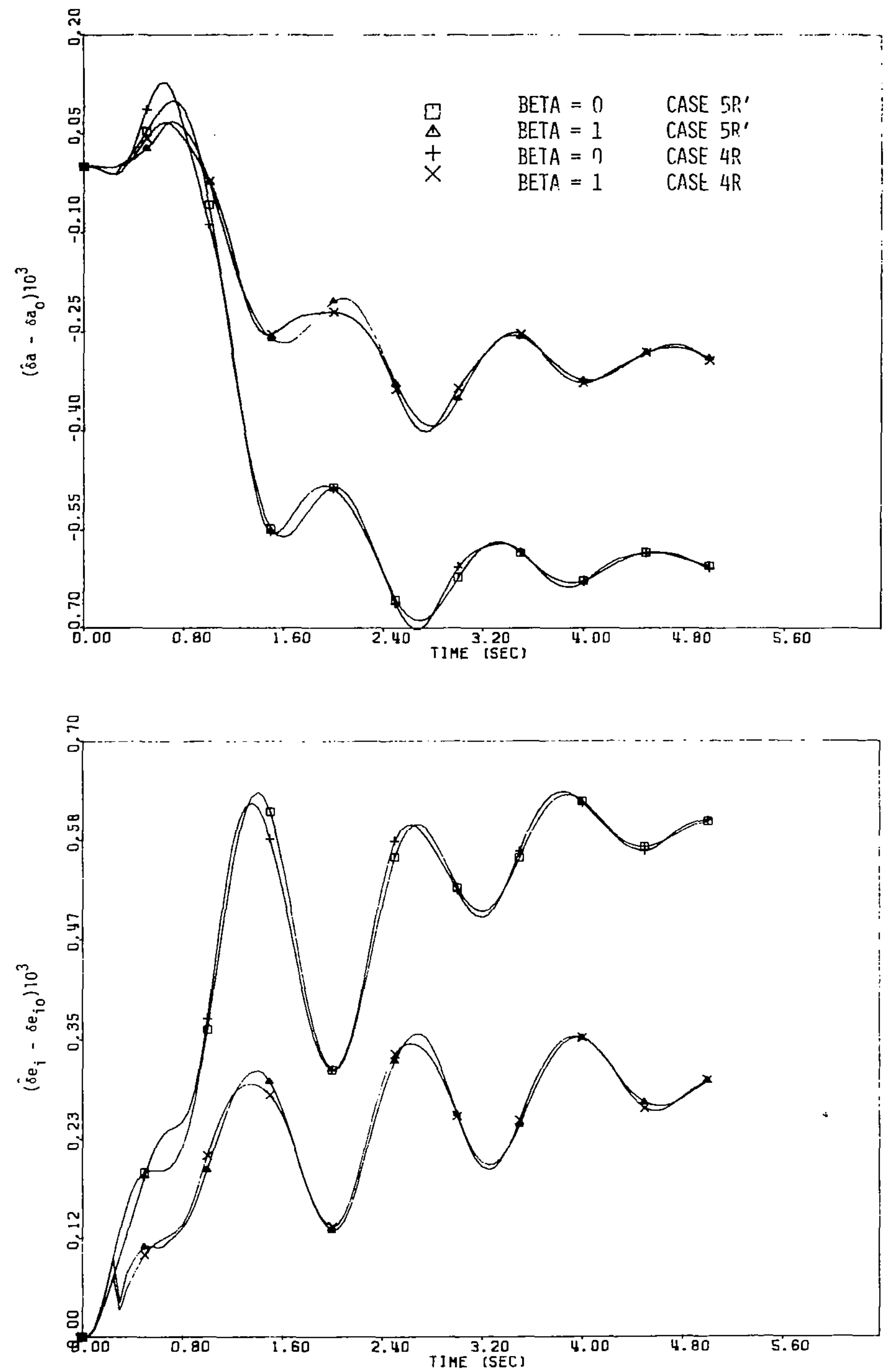


Figure 20b. Case 5R' and Case 4R Responses to $u_2 = 0.02$ for $\bar{q}_f = M_{W_f} = 1.0$, $\omega_f = 0.75$

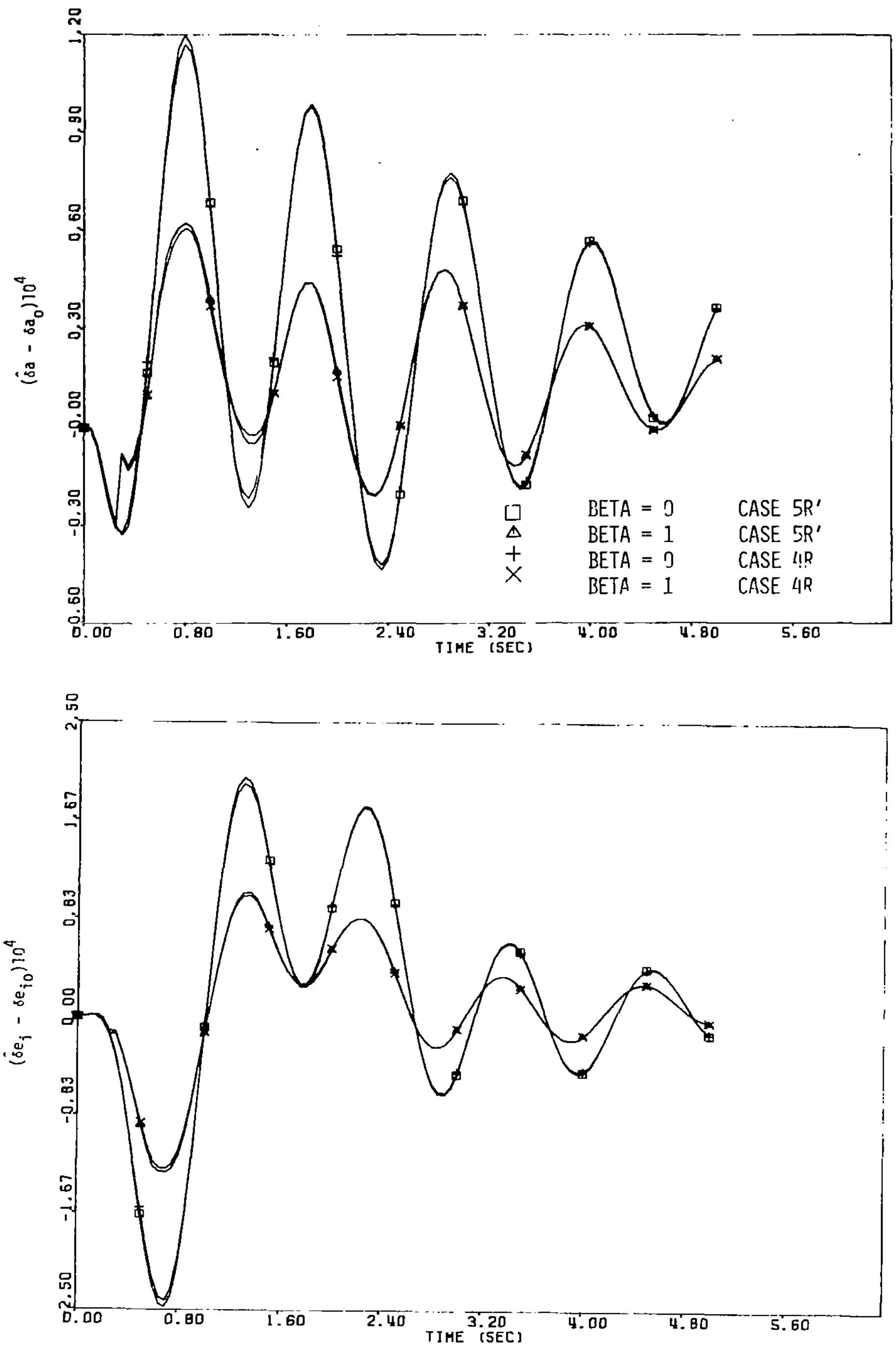


Figure 21a. Case 5R' and Case 4R Responses to $u_1 = 0.02$ for $\bar{q}_f = 1.25$, $\omega_f = M_{w_f} = 1.0$

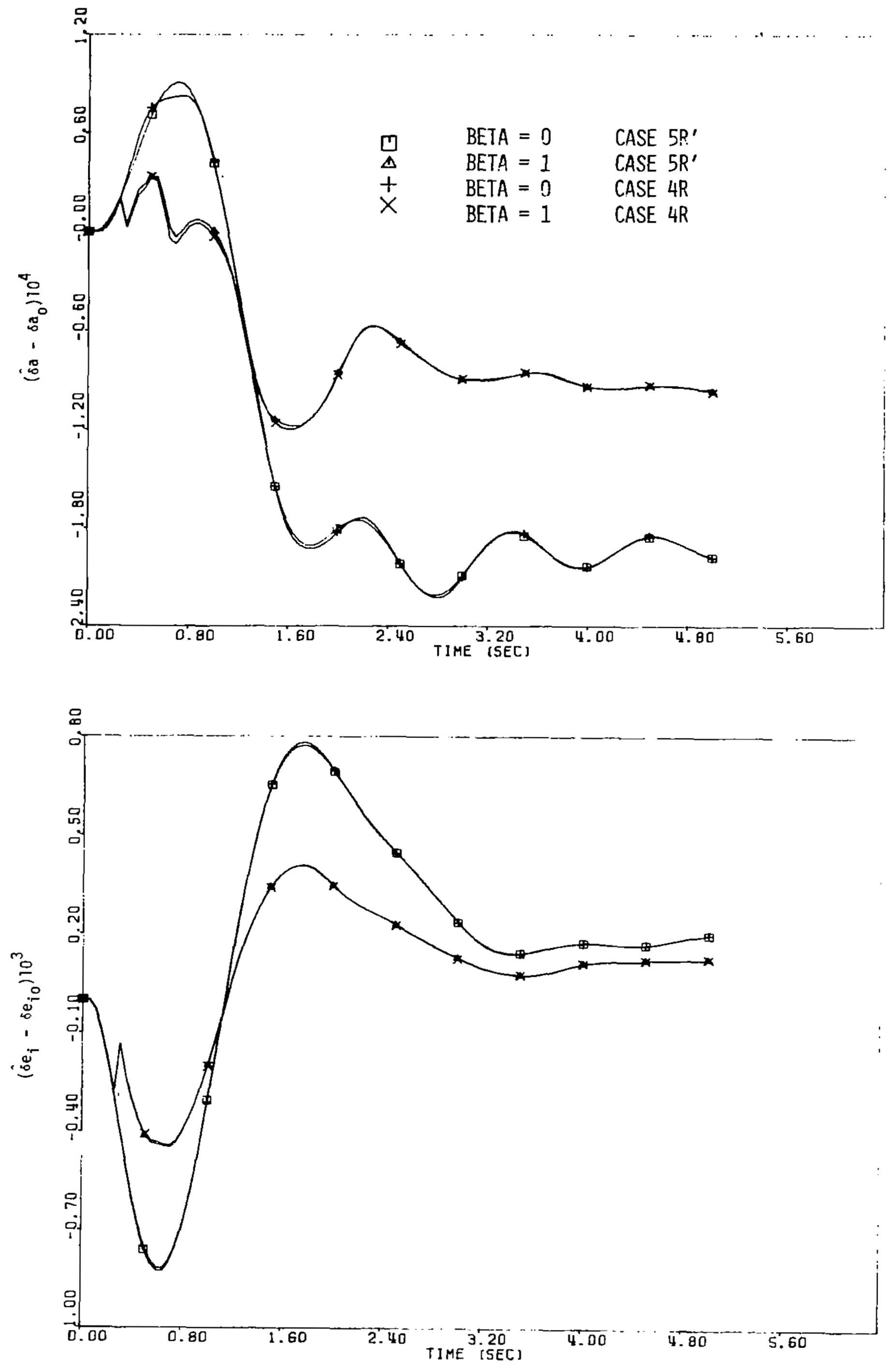
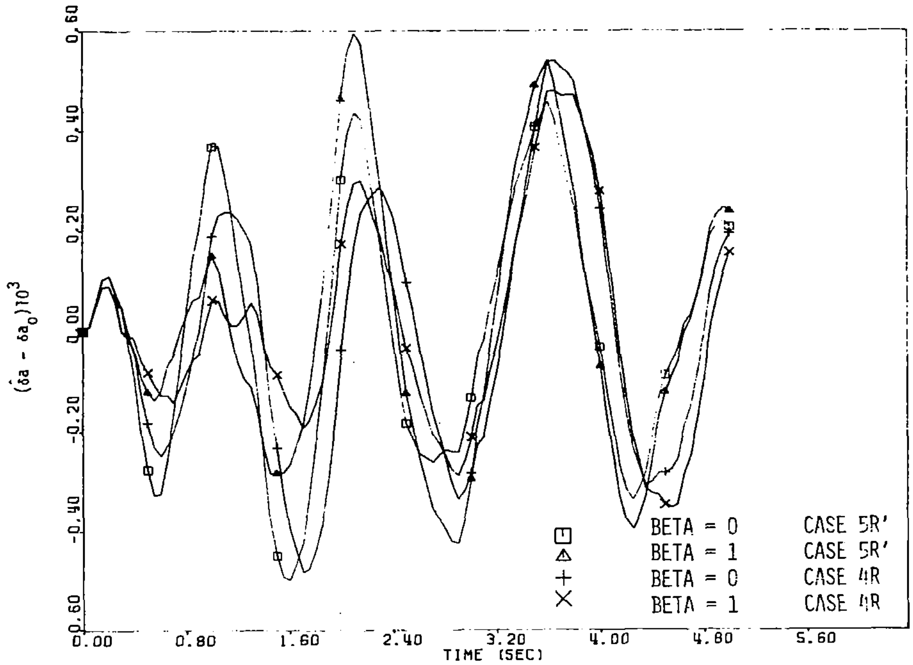


Figure 21b. Case 5R' and Case 4R Responses to $u_2 = 0.02$ for $\overline{q}_f = 1.25$, $\omega_f = M_{w_f} = 1.0$



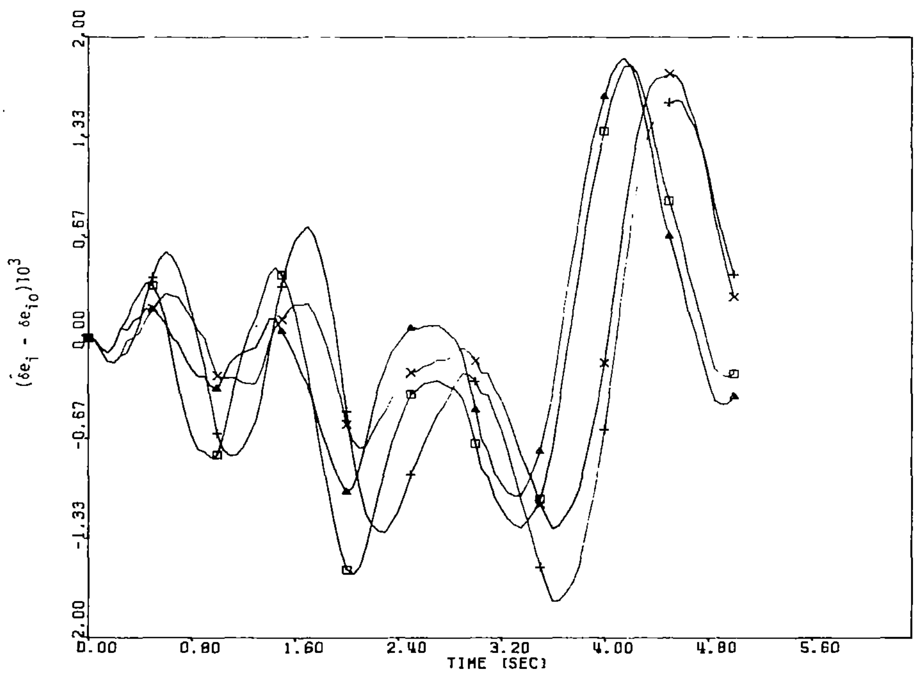


Figure 22. Case 5R' and Case 4R Gust Responses for $\bar{q}_f = \omega_f = 1.0$, $M_{w_f} = 0.8$

