Applied Routh Approximation

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SUMMARY

Models of physical processes are often too complex to be handled directly. Approximations of these models that are of reduced complexity but that maintain the essential model characteristics are often used for analysis rather than the original model. One way to generate these approximations is the Routh approximation technique. In this report, this technique is programmed and applied in the frequency domain to a 16th-order, state-variable model of the F100 engine and to a 43rd-order, transfer-function model of a launch vehicle boost-pump pressure regulator. The accuracies of reduced-order approximants obtained for both models are compared with the original model. The results indicate that the Routh method is an excellent technique for linear system reduction in the frequency domain. Therefore, the frequency-domain formulation was extended to the time domain in order to handle the state-variable model directly. The time-domain formulation is derived herein and a new characterization that specifies all possible Routh similarity transformations is given. This characterization is computed by using eigenvector-eigenvalue techniques and is therefore quite accurate. The application of the time-domain Routh technique to the state-variable engine model is described, and reduced-order approximation results are given. Additionally, an optimization procedure that can improve the approximation accuracy by taking advantage of the transformation characterization is applied to the engine example. Also, the limitations of applying the Routh technique to the fixed-dimension, multiple-input - multiple-output reduction problems are discussed.

INTRODUCTION

In control system design the system to be controlled is often represented by a complex mathematical model. Practically, this complex model may be difficult to use for design purposes. Additionally, the resultant control design may be too complex to implement. To eliminate these problems, model reduction methods are often used before
designing the control system in order to reduce the complexity of the original model while maintaining its important characteristics. Alternatively, model reduction methods could be applied to the reduction of complex control designs. In either case the model reduction process would be the same.

One philosophy adopted in reduction methods in dominant-mode approximation (ref. 1). In this report, a dominant-mode-approximation technique, the Routh approximation technique (refs. 2 and 3), is applied in both the frequency and time domains to simplify a state-variable turbofan engine model, and in the frequency domain to simplify a transfer-function model of a launch vehicle boost-pump pressure regulator. Hutton and Rabins (ref. 4) have shown the frequency-domain formulation of the Routh approximation technique to be an excellent reduction method for many mechanical systems. This formulation is applicable to turbofan engine and boost-pump regulator models as well and is therefore a strong motivation for the work of this report.

The objectives considered included the application of the Routh approximation method in the frequency domain to the F100 turbofan engine. This application demonstrated that adequate, reduced-complexity models can be determined by the Routh technique. Next, the applicability of the frequency-domain approach to high-order systems was studied by using the boost-pump pressure regulator model. Finally, the frequency-domain Routh formulation was extended to the time domain. This extension allows the usual engine time-domain formulation to be handled directly. The time-domain Routh approximation formulation also allows additional flexibility in the choice of the approximate system. By using the additional flexibility of the time-domain formulation, optimized approximations can be readily generated. The time-domain formulation was applied to the engine model, and some results are given. Optimization results and the application of the Routh method to deriving fixed-dimension realizations of the F100 engine are also discussed.

**ROUTH APPROXIMATION**

Many physical processes can be represented as multiple-input–multiple-output (MIMO) dynamic systems by using linear, constant-coefficient, differential equations to model small variations at various operating-point conditions. A transfer-function representation of these equations can be used as input information in a classical or multivariable-frequency-response control system design. The complexity of these transfer functions can cloud the design process with unnecessary detail or difficult computational problems. Additionally, control systems may be made more complex by unnecessary model complexity. Thus, there is a strong motivation to reduce the complexity of system and control models.
Frequency Domain

The Routh approximation method (refs. 2 and 3) is a dominant-mode reduction technique. In the frequency domain the method incorporates parameters obtained from a Routh stability table analysis in the reduction process. These parameters, called alpha and beta parameters, are obtained from the coefficients of the MIMO transfer function

$$W(s) = \frac{B_1 s^{n-1} + \ldots + B_n}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n}$$  \hspace{1cm} (1)

where the $a_i$ are scalars and the $B_i$ are $l \times m$ matrices. (All symbols are defined in appendix A.) Algorithmically, the $n$ alpha parameters are computed from the denominator of equation (1) as (ref. 4)

$$a_j^0 = a_{n-j} \hspace{1cm} j = 0, 2, 4, \ldots, \leq n$$

and

$$a_j^1 = a_{n-j-1}$$

and

$$\alpha_i = \frac{a_i^{i-1}}{a_i^0} \hspace{1cm} i = 1, 2, \ldots, n - 1$$

$$a_j^{i+1} = a_j^{i+2} - \alpha_i a_j^{i+2} \hspace{1cm} j = 0, 2, 4, \ldots, \leq n - i - 1$$

where

$$a_j^i = 0 \hspace{1cm} \text{for} \hspace{1cm} i + j > n$$

The coefficients $a_j^i$ can be arranged as entries in a Routh table (ref. 2). This tabular construction suggests the name Routh approximation.

