General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)
ON N-TH ROOTS OF POSITIVE OPERATORS
BY
D.R. BROWN & M.J. O'MALLEY

PREPARED FOR
EARTH OBSERVATION DIVISION, JSC
UNDER
CONTRACT NAS-9-15009
ON \textsuperscript{th} ROOTS OF POSITIVE OPERATORS

D.R. Brown and M.J. O'Malley
Department of Mathematics
University of Houston
Houston, Texas 77004

Report #68
February 1978
ON N\textsuperscript{th} ROOTS OF POSITIVE OPERATORS

by D.R. Brown and M.J. O'Malley

A bounded operator $A$ on a Hilbert space $H$ is positive provided $<Ax,x> \geq 0$ for all $x \in H$. These operators are symmetric, and as such constitute a natural generalization of non-negative real diagonal matrices. The following result is thus both well known and not surprising:

\textbf{Theorem:} A positive operator has a unique positive square root (under operator composition).

This may be established by integration of the correct function, invoking the spectral theorem for self-adjoint operators. A more accessible argument for those not acquainted with the mysteries of spectral measures may be found in [1,p.317].

While square roots and their iterates seem to provide a sufficient analytic tool for most purposes, it is also a (folk) theorem that positive operators possess unique positive $n$\textsuperscript{th} roots for every positive integer $n$. As in the $n = 2$ case, existence follows from an application of the spectral theorem; however, we give an argument in the spirit of [1]. The purpose in so doing is not to exercise the reader's knowledge of induction, but rather to illustrate another use of the Law of the Mean as a motivational instrument.

1) Both authors received partial support under NASA contract NAS-9-15000.
Let \( I \) be the identity operator on \( \mathcal{H} \), and let \( \mathcal{B}(\mathcal{H}) \) denote the set of bounded operators on \( \mathcal{H} \). We will need the following properties of positive operators:

1. the relation on positive operators defined by \( A \preceq B \) if and only if \( B - A \) is positive, is reflexive, transitive, and consistent with the notation \( 0 \preceq A \) for any positive \( A \); moreover, this relation is preserved by operator addition and positive real scalar multiplication, and reversed by negative scalar multiplication.

2. If \( A \) and \( B \) are positive and if \( AB = BA \), then \( AB \) is positive.

3. If \( 0 \preceq A \preceq I \), then \( 0 \preceq I - A \preceq I \).

4. If \( 0 \preceq A \), then \( A \preceq |A|I \), so that \( (|A|)^{-1}A \preceq I \), if \( A \neq 0 \).

5. If \( 0 \preceq A \preceq I \), then \( A^n \preceq A \) for all positive integers \( n \).

We also require:

Lemma. If \( \{S_n\} \) is a sequence in \( \mathcal{B}(\mathcal{H}) \) such that \( 0 \preceq S_n \preceq S_{n+1} \preceq I \), then there exists \( S \in \mathcal{B}(\mathcal{H}) \) such that \( \{S_n\} + Su \) for all \( u \in \mathcal{H} \).

All of the conclusions above are verified by straightforward arguments in [1, pp. 317-320].

Theorem: Let \( A \in \mathcal{B}(\mathcal{H}) \), \( 0 \preceq A \), and let \( k \) be a positive integer. Then there exists a unique positive operator \( B \) such that \( B^k = A \).

Proof: By (4) above, we need only consider the case in which \( A \preceq I \).
We first prove the existence of $B$. Since the theorem is a tautology for all operators when $k = 1$, we assume the existence of positive $(k-1)$-st roots for all positive operators.

Under the momentary supposition that $B$ exists, let $R = I - A$ and $S = I - B$. Then $(I - S)^k = I - R$, so that

\[(*) \quad S = (1/k) \left[ R + \sum_{r=2}^{k} \binom{k}{r} (-1)^r S^r \right]. \]

Clearly the existence of a positive operator satisfying this implicit relation is necessary and sufficient to establish the existence of the desired operator $B$. To this end, we define a sequence of operators by $S_0 = 0$, $S_{n+1} = (1/k) \left[ R + \sum_{r=2}^{k} \binom{k}{r} (-1)^r S^r \right]$. In order to show $S_n \leq S_{n+1}$, it suffices to show, under the assumption $0 \leq S_{n-1} \leq S_n \leq I$, that $0 \leq S_{n+1} - S_n = (1/k) \left[ \sum_{r=2}^{k} \binom{k}{r} (-1)^r (S_n^r - S_{n-1}^r) \right]$. 

To accomplish this, we digress to a consideration of the polynomial $f(x) = \sum_{r=2}^{k} \binom{k}{r} (-1)^r x^r = (1-x)^k + kx - 1$. Since $f'(x) = k \left[ \frac{1}{1 - (1-x)^{k-1}} \right] \geq 0$ on $[0,1]$, clearly $f$ is increasing on this interval. To translate this to operators, it is necessary to examine the situation more carefully. By the Mean Value Theorem, given $0 \leq y < z \leq 1$, there exists a (unique) number $c \in (y,z)$ such that

\[(**) \quad f(z) - f(y) = f'(c)(z - y). \]

Upon solving, $c = 1 - \left[ (1/k) \sum_{r=0}^{k-1} (1 - y)^{k-r-1}(1 - z)^r \right]^{1/(k-1)}$. 