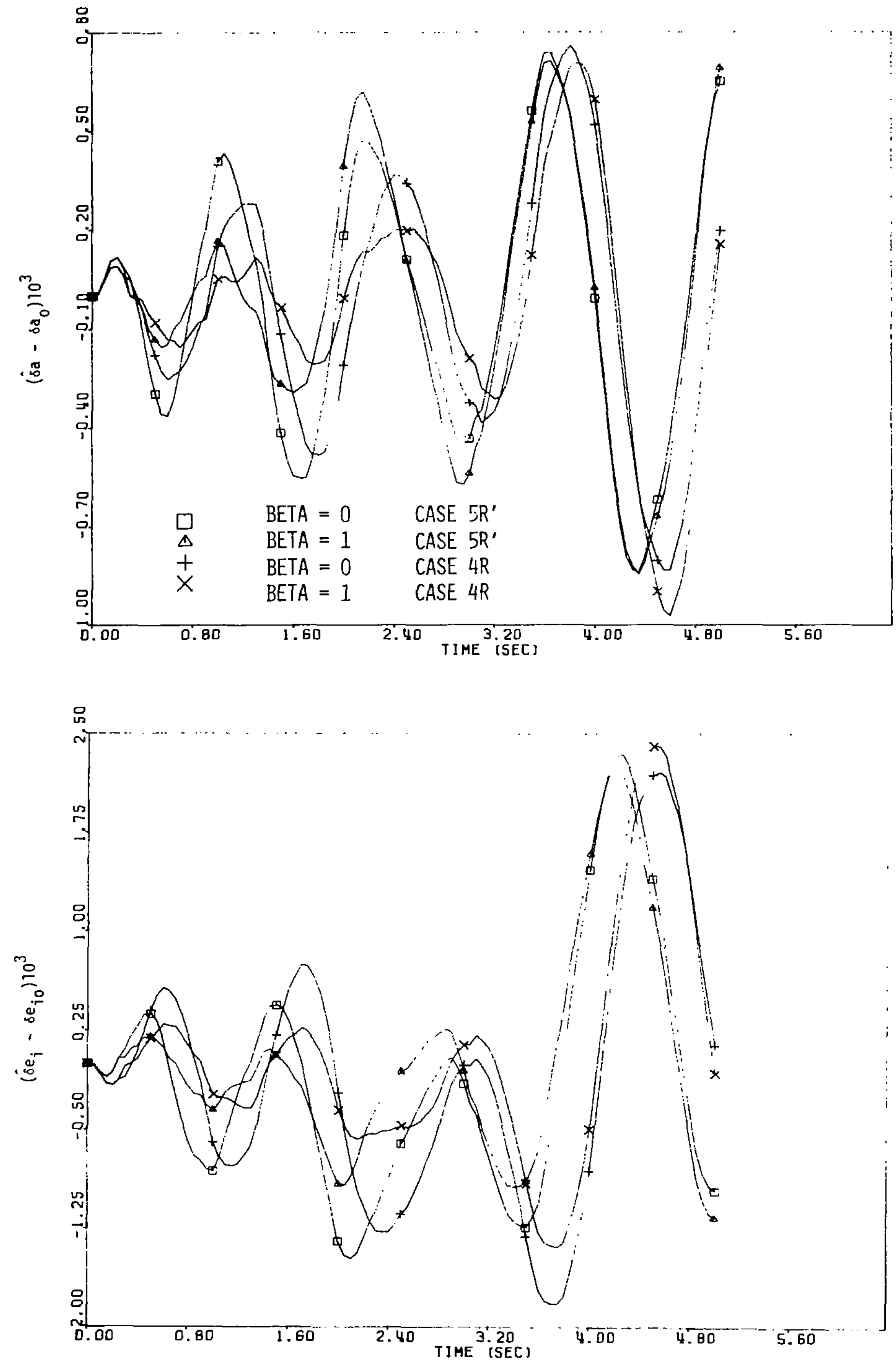


Figure 23. Case 5R' and Case 4R Gust Responses for $\bar{q}_f = M_{w_f} = 1.0, \ \omega_f = 0.75$

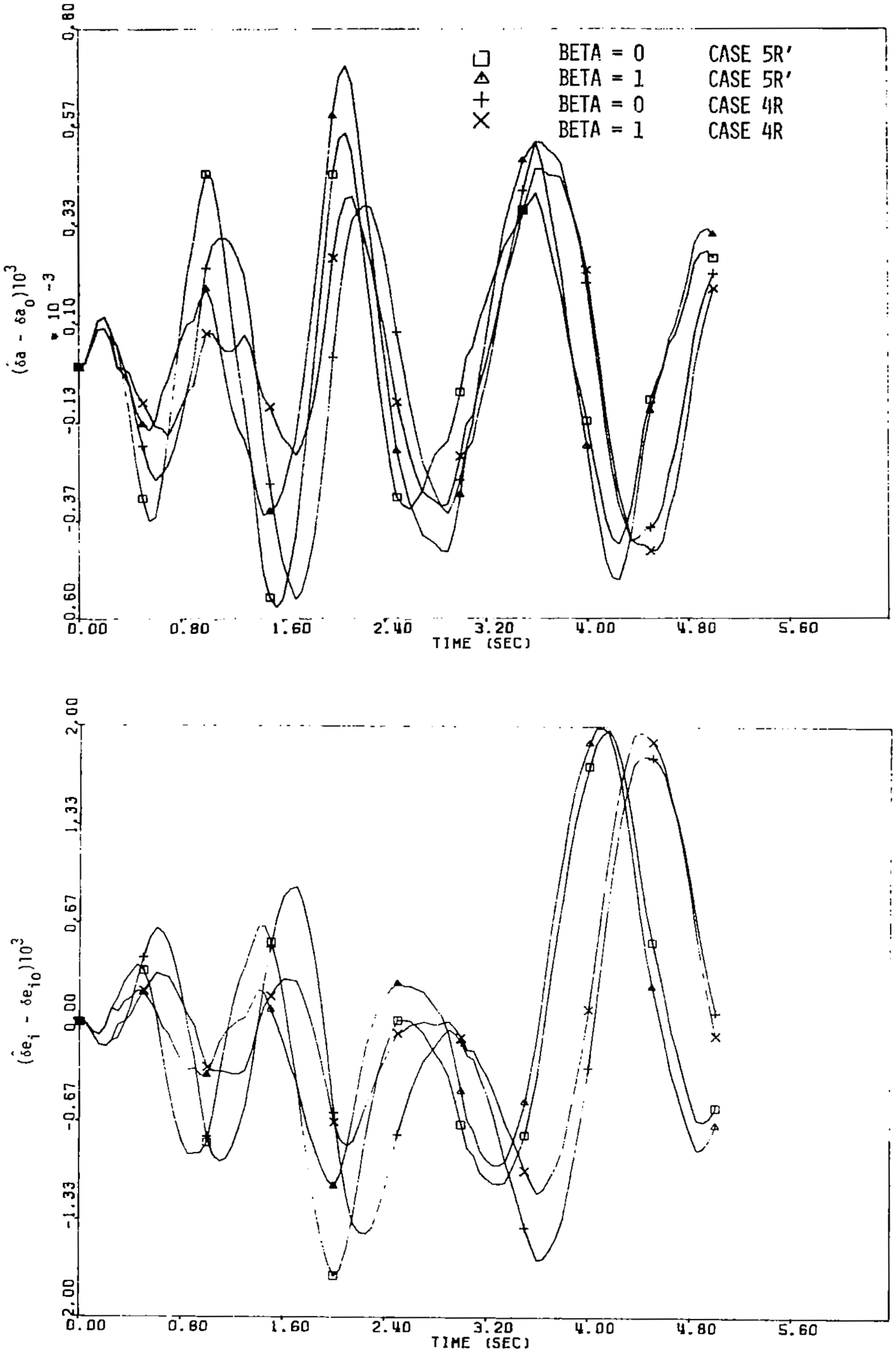


Figure 24. Case 5R' and Case 4R Gust Responses for $\overline{q}_f = 1.25$, $\omega_f = M_{w_f} = 1.0$

unmodeled dynamics have very little effect. The greatest effect appears to be the change in the gust sample caused by the added filtering.

CONCLUSIONS AND RECOMMENDATIONS

The performance of the compensator generally lives up to theoretical predictions. There are four major areas where the current formulation is deficient for an aircraft flight control application such as the C-5A. They are

1. Reduction of sensitivity of arbitrary responses of interest,
2. Application to a stationary system,
3. Adequate treatment of nonlinear dependence on parameters, and
4. Severity of computational requirements.

In the C-5A example, insensitivity of many responses and particularly bending and torsion moments are desired. Increases in these responses were caused by the compensator in most instances. Presumably, this would have been avoided if they were used as the outputs. But in this example, it would have required an internally unstable compensator. The formulation should be modified to include insensitivity to such responses.

The recycling was an ad hoc attempt to modify the compensator to account for stationarity. It was not completely satisfactory; alternative formulations should be considered.

The nonlinear dependence on parameters is significant for the C-5A example and the theory should be modified to encompass this phenomena. In this example, this effect and the stationarity effect were coupled by the choice of inputs used. Other inputs or additional inputs could alleviate this coupling.

The computational requirements associated with the implementation were significant for the seventh order design model with the limited number of inputs used in construction of the finite dimensional inverse. This was not a major concern in performing an evaluation of the concept, but it would be for an operational system.

Thus, although the concept lived up to expectations, further development is required before the technique could lead to operational systems for an application such as the C-5A.

SECTION IV

INSENSITIVE CONTROL SYSTEM DESIGN VIA AN INFORMATION MATRIX APPROACH

The technique described in this section was developed by contract consultant Professor David L. Kleinman of the University of Connecticut. The technique is based on the utilization of the Fisher Information Matrix[4] which is a fundamental feature of maximum likelihood parameter identification. In identification applications, it is desirable to minimize in some sense the inverse of the Information Matrix, or the dispersion matrix, in order to enhance the identifiability of a set of system parameters. With respect to sensitivity, given a set of responses, it was hypothesized that minimizing the Information Matrix itself would reduce the identifiability of system parameters and, consequently, the sensitivity of the system response to variations in those system parameters.

The evaluation of control systems designed with the technique consisted of a preliminary evaluation of the effect of adjusting design parameters on a single system response of the C-5A test example and a full system evaluation for the 15th order model as was done in Reference 1.

MATHEMATICAL FORMULATION

Consistent with the assumptions given in Reference 1, the system to be controlled may be represented by a set of linear constant coefficient differential equations,

$$\underline{\dot{x}} = F(\underline{\alpha})\,\underline{x} + G_1\underline{u} + G_2\eta \tag{66}$$

where

$\underline{x}$ is an n_x state vector

$\underline{u}$ is an n_u control vector

η is a scalar white noise with N(0, 1)

α is an n_p parameter vector

The n_r system responses may be represented by

$$\underline{r} = H(\underline{\alpha})\,\underline{x} + D(\underline{\alpha})\underline{u} \tag{67}$$

In Equations (66) and (67), the matrices F, G_1, G_2, H, D have the appropriate dimensions. As in Reference 1, only F, H, D are assumed to depend on the parameters, $\underline{\alpha}$. However, G_1 and G_2 can also depend upon $\underline{\alpha}$, in general.

The parameter vector is assumed to be bounded:

$$\underline{\alpha} \;\epsilon\, [\underline{\alpha}_L, \underline{\alpha}_H]$$

Since the parameters are regarded as uncertainty factors (as opposed to absolute deviations), the matrices F, H, and D about any operating point $\underline{\alpha}$ can be expressed as

$$F(\underline{\alpha}) = F_o + \sum_{i=1}^{p} (\alpha_i - 1)F_i \tag{68a}$$

$$H(\underline{\alpha}) = H_o + \sum_{i=1}^{p} (\alpha_i - 1)H_i \tag{68b}$$

$$D(\underline{\alpha}) = D_o + \sum_{i=1}^{p} (\alpha_i - 1)D_i \tag{68c}$$

where F_o, H_o, and D_o are matrices at a nominal point, i.e., by definition where $\alpha_i = 1$. The matrices F_i, etc., are the gradient matrices evaluated at the nominal point,

$$F_i \triangleq \nabla_i F(\underline{\alpha}) \Big|_{\underline{\alpha} = 1} = \frac{\partial F}{\partial \alpha_i}\Big|_{\underline{\alpha} = 1} = \text{grad}_i\,(F) \tag{69}$$

$$H_i = \text{grad}_i\,(H); \quad D_i = \text{grad}_i\,(D)$$

For the system described by Equations (66) through (68), the problem is to determine a feedback control law $\underline{u} = K\underline{x}$ such that the system is "insensitive" to parameter variations and satisfies representative performance criteria. For this application, latter criteria are expressed via the minimization of a quadratic criterion

$$J_1 = E\{\underline{r}'\, Q\underline{r}\} \tag{70}$$

where the weighting matrix has been selected to meet specifications at a nominal design point. The minimization of J_1 yields a nominal feedback control

$$\underline{u} = K_o\, \underline{x} \tag{71}$$

Of course, this set of gains K_o has been determined with the neglect of explicit sensitivity criteria.

Use of the Information Matrix

The Fisher Information Matrix for the parameter set $\underline{\alpha}$ and the responses $\underline{r}(t)$ in Equation (67) is given (approximately)* by

$$M = E\left\{\int_0^T \left(\frac{\partial \underline{r}}{\partial \underline{\alpha}}\right)' S \left(\frac{\partial \underline{r}}{\partial \underline{\alpha}}\right) dt\right\} \tag{72}$$

where T is the observation time or measurement interval, S is a weighting matrix to be discussed later, and $\partial \underline{r}/\partial \underline{\alpha}$ is an n_r x p sensitivity matrix with elements

$$\left(\frac{\partial \underline{r}}{\partial \underline{\alpha}}\right)_{ij} = \frac{\partial r_i}{\partial \alpha_j}$$

Thus, the i-th column of the sensitivity matrix reflects the sensitivity of $\underline{r}$ with respect to the i-th parameter.

The use of the information matrix for optimal input design to enhance the identification of unknown parameters from the measurement set $\underline{r}$ (assuming that an efficient, unbiased estimator of $\underline{\alpha}$ exists) is well known.

This technology has served as the motivation behind the present work. The major points relevant to the discussion are the following:

---

*The approximation used is to replace $\hat{\underline{x}}$ by $\underline{x}$, i.e., we omit the Kalman filter innovations representation in the Maximum Likelihood formulation. This greatly simplifies the problem while retaining the essential parameter sensitivities.

1. The inverse of the information matrix, called the dispersion matrix, gives a bound on the parameter error covariance

$$E\{(\underline{\alpha} - \hat{\underline{\alpha}})(\underline{\alpha} - \hat{\underline{\alpha}})'\} \geq M^{-1} \stackrel{\Delta}{=} \text{(Cramer-Rao lower bound)}^{4}$$

Here, $\underline{\alpha}$ is the actual parameter vector and $\hat{\underline{\alpha}}$ is the estimate. The idea behind optimal input design is to find $\underline{u}$ to maximize a metric on the information matrix (e.g., det (M), tr (M), max λ (M)) or equivalently to minimize a measure of the dispersion matrix. The net effect will be that one can place more confidence in the parameters estimated from the input-output data.

2. Obviously, maximizing tr(M) would lead to increased sensitivity of $\underline{r}(t)$ with respect to the unknown parameters, inasmuch as M is a function of the output response sensitivities.

With respect to sensitivity, the problem is precisely the opposite to optimal input design, i.e., to determine a control input such that $\underline{r}(t)$ is least sensitive to parameter variations. To solve this "inverse" problem, we seek the "worst" input from an identification viewpoint, i.e., one that makes the parameter set as unidentifiable as possible. Therefore, it is natural to seek to minimize a measure of the information matrix.

Determining which measure to use poses a problem. In the identification literature, it is reported that choosing optimal inputs to minimize the trace of the dispersion matrix gives the most accurate estimates of $\bar{\underline{\alpha}}$. Therefore, it is conjectured that minimizing the trace, or weighted trace, of the information matrix should be the "best" criterion for choosing a desensitive feedback control.

As the observation interval $T \to \infty$ in Equation (72), the norm of the information matrix approaches infinity. Thus, since we are dealing with a steady-state optimization problem, it is more appropriate to consider the information matrix per unit time, or the average information matrix, M_∞. In the steady state, we have approximately

$$M_\infty = E\left\{\left(\frac{\partial \underline{r}}{\partial \underline{\alpha}}\right)' S\left(\frac{\partial \underline{r}}{\partial \underline{\alpha}}\right)\right\} \tag{73}$$

The weighting matrix, S, is the inverse of the measurement noise covariance. Since the problem formulation does not include measurement noise (note that it could), S will be interpreted in terms of a "pseudo" measurement noise injected onto r(t). We select S to be diagonal with elements

$$s_i = (\rho \sigma^2_{r_i})^{-1} \qquad i = 1, \ldots, n_r \tag{74}$$

The scaling of the measurement noises with the associated RMS response is a common practice. It is further motivated by the form of a human's "observation" noise in man-machine studies. The noise-to-signal ratio ρ is an overall scale factor; thus, its actual value is not of large concern. However, a value $\rho = 0.01\,\pi$ has been selected from previous experience in man-machine systems to represent a nominal noise level.* The variances $\sigma^2_{r_i}$ are picked at the nominal point K_o and held constant at these values throughout the analysis.

In any specific design it may or may not be necessary for all responses $r_1, \ldots, r_{n_r}$ to be insensitive to parameter variations. Thus, it is desirable to include within M_∞ only those responses appropriate for desensitive design. The s_i easily serve this purpose through setting $s_i = o$ for these responses. Thus,

$$s_i = \begin{cases} (\rho\sigma^2_{r_i})^{-1} & \text{for desensitive } r_i \\ o & \text{otherwise} \end{cases}$$

The weighted trace of the (average) information matrix is

$$\begin{aligned} J_2 = \operatorname{tr}\,[WM_\infty] &= E\left\{ \sum_{i=1}^{p} w_i \left(\frac{\partial \underline{r}}{\partial \alpha_i}\right)' S \left(\frac{\partial \underline{r}}{\partial \alpha_i}\right) \right\} \\ &= E\left\{ \sum_{i=1}^{p} \sum_{j=1}^{n_r} w_i\, s_j \left(\frac{\partial r_j}{\partial \alpha_i}\right)^2 \right\} \end{aligned} \tag{75}$$

where W is a diagonal p x p weighting matrix, $W = \operatorname{diag}(w_i)$. The weightings w_i are selected to normalize the information

*The case where the ρ_i are not all equal could be considered.

matrix at the nominal condition, i.e., at $\alpha_i = 1$, and to keep the analysis in terms of relative sensitivity (with respect to the nominal). Thus, it has been found convenient to select

$$w_i = \gamma_i \cdot \frac{\min\limits_i (M_\infty^o)_{ii}}{(M_\infty^o)_{ii}} \tag{76}$$

where M_∞^o is the information matrix evaluated using the nominal matrices F_o, H_o, D_o and the nominal feedback gains K_o. The additional scaling factors γ_i can be selected to reflect

1. The relative importance of α_i to the design problem as noted through experience or experimentation, or

2. The relative probability or frequency of occurrence of variations in the parameter α_i.

In the present analysis we set

$$\gamma_i = 1 \qquad i = 1, \ldots, p$$

to indicate that all parameters are equally important and equally likely to vary.

Thus, unlike some other methods of desensitive controller design, the choice of weighting parameters S and W is fairly straightforward. In the next section, J_2 is appended to the original performance cost functional J, given in Equation (70).

Design Method

When the system parameters are subject to variation, a design method based upon optimal control theory involves the following two steps, either separately or in combination:

1. Selection of a design point, i.e., a parameter set $\underline{\alpha}_d$ at which the design is done. Note that $\underline{\alpha}_d$ need not be the nominal point $\underline{\alpha}_o$.

2. Selection of a set of optimal gains K at the design point.

Constraints that are imposed by the physical system, such as limited control effort, maximum allowable deviations, etc., are to be satisfied in the selection process.