The beta parameters are computed from both the numerator and the denominator of the given transfer function. Consider the single-input - single-output (SISO) transfer function from input $q$ to output $p$ implicitly given in equation (1), $W_{pq}(s)$. The beta parameters for this SISO transfer function are

$$b_j^1 = b_{n-j} \hspace{1cm} j = 0, 2, 4, \ldots, \leq n$$

$$b_j^2 = b_{n-j-1}$$

(4)
and

\[
\begin{align*}
\beta_1 &= \frac{b_0^i}{a_0^i} \\
\beta_i &= \frac{b_0^{i-2}}{a_0^{i-2}} \\
b_i^{i+2} &= b_{j+2}^{i+2} - \beta_1 a_j^{i+2}
\end{align*}
\]  

(5)

where

\[
b_j^i = 0 \quad \text{for} \quad i + j > n
\]

and \(b_r\) is the \((qp)^{th}\) element of the matrix \(B_r\), where \(r = 1\) to \(n\). In this way, beta parameters for the \(l \times m\) possible SISO transfer functions of equation (1) can be found. Again, considering the SISO transfer function \(W_{pq}(s)\), the computation of the \(k^{th}\) approximant from the first \(k\) alpha and beta parameters is given recursively as

\[
\begin{align*}
a_0^i &= 1 \\
a_0^{i-1} &= \alpha_i a_0^i \\
b_i^0 &= \beta_i a_0^i \\
a_j^{i-1} &= \alpha_i a_j^{i} + a_j^{i+1} \\
b_j^i &= \beta_i a_j^{i} + b_j^{i+2}
\end{align*}
\]

(6)

and

\[
\begin{align*}
c_{k-j}^0 &= a_j \\
c_{k-j-1}^1 &= a_j \\
d_{k-j}^1 &= b_j \\
d_{k-j-1}^2 &= b_j
\end{align*}
\]

(7)

(8)

where

\[
a_j^i = b_j^i = 0 \quad \text{for} \quad i + j > k
\]

Note that, for different output-input pairs, different approximant orders \(k\) can be chosen. The \(k^{th}\)-order approximant for the \((pq)^{th}\) output-input pair is written as
Properties. - The Routh approximation method exhibits several very useful properties. Each is important in the reduction problem and is described briefly here.

Stability: If the original transfer function is asymptotically stable, the Routh approximant, of any order, will be asymptotically stable.

Pole-zero locations: The poles and zeros of the approximants approach the poles and zeros of the original function as the order of the approximation is increased.

Impulse response energy: If \( h(t) \) is the impulse response of \( W_{pq}(s) \), and if the impulse response energy is defined as

\[
E = \int_0^\infty h^2(t) \, dt
\]

then

\[
E = \sum_{i=1}^{n} \frac{\beta_i^2}{2\alpha_i}
\]

Also, if \( h_k(t) \) is the impulse response of the \( k^{th} \)-order approximant of \( W_{pq}(s) \), then

\[
E_{k+1} = E_k + \frac{\beta_{k+1}^2}{2\alpha_{k+1}}
\]

Since \( W_{pq}(s) \) is assumed to be asymptotically stable, \( \alpha_i > 0 \) for \( i=1, \ldots, n \) and

\[
0 < E_1 < E_2 < E_3 < \ldots < E_n = E
\]

The ratio \( E_k/E \) gives an indication of the approximation accuracy in terms of the percentage of total energy accounted for by the \( k^{th} \) approximation.

Derivatives. - The \( k^{th} \) Routh approximant satisfies the following derivative condition:

\[
\left. \frac{d^i}{ds^i} [W(s)] \right|_{s=0} = \left. \frac{d^i}{ds^i} [W_k(s)] \right|_{s=0} \quad i = 0, 1, \ldots, k - 1
\]

Initial and final values. - The initial and final values of the step response trajecto-
ries for the original system and its \( k \)-th order Routh approximant are the same. This is true even though the approximant is a strictly proper transfer function for all \( p \) and \( q \).

Time Domain

The Routh approximation method was extended to the time domain in order that Routh approximants could be directly determined from a state-space model formulation. The alternative would be to determine the \( p \times q \) transfer functions for the MIMO system and then reduce each function individually. For a typical engine model with \( p = 16 \) and \( q = 5 \), this alternative would involve reducing 80 transfer functions at each operating point. Computationally, this is undesirable. Hutton (ref. 3) outlines a method for directly determining the alpha and beta parameters from the state-space formulation. However, computationally, this method failed for the 16th-order engine example of this report. Thus, an alternative computational procedure was devised. It incorporates eigenvalue-eigenvector solutions that are well known and for which excellent computational solutions exist (ref. 5).

Given the state-space representation \( \Sigma_1 \),

\[
\begin{align*}
\dot{x} &= \hat{A}x + \hat{B}u \\
y &= \hat{C}x
\end{align*}
\]  

(15)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^l \), define a reciprocal system, \( \Sigma_2 \), such that

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]  

(16)

where

\[
\begin{align*}
A &= \hat{A}^{-1} \\
B &= -A\hat{B} \\
C &= \hat{C}
\end{align*}
\]  

(17)

The system \( \Sigma_1 \) is assumed to be stable and have distinct eigenvalues. For the jet engine system studied herein, this is a realistic assumption. Note that the reciprocal transformation of \( \Sigma_2 \) is \( \Sigma_1 \). This transformation will always exist since \( A \) is assumed to be stable. The transformation is required to preserve the dominant or low-frequency information of \( \Sigma_1 \) during the reduction process (ref. 3). Now consider a
nonunique similarity transformation of $\Sigma_2$, called the Routh transformation, such that
\[ ATR = TRR \] (18)
where $R$ is the Routh stability matrix. That is,
\[ R = \Lambda_R^{-1} \Gamma \] (19)

\[
\begin{bmatrix}
-1 & -1 \\
1 & 0 & -1 \\
0 & 1 & 0 & -1 \\
\end{bmatrix}
\]

\[ \Gamma = \begin{bmatrix} \vdots \end{bmatrix} \]

\[ \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \]

and
\[ \Lambda_R = \text{diag}[\alpha_i] \quad i = 1, \ldots, n \] (21)