**REPRODUCIBILITY OF THE ORIGINAL PAGE IS POOR**
Returning to our operator problem, we wish to apply this
information to the sequence \( \{S_n\} \). Since all members of this
family are polynomials in \( R = I - A \), any two of them commute.
This is a property sufficient to permit imitation of equation (**)
with operators; let \( z = S_n \), \( y = S_{n-1} \). In this format, we use \( C \)
to represent the operator \( I - J \), where \( J \) is (any) positive
\((k-1)\)st root of the operator \( (1/k) \sum_{r=0}^{k-1} (I - S_{n-1})^{k-r-1}(I - S_n)^r \).
The following chain of equalities is easily calculated:

\[
S_{n+1} - S_n = \frac{1}{k} \cdot (f(S_n) - f(S_{n-1}))
\]

\[
= \frac{1}{k} \cdot (k[I - (I - C)^{k-1}]) \cdot (S_n - S_{n-1})
\]

\[
= [I - (I - C)^{k-1}] \cdot (S_n - S_{n-1})
\]

\[
= [I - J^{k-1}] \cdot (S_n - S_{n-1})
\]

\[
= [I - (1/k) \sum_{r=0}^{k-1} (I - S_{n-1})^{k-r-1}(I - S_n)^r] \cdot (S_n - S_{n-1})
\]

By application of remarks (2), (3) and (5), the assumption of
existence of \((k-1)\)st roots, and the inductive hypothesis \( S_{n-1} \leq S_n \),
the latter operator product exists and is positive. Hence \( S_n \leq S_{n+1} \),
and the sequence \( \{S_n\} \) is increasing. Of course, the Law of the Mean
is not applicable in this setting, nor is it used other than to motivate
the choice of \( C \). Indeed, the discerning reader will note that the
extremes of the chain above may be shown to be equal without the
introduction of \( C \). However, the rather unusual factorization of
\( S_{n+1} - S_n \) would be more difficult to discover without the example
To invoke the Lemma and complete the proof of existence of $k$th roots, it remains to show $S_n \leq I$ for all $n$. Assuming $0 \leq S_m \leq I$, we have $kS_{m+1} = R + \frac{k}{r+2}(1)^r(-1)^rS_m = R - I + kS_m + (I - S_m)^k$.

By remark (5), $(I - S_m)^k \leq I - S_m$; therefore

$$R + kS_m - I + (I - S_m)^k \leq R + kS_m - I + (I - S_m)^k \leq I + (k-1)S_m \leq kI.$$ Hence

$kS_{m+1} \leq kI$ and $S_{m+1} \leq I$, as desired. Thus, the Lemma gives an operator as in (*) and $I - S = B$ is a $k$th root of $A$.

In order to prove the uniqueness of a positive $k$th root of $A$, we first observe that if $T$ is any positive $k$th root of $A$, then $T$ must perforce commute with $A$, hence with $I - A = R$, hence with each $S_n$, and thus with $S$ and $I - S = B$. Let $u \in H$, $v = (B-T)u$. Then $0 = \langle (B-T)^k u, v \rangle = \sum_{r=0}^{k-1} B^{k-r-1}T^r (B-T)u, v \rangle = k_{k-1} \langle B^{k-r-1}T^r v, v \rangle.$

Since $B$ and $T$ commute, $0 \leq B^{k-r-1}T^r$, whence $\langle B^{k-r-1}T^r v, v \rangle = 0$, $r = 0, 1, \ldots, k-1$. Let $F_r$ be any positive (hence symmetric) square root of $B^{k-r-1}T^r$. Then $||F_r v||^2 = \langle F_r v, F_r v \rangle = \langle F_r^2 v, v \rangle = 0$, so that $F_r v = 0$ and $B^{k-r-1}T^r v = F_r^2 v = 0$. Therefore $B^{k-r-1}T^r (B-T)u = 0$.

or $B^{k-r-1}T^r u = B^{k-r-1}T^r_{r+1} u$, $r = 0, 1, \ldots, k-1$. In particular, for $r = k-1$, $B^{k-r-1} = T$. Multiplying by $T$, we have $B^{k+1} = BA = BT^k = T^{k+1}$.

If $k = 2$, the argument above shows $Bv = 0 = Tv$, whence $||B(T)v||^2 = \langle B(T)^2 u, v \rangle = \langle B(T)v, u \rangle = 0$. Hence $Bu = Tu$ for all $u \in H$, and $B$ is thus unique. Now assume all positive roots, of order less than $k$, for positive operators are unique. If $k = 2j$, then $(B^j)^2 = B^{2j} = B^k = T^k = (T^j)^2$, whence $B^j = T^j$ and thus $B = T$. If $k$ is odd, we have shown above that $B^{k+1} = T^{k+1}$, so, by the even
exponent argument, again $B = T$. This completes the proof.

REFERENCE


University of Houston
Houston, Texas, 77004