It would be desirable to achieve both objectives through solving one optimization problem with a generalized quadratic cost functional. The cost functional should reflect the dual goals of performance and desensitivity, and so an intuitive choice is

$$J = \beta_1 J_1 + \beta_2 J_2$$

$$= \beta_1 E\{\underline{r}' Q \underline{r}\} + \beta_2 \operatorname{tr} [W M_\infty] \qquad (77)$$

This is a weighted combination of performance and sensitivity "costs." The selection of both $\underline{\alpha}_d$ and K could be accomplished by solving a minimax problem, viz,

$$J^* = \min_K \max_{\underline{\alpha}} J(\underline{\alpha}, K)$$

However, this represents a problem of immense difficulty. An alternate approach has been suggested, based on the concept of "maximum difficulty":

1. Determine the design point $\underline{\alpha}_d$ on the basis of a maximum difficulty criterion. This criterion is dependent on open-loop system properties, and so $\underline{\alpha}_d$ can be found independent of K.

2. Determine K by solving an ordinary minimization problem, min $J(\underline{\alpha}_d, K)$.

This two-step procedure is feasible from a computational viewpoint. The first step is discussed in Appendix B. The second problem, finding K, is the subject of this effort. But since $\underline{\alpha}_d$ has not been selected, the optimization with respect to K will consider $\underline{\alpha}_d$ as fixed, but arbitrary. As a starting point we will pick $\underline{\alpha}_d = \underline{\alpha}_o$ = nominal point. Thus, we seek

$$K^* = \arg \min_K J(\underline{\alpha}_d, K)$$

Another interpretation of the above cost functional is that J_1 seeks a K to minimize performance at the design point $\underline{\alpha}_d$. The second term J_2 seeks to enlarge the region about $\underline{\alpha}_d$ in which the gains K remain useful. The relative weightings β_1 and β_2 have $\beta_1 = 1$ and β_2 chosen so that (after finding K*) the resulting J_1 does not exceed its minimum value by more than a preselected factor $1 + \varepsilon$, where

$$1 + \varepsilon = \frac{J_1(\alpha_o, K^*)}{J_1(\alpha_o, K_o)} \tag{78}$$

Thus, we trade off a fraction $\epsilon > 0$ of performance cost for desensitivity. Note that this trade-off need not necessarily be on J_1; it could be on $E\{u_i^2\}$, $E\{\dot{u}_i^2\}$, etc.

Consideration has been given to including a third term

$$J_3 = E\{(\underline{r} - \underline{r}_o)'P(\underline{r} - \underline{r}_o)\}$$

within the cost functional J. This would tend to minimize the deviations in responses from the original design. Also, it would add terms of the form $(K - K_o)$, thereby placing constraints on the control gains and indirectly on the control effort. It is similar to the uncertainty weighting design of Reference 1, but in a more meaningful closed-loop context. The equations that result from appending $\beta_3 J_3$ have been developed in detail. But since they are more complex than those for J_1 and J_2 alone, they will not be included in the following sections; in the following sections, we assume $\beta_3 = 0$.

PROBLEM SOLUTION

In this subsection a closed-form expression is obtained for the gradient matrix $\partial J / \partial K$ that will be used in the subsequent numerical optimization.

An Expression for the Cost Functional

The cost functional J, Equation (77), can be rewritten to combine the two terms J_1 and J_2. The result is

$$J = E\{\underline{x}' Q_{xx} \underline{x} + \sum_{i=1}^{p} \underline{\sigma}_i' Q_{\sigma_i \sigma_i} \underline{\sigma}_i + 2 \sum_{i=1}^{p} \underline{x}' Q_{x\sigma_i} \underline{\sigma}_i\} \tag{79}$$

where

$$\underline{\sigma}_i = \frac{\partial \underline{x}}{\partial \alpha_i}$$

is the closed-loop sensitivity vector at the point $\underline{\alpha} = \underline{\alpha}_d = \underline{\alpha}_o$. Defining

$$\bar{F} = F + G_1 K \tag{80}$$

as the actual closed-loop system matrix, $\underline{\sigma}_i$ is seen to satisfy the differential equation

$$\dot{\underline{\sigma}}_i = \bar{F} \underline{\sigma}_i + F_i \underline{x} ; \quad i = 1, \ldots, p \tag{81}$$

where we recall that G_1 and G_2 are not functions of $\underline{\alpha}$.

The weighting matrices in Equation (79) are

$$Q_{xx} = \beta_1 (H+DK)' Q (H+DK) + \beta_2 \sum_{i=1}^{p} (H_i+D_iK)' S_i (H_i+D_iK) \tag{82a}$$

$$Q_{\sigma_i \sigma_i} = \beta_2 (H+DK)' S_i (H+DK) \tag{82b}$$

$$Q_{x\sigma_i} = \beta_2 (H_i+D_iK)' S_i (H+DK) \tag{82c}$$

where the matrices

$$S_i \stackrel{\Delta}{=} w_i S \qquad i = 1, \ldots, p \tag{83}$$

The state vector, $\underline{x}$, and sensitivity vectors $\underline{\sigma}_i$ can be combined into an $(n_p + 1) \cdot n_x$ augmented system. Defining

$$\underline{x}_A = [\underline{x}, \underline{\sigma}_1, \underline{\sigma}_2, \ldots, \sigma_p]'$$

as the augmented state, the resulting system equation is

$$\dot{\underline{x}}_A = \bar{F}_A \underline{x}_A + G_{2A} \eta \tag{84}$$

where

$$\bar{F}_A = \begin{bmatrix} \bar{F} & 0 & \ldots & & 0 \\ F_1 & \bar{F} & & & \\ F_2 & 0 & \bar{F} & & \\ \vdots & & & & \\ F_{n_p} & 0 & \ldots & 0 & \bar{F} \end{bmatrix} ; \qquad G_{2_A} = \begin{bmatrix} G_2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

It is important to point out that this augmentation is done purely for analytic simplicity. Fortunately, it will not be necessary to solve large dimensional linear (or nonlinear) matrix equations in the ensuing optimization process.

In terms of $\underline{x}_A$, the cost functional J can be written as

$$J = E\{\underline{x}_A' Q_A \underline{x}_A\} \tag{85}$$

where

$$
Q_A = \begin{bmatrix} Q_{xx} & Q_{x\sigma_1} & \cdots & & Q_{x\sigma_p} \\ Q'_{x\sigma_1} & Q_{\sigma_1\sigma_1} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ Q'_{x\sigma_p} & 0 & & & Q_{\sigma_p\sigma_p} \end{bmatrix}
$$

Note that there are no cross-terms between σ_i and σ_j. We define

$$
C_A = \begin{bmatrix} C_{xx} & C_{x\sigma_1} & C_{x\sigma_2} & \cdots & C_{x\sigma_p} \\ C'_{x\sigma_1} & C_{\sigma_1\sigma_1} & C_{\sigma_1\sigma_2} & \cdots & C_{\sigma_1\sigma_p} \\ C'_{x\sigma_2} & C'_{\sigma_1\sigma_2} & C_{\sigma_2\sigma_2} & & \\ C'_{x\sigma_p} & \vdots & & & C_{\sigma_p\sigma_p} \end{bmatrix}
$$

as the covariance matrix of the augmented state,

$$
C_A = \mathrm{COV}[\underline{x}_A] = E\{\underline{x}_A \underline{x}_A\} \tag{86}
$$

It is a full matrix with

$$
C_{xx} = E\{\underline{x}\ \underline{x}'\} \tag{87a}
$$

$$
C_{x\sigma_i} = E\{\underline{x}\ \sigma_i\} \tag{87b}
$$

$$
C_{\sigma_i\sigma_j} = E\{\underline{\sigma}_i \underline{\sigma}_j'\} \tag{87c}
$$

Using the cyclic property of the trace in Equation (85) and substituting Equation (86) yields

$$J = \mathrm{tr}\,[Q_A C_A] \tag{88}$$

From Equation (84) it is seen that C_A satisfies the Lyapunov equation

$$0 = \bar{F}_A C_A + C_A \bar{F}_A + G_{2A} G_{2A}' \tag{89}$$

or, equivalently,

$$C_A = \int_0^\infty e^{\bar{F}_A t}\, G_{2A} G_{2A}'\, e^{\bar{F}_A' t}\, dt \tag{90}$$

Incorporating Equation (90) into Equation (88) and manipulating terms gives

$$J = \mathrm{tr}\,[L_A\, G_{2A}\, G_{2A}'] \tag{91}$$

where

$$L_A = \int_0^\infty e^{\bar{F}_A' t}\, Q_A\, e^{\bar{F}_A t}\, dt$$

satisfies the linear equation

$$\bar{F}_A'\, L_A + L_A\, \bar{F}_A + Q_A = 0 \tag{92}$$

Equation (91) is better to use in the analysis than is Equation (88) since both Q_A and C_A in Equation (88) depend on K, whereas only L_A in Equation (91) depends on K.

Equations (91) and (92) may be further simplified by investigating the special block structure and form of L_A:

$$L_A = \begin{bmatrix} L_{xx} & L_{x\sigma_1} & L_{x\sigma_2} & \cdots & L_{x\sigma_p} \\ L'_{x\sigma_1} & L_{\sigma_1\sigma_1} & 0 & \cdots & 0 \\ L'_{x\sigma_2} & 0 & L_{\sigma_2\sigma_2} & & \\ \vdots & \vdots & & \ddots & \\ L'_{x\sigma_p} & 0 & & & L_{\sigma_p\sigma_p} \end{bmatrix}$$

It is easy to show, by expanding Equation (92), that the cross-terms $L_{\sigma_i\sigma_j} = 0$. The equations for the nonzero components are

$$0 = \bar{F}'L_{\sigma_i\sigma_i} + L_{\sigma_i\sigma_i}\bar{F} + Q_{\sigma_i\sigma_i} \qquad i = 1, \ldots, p \tag{93a}$$

$$0 = \bar{F}'L_{x\sigma_i} + L_{x\sigma_i}\bar{F} + Q_{x\sigma_i} + F_i'L_{\sigma_i\sigma_i} \qquad i = 1, \ldots, p \tag{93b}$$

$$0 = \bar{F}'L_{xx} + L_{xx}\bar{F} + Q_{xx} + \sum_{i=1}^{p} (L_{x\sigma_i}F_i + F_i'L'_{x\sigma_i}) \tag{93c}$$

These equations are solved in the order given. We first solve for $L_{\sigma_i\sigma_i}$,* then for $L_{x\sigma_i}$, and then for L_{xx}. The matrix L_{xx} is all that is needed in evaluating J because of the sparse form of $G_{2A}G'_{2A}$. Thus, from Equation (91),

---

*Note $L_{\sigma_i\sigma_i} = \frac{wi}{wj} L_{\sigma_j\sigma_j}$ so that only the equation for σ_i need be solved.

$$J = \mathrm{tr}\,[L_{xx}\, G_2 G_2'] \qquad (94)$$

The sequence of lower order matrix equations to compute the covariance terms (62) and (63) is given by

$$0 = \bar{F} C_{xx} + C_{xx} \bar{F}' + G_2 G_2' \qquad (95a)$$

$$0 = \bar{F} C_{x\sigma_i} + C_{x\sigma_i} \bar{F}' + C_{xx} F_i' \qquad i = 1, \ldots, p \qquad (95b)$$

$$0 = \bar{F} C_{\sigma_i \sigma_j} + C_{\sigma_i \sigma_j} \bar{F}' + F_i C_{x\sigma_j} + C_{x\sigma_i}' F_j' \qquad (95c)$$

$$\text{for } i = 1, \ldots, p; \quad j = i, \ldots, p$$

Thus, on the basis of Equations (93) and (94), only (p + 2) linear equations of order n_x need be solved to obtain the value of J at a given K. Moreover, all of these equations involve the same matrix $\bar{F}$; the covariance equations (95) all involve $\bar{F}'$. It is shown below that a modified Bartels-Stewart algorithm[5] can be used to reduce greatly the computational burden associated with evaluating Equations (93) through (95).

The optimization problem is therefore to find the constant gain matrix K* to minimize $\mathrm{tr}\,[L_{xx}\ G_2 G_2']$ (at the nominal point $\underline{\alpha}_d = \underline{\alpha}_o$), such that

$$J_1(\underline{\alpha}_o, K^*) \leq (1 + \epsilon)\, J_1(\underline{\alpha}_o, K_o)$$

Note that this approach constrains the control $\underline{u}(t)$ to be of the form

$$\underline{u}(t) = K\, \underline{x}(t)$$

There is no feedback of the sensitivity vectors $\underline{\alpha}_i$ as in the sensitivity vector approach.

Gradient Expressions

It is not possible to find a closed-form expression for the gains that minimize J, except when $\beta_2 = 0$. For this reason, the optimization will be approached via some form of gradient algorithm. Closed-form expressions for the gradient matrix $\partial J/\partial K$ will be of great advantage in this process, since numerical evaluation of gradients would be extremely time-consuming.

The gradient $\nabla_K J = \partial J/\partial K$ is evaluated using a technique of Kleinman for derivatives of trace functions.[6] From (93c), the first order variation in L_{xx} to a change $K \rightarrow K + \delta K$ is (note that $\bar{F}$ is a function of K),

$$0 = \bar{F}'\delta L_{xx} + \delta L_{xx}\bar{F} + \delta K'G_1'L_{xx} + L_{xx}G_1\delta K + \delta Q_{xx}$$

$$+ \sum_{i=1}^{p} (\delta L_{x\sigma_i} F_i + F_i' \, \delta L'_{x\sigma_i}) \tag{96a}$$

where

$$0 = \bar{F}'\,\delta L_{x\sigma_i} + \delta L_{x\sigma_i}\bar{F} + \delta K'G_1' L_{x\sigma_i} + L_{x\sigma_i} G_1\delta K + \delta Q_{x\sigma_i}$$

$$+ F_i' \, \delta L_{\sigma_i\sigma_i} \tag{96b}$$

and,

$$0 = \bar{F}'\delta L_{\sigma_i\sigma_i} + \delta L_{\sigma_i\sigma_i}\bar{F} + \delta K'G_1' L_{\sigma_i\sigma_i} + L_{\sigma_i\sigma_i} G_1\delta K + \delta Q_{\sigma_i\sigma_i} \tag{96c}$$

for $i = 1, \ldots, p$. The first order variations in the components of Q_A are given by

$$\delta Q_{xx} = \beta_1[\delta K'D'Q(H + DK) + (H + DK)'QD\delta K] + \beta_2 \sum_{i=1}^{p} [\delta K'D_i'S_i(H_i + D_iK) + (H_i + D_iK)'S_iD_i\delta K] \tag{97a}$$

$$\delta Q_{x\sigma_i} = \beta_2 [(H_i + D_iK)'S_iD\delta K + \delta K'D_i'S_i(H + DK)] \tag{97b}$$

$$\delta Q_{\sigma_i\sigma_i} = \beta_2 [\delta K'D'S_i(H + DK) + (H + DK)'S_i D\delta K] \tag{97c}$$

The matrix $\bar{F}$ is the closed-loop system matrix which is required to be stable for any choice of K. Equations (96a) through (96c) can therefore be written as equivalent integral expressions,

$$\delta L_{\sigma_i\sigma_i} = \int_o^\infty e^{\bar{F}'\gamma}[\delta K'G_1'L_{\sigma_i\sigma_i} + L_{\sigma_i\sigma_i}G_1\delta K + \delta Q_{\sigma_i\sigma_i}]e^{\bar{F}\gamma}d\gamma$$

$$\delta L_{x\sigma_i} = \int_o^\infty e^{\bar{F}'t}[\delta K'G_1'L_{x\sigma_i} + L_{x\sigma_i}G_1\delta K + \delta Q_{x\sigma_i} + F_i'\delta L_{\sigma_i\sigma_i}]e^{\bar{F}t}dt$$

where a similar expression for δL_{xx} can be written directly from Equation (96a).

Since $\delta J = tr[\delta L_{xx}G_2G_2']$, we substitute for δL_{xx} its integral expression. Substituting further the integral expressions for $\delta L_{\sigma_i\sigma_i}$ and $\delta L_{x\sigma_i}$, using the cyclic properties of the trace and Kleinman's lemma,[6] one obtains, after tedious manipulation,

$$\nabla_K J = 2G_1'L_{xx}C_{xx} + 2\beta_1 D'Q(H + DK)C_{xx} + 2G_1' \sum_{i=1}^{p} [L_{x\sigma_i}C_{x\sigma_i}' + L_{x\sigma_i}'C_{x\sigma_i} + L_{\sigma_i\sigma_i}C_{\sigma_i\sigma_i}]$$

$$+ 2\beta_2 \sum_{i=1}^{p} [D_i'S_i(H + DK)C'_{x\sigma_i} + D'S_i(H_i + D_iK)C_{x\sigma_i}$$

$$+ D'S_i(H + DK)C_{\sigma_i\sigma_i} + D_i'S_i(H_i + D_iK)C_{xx}] \tag{98}$$

Equation (98) will serve as the basis of a gradient algorithm to minimize J. The computational requirements to compute $\nabla_K J$ and J at a given K are now the major issues.

COMPUTATIONAL ALGORITHMS

This section describes the numerical schemes for computing J, $\nabla_K J$, and M_∞ for a given feedback.

General Overview

In order to numerically evaluate the cost functional J we must solve (p + 2) linear matrix equations (93a) through (93c). To evaluate the gradient $\nabla_K J$, an additional (2p + 1) equations need be solved for C_{xx}, $C_{x\sigma_i}$, and $C_{\sigma_i\sigma_i}$ as in Equations (95a) through (95c). Note that the cross-correlations $C_{\sigma_i\sigma_j}$ $i \neq j$ are not needed to compute $\nabla_K J$. Examination of these 3p + 3 equations reveals the following:

1. All equations involve $\bar{F}$ or $\bar{F}'$. Half of the linear equations are adjoint to the other half. In particular, Equation (95) is adjoint to Equation (93).

2. A total of p + 3 equations have symmetric solutions, whereas the other 2p equations for $L_{x\sigma_i}$ and $C_{x\sigma_i}$ have nonsymmetric solutions.