Also, consider the modal transformation of $\Sigma_2$, where
\[ AT_mD_m = T_mD_mA \] (22)
and
\[ \Lambda = \text{diag}[\lambda_i] \quad i = 1, \ldots, n \] (23)

Here the $\lambda_i$ are eigenvalues of $A$ and the $i^{th}$ column of $T_m$ is the corresponding eigenvector. Note that $T_m$ will be unique with respect to a scaling convention and a specific order of the $\lambda_i$'s. The scaling convention is represented by $D_m$, an arbitrary, full rank, diagonal matrix with diagonal elements $d_i$ that may be complex.

Since $R$ is similar to $A$, 


Again $T_z$ is a modal matrix for $R$ and $\Lambda$, and it is unique with respect to a scaling convention represented by $D_z$. From equations (18), (22), and (24) and the definition

$$D = D_m D_z^{-1}$$

all possible Routh transformations can be characterized as

$$T_R = T_m D T_z^{-1}$$

Recall that a fixed ordering of eigenvalues is given. Permutations of this ordering can be represented by a permutation matrix $E_p$, where

$$T_{R, p} = E_p T_R$$

and $E_p$ is the identity matrix with appropriate columns interchanged. Although this permutation does not affect the input-output transfer relationship of the original system, it will affect the input-output characteristics of the approximate system. Undoubtedly, one of the $n!$ possible state permutations will yield a better approximation in some sense than another, but this problem is not considered herein.

From equation (18) a Routh canonical system $\Sigma_3$ can be written as

$$\dot{x}_R = R x_R + G u$$

$$y = H x_R$$

where

$$x = T_R x_R$$

The Routh approximation procedure starts by assuming that

$$\left( x_R \right)_2 = 0$$

where

$$x_R = \begin{bmatrix} \left( x_R \right)_1 \\ \left( x_R \right)_2 \end{bmatrix}$$
and \( \left( x_R \right)_1 \in \mathbb{R}^k \), and \( \left( x_R \right)_2 \in \mathbb{R}^{n-k} \). The system \( \Sigma_2 \) that incorporates the assumption of equation (30) becomes \( \Sigma_2' \), or

\[
\begin{align*}
\dot{x}_1 &= T_{R11} R_{11} T_{R11}^{-1} x_1 + T_{R11} G_1 u \\
y &= H_1 T_{R11}^{-1} x_1
\end{align*}
\]

(32)

where

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

and \( x_1 \in \mathbb{R}^k \) and the subscripts indicate conformable partitioning of the appropriate matrices. The desired approximation then is the reciprocal transformation of \( \Sigma_2' \), called \( \Sigma_2'' \).

Examination of equation (32) shows that the reduced-order system matrix, \( T_{R11} R_{11} T_{R11}^{-1} \), is simply a similarity transformation of the truncated Routh stability matrix. Truncating an \( n \)th-order Routh matrix \( R \) yields another Routh matrix of \( k \)th order \( R_{11} \). Since the original system is stable, the alpha parameters of \( R \), \( \alpha_i \) (i = 1 or \( n \)), are positive. Therefore, \( R_{11} \) is also stable. The reduction process implied in equation (32) exhibits the same pole property as the frequency-domain approach.

Hutton (ref. 3) has shown that if the system given by equation (16) is a single-input-multiple-output system and \( T_R \) is selected such that

\[
T_R^{-1} B = \begin{bmatrix}
-1/\alpha_1 \\
0 \\
\cdot \\
\cdot \\
0
\end{bmatrix}
\]

(33)

the elements of the rows \( h_i \) of \( H \) become the beta parameters for the output considered

\[
h_i = (\beta_{1i}, \beta_{2i}, \ldots, \beta_{ni})
\]

(34)
and the properties of the approximation described previously hold. It can be shown that if $D$ is selected as $D = \text{diag}[\theta]$, where $\theta = T_{m}^{-1}B$, equation (33) is satisfied.

**COMPUTATIONAL ASPECTS OF ROUTh APPROXIMATION**

Computational aspects of both the frequency- and time-domain formulations are discussed in this section.

**Frequency Domain**

In the frequency domain the Routh approximation problem is solved by calculating the alpha and beta parameters for a given transfer function. Straightforward programming of equations (3), (5), and (6) as detailed in subroutines DROUTH and DTHCNV in appendix B gives these parameters (DROUTH) and the Routh approximants (DTHCNV). Also, the impulse response energy property of equation (11) was programmed in subroutine DIMPLS. Implementing this Routh tabular procedure is straightforward, and computationally the procedure is an efficient and accurate method of reducing SISO transfer functions in the frequency domain.

**Time Domain**

Computationally, given the state-space formulation of equation (15), time-domain Routh approximation requires (1) the alpha coefficients and (2) the Routh transformation matrix. The alpha coefficients can be found readily from equation (3) if the system-characteristic-equation coefficients can be found. A program that incorporates the method of Danielevski (described in ref. 6) to find the characteristic equation of the system matrix $\hat{A}$ was used in this report. The alpha coefficients were found by applying subroutine DROUTH.