A significant reduction in computation time can be brought about by several modifications to the linear equation algorithm of Bartels and Stewart. This algorithm is well suited to the efficient sequential solution of

$$A'X + XA = C_i \tag{99}$$

with different right-hand sides C_i. The existing algorithms and available programs are geared for symmetric C_i, and hence, symmetric X. By a slight modification a skew-symmetric C_i can be handled as well. Thus, the two major objectives for sensitivity design are

1. To solve Equation (99) when C_i (and therefore, X) is nonsymmetric, and still take advantage of the saving in computer time and efficiency afforded by a symmetric problem, and

2. To solve the adjoint equation

$$AX + XA' = C_i \tag{100}$$

using a computer program written to solve Equation (99).

Modifications to the Bartels-Stewart Algorithm

The Bartels-Stewart algorithm for solving Equation (99) in the symmetric (or skew-symmetric) case first reduces A to an upper Schur from via an orthogonal transformation Q. The resulting matrix $\tilde{A}$ is of the form

$$\tilde{A} = Q' A Q = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1p} \\ 0 & \tilde{A}_{22} & \cdots & \tilde{A}_{2p} \\ & & \ddots & \\ 0 & & & \tilde{A}_{pp} \end{bmatrix}$$

where each submatrix $\tilde{A}_{ii}$ is at most a 2 x 2. If

$$\tilde{C}_i = Q' C_i Q = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \cdots & \tilde{C}_{1p} \\ \tilde{C}_{21} & \tilde{C}_{22} & \cdots & \tilde{C}_{2p} \\ \vdots & & & \vdots \\ \tilde{C}_{p1} & & & \tilde{C}_{pp} \end{bmatrix}$$

and

$$\tilde{X} = Q' X Q = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \cdots & \tilde{X}_{1p} \\ \tilde{X}_{21} & \tilde{X}_{22} & \cdots & \tilde{X}_{2p} \\ \vdots & & & \\ \tilde{X}_{p1} & \cdots & & \tilde{X}_{pp} \end{bmatrix}$$

then Equation (99) is equivalent to

$$\tilde{A}' \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}_i \tag{101}$$

Since the partitions of $\tilde{C}_i$ and $\tilde{X}$ are conformal with $\tilde{A}$, expanding Equation (101) gives

$$\tilde{A}'_{kk} \tilde{X}_{k\ell} + \tilde{X}_{k\ell} \tilde{A}_{\ell\ell} = \tilde{C}_{k\ell} - \sum_{j=1}^{k-1} \tilde{A}'_{kj} \tilde{X}_{j\ell} - \sum_{i=1}^{\ell-1} \tilde{X}_{k_i} \tilde{A}_{i\ell} \tag{102}$$

$$\ell = 1, 2, \ldots, p; \quad k = \ell, \ldots, p$$

These equations can be solved sequentially for $\tilde{X}_{11}$, $\tilde{X}_{21}$, $\tilde{X}_{22}, \ldots, \tilde{X}_{p1}$, $\ldots, \tilde{X}_{pp}$; we then fill in the upper part of $\tilde{X}$ by symmetry

$$\tilde{X}_{\ell k} = \tilde{X}'_{k\ell}$$

or skew-symmetry

$$\tilde{X}_{\ell k} = -\tilde{X}'_{k\ell}$$

as the case may be. The "mini-systems" of Equation (102) are solved via a simple algebraic equation program. Thus, with $\tilde{X}$ calculated, the solution X to Equation (99) is

$$X = Q \tilde{X} Q'$$

Once the real Schur form of A and Q have been calculated, they may be saved and reused to solve the same equation (99) with different C_i. They are also used in the iterative refinement of the computed solution. This is the forte of the Bartels-Stewart algorithm.

Solution of the Nonsymmetric Case

Any general C_i can be written as a sum of a symmetric part $C_i^{(1)}$ and a skew symmetric part $C_i^{(2)}$ where

$$C_i^{(1)} = \tfrac{1}{2}(C_i + C_i')$$

$$C_i^{(2)} = \tfrac{1}{2}(C_i - C_i')$$

For a general matrix C_i, the linear equation (99) can be solved by adding the solutions of the two equations

$$A'X^{(1)} + X^{(1)}A = C_i^{(1)} \tag{103a}$$

$$A'X^{(2)} + X^{(2)}A = C_2^{(2)} \tag{103b}$$

Thus, $X = X^{(1)} + X^{(2)}$, where we note that $X^{(1)}$ is symmetric and $X^{(2)}$, is skew-symmetric.

Once an upper Schur form of A is available, say from a previous solution of Equation (99), we can call the computer program twice and solve for a nonsymmetric solution. The time required to solve the linear equation (99) once the matrices $\tilde{A}$ and Q have been found is about 40 percent of that required to solve the equation for the first time. Therefore, a nonsymmetric solution is obtained in about 80 percent of the time needed to solve the equation once.

Solution of the Adjoint Case

The adjoint equation (100) could be solved rapidly if somehow we could obtain $\tilde{A}'$ in upper Schur form from $\tilde{A}$. Consider Equation (100) where Q and the upper Schur form are given. Pre- and post-multiply this equation by Q' and Q, and note that $QQ' = I$ gives

$$Q'AQ\,(Q'XQ) + (Q'XQ)\,Q'A'Q = Q'C_iQ$$

or

$$\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{A}' = \tilde{C}_i \tag{104}$$

where $\tilde{A}'$ is in lower Schur form. Now we need $\tilde{A}'$ in upper Schur form to use a program written to solve Equation (99). The matrix $\tilde{A}'$ can be transformed to upper Schur form by applying a symmetric, orthogonal transformation T to Equation (104) where

$$T = \begin{bmatrix} & 0 & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & 0 & \end{bmatrix}$$

Pre- and post-multiplying Equation (104) by T gives

$$T\tilde{A}T(T\tilde{X}T) + (T\tilde{X}T)T\tilde{A}'T = T\tilde{C}_iT$$

or

$$\tilde{A}_1'\tilde{X}_1 + \tilde{X}_1\tilde{A}_1 = T\tilde{C}_iT \tag{105}$$

where $\tilde{A}_1$ is now in upper Schur form as required. A summary of the steps needed to solve Equation (100) is as follows:

1. Transpose $\tilde{A}$.
2. Obtain $\tilde{A}_1 = T\,\tilde{A}'T$, so $\tilde{A}_1$ is $\tilde{A}'$ with its rows and columns written in reverse order.
3. Obtain $T\,\tilde{C}_i\,T$.
4. Solve for $\tilde{X}_1$ using the same algorithm as for Equation (99).
5. Obtain $X = T(Q\,\tilde{X}_1\,Q')T$.

A general-purpose computer program, AXPTA, has been written to solve the linear matrix equation via the Bartels-Stewart approach. It has the features to solve the symmetric, skew-symmetric, and adjoint cases taking advantage of previously obtained $\tilde{A}$ and Q.

<u>Computational Requirements for J, $\nabla_K J$, and M_∞</u>

As noted above, p + 3 symmetric and 2p nonsymmetric solutions are needed in evaluating J and $\nabla_K J$. Taking advantage of the algorithm modifications, these symmetric equations can be solved in an "equivalent" computational time of $1 + 0.4(p + 2) = 0.4p + 1.8$ linear equations. The remaining 2p equations with nonsymmetric right-hand sides can be solved in the equivalent of $2p \times 0.8 = 1.6p$ linear equations. The total computation time is then $\approx 2.0p + 1.8$ linear equations. Thus, to obtain J and $\nabla_K J$ for $p = 3$, we need solve the equivalent of roughly eight linear equations. This is comparable to the time required to solve one (n_x-dimensional) Riccati equation. Thus, the computations of J and its gradient (at each iteration) are not excessive.

The unit time information matrix M_∞ is not required explicitly in the optimization algorithm. However, it is useful to monitor M_∞ and the dispersion matrix

$$D_M \triangleq M_\infty^{-1} \tag{106}$$

as the algorithm proceeds through the iterations to see how the uncertainty regions for α_i increase. Certainly, one would wish to compare M_∞ or D_M at the optimal point K* with their initial values at K_o. In addition, the (initial) diagonal elements of M_∞ are used in forming the weighting factors w_i in Equation (52).

The p x p information matrix M_∞ is

$$M_\infty = E\left\{\left(\frac{\partial \underline{r}}{\partial \alpha_1}\ \frac{\partial \underline{r}}{\partial \alpha_2} \cdots \frac{\partial \underline{r}}{\partial \alpha_p}\right)' S \left(\frac{\partial \underline{r}}{\partial \alpha_1}\ \frac{\partial \underline{r}}{\partial \alpha_1} \cdots \frac{\partial \underline{r}}{\partial \alpha_p}\right)\right\} \tag{107}$$

The response sensitivities are

$$\frac{\partial \underline{r}}{\partial \alpha_i} = (H_i + D_i K)\,\underline{x} + (H + DK)\,\underline{\sigma}_i$$

Substituting these into Equation (107) gives for the ij element of M_∞,

$$(M_\infty)_{ij} = \mathrm{tr}\,[(H + DK)'\,S\,(H + DK)\,C_{\sigma_i\sigma_j} + (H_i + D_iK)'S(H_j + D_jK)C_{xx}$$

$$+ (H + DK)'\,S\,(H_j + D_jK)C_{x\sigma_i} + (H_i + D_iK)'S(H + DK)C'_{x\sigma_j}]$$

Notice that C_{xx}, $C_{x\sigma_i}$ and $C_{\sigma_i\sigma_i}$ will have already been evaluated while computing $\nabla_K J$. Therefore, it is only necessary to determine $C_{\sigma_i\sigma_j}$ for $i \neq j$ from the Equation (95c). Only the terms for $j > i$ need be computed because of symmetry. This requires solving an additional $p(p - 1)/2 = 3$ linear equations, all with the same system matrix $\bar{F}$.

Iterative Algorithm for Finding K*

Returning to the optimization problem for K*, two gradient algorithms are proposed in this section. The first algorithm was tried first, primarily to check the validity of the overall approach to the sensitivity problem. The second method is based on the conjugate gradient algorithm of Fletcher-Reeves.[7] As noted earlier, the computer programs solve for K* at a given design point $\underline{\alpha}_d$, assumed here to be the same as $\underline{\alpha}_o$.

Successive Substitutions Scheme

This is a simple iterative scheme that has no proven convergence properties. It is motivated by old algorithms for the optimal output feedback problem. The idea is to find a set of gains $K^{(\ell + 1)}$ that would result in $\nabla_K J = 0$. The choice is based on matrices computed at iteration ℓ. The algorithm is as follows:

1. Set ℓ = o select an arbitrary initial (stabilizing) gain $K^{(o)}$. Usually $K^{(o)} = K_o$. Set $J^{(-1)} = \infty$.

2. Compute $L_{\sigma_i\sigma_i}$, $L_{x\sigma_i}$, L_{xx} and $J(K^{(\ell)}) \stackrel{\Delta}{=} J^{(\ell)}$.

3. Check stopping condition $|J^{(\ell)} - J^{(\ell - 1)}| < TOL$
   If satisfied, stop.

4. Compute C_{xx}, $C_{x\sigma_i}$, $C_{\sigma_i\sigma_i}$ and the gradient $\nabla_j J^{(\ell)}$.

5. Compute the gain increment $\Delta K^{(\ell)}$ that would make $\nabla_K J^{(\ell + 1)} = 0$ assuming all other matrices remained constant. As seen from Equation (98) this is a near-impossible task. Thus, the gains K in the summation term are set to $K^{(\ell)}$; i.e., they are fixed, and we find

$$\Delta K^{(\ell)} = -\frac{1}{2\beta_1}\,(D'Q\,D)^{-1}\left\{\nabla_K J^{(\ell)}\right\} C_{xx}^{-1}$$

6. Select new gains $K^{(\ell+1)} = K^{(\ell)} + b\,\Delta K^{(\ell)}$ where $b = \frac{1}{2}$ initially. If the cost $J^{(\ell+1)}$ does not decrease, a smaller step is taken by reducing b.

7. Set $\ell = \ell + 1$ and return to Step 2.

The above algorithm is essentially a successive substitution scheme for solving an equation of the form $x = f(x)$. Such a scheme is convergent only if the slope of f is < 1. Thus, the given algorithm is expected to converge when the optimum gain K^* is close to the initial gain $K^{(o)}$. Unfortunately, the convergence rate of this method was found to be very slow, with considerable oscillation in K when near the optimum. Convergence, to less than 1 percent error, was usually attained in 10 to 18 iterations.

Conjugate Gradient Method

A conjugate gradient scheme was picked as an alternative to the above method. The steps are as outlined:

1. Set $\ell = o$ select initial gain $K^{(o)}$. Usually, we set $K^{(o)} = K_o$. Pick $M = p$ as recycle index.

2. Compute $J^{(o)}$.

3. Compute the gradient $\nabla J^{(\ell)}$ at the current gain $K^{(\ell)}$. If J is too small, then stop.

4. Compute the current descent direction

$$s^{(\ell)} = - J^{(\ell)} + a^{\ell} s^{(\ell - 1)}$$

where $a^o = 0$, and on subsequent iterations

$$a^{\ell} = \frac{tr\{[J^{(\ell)} - J^{(\ell - 1)}]' [J^{(\ell)}]\}}{tr\{[J^{(\ell)} - J^{(\ell - 1)}]' s^{(\ell - 1)}\}}$$

Normalize $s^{(\ell)}$ so that $\| s^{(\ell)} \| = 1$.

5. Compute the current step size b^{ℓ} using a one-dimensional search

$$b^{\ell} = \arg \min_{b} J [K^{(\ell)} + b\, s^{(\ell)}]$$

This is done via a quadratic interpolation scheme.

6. Compute new gains

$$K^{(\ell + 1)} = K^{(\ell)} + b^{\ell} s^{\ell}$$

and the cost $J^{(\ell + 1)}$.

7. Check convergence tests. If passed, then stop.

8. If $\ell < M$ set $\ell = \ell + 1$. Otherwise, set $K^{(o)} = K^{(\ell + 1)}$, $\ell = o$ Go to Step 3.

The convergence rate of the conjugate gradient method has proved superior to the first noted algorithm.* Typically, three to eight iterations have been needed for convergence, CPU time on 360/65 of about 6 to 7 minutes. The critical parameter for convergence is the initial guess for b in the one-dimensional search (Step 6). The present ad hoc guess is

$$b = \frac{\|K^{(\ell)}\|}{2} \cdot \frac{\text{Min Abs element in } K^{(\ell)}}{\text{Max Abs element in } K^{(\ell)}}$$

PRELIMINARY EVALUATIONS

The 15-state residualized model of the C-5A longitudinal dynamics as described in Reference 1 was used for evaluation purposes. The response vector used for the design of the nominal controller is ninth order and is defined by

r_1 = Bending moment at wing root

r_2 = Torsion moment at wing root

r_3 = Bending moment rate

r_4 = Torsion rate

r_5 = Aileron displacement

r_6 = Inboard elevator displacement

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*A combination of the two algorithms to give $\Delta K^{(\ell)}$ was tried. However, it gave mixed results and so was not pursued.

r_7 = Modified aileron rate

r_8 = Inboard elevator rate

r_9 = Control follower response

The weighting matrix Q used in the quadratic synthesis design is diagonal with

$$Q = \{0.1E-9,\ 0.1E-9,\ 0.1E-10,\ 0.32E+8,\ 0.0,\ 0.1E+7,\ 0.1E+5,\ 0.2E+6\}$$

The parameter p vector $\underline{\alpha}$ for the C-5A example is

$$\underline{\alpha} = (\bar{q}_f,\ \omega_f,\ M_{W_f})$$

where

$\bar{q}_f$ = dynamic pressure uncertainty factor

ω_f = structural frequency uncertainty factor

M_{W_f} = stability derivative uncertainty factor

The design process is conducted at the nominal condition, $\alpha_i = 1$. The gradient matrixes F_i, H_i, D_i for i = 1, 2, 3 were computed numerically. They are assumed to be constant within the range of parameter variation $[\bar{\alpha}_L, \bar{\alpha}_H]$.

The sensitivity reduction problem formulated via the Information Matrix approach leaves very few free parameters to be chosen. This is by design, since we have elected to minimize the free parameters in order to minimize the number of design iterations. As a result, one needs only select

1. Which of the n_r responses are to be desensitized, i.e., which responses $r_i(t)$ have $s_i \neq 0$, and

2. The value of β_2, giving a relative weighting to the sensitivity cost versus the performance cost.

As a first step in evaluating the technique, the effect of varying β_2 on the bending moment response, r_1, was investigated. In other words, $s_1 = 1$, $s_j = 0$, $j = 2, 9$. W_i's were set to unity to reflect equal importance on all uncertain parameters. Note that the weighted trace of the information matrix in this case is

$$\sum_{i=1}^{p} E\left\{\left(\frac{\partial r_1}{\partial \alpha_i}\right)^2\right\} s_1$$

As a means of evaluating the effect of varying β_2, a measure of local sensitivity was defined as

$$S_{\alpha_i}^{r_1} = \left[\frac{E\left\{\frac{\partial r_1}{\partial \alpha_i}^2\right\}}{E\left\{r_{1,o}^2\right\}}\right]^{\frac{1}{2}}$$

where the term $r_{1,o}$ refers to the bending moment response at the nominal condition. This is slightly in variance to the more classical definition of sensitivity which is given by

$$S_{\alpha_i}^{\sigma_r^2} = \frac{1}{E\{r^2\}} \frac{\partial E(r^2)}{\partial \alpha_i}$$

$$= \frac{1}{E\{r^2\}} \quad 2E\left\{r \frac{\partial r}{\partial \alpha_i}\right\}$$

$$= \frac{2\rho E\{r^2\}^{\frac{1}{2}}}{E\{r^2\}} \quad E\left\{\left(\frac{\partial r}{\partial \alpha_i}\right)^2\right\}^{\frac{1}{2}}$$

$$= 2\rho \left[\frac{E\left\{\left(\frac{\partial r}{\partial \alpha_i}\right)^2\right\}}{E\{r^2\}}\right]^{\frac{1}{2}}$$

where ρ is the correlation coefficient between r and $\frac{\partial r}{\partial \alpha}$ at the nominal point. Hence, the measure used is proportional to the more classical definition.

Figure 25 presents a plot of $S_\alpha^{r_1}$ versus β_2. As can readily be seen, there is a reduction in sensitivity, particularly with respect to structural frequency uncertainties.

The effects of varying β_2 were also ascertained with respect to J_1, the performance cost index, control activity measured in terms of aileron displacement, and the identifiability of the uncertain parameters measured in terms of standard deviations of the estimates. These results are plotted in Figures 26, 27, and 28, respectively. As expected, the results are as follows:

- Performance as measured by the cost index J_1 increases with β_2.
- Decreased sensitivity requires increased control activity.
- Uncertain parameters are harder to identify as β_2 is increased.

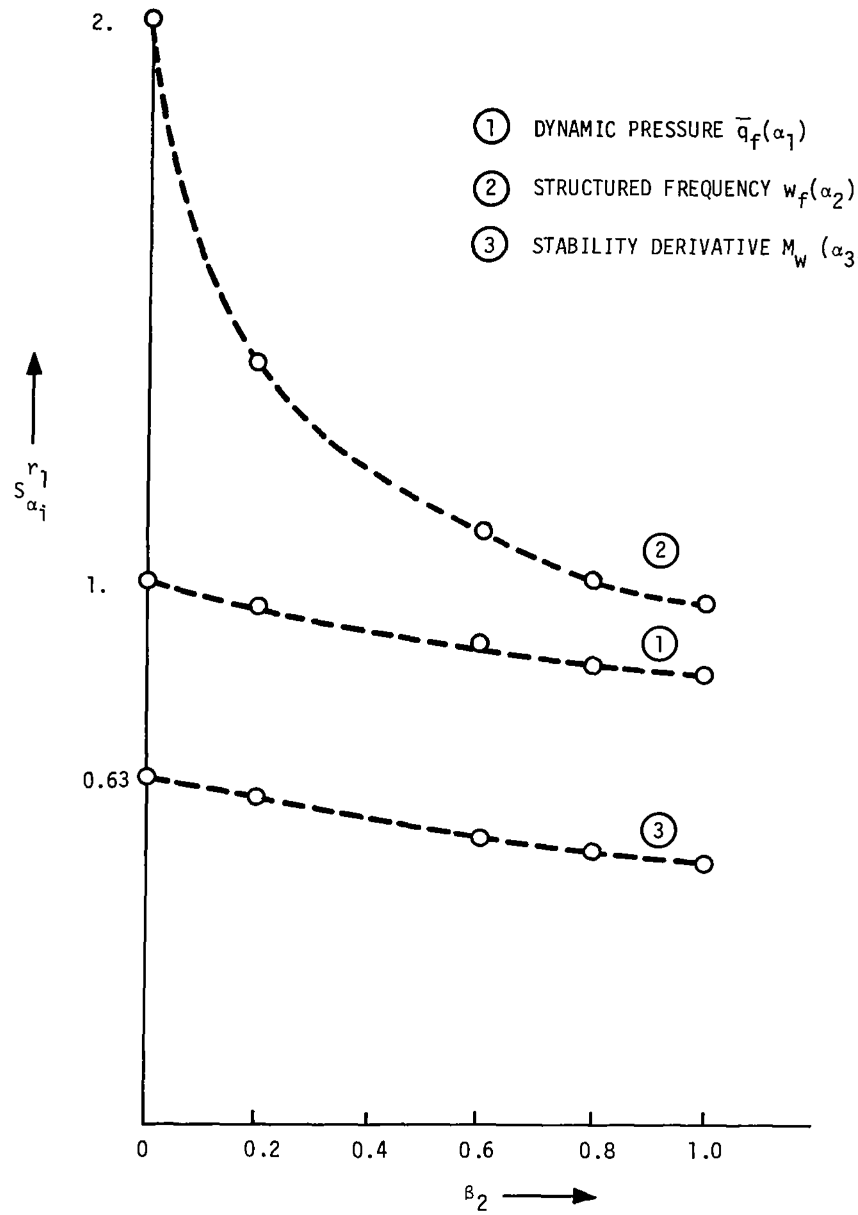

Figure 25. r_1 Sensitivity versus β_2

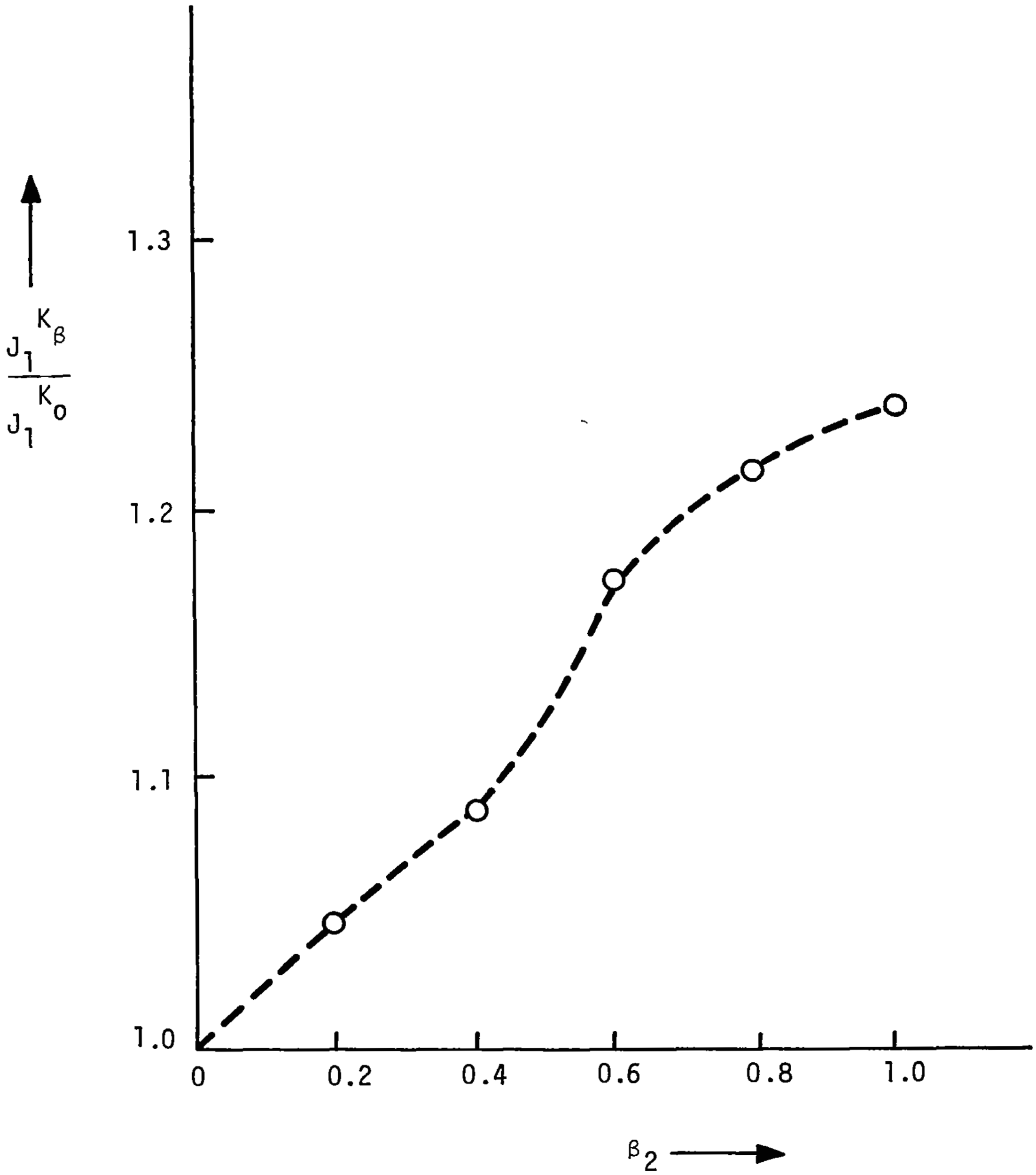

Figure 26. Cost Ratio versus β_2

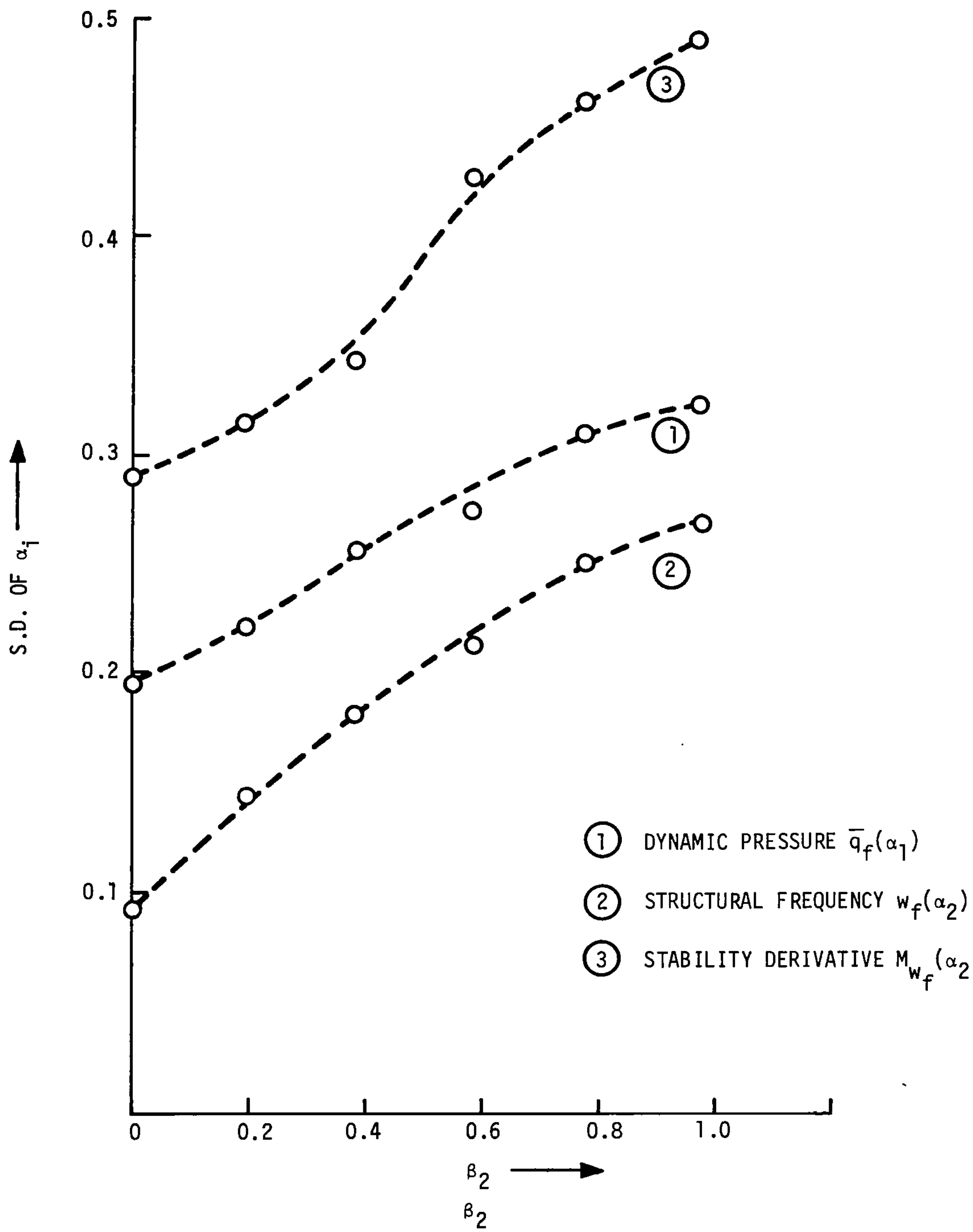

Figure 27. Aileron Control Activity versus β_2

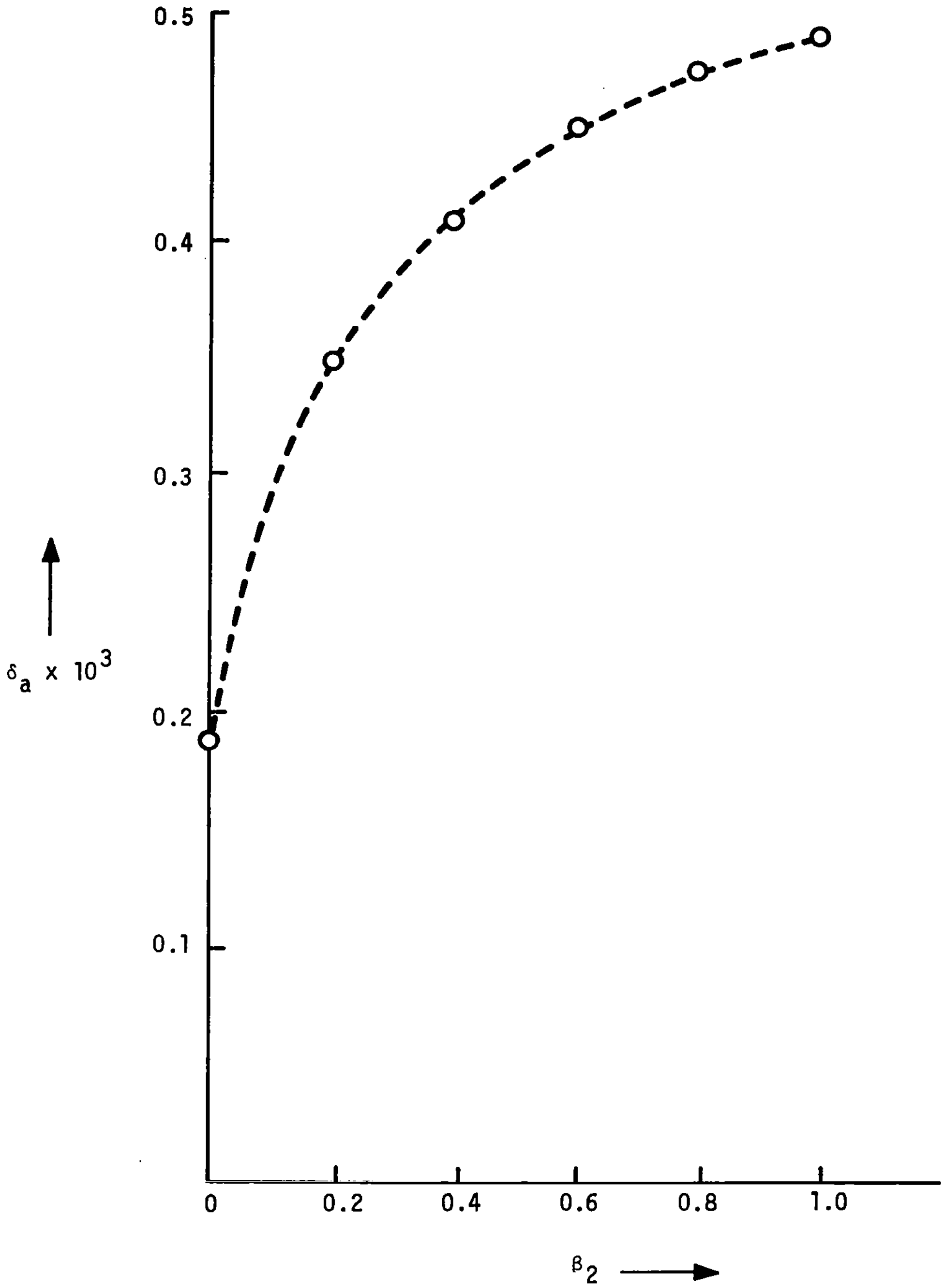

Figure 28. Standard Deviation of Uncertain Parameters versus β_2

Since the sensitivity measure employed is only valid over small variations, a linearity test was also performed. The parameters α_i were varied independently from -0.2 to 0.2 in steps of 0.05. The normalized incremental rms response $\frac{\Delta\sigma_{r_1}}{\sigma_{r_{1,o}}}$ and linear "prediction" $\frac{\partial\sigma_{r_1}}{\partial\alpha_i}\ \Delta\alpha_i/\sigma_{r_{1,o}}$ are listed in Table 7. The results are shown for parameters 1 and 3 only. For α_2, the analysis showed that the correlation between r_1 and $\partial r_1/\partial\alpha_2$ was very small. Thus, although $E\left\{\left(\frac{\partial r_1}{\partial\alpha_2}\right)^2\right\}$ is large, the effect of variations in α_2 on the actual response is small. As seen in Table 7, the changes are approximately linear.

TABLE 7. INCREMENTAL VERSUS LINEARIZED PARTIALS COMPARISON

Parameter Variation	Actual Performance Change		"Predicted" Change	
$\Delta\alpha_i$	$\frac{\Delta\sigma_{r_1}}{r_{1,o}}$		$\frac{\partial\sigma_{r_1}}{\partial\alpha_i}\ \Delta\alpha_i/r_{1,o}$	
	i=1	i=3	i=1	i=3
0.05	0.031	0.015	0.031	0.0142
0.10	0.061	0.031	0.059	0.028
0.15	0.090	0.049	0.086	0.042
0.20	0.1193	0.068	0.1111	0.056
-0.05	-0.0327	-0.0137	-0.0322	-0.0144
-0.10	-0.0679	-0.0262	-0.0654	-0.0291
-0.15	-0.1055	-0.038	-0.1009	-0.044
-0.20	-0.1460	-0.0483	-0.1357	-0.0591

With the effects of varying β_2 established, a refined design was undertaken and evaluated. This is discussed in the following section.

C-5A CONTROLLER DESIGN EVALUATION

As discussed in the preceding section, the Information Matrix design technique was formulated to limit the number of free parameters which the control system designer must manipulate. As presented here, the designer has freedom to vary the scalar term β_2 which weights the sensitivity reduction. The designer also may select which of the system responses he wishes to desensitize by manipulation of the binary variable s_i. The effect of varying β_2 on one system response was discussed in the previous section. This section will discuss the effect of varying both β_2 and s_i with the purpose of obtaining an insensitive C-5A control system.