A method for computing the Routh transformation matrix of equation (18) has been proposed by Datta (ref. 5). This method was programmed and applied to the 16th-order jet engine model. Numerical results were unacceptable because of large computational errors, and so another technique was sought. Since accurate and efficient eigenvalue-eigenvector techniques are well known (for the case of distinct eigenvalues), the computation of the Routh transformation matrix was reformulated as the solution of two eigenvalue-eigenvector problems. These two problems are represented by equations (22) and (24) and are solved by using the computational methods of reference 7. Since
A and R are similar, they have the same eigenvalues. The Routh matrix R is also known from equation (19) since the alpha parameters have been found.

The transformation matrices of equations (22) and (24) may have elements in the complex field. Thus, from equations (2) to (6), $T_R$ can in general be complex. However, a real $T_R$ is desired to yield a physically realizable approximation and to facilitate computer computations. The matrix $T_R$ can be constrained to be real by the proper selection of the elements $d_i$ of $D$. To determine these elements, consider the eigenvector solution of equation (24) by means of real-number computer operations. Given A and its eigenvalues, the modal matrix of A can be written as

$$T_{mm} = T_m V_m$$

where $V_m$ is the block diagonal and $T_{mm}$, the modified modal matrix of A, is a matrix of real numbers. If the eigenvalues of A were all real, $V_m$ would equal I. However, if $\lambda_i$ and $\lambda_{i+1}$ were a complex conjugate pair, the $i^{th}$ block of $V_m$, $(V_m)_i$, would be defined as

$$\left( V_m \right)_i = \frac{1}{2} \begin{bmatrix} 1-j & 1+j \\ 1+j & 1-j \end{bmatrix}$$

Likewise, for equation (24),

$$T_{z,m} = T_z V_m$$

Now from equations (26), (33), and (35) and the definition

$$D_B = V_m^{-1} D V_m$$

$T_R$ can be written

$$T_R = T_{mm} D_B T_{z,m}^{-1}$$

Now if $T_R$ is real, $D_B$, a block diagonal matrix, must be real. This can be assured by selecting the elements of $D$, $d_i$, such that

1. $d_i$ is real if $\lambda_i$ is real
2. $d_i, d_{i+1}$ are a complex conjugate pair if $\lambda_i, \lambda_{i+1}$ are a complex conjugate pair.

The reciprocal transformation of equation (17) does not represent a significant increase in computations since the modal transformation is the same for $A$ and $\hat{A}$ and the eigenvalues are simply reciprocals.

The matrix inversion of equation (26) can be eliminated by noting that
\[ r = T^T I T \tilde{T} \]

where

\[ \tilde{T} = \text{diag}[(-1)^j] \]

From equation (19) then

\[ A_R \tilde{T} R = R^T A_R \tilde{T} \]

Now from equations (24), (26), and (40)

\[ T_R = T_m D_m T_z \tilde{T} A_R \]

A program that solves for \( T_R \), the Routh transformation in equation (43), was written. This program, called DRTTRC, not only finds \( T_R \), but also the transformed matrices of equation (28), \( G \) and \( H \). The FORTRAN listing of DRTTRC is given in appendix B. Additionally, a program that finds the reduced system of equation (32), called DTYPEI, was written and is given in appendix B.

APPLICATION TO ENGINE MODEL

The frequency-domain formulation of the Routh approximation technique was applied to the state-variable model of the F100 engine at the intermediate-power operating point and to the transfer-function model of the boost-pump pressure regulator design. The time-domain formulation was applied only to the engine model.

Frequency-Domain Applications

As an initial experiment the frequency-domain formulation was applied to two 16th-order SISO transfer functions. These transfer functions represent dynamics of the turbobfan engine from the input \( w_f \) (fuel flow) to the outputs \( N_c \) (compressor speed) and \( T_t \) (turbine inlet temperature). The coefficients for these transfer functions were calculated and are given in table I. Table II shows the impulse response ratios for Routh approximations of increasing order for the two transfer functions. These ratios were calculated from the alpha and beta coefficients as outlined in equation (12). From these ratios an acceptable approximation order can be estimated.

The approximant order was selected by first choosing a minimum acceptable level
of accuracy, as defined by the impulse response energy ratio. The level chosen cor-
responds to a ratio of 0.81. This level of accuracy was assumed to be adequate for the 
purpose of this report. Next the order that most nearly corresponded to the selected 
ratio was chosen as the order of the Routh approximant. For the energy ratios given 
in table II, the Routh approximant orders, as found by these criteria, are

\[ k_1 = 3 \quad (y_1 = N_c) \]
\[ k_2 = 9 \quad (y_2 = T_t) \]

The Routh approximants for \( N_c \) and \( T_t \) were calculated by using the approximant 
orders \( k_1 \) and \( k_2 \) and the algorithm given in equations (6) to (9). The results are 
summarized in table III. Comparisons of pole locations and step responses for exact 
and approximate transfer functions were made to evaluate the adequacy of the approxi-
mations. Table IV gives the pole locations, and figure 1 the step response comparisons. 
The pole comparison shows a good correlation between actual and approximate system 
poles. Likewise, the step responses of figure 1 indicate an excellent agreement between 
actual and approximate model representations. Based on these results it was concluded 
that the Routh approximation technique is a viable approach to reducing the complexity of 
frequency-domain models of jet engine dynamics. Also, the accuracy level selected in 
this initial example may be too stringent, based on the step response comparison.