Design Approach

In order to limit the freedom on selection of β_2, two additional constraints were imposed:

1. The value of J_1 of the insensitive controller must be less than 1.2 times the J_1 of the nominal controller.

2. Aileron and elevator control activity for the insensitive controller must be less than two times the controller activity of the nominal.

With these constraints imposed, computer runs were made to study the effects of

1. Sensitivity weights on rates $\dot{B}_1$, $\dot{T}_1$ to aid in desensitizing the bending and torsion moment responses,

2. Weighting aileron control responses r_5 to keep control effort from rising too rapidly,

3. Weighting the controller follower response r_9 as an attempt to desensitize ω_{sp}, ζ_{sp}, and

4. Various choices of β_2.

A total of 15 different cases were studied, including the nominal. The cases, described in Table 8, may be grouped into five categories:

1. Variations in β_2: Case 2A, 2B, 2C; Case 3, 3A, 4; Case 7, 8

2. Sensitivity weights on $\dot{B}_1$, $\dot{T}_1$: Case 7, 8

3. Sensitivity weight on δa: Case 2A-2F, 3, 3A, 4

4. Sensitivity weights on δ_{cF}: Case 2A-2F, 6

5. Variations in the (pseudo) noise/signal ratio φ on δa to study more closely effect of control: Case 2C-2F.

TABLE 8. PARAMETER ESTIMATE STANDARD DEVIATIONS AND OTHER PERTINENT INFORMATION FOR DIFFERENT GAINS

Run #	β_2	Diagonal elements of M_∞ with nominal gain K^o	Weights on trace of M_∞	Initial estimates of S-D of parameters with K^o	Lower Bounds On: Final estimates of S-D of parameters with K^*	Lower Bounds On: Volume of uncertainty ellipsoid at $K^o \times 10^7$	Lower Bounds On: Volume of uncertainty ellipsoid at $K^* \times 10^7$	Lower Bounds On: Cost, J_1 (Ratio)	Ratios of responses with K^* and K^o: $\frac{r_1 K^*}{r_1 K^o}$	Ratios of responses with K^* and K^o: $\frac{r_2 K^*}{r_2 K^o}$	Any weights on r_3 and r_4 i.e., B_1 and γ	Any weights on control response i.e., r_5	Any weight on control followed response	Signal/noise ratio on control, r_5 (% pseudo noise)
1 (Nominal)	NA	NA	NA	NA	NA	NA	NA	88.7 (1.0)	1.0	1.0	NA	NA	NA	NA
2A	0.4	116.2 538.2 49.1	0.40 0.08 0.95	0.0933 0.0431 0.1436	0.1146 0.0499 0.1551	3.36	8.035	91.0 (1.026)	0.977	0.944	No	Yes	Yes	31.8 (1.0)
2B	0.6	120.8 559.6 50.8	0.42 0.09 1.00	0.0913 0.0423 0.1409	0.1269 0.0590 0.1512	4.82	12.518	93.6 (1.055)	0.940	0.939	No	Yes	Yes	31.8 (1.0)
2C	0.5	120.8 559.6 50.8	0.42 0.09 1.00	0.0913 0.0423 0.1409	0.1110 0.0473 0.1571	4.82	6.770	93.6 (1.055)	0.952	0.924	No	Yes	Yes	31.8 (1.0)
2D	0.5	78.0 253.6 40.8	0.64 0.20 1.00	0.1155 0.0634 0.1432	0.1229 0.0670 0.1569	10.52	16.434	92.1 (1.038)	0.940	0.943	No	Yes	Yes	3.18 (10.0)
2E	0.5	87.0 325.2 48.6	0.558 0.149 1.0	0.1086 0.0557 0.1448	0.1177 0.0583 0.1548	7.47	11.21	90.38 (1.020)	0.954	0.955	No	Yes	Yes	10.6 (3.0)
2F	0.5	94.9 381.8 48.8	0.558 0.149 1.0	0.1036 0.0513 0.1443	0.1298 0.0630 0.1593	5.79	15.937	91.72 (1.034)	0.947	0.949	No	Yes	Yes	13.2 (2.0)
3	0.6	104.9 497.7 41.2	0.40 0.09 0.95	0.0989 0.0449 0.1579	0.1372 0.0568 0.1968	4.90	23.49	95.4 (1.076)	0.937	0.910	No	Yes	No	31.8 (1.0)
3A	0.7	104.9 497.7 41.2	0.40 0.09 0.95	0.0989 0.0449 0.1579	0.1448 0.0688 0.1995	4.90	38.90	94.47 (1.065)	0.9126	0.9026	No	Yes	No	31.8 (1.0)

TABLE 8. PARAMETER ESTIMATE STANDARD DEVIATIONS AND OTHER PERTINENT INFORMATION FOR DIFFERENT GAINS (concluded)

Run #	β_2	Diagonal elements of M_∞ with nominal gain K^o	Weights on trace of M_∞	Initial estimates of S-D of parameters with K^o	Lower Bounds On: Final estimates of S-D of parameters with K^*	Lower Bounds On: Volume of uncertainty ellipsoid at $K^o \times 10^7$	Lower Bounds On: Volume of uncertainty ellipsoid at $K^* \times 10^7$	Lower Bounds On: Cost, J_i (Ratio)	Ratios of responses with K^* and K^o: $\frac{r_1 K^*}{r_1 K^o}$	Ratios of responses with K^* and K^o: $\frac{r_2 K^*}{r_2 K^o}$	Any weights on r_3 and r_4 i.e., B_1 and γ	Any weights on control response i.e., r_5	Any weight on control followed response	Signal/noise ratio on control, r_5 (% pseudo noise)
4	0.4	104.1 497.7 41.2	0.40 0.08 0.95	0.0989 0.0499 0.1597	0.1296 0.0546 0.1867	4.90	17.43	91.4 (1.03)	0.961	0.931	No	Yes	No	31.8 (1.0)
5	0.2	59.8 169.3 40.3	0.68 0.20 0.80	0.1385 0.0796 0.1632	0.1452 0.0921 0.1780	15.03	50.78	90.1 (1.019)	0.931	0.967	No	No	No	NA
6	0.2	67.1 290.6 41.5	0.68 0.20 0.80	0.1258 0.0603 0.1555	0.1277 0.0824 0.1573	13.14	25.54	90.0 (1.018)	0.960	0.958	No	No	Yes	NA
7	0.2	127.2 1432.8 101.6	0.800 0.043 0.700	0.0942 0.0281 0.0992	0.1008 0.0331 0.1065	0.689	1.259	91.2 (1.027)	0.920	0.955	Yes	No	No	NA
8	0.1	127.2 1432.8 101.6	0.800 0.043 0.700	0.0942 0.0281 0.0992	0.0978 0.0306 0.1036	0.689	0.960	89.5 (1.0009)	0.945	0.978	Yes	No	No	NA
9	0.3	59.8 169.3 40.3	0.612 0.237 0.800	0.1385 0.0796 0.1632	0.1484 0.0994 0.1864	15.03	70.0	91.3 (1.029)	0.922	0.941	No	No	No	NA

The performance of the resulting 14 controllers plus the nominal controller are presented in Table 9. The 14 controllers were then compared to determine which one would be evaluated with the design criteria defined in Reference 1. The Case 3A controller was chosen based on conditions that

1. All the design specs were satisfied plus the additional imposed constraints on J_1 and control activity, and

2. The identifiability of the three uncertain parameters ($\overline{q}_f$, w_f, M_{W_f}) was reduced the most. This was measured by the volume of the uncertainty ellipse (which is approximately the determinant of the dispersion matrix) and the standard deviations of the uncertain parameters.

The gains for Case 3A are given in Table 10. With respect to the other variations that were investigated, it was found that

1. Weighting the control follower response offers no advantage. This is to be expected since the control follower response is only valid at the nominal condition. Its purpose is to achieve a specific control configuration (i.e., $\delta e_c = 0.5q$) which at the nominal produces desirable short period frequency and damping characteristics.

   At other than the nominal condition, the control follower response will not produce the described short period frequency and damping characteristics.

TABLE 9. FEEDBACK CONTROLLER PERFORMANCE--CASE 4R
NOMINAL (q_f = 1.0, ω_f = 1.0, M_{w_f} = 1.0)

	Specification Description		Criteria	Run Number														
				1 (Nom)	2A	2B	2C	2D	2E	2F	3	3A	4	5	6	7	8	9
$\times 10^{-6}$	RMS Responses	B_1	NA	0.721	0.705	0.677	0.687	0.678	0.688	0.683	0.676	0.658	0.692	0.671	0.692	0.663	0.681	0.672
		T_1		0.109	0.103	0.103	0.100	0.103	0.104	0.104	0.0993	0.099	0.101	0.106	0.105	0.104	0.107	0.106
$\times 1$	Handling Qualities	ω_{sp}	> 1.6 rad/s	2.10	2.20	2.36	2.22	2.26	2.23	2.50	2.27	2.36	2.18	2.24	2.24	2.29	2.22	2.27
		ζ_{sp}	0.7-0.8 sec^{-1}	0.710	0.711	0.774	0.682	0.728	0.726	0.831	0.685	0.724	0.611	0.731	0.738	0.743	0.731	0.741
$\times 10^{2}$	Surface Activity	δ_a	NA	0.0193	0.0207	0.0312	0.0277	0.0289	0.0213	0.0249	0.0258	0.022	0.0176	0.0275	0.0254	0.0303	0.0273	0.0282
	RMS (rad, rad/s)	$\dot{\delta}_a$		0.113	0.100	0.146	0.130	0.162	0.117	0.129	0.118	0.101	0.0895	0.162	0.158	0.179	0.159	0.172
		δ_e		0.150	0.167	0.176	0.175	0.172	0.169	0.174	0.184	0.19	0.173	0.169	0.164	0.172	0.163	0.169
		$\dot{\delta}_e$		0.371	0.384	0.418	0.378	0.367	0.382	0.429	0.409	0.48	0.400	0.416	0.379	0.441	0.404	0.432

TABLE 10. GAIN MATRIX FOR RUN #3A

11	-1.5756E-05	12	5.5747E-06	13	3.2549E-05	14	2.0883E-05	15	-2.0162E-04
16	-7.0419E-04	17	-2.1381E-01	18	1.1257E-03	19	2.5313E-05	110	-2.4873E-04
111	8.9334E-05	112	-8.5690E-06	113	-1.9357E-04	114	1.3001E-04	115	-4.5946E-07
21	1.9712E-04	22	5.0306E-04	23	-2.8724E-04	24	2.6991E-04	25	4.7223E-04
26	-6.2514E-03	27	3.4040E-03	28	-1.5226E-01	29	6.7437E-06	210	6.7847E-04
211	-1.2334E-03	212	-3.1175E-05	213	4.4720E-04	214	8.4716E-04	215	1.3936E-03

2. There is no significant change in system response as the signal/noise ratio on δ_a is varied.

3. The effect of including $\dot{B}_1$ and $\dot{T}_1$ in the sensitivity computations appears to have little effect on the resultant B_1 and T_1 responses.

Design Evaluation

The Case 3A Information Matrix controller was evaluated on the 15-state Case 4R residualized C-5A model at the six evaluation conditions. These conditions, chosen in the Reference 1 study, are as follows:

1. Nominal condition: $\underline{\alpha} = (1.0, 1.0, 1.0)$
2. Worst Case 1: $\underline{\alpha} = (1.25, 0.75, 0.8)$
3. Worst Case 2: $\underline{\alpha} = (0.5, 1.0, 1.2)$
4. Independent Variation 1: $\underline{\alpha} = (1.0, 1.0, 0.8)$
5. Independent Variation 2: $\underline{\alpha} = (1.0, 0.75, 1.0)$
6. Independent Variation 3: $\underline{\alpha} = (1.25, 1.0, 1.0)$

The performance of the Information Matrix controller is tabulated in Table 11 at the six evaluation conditions. Figures 29 through 33 graphically portray the tabulated data for each of the design specifications versus the performance of the nominal controller.

The three criteria defined in Reference 1 were used for evaluating the Information Matrix controller. The criteria may be briefly described as follows:

TABLE 11. INFORMATION MATRIX CONTROLLER PERFORMANCE EVALUATION MODEL--CASE 4R

Specification Description		Criteria	Nom $\bar{q}_f$ =1.0 ω_f =1.0 M_{w_f} =1.0	WC1 $\bar{q}_f$ =1.25 ω_f =0.75 M_{w_f} =0.8	WC2 $\bar{q}_f$ =0.5 ω_f =1.0 M_{w_f} =1.2	P1 $\bar{q}_f$ =1.0 ω_f =1.0 M_{w_f} =0.8	P2 $\bar{q}_f$ =1.0 ω_f =0.75 M_{w_f} =1.0	P3 $\bar{q}_f$ =1.25 ω_f =1.0 M_{w_f} =1.0
Maneuver Load % Change	B	< -30%	-45.4%	-32.1%	-73.9%	-45.7%	-45.3%	-32.5%
Gust Load Alleviation % Change	B	< -30%	-40.6%	-29.1%	-63.3%	-37.7%	-40.1%	-32.0%
	T	< + 5%	-37.9%	-20.7%	-59.6%	-32.9%	-36.3%	-29.8%
Handling Qualities	ω_{sp}	> 1.5 rad/sec	2.36	3.01	1.17	2.26	2.30	3.08
	ζ_{sp}	0.7-0.8 sec^{-1}	0.72	0.86	0.677	0.755	0.768	0.728
Stability Margins Gain:	aileron	> 6db	∞	∞	∞	∞	∞	∞
	elevator		32db	18db	*	32db	21db	27db
Phase:	aileron	> 45°	∞	∞	∞	∞	∞	∞
	elevator		118°	118°	125°	115°	125°	120°
Surface Activity RMS (rad rad/sec)	δa	NA	0.00022	0.00023	0.00024	0.00022	0.00022	0.00022
	δa		0.0010	0.0012	0.0011	0.0010	0.0011	0.0010
	δe		0.0019	0.0023	0.0018	0.0021	0.0020	0.0019
	δe		0.0048	0.0061	0.0036	0.0048	0.0053	0.0053

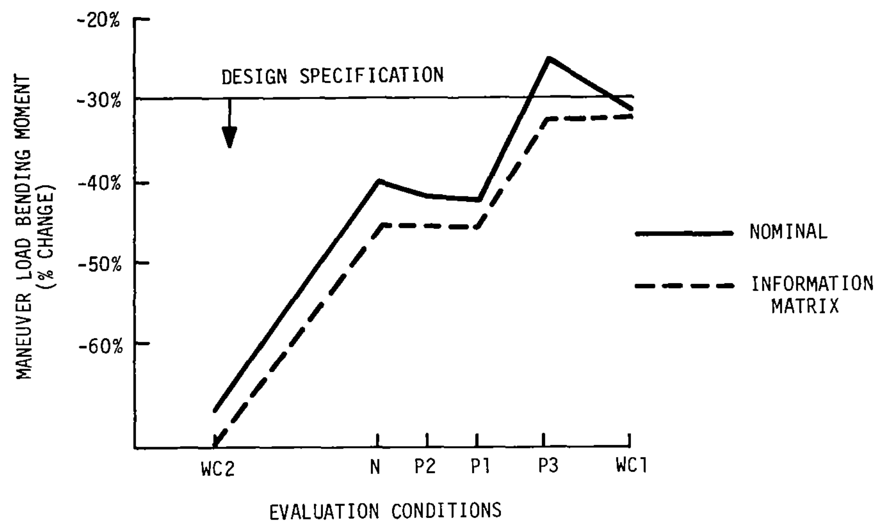

Figure 29. Case 4R Maneuver Load Performance

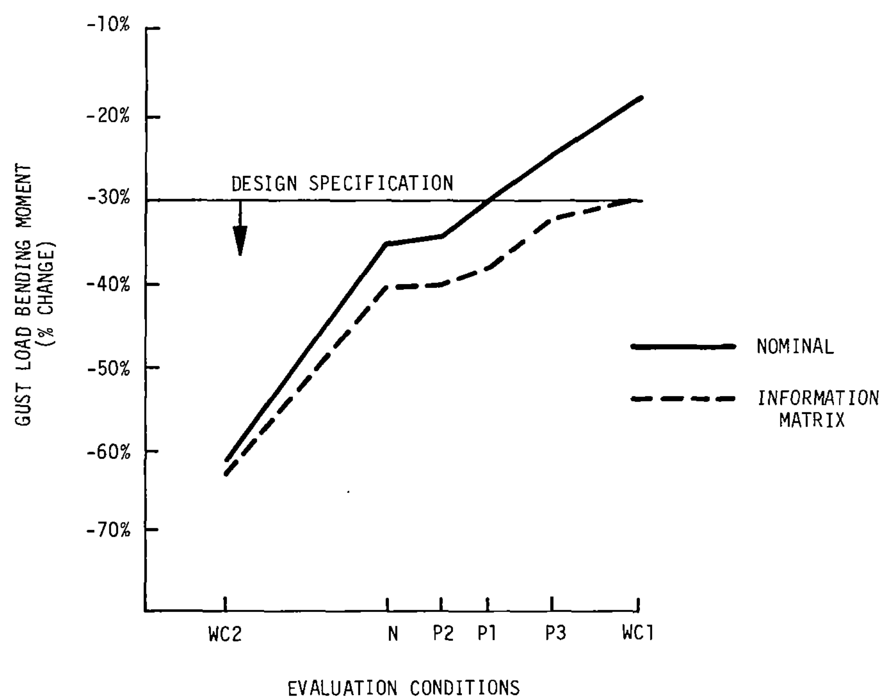

Figure 30. Case 4R Gust Load Performance (Bending Moment)

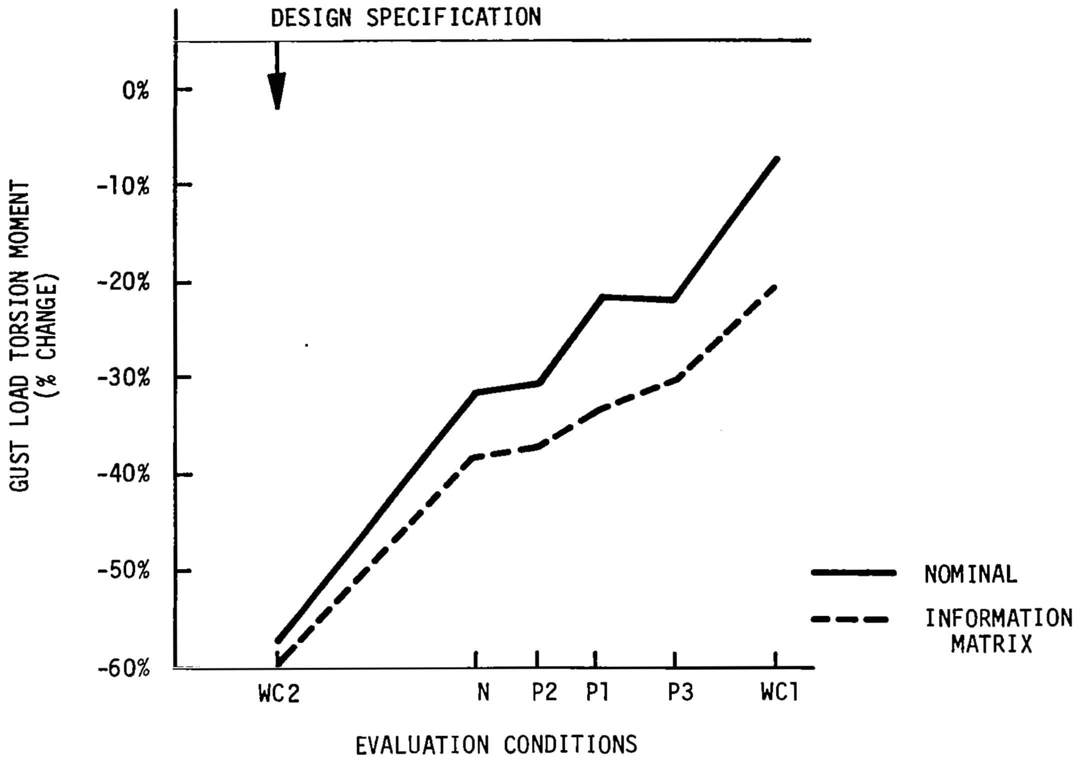

Figure 31. Case 4R Gust Load Performance (Torsion Moment)

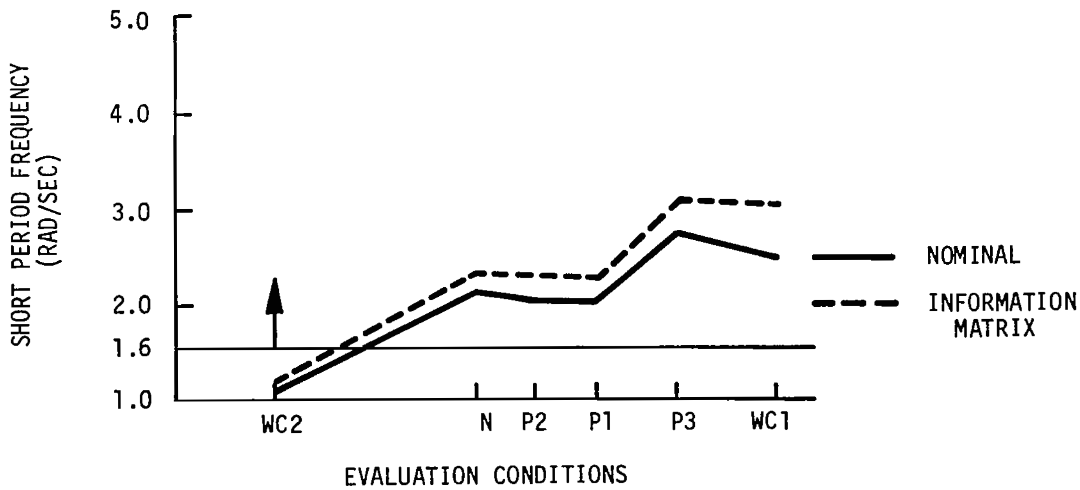

Figure 32. Case 4R Short Period Frequency

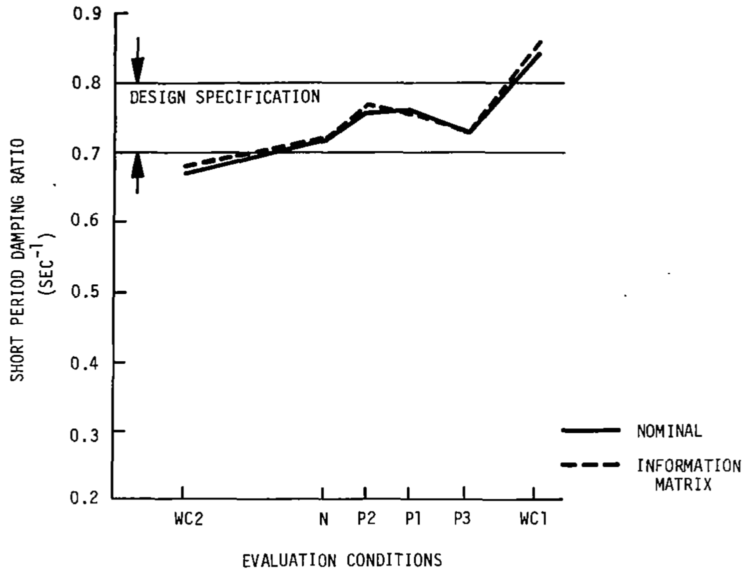

Figure 33. Case 4R Short Period Damping

1. Overall Relative Score--Coarse measure of the performance of the nominal controller with respect to each specification.

$$\text{ORS} \approx \frac{\text{Ideal Score - Score of Insensitive Controller}}{\text{Ideal Score - Score of Nominal Controller}}$$

2. Normalized Performance/Range--The normalized performance is the average of the performance of the nominal controller divided by the average of the performance of the insensitive controller for each design specification.

$$\|P\| = \frac{\frac{1}{N} \sum_i P_{i_{1C}}}{\frac{1}{N} \sum_i P_{x_N}} \qquad \begin{array}{l} i = 1,\ N \\ N = \text{number of evaluation conditions} \end{array}$$

The normalized range is the range of the insensitive controller divided by the range of the nominal controller

$$\|R\| = \underset{\text{(no. of criteria)}}{\mathrm{RSS}} \quad \frac{\mathrm{MAX}_i(P_{1C} - \mathrm{MIN}_i(P_{1C})}{\mathrm{MAX}_i(P_N) - \mathrm{MIN}_i(P_N)} \qquad i = 1, N$$

3. Normalized Specification Violation--Total spec violations for each insensitive controller for all evaluation conditions normalized by the maximum spec violation

$$\|SV\| = \underset{\text{(no. of criteria)}}{\mathrm{RSS}} \quad \frac{\sum_i SV_i}{\underset{j}{\mathrm{MAX}} \sum_i SV_i} \qquad \begin{array}{l} i = 1, N \\ J = 1, M \\ M = \text{no. of insensitive controllers} \end{array}$$

The criteria are described in more detail in Reference 1. As they have been defined, the lower the numerical rating the better the performance of the controller.

Figure 34 shows the performance of the Information Matrix controller as measured by the overall relative score versus the eight controllers evaluated in Reference 1. Figures 35 and 36 show similar comparisons for the Information Matrix controller performance as measured by the Normalized Performance/Range score and the Normalized Spec Violation score, respectively. Table 12 presents a summary of the rankings of the Information Matrix controller versus the eight controllers evaluated in Reference 1.

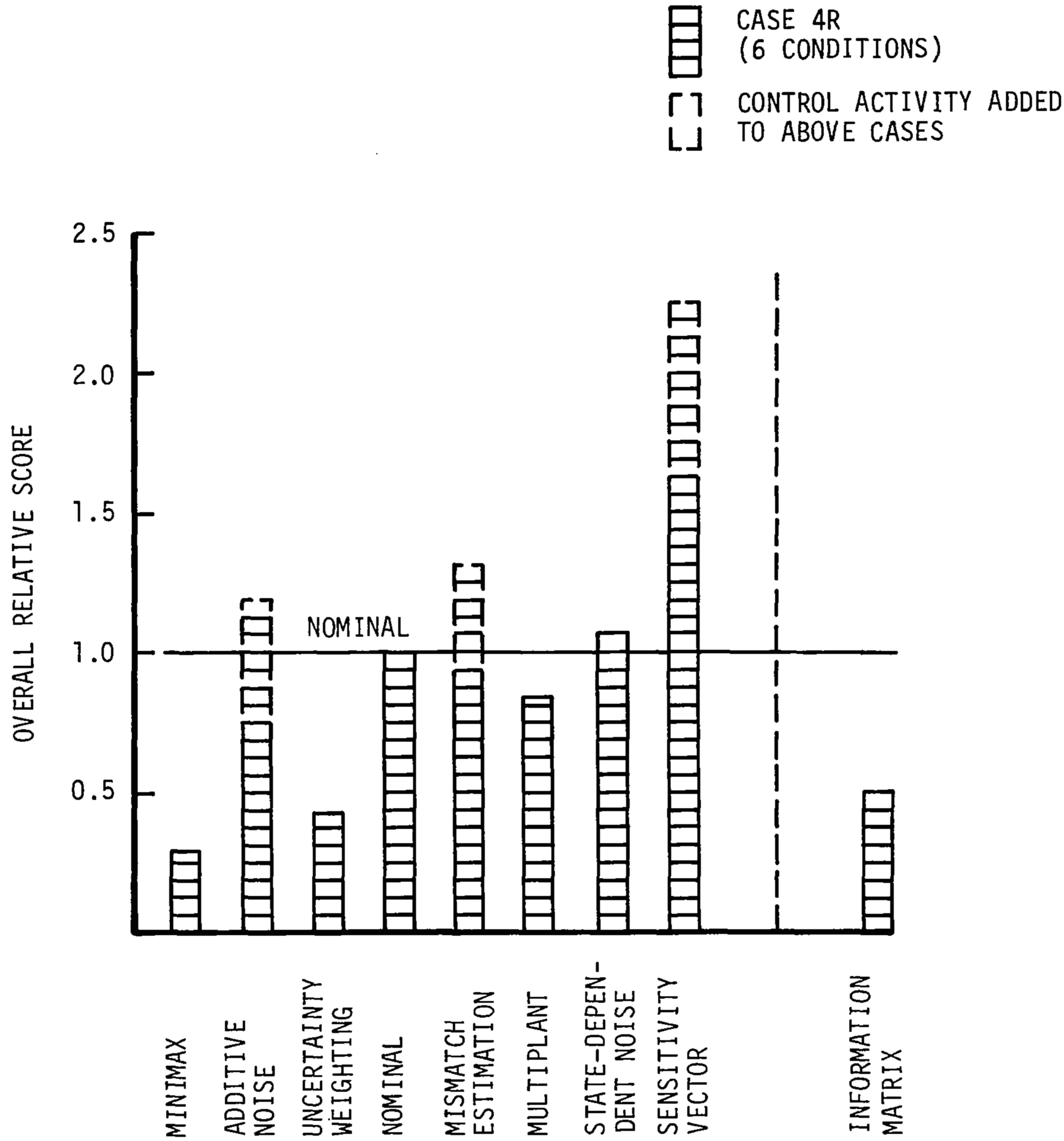

Figure 34. Overall Relative Score Comparison

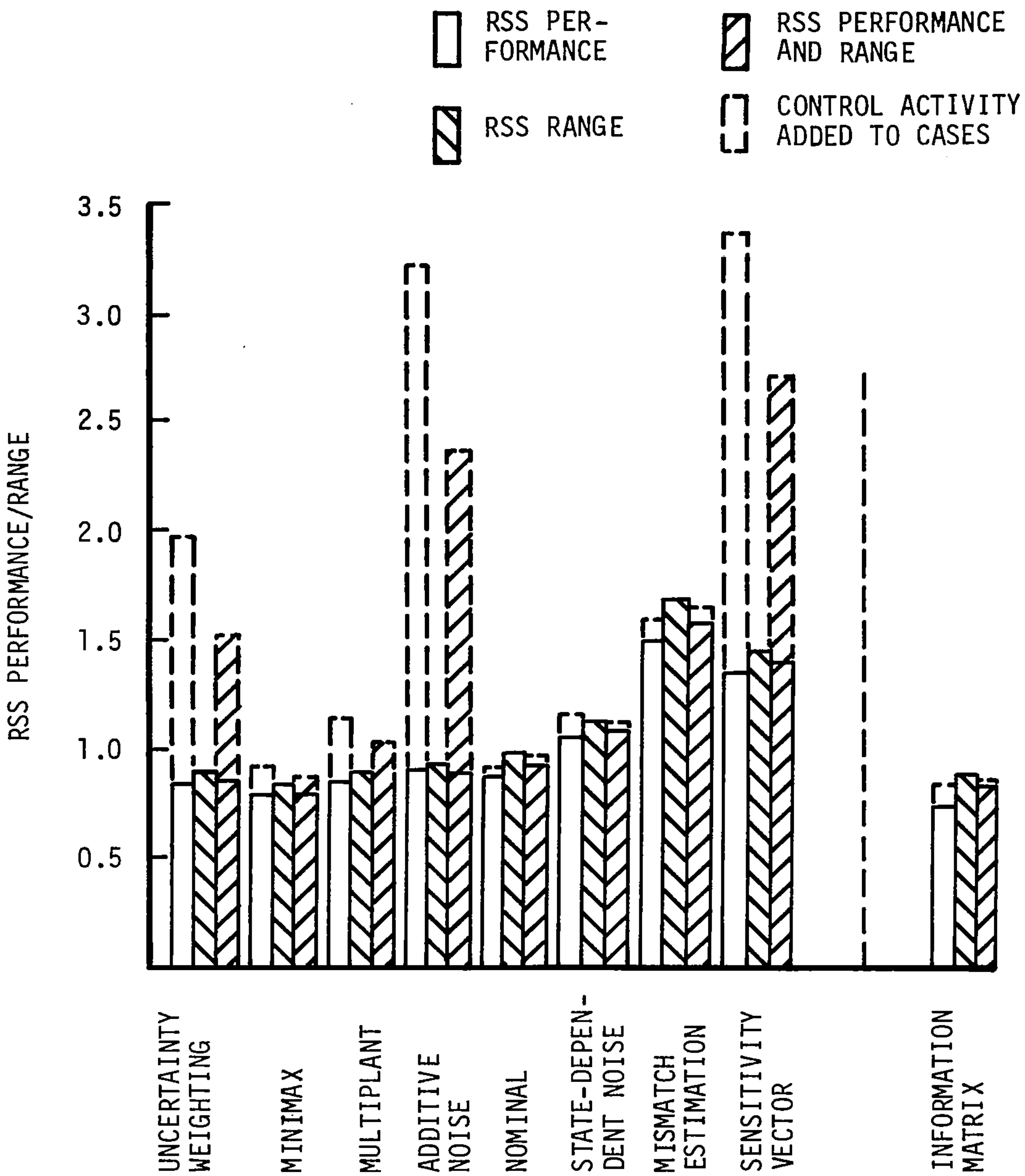

Figure 35. Normalized Performance/Range Comparison

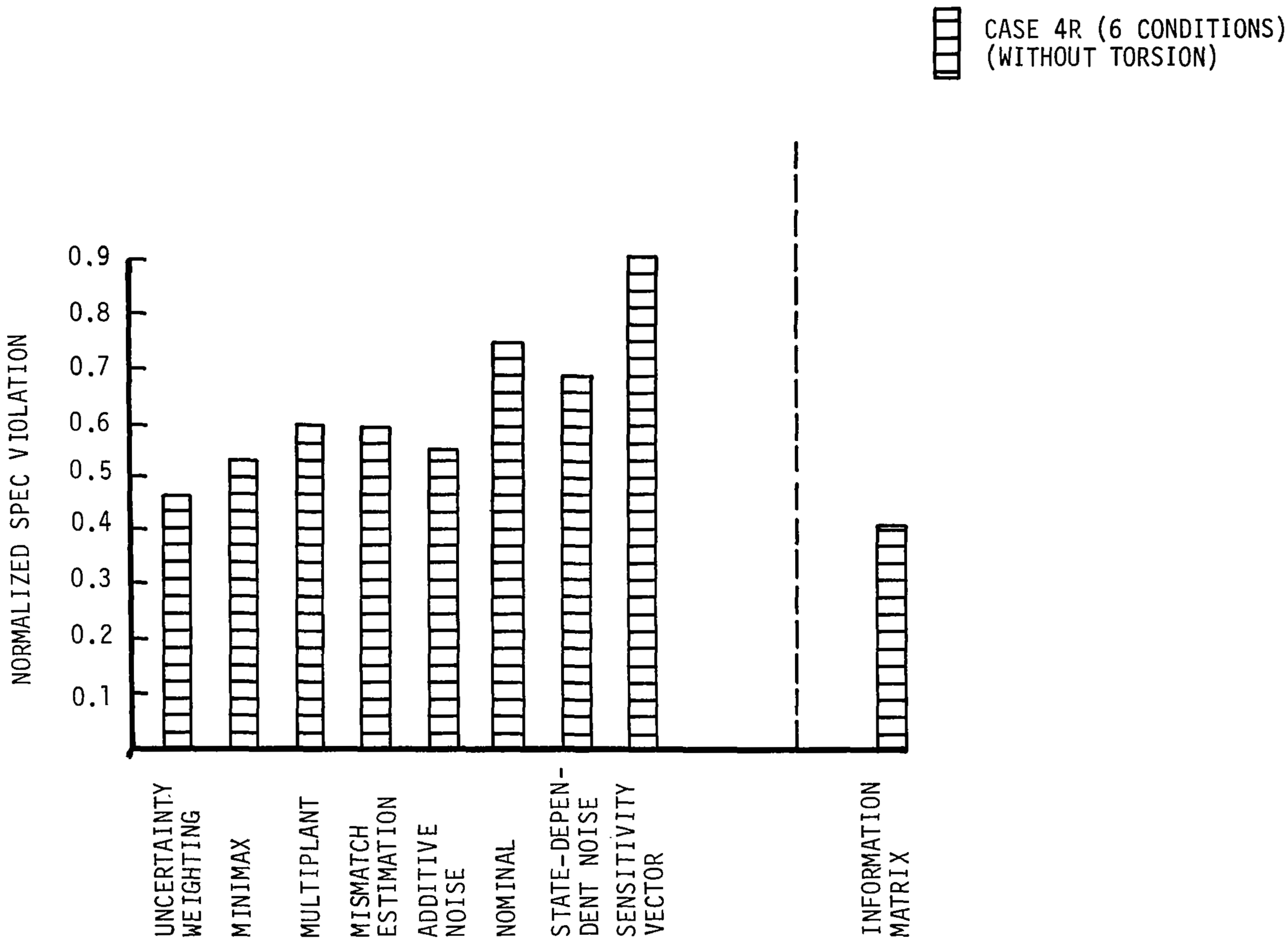

Figure 36. Normalized Spec Violation Comparison

TABLE 12. RANKING OF INSENSITIVE CONTROLLERS INCLUDING THE INFORMATION MATRIX APPROACH

Controller	Overall Relative Scoring	Overall Performance/Range	Overall Specification Violation	Sum
Information Matrix	3	1	1	5
Minimax	1	2	3	6
Uncertainty Weighting	2	3	2	7
Additive Noise	4	5	4	13
Multiplant	5	4	6	15
Mismatch Estimation	6	9	5	20
Nominal	7	6	8	21
State-Dependent Noise	8	7	7	22
Sensitivity Vector	9	8	9	26

Comparisons and Conclusions

Although the Information Matrix controller does result in improved performance over the top-ranked minimax and uncertainty weighting of Reference 1, it is premature to state that the Information Matrix technique is in some way better than the others. The fact that the controller designed with the Information Matrix technique did do well, though, indicates significant potential for the approach. It is extremely difficult to extend a theoretical concept to practical worthiness with one application. It is felt that further investigation into increased values of β_2 would have resulted in an even better performance. It is also felt that a reformulation of the response vector to better control short period frequency and damping would have improved performance. (This actually refers to the evaluation of the insensitive controllers in Reference 1.) Some definite advantages that can be stated at this time include:

1. No a priori range of parameter variations is required since the design is done at the nominal. One needs only the partial derivatives of the system matrices at the design point.