For comparison, the Routh approximation technique was applied to all 80 (\( l = 16, 
m = 5 \)) possible transfer functions of the given engine example. The impulse response 
energy criterion for model order was used to determine the order for each of the ap-
proximants. These reduced orders are given in table V. Examination of this table gives 
dynamically fast (\( x_3, x_4 \), e.g.) and slow (\( x_1, x_2, x_{10} \), e.g.) states as well as reduced 
order. The large difference in reduced order for transfer functions with the same input 
indicates that modeling of the fast states as constants with respect to the slow states 
would be acceptable.

The frequency response technique was also applied to a high-order (43rd order) 
model of the boost-pump pressure regulator, which represents the linearized line dy-
namics of a liquid-oxygen supply system in a rocket engine (ref. 8). The model input 
represents a pressure error signal; the output represents an actuator piston-commanded 
velocity. The transfer function (time scaled by a factor of 100) for this system is given 
in table VI. This application tests the accuracy and the computational feasibility of the 
programs DROUTH, DTHCNV, and DIMPLS for high-order problems. Again, straight-
forward application of the programs was all that was required to obtain Routh approxi-
mants to the boost-pump pressure regulation system. Additionally, it was determined 
that a 43rd-order implementation of this pressure control would be impractical and that 
a study of lower order implementations would be desirable. Thus, this application is
of more than numerical significance.

For the 81-percent impulse response energy criterion, a 26th-order Routh approximant was selected. The poles of the original and approximant system are compared in table VII; the zeros are compared in table VIII. Both show good comparability between original and approximant values. For reference the 26th-order approximant is given in table IX. Figure 2 compares the frequency responses for the original and approximant systems. The comparison is exact for both magnitude and phase for frequencies less than 20 hertz. The approximation is still good for higher frequencies, and the loss of frequency information due to the approximation is clearly shown.

Time-Domain Application

The time-domain formulation of the Routh application technique was applied to the 16th-order, state-variable F100 engine model. This model represents the dynamic response of the F100 engine for small perturbations about the intermediate-power operating point. The normalized system matrices are given in table X and the state, input, and output vector definitions in table XI. Calculating the Routh transformation matrix $T_R$ poses the only significant computational problem in applying the time-domain formulation. Thus, as a test, the alpha parameters calculated from the characteristic equation of $\hat{A}$ (Routh table approach) and from the similarity transformation $T_R^{-1}A_T R$ are compared in table XII. Clearly, calculating $T_R$ with DRTTRC is accomplished with a high degree of accuracy. Therefore, the tools exist to accurately apply the time-domain formulation to high-order systems.

Examining table V shows that a fifth-order approximation for each input would represent an adequate tradeoff between accuracy and complexity. Reduced approximants were calculated by using DRTTRC and DTYPEI for this order. Step response trajectories comparing the original and approximate systems were generated. The $T_t$ result indicates the degradation of accuracy obtained with a lower order model. Typical step responses for one input (fuel flow) and two outputs (compressor speed and turbine inlet temperature) are given in figure 3.

With this time-domain formulation, fifth-order models were calculated for each input that exactly match those that would have been determined with a frequency-domain analysis. However, the approximants were determined directly from the state-variable model without first calculating the frequency transfer function for each input-output pair. All the properties of the frequency-domain approach hold for the time-domain approach when considering the state-variable formulation one input at a time.
Fixed-Dimension Approximant

For a five-input system approximated by fifth-order models, a total reduced system realization may require a state vector dimension of 25. In fact, the realization dimension will probably vary for different numbers of inputs and orders of approximation. Often, however, the allowable dimension of the total reduced approximation is fixed by some engineering or economic constraint. Constructing a fixed-dimension realization of the total system from the Routh approximation can be a very difficult computational problem, especially if the input and output dimensions are large. Thus, the technique as outlined does not directly handle the fixed-dimension problem. However, if the same $T_R$ matrix is used in the time-domain reduction process for each input, the total approximation can be realized by a system of $k^{th}$ order, where $k$ is the number of alpha parameters retained, and the fixed-dimension approximation problem is solved. Selecting the $T_R$ matrix that will give the best $k^{th}$-order approximation is therefore an interesting problem.

In this regard the somewhat arbitrary selection of $D$ can be used to good advantage. Many different transformations can be found very quickly once the original two eigenvector problems have been solved. Indeed, the selection process can be automated by optimizing some function of error between the original and total approximant systems over the $n$-parameter space of $D$. Such an optimization scheme was tried on the example given in this report. Two different error functions were used for comparison. The first was a weighted sum of the differences in system and approximant step response energies. The second was a weighted sum of squares of the differences in system and approximant steady-state values. Significant minimization of each error function was easily achieved in relatively small amounts of computer time by using a conjugate direction optimization scheme. Thus, the general optimization procedure would appear to be a good way to improve the accuracy of approximants for certain systems while maintaining a fixed order of realization. However, the time-domain Routh approximation procedure, when constrained to yield a fixed-dimension realization in the multiple-input case, does not exhibit the final-value property of the single-input case. For the engine example posed, the significant improvement in the approximation gained by function optimization was overshadowed by these final-value errors for the multiple-input case.

Based on this observation the time-domain formulation was modified to ensure that the final-value property would be met by a fixed-dimension, multiple-input Routh approximant. In the original formulation the difficulty with final values in the multiple-input, fixed-dimension problem can be traced to the original assumption of the reduction process (eq. (30)). If the initial assumption were changed to

$$\langle x_R \rangle_2 = 0$$

(44)
the approximation would force the final-value property. Unfortunately, the assumption of equation (44) when applied to the original system may yield an unstable approximant for a given stable system. This was the case for the engine example and, thus, the modification of the Routh procedure was rejected.

CONCLUSIONS

Models of physical processes are often too complex to handle directly. Approximations of these models that are of reduced complexity but that maintain essential model characteristics are often used for analysis rather than the original model. One way to generate these approximations is the Routh approximation technique. The frequency-domain formulation of the Routh technique was applied to transfer-function models of an F100 engine and a launch vehicle boost-pump pressure regulator. Also, the Routh approximation process was reformulated in the time domain. A new characterization of the nonunique Routh similarity transformation was derived that describes all possible Routh transformations. This characterization casts the computation of the Routh transformation into two eigenvector-eigenvalue problems that are easily solved. The application of the time-domain formulation to a 16th-order state-variable description of a turbobfan engine was described, and the results were given. These results indicate that the time-domain Routh approximation technique can be valuable in reducing engine model complexity when dealing with the model on a single-input basis. An optimization procedure was discussed that can significantly improve the approximation in a computationally efficient manner by taking advantage of the new time-domain characterization.

Lewis Research Center,
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505-05.
APPENDIX A

SYMBOLS

\( A \) reciprocal system matrix
\( \hat{A} \) original system matrix
\( a \) coefficient of characteristic equation
\( B \) reciprocal control matrix
\( \hat{B} \) original control matrix
\( B_i \) matrix of numerator coefficients
\( b \) element of \( B_i \)
\( C \) reciprocal output matrix
\( \hat{C} \) original output matrix
\( c \) coefficient of characteristic equation for reduced system
\( D \) diagonal parameter matrix
\( d \) coefficient of numerator polynomial for reduced system
\( \frac{d^i}{ds^i} \) \( i \)th-order differentiation operator
\( E \) impulse response energy
\( E_p \) permutation matrix
\( G \) Routh control matrix
\( H \) Routh output matrix
\( h \) impulse response
\( h_i \) number of rows
\( K \) reduced-system order
\( k \) Routh approximant order
\( l \) number of outputs
\( m \) number of inputs
\( N_c \) compressor speed
\( n \) original-system order
\( R \) Routh-system matrix
\( s \) Laplace transform variable
T  similarity transformation matrix
T_t  turbine inlet temperature
t  time, sec
u  control vector
V  block diagonal modification matrix
W(s)  matrix transfer function
w_f  engine fuel flow
x  reciprocal state vector
\mathbf{x}  original state vector
y  output vector
\alpha  alpha parameter
\beta  beta parameters
\Gamma  gamma matrix
\lambda  diagonal parameter matrix
\Sigma  state-space system

Subscripts:

k  \textsuperscript{k}\text{-order approximant}
m  modal
\text{mm}  modified modal
p  output number
q  input number
R  Routh transformation
z  Routh transformed system
APPENDIX B

SUBROUTINES USED IN ROUTH APPROXIMATION

SUBROUTINE ROUTH

PURPOSE
TO CALCULATE THE ALPHA-BETA EXPANSION
COEFFICIENTS OF A LINEAR, TIME-ININVARIANT
TRANSFER FUNCTION (SISO)

METHOD
ROUTH TABLE

SUBROUTINE DROULTH(71,22,ALPHA,BETA,RTA,RTB,N,NMAX)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION Z1(I),Z2(I),ALPHA(I),BETA(I),RTA(NMAX,1),RTB(NMAX,1)
KK=0
DO 5 I=1,N
DO 5 J=1,N
RTA(I,J)=0.
RTB(I,J)=0.
5 CONTINUE

INITIALIZE THE ALPHA AND BETA ARRAYS

IF(MOD(N,2).EQ.0) N1=N/2
IF(MOD(N,2).EQ.1) N1=(N-1)/2
DO 10 J=1,N1
DO 10 I=1,N
NN=N-KK
RTA(I,J)=Z1(NN)
RTB(I,J)=Z2(NN)
KK=KK+1
10 CONTINUE

N2=N1+1
RTA(1,N2)=1.
RTB(1,N2)=1.
IF(NN.EQ.0) RTA(2,N2)=0.
IF(NN.EQ.1) RTA(1,N2)=Z1(1)
IF(NN.EQ.2) RTA(1,N2)=Z1(1)
RTB(2,N2)=0.