2. The method treats nicely the response uncertainties.

3. Since the control is assumed to be in the form $\underline{u} = K\underline{x}$, only n_x-dimensional equations need be solved. No extra modeling or filters are necessary.

4. It treats the actual closed-loop sensitivity, unlike the sensitivity vector augmentation or uncertainty weighting methods.

5. The technique can be extended easily to limited-state feedback and possibly observer/Kalman filter cases.

6. With the modified Bartels-Stewart algorithm, only a modest eight linear equations (equivalent) need be solved per iteration to get the cost and gradient for $p = 3$.

7. The approach provides an intuitive feel and insight to the design problem. It indicates clearly the improvement in system sensitivity (in terms of the dispersion matrix) and the price paid in terms of performance J_1. The key parameter β_2 controls the trade-off between sensitivity and performance.

8. The technique can be tuned to weight the relative importance of different parameters.

SECTION V

CONCLUSIONS AND RECOMMENDATIONS

The objective of this study, to develop useful synthesis techniques from the two advanced theoretical concepts created in the previous study, has been satisfied. This study has shown that

- The insensitive controller synthesis technique based on the Finite Dimensional Inverse (FDI) concept is impractical for flight control system design in its current formulation. This is due to the time-varying nature of the resultant controller, which is more amenable to trajectory-type applications.
- Despite severe computational requirements, FDI controller synthesis and implementation are feasible. Experiments with recycling stored data to alleviate storage requirements produced apparently satisfactory results following some initialization transients that could be reduced with filtering.
- The FDI technique provides an on-line identification capability that could be useful for many applications.
- The Information Matrix (IM) synthesis technique is definitely applicable to flight control problems because the resultant insensitive controller has constant gains.
- The IM controller performs as well as the top-ranked uncertainty weighting and minimax controllers of the previous study. As in the previous study, it will be necessary to

qualify the results of the evaluation. It should be emphasized that good performance on one example with one set of criteria does not imply universal goodness. The IM approach, however, with its design feature of weighting performance versus sensitivity without specifying the range of uncertain parameters, together with the evaluation results indicate a worthwhile development.

Based on the results of this study, we recommend the following areas for further research:

- Formulation of the FDI approach to handle stationary problems.
- Development of FDI capability to handle nonminimum phase systems.

  This capability is needed, as demonstrated in the C-5A example, when the design responses used result in unstable compensation.
- Further refinement of the IM methodology to quantify the import of modulation of controller design parameters.

APPENDIX A

THE SEVENTH-ORDER MODEL

The numerical data for the seventh-order model for six parameter values is given. The usual state variable representation is used:

$$\dot{x} = Fx + G_1 u + G_2 \eta$$

$$r = Hx + Du$$

with

$$x^T = [w,\ q/\dot{n}_2,\ \dot{\eta}_1,\ \eta_1,\ \delta a,\ \delta e_i,\ w_g]$$

$$u^T = [\delta a_c,\ \delta e_{ic}]$$

$$r^T = [B_1,\ T_1,\ \dot{B}_1,\ \dot{T}_1,\ \delta a,\ \delta e_i,\ \dot{\delta a}_m,\ \dot{\delta e}_i,\ r_{cf}]$$

The G_1 and G_2 matrices are the same for all parameter values. They are:

$$G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 6 & 0 \\ 0 & 7.5 \\ 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.861 \end{bmatrix}$$

The last four rows of F and the last five rows of H and D are also independent of the parameters. Thus, we may write F, H, and D as:

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \qquad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

where

$$F_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.371 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.5 & 0 \\ 0 & h_{9,2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad h_{9,2} = -2.2748E-03$$

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6 & 0 \\ 0 & 7.5 \\ 0 & 7.5 \end{bmatrix}$$

The matrices F_1, H_1, and D_1 are shown below for the six parameter values in Tables A1 through A6.

TABLE A1. NOMINAL CONDITION MATRICES

F_1

Row 1	-6.9991E-01	3.2724E+00	-3.5667E-02	-6.6031E-01	-2.0358E+02	-2.0256E+02	-8.6914E+00
Row 2	-4.0476E-01	-1.0959E+00	3.1639E-02	-1.8709E-01	-5.2655E+02	-2.3074E+03	-9.4898E+00
Row 3	-1.6852E+00	1.8052E-02	-9.7983E-01	-3.0030E+01	-3.0981E+03	1.2357E+03	-1.8459E+01

H_1

Row 1	-2.8733E+04	-1.5687E+03	1.3174E+04	1.2412E+06	3.2575E+07	3.5344E+06	-3.3905E+05
Row 2	-2.2396E+04	-3.4449E+03	-1.6286E+03	-8.6938E+04	8.6275E+06	-2.4293E+06	-2.7210E+05
Row 3	-1.5764E+04	-1.4707E+04	1.2510E+06	-3.3848E+05	-2.5627E+07	1.2064E+07	-1.4803E+05
Row 4	5.4500E+03	-2.7842E+04	-5.8879E+04	-6.6787E+03	1.3327E+07	2.6957E+06	7.1959E+04

D_1

Row 1	0.	0.
Row 2	0.	0.
Row 3	-8.4864E+06	3.1845E+06
Row 4	-1.2719E+07	2.4720E+05

TABLE A2. WORST CASE 1 MATRICES

F_1

Row 1	-9.5644E-01	4.0757E+00	-5.9648E-02	-1.0294E+00	-1.6296E+02	-2.6206E+02	-1.1848E+01
Row 2	-4.9662E-01	-1.2354E+00	4.0758E-03	-5.9496E-01	-5.2179E+02	-2.5402E+03	-1.1194E+01
Row 3	-2.9321E+00	-2.1624E-01	-1.2407E+00	-2.394E+01	-2.9521E+03	1.2809E+03	-3.3386E+01

H_1

Row 1	-3.7187E+04	-3.1421E+03	1.5809E+04	1.4645E+06	3.9282E+07	1.5362E+06	-4.4309E+05
Row 2	-3.0729E+04	-5.0614E+03	-2.1960E+03	-7.0371E+04	1.3523E+07	-4.0092E+06	-3.7416E+05
Row 3	-3.4336E+04	-2.6231E+04	1.4743E+06	-3.2140E+05	-2.8624E+07	1.6160E+07	-3.6035E+05
Row 4	8.1625E+03	-4.3576E+04	-2.9472E+04	-7.6219E+03	1.5514E+07	4.4698E+06	1.0772E+05

D_1

Row 1	0.	0.
Row 2	0.	0.
Row 3	-1.0608E+07	3.9806E+06
Row 4	-1.5899E+07	3.0900E+05

TABLE A5. ω PERTURBATION MATRICES

$(\omega = 0.75\ \omega_{NOM})$

F_1

Row 1	-7.3843E-01	3.2656E+00	-4.2888E-02	-7.5762E-01	-1.6024E+02	-2.0636E+02	-9.1547E+00
Row 2	-5.0739E-01	-1.0219E+00	1.3385E-02	-3.6874E-01	-4.5964E+02	-2.1215E+03	-9.0558E+00
Row 3	-2.0814E+00	-1.0020E-01	-9.8431E-01	-2.1095E+01	-2.6579E+03	1.1008E+03	-2.3423E+01

H_1

Row 1	-2.9371E+04	-2.1935E+03	1.2852E+04	1.1860E+06	3.1797E+07	2.0052E+06	-3.4887E+05
Row 2	-2.3701E+04	-3.8144E+03	-1.6975E+03	-6.1585E+04	9.9185E+06	-2.9214E+06	-2.8838E+05
Row 3	-1.9502E+04	-1.6019E+04	1.1962E+06	-2.3069E+05	-2.0137E+07	1.0730E+07	-1.9624E+05
Row 4	5.3642E+03	-2.7868E+04	-3.0736E+04	-4.9718E+03	1.2773E+07	2.7675E+06	7.0568E+04

D_1

Row 1	0.	0.
Row 2	0.	0.
Row 3	-8.4864E+06	3.1845E+06
Row 4	-1.2719E+07	2.4720E+05

TABLE A6. $\bar{q}$ PERTURBATION MATRICES

$(\bar{q} = 1.25\bar{q}_{NOM})$

F_1

Row 1	-8.8911E-01	4.0881E+00	-4.7302E-02	-8.6178E-01	-2.3841E+02	-2.5439E+02	-1.1035E+01
Row 2	-6.4055E-01	-1.3372E+00	3.2041E-02	-3.0658E-01	-6.3226E+02	-2.8037E+03	-1.1641E+01
Row 3	-2.2559E+00	-2.3891E-02	-1.1964E+00	-3.2267E+01	-3.7075E+03	1.4896E+03	-2.4954E+01

H_1

Row 1	-3.6171E+04	-2.2256E+03	1.6183E+04	1.5200E+06	4.0385E+07	3.7692E+06	-4.2780E+05
Row 2	-2.8484E+04	-4.4490E+03	-1.9753E+03	-9.3611E+04	1.1260E+07	-3.2345E+06	-3.4624E+05
Row 3	-2.6015E+04	-2.3634E+04	1.5296E+06	-4.4507E+05	-3.9464E+07	1.9481E+07	-2.5615E+05
Row 4	8.3336E+03	-4.3517E+04	-5.7628E+04	-1.1698E+04	1.6277E+07	4.4470E+06	1.0918E+05

D_1

Row 1	0.	0.
Row 2	0.	0.
Row 3	-1.0608E+07	3.9806E+06
Row 4	-1.5899E+07	3.0900E+05

APPENDIX B

MAXIMUM DIFFICULTY METRIC

This appendix summarizes work performed (on the maximum difficulty metric) in determining the design point in parameter space at which control is most difficult.

Recall that the basic idea was to find the point $\underline{\alpha}$ in parameter space to maximize the difficulty metric

$$J = \mathrm{tr}[\hat{H}' (\hat{H} W \hat{H}')^{-1} \hat{H}] \stackrel{\Delta}{=} \mathrm{tr}\, D_\alpha \qquad \text{(B-1)}$$

The approach taken is to get an analytic expression for the gradient $\partial J/\partial\alpha$ that could be used as a basis for a numerical optimization scheme.

In Equation (B-1)

$$\hat{H} = He^{AT} \qquad \text{(B-2a)}$$

$$W = \int_0^T e^{-At} BR^{-1} B' e^{-A't}\, dt \qquad \text{(B-2b)}$$

where T is an arbitrary parameter. Also, for notational convenience

$$\underline{X} \stackrel{\Delta}{=} (\hat{H} W \hat{H}')$$

The matrices A, H are subject to parameter uncertainty

$$A \to A + \sum_{i=1}^{p} \delta\alpha_i A_i \qquad \text{(B-3a)}$$

$$H \to H + \sum_{i=1}^{p} \delta\alpha_i H_i \tag{B-3b}$$

Consider now a first order pertubation in $\alpha_i \to \alpha_i + \delta\alpha_i$ and the resulting change in J:

$$J + \delta J = \mathrm{tr}[(\hat{H} + \delta\hat{H})' \{\underline{\bar{X}} + \delta\underline{\bar{X}}\}^{-1} (\hat{H} + \delta\hat{H})]$$

expanding to first order, keeping terms of $o(\delta)$ only and using

$$(\underline{\bar{X}} + \delta\underline{\bar{X}})^{-1} \sim \underline{\bar{X}}^{-1} - \underline{\bar{X}}^{-1} (\delta\underline{\bar{X}}) \underline{\bar{X}}^{-1}$$

we obtain

$$\delta J = 2\,\mathrm{tr}\,(\hat{H}'\underline{\bar{X}}^{-1} \delta\hat{H}) - \mathrm{tr}\,(\underline{\bar{X}}^{-1} \hat{H} \hat{H}'\underline{\bar{X}}^{-1} \delta\underline{\bar{X}}) \tag{B-4}$$

So we need only to get $\delta\hat{H}$ and $\delta\underline{\bar{X}}$ in terms of $\delta\alpha_i$.

First consider $\delta\hat{H}$.

$$\delta\hat{H} = (\delta H)\, e^{AT} + H\,(\delta e^{AT})$$

$$= \delta\alpha_i H_i e^{AT} + H\,[e^{(A + \delta\alpha_i A_i)T} - e^{AT}]$$

Using a result from Bellman,

$$e^{(A + \delta\alpha_i A_i)T} \sim e^{AT} + \delta\alpha_i \int_0^T e^{A(T-t)} A_i e^{At}\, dt \tag{B-5}$$

$$= e^{AT} + \delta\alpha_i e^{AT} \Gamma(T) \tag{B-5a}$$

Thus,

$$\frac{\partial\hat{H}}{\partial\alpha_i} = \frac{\delta\hat{H}}{\delta\alpha_i} = H_i e^{AT} + \hat{H}\,\Gamma(T) \tag{B-6}$$

where

$$\Gamma(T) = \int_o^T e^{-AT} A_i e^{AT} dt \tag{B-7}$$

The process of computing $\delta \underline{\bar{X}}$ proceeds in a similar manner.

$$W + \delta W = \int_o^T e^{-(A + \delta\alpha_i A_i)t} BR^{-1} B' e^{(A + \delta\alpha_i A_i)'t} dt$$

It is necessary to work with $\underline{\bar{X}}$, so with a change of variable,*

$$\underline{\bar{X}} + \delta \underline{\bar{X}} = (H + \delta\alpha_i H_i) [\int_o^T e^{(A + \delta\alpha_i A_i)\sigma} BR^{-1}B' e^{(A + \delta\alpha_i A_i)'\sigma} d\sigma]$$

$$(H + \delta\alpha_i H_i)'$$

Expanding the integral term, using Equation (B-5), gives

$$\frac{\partial \underline{\bar{X}}}{\partial \alpha_i} = H [\int_o^T e^{A\sigma} BR^{-1} B' \Gamma'(\sigma) e^{A'\sigma} d\sigma + \int_o^T e^{A\sigma} \Gamma(\sigma) BR^{-1} B' e^{A'\sigma} d\sigma] H'$$

$$+ \hat{H} W e^{A'T} H_i' + H_i e^{AT} W \hat{H}'$$

Substituting Equations (B-6) and (B-8) into Equation (B-4) gives

$$\frac{\partial J}{\partial \alpha_i} = 2 \text{ tr } (\hat{H}' \underline{\bar{X}}^{-1} \frac{\partial \hat{H}}{\partial \alpha_i}) - \text{tr } (\underline{\bar{X}}^{-1} \hat{H} \hat{H}' \underline{\bar{X}}^{-1} \frac{\partial \underline{\bar{X}}}{\partial \alpha_i})$$

$$= 2 \text{ tr } [\hat{H}' \underline{\bar{X}}^{-1} H_i e^{AT} + D_\alpha \Gamma(T) - D_\alpha^2 W_1(T) - D_\alpha \hat{H}' \underline{\bar{X}}^{-1} H_i e^{AT} W]$$

---

* $\hat{H} W \hat{H}' = H \int_o^T e^{A(T-t)} BR^{-1}B' e^{A'(T-t)} dt H' = H \int_o^T e^{A\sigma} BR^{-1} B' e^{A'\sigma} d\sigma H'$

where D_α is the difficulty matrix and

$$W_1(T) = \int_0^T e^{-At} BR^{-1} B' \Gamma'(T - t) e^{-A't} dt$$

$$= \int_0^T e^{-At} BR^{-1} B' e^{-A't} [\int_t^T e^{A'\xi} A_i' e^{-A'\xi} d\xi] dt$$

Computational methods for evaluating $W_1(T)$ have been considered, but thus far nothing simple has come up. A straightforward numerical evaluation

$$\frac{\partial J}{\partial \alpha_i} = \frac{J(\alpha + \delta\alpha_i) - J(\alpha - \delta\alpha_i)}{2\delta\alpha_i}$$

may be easier.

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