CALCULATE THE ALPHA AND BETA ARRAYS

DO 30 I=1,N
DO 30 J=1,N1
ALPHA(I)=RTA(I,1)/RTA(I+1,1)
BETA(I)=RTB(I,1)/RTA(I+1,1)
RTA(I+2,J)=RTA(I,J+1)-ALPHA(I)*RTA(I+1,J+1)
RTB(I+2,J)=RTB(I,J+1)-BETA(I)*RTA(I+1,J+1)
30 CONTINUE

RETURN
END
SUBROUTINE IMPULS

PURPOSE
TO COMPUTE THE ENERGY ASSOCIATED WITH EACH
ROUTH APPROXIMANT ASSUMING THE SYSTEM IS
DRIVEN WITH AN IMPULSE

SUBROUTINE DIMPS(ALPHA,BETA,RMS,N)
IMPLICIT DOUBLE PRECISION (A-H.O-Z)
DIMENSION ALPHA(1),BETA(1),RMS(1)
SUP=0.0
DO 10 J=1,N
SUP=BETA(J)**2/2.*ALPHA(J)+SUM
IF (SUP.LE.0.0) RETURN
RMS(J)=DSQRT(SUM)
10 CONTINUE
RETURN
END

SUBROUTINE MRTRC

PURPOSE
TO FORM THE STATE SPACE ROUGH APPROXIMATE
OF THE ORDER REDUCED FROM THE ORIGINAL RECIPROCAL
SYSTEM AND THE ROUGH TRANSFORMATION MATRIX

SUBROUTINE MRTRC(ALFA,XI,F,C,T,TZ,TR,N,BHT,CHT,WA,WR,WC,
1 N,MM,NHAT)
IMPLICIT DOUBLE PRECISION (A-H.O-Z)
DIMENSION ALFA(N),F(N,N),C(N,N),BHT(N,N,N),CHT(N,N,N),
1 WA(N,N),WR(N,N),WC(N,N,N),T(N,N),TZ(N,N),TR(N,N),DIM(N),XI(N)

FORM THE ROUGH TRANSFORMATION MATRIX TR

IJ=N
DO 1 J=1,N
IF(IJ.EQ.1) GO TO 6
IF(XI(IJ).NE.C.C) GO TO 5
DO 11 I=1,N
WA(I,J)=TZ(I,J)
11 WA(I,J)=7(I,J)*DI(J)
GO TO 1
5 DO 12 I=1,N
WI(I,J)=TZ(I,J+1)
W(I,J+1)=TZ(I,J)
WI(I,J+1)=TZ(I,J+1)
WAIT,J)=D(I,J)+T(I,J+1)*DI(J+1)
12 WA(I,J+1)=T(I,J)*DI(J+1)+T(I,J+1)*DI(J)
IJ=I
GO TO 1
6 IJ=0
1 CONTINUE
CALL OTMULT(WA,WR,TR,N,N,N)
IJK=-1
DO 2 J=1,N
IJK=IJK
ALFD=IJK*ALFA(J)
2 IJK=J
CONT
DO 2 I=1,N
2  TR(I,J)=TR(I,J)*ALFD
C C FORM CHAT
C CALL DMULT(C,TR,WC,MM,N,N)
DO 3 I=1,MM
DO 3 J=1,NHAT
3 CHII(J)=WC(I,J)
C C FIND THE INVERSE OF THE TR MATRIX
C IJ=0
CALL DMMPY(WA,WB,I,DO,N,N,N)
DO 7 J=1,N
IF(IJ.EQ.1) GO TO 72
IF(XI(J).EQ.C.0) GO TO 7
DO 71 I=1,N
WB(I,J)=WAIT(I,J+1)
71 WB(I,J+1)=WAIT(I,J)
IJ=1
GO TO 7
72 IJ=0
7 CONTINUE
CALL DMMT(WB,TZ,WA,N,N,N)
IJK=-1
DO 8 J=1,N
IJK=-IJK
ALFD=ALFA(J)*IJK
DO 8 I=1,N
8 WAIT(J)=WAIT(I,J)*ALFD
CALL DMULT(WA,TZ,WPN,N,N,N)
IDC=6
CALL DINV1F(WB,N,W,TDGT,WC,IER)
CALL DMULT(TZ,WA,WBN,N,N)
C C FORM B HAT
C CALL DMULT(WB,P,WAN,N,N,N)
DO 4 I=1,NHAT
DO 4 J=1,M
4 BHII(J)=WAIT(I,J)
RETURN
END

C C SUEROUTINE RTHCNV

C C PURPOSE
C TO COMPUTE THE COEFFICIENTS OF THE ROUTH CONVERGENT APPROXIMANTS
C C SUEROUTINE RTHCNV(ALPHA,PETA,NMNTX,DNMTX,N,NMAX)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DOUBLE PRECISION NMNTX
DIMENSION ALPHA(NMAX),BETA(NMAX),NMNTX(NMAX,1),DNMTX(NMAX,1)
DO 10 I=1,NMAX
DO 10 J=1,NMAX
NMTX(I,J):=0.
NMTX(1,J):=1.
10 CONTINUE

THE DENOMINATOR CONVERGENTS ARE IN DNMTX
THE NUMERATOR CONVERGENTS ARE IN NMTX

DO 20 I=1,N
DO 20 J=1,N
NMTX(I,J):=DNMTX(I-1,J-1)+ALPHA(I)*DNMTX(I-1,J-1)
NMTX(I,J):=NMNTX(I-1,J-1)+ALPHA(I)*NMNTX(I-1,J-1)
DO 30 I=1,N
DO 30 J=1,N
DNMTX(I,J):=DNMTX(I,J-1)*ALPHA(I)+DNMTX(I,J-2)*ALPHA(I)
DNMTX(I,J):=NMNTX(I,J-1)*ALPHA(I)+NMNTX(I,J-2)*ALPHA(I)
30 CONTINUE
DO 40 I=1,N
DO 40 J=1,N
DNMTX(I,J):=DNMTX(I,J-1)*ALPHA(I)+DNMTX(I,J)
40 CONTINUE
DNMTX(N,N-1):=0.
NMNTX(N,N-1):=0.
PFC0=1.
DO 50 I=1,N
50 CONTINUE
DNMTX(N,N):=PROC
NMNTX(N,N):=BETA(I)*PROC/ALPHA(I)
RETURN
END

SUBROUTINE DTYPEF

PURPOSE
TO FIND THE KTH ORDER TYPE I DOMINANT MODE
APPROXIMATION OF AN NTH ORDER SYSTEM

THE ORIGINAL SYSTEM
ZDOT = T*INV(T) + Z + G*U
Y = H*INV(T)*Z

SUBROUTINE DTYPEF(A,B,C,T,INV,R11,C,H,DKK1,DKK2,DNN1,N,M,L,K)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
DIMENSION AK(K,K),BK(K,K),CK(K,K),TN,N1,R11(K,K),G(N,M),H(L,L)
IF (M.GT.K) WRITE(6,1000)

1000 CONTINUE
**Fortran Code:**

```fortran
C
C     FIND THE INVERSE OF THE K X K BLOCK OF T, T11
C
DO 1 I=1,K
DO 1 J=1,K
1   DKK1(I,J)=T(I,J)
   CALL DMTPY(DKK1, DKK2, 1, DO, K, K, K)
   CALL DLNV1F(DKK2, K, K, A, IDGT, B, IEP)

C
C     FIND THE CONTROLL MATRIX P
C
   CALL DMULT(TINV, G, DNN1, N, M, N)
   DO 2 I=1,K
   DO 2 J=1,M
2   DKK2(I,J)=DNN1(I,J)
   CALL DMULT(DKK1, DKK2, P, K, M, K)

C
C     FIND THE OUTPUT MATRIX C
C
   CALL DMULT(T, A, DNN1, N, K, K)
   CALL DMULT(H, DNN1, C, L, K, N)

C
C     FIND THE SIMILARITY TRANSFORM T11*R11*INV(T11)
C
   CALL DMULT(P11, A, DKK2, K, K, K)
   CALL DMULT(DKK1, DKK2, A, K, K, K)
RETURN
END
```
REFERENCES


<table>
<thead>
<tr>
<th>Coefficient, $i$</th>
<th>Coefficient of characteristic equation, $a_i$</th>
<th>Compressor speed, $N_c$ \ (p = 1, q = 1)</th>
<th>Turbine inlet temperature, $T_t$ \ (p = 2, q = 1)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.0000</td>
<td>0.0000</td>
<td>5.7270</td>
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<td>1.0638 x 10^3</td>
<td>0.1140</td>
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<td>2.9209 x 10^8</td>
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<td>-4.2414 x 10^3</td>
<td>2.6079 x 10^10</td>
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<td>5</td>
<td>4.7772 x 10^12</td>
<td>-8.2163 x 10^7</td>
<td>1.4142 x 10^15</td>
</tr>
<tr>
<td>6</td>
<td>2.2150 x 10^14</td>
<td>2.2542 x 10^10</td>
<td>2.5027 x 10^16</td>
</tr>
<tr>
<td>7</td>
<td>7.1949 x 10^15</td>
<td>2.8702 x 10^12</td>
<td>3.0507 x 10^17</td>
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<tr>
<td>8</td>
<td>1.6584 x 10^17</td>
<td>1.5789 x 10^14</td>
<td>2.5208 x 10^18</td>
</tr>
<tr>
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<td>2.7151 x 10^18</td>
<td>5.0230 x 10^15</td>
<td>1.3685 x 10^19</td>
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<td>3.1255 x 10^19</td>
<td>1.0028 x 10^17</td>
<td>4.6358 x 10^19</td>
</tr>
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<td>2.4739 x 10^20</td>
<td>1.2850 x 10^18</td>
<td>9.0249 x 10^19</td>
</tr>
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<td>1.2973 x 10^21</td>
<td>1.0446 x 10^19</td>
<td>8.7170 x 10^19</td>
</tr>
<tr>
<td>13</td>
<td>4.2578 x 10^21</td>
<td>5.1525 x 10^19</td>
<td>7.4230 x 10^19</td>
</tr>
<tr>
<td>14</td>
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<td>1.4079 x 10^20</td>
<td>2.9283 x 10^19</td>
</tr>
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<td>7.4304 x 10^21</td>
<td>1.7841 x 10^20</td>
<td>2.9283 x 10^19</td>
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<td>16</td>
<td>2.4119 x 10^21</td>
<td>7.4230 x 10^19</td>
<td>2.9283 x 10^19</td>
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</table>
TABLE III. - ROUTH APPROXIMANT COEFFICIENTS FOR TWO TRANSFER FUNCTIONS

<table>
<thead>
<tr>
<th>Coefficient, $i$</th>
<th>Compressor speed, $N_c$ ($p = 1, q = 1$)</th>
<th>Turbine inlet temperature, $T_t$ ($p = 2, q = 1$)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$c_i$</td>
<td>$d_i$</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>0</td>
<td>1.0000</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>2.6132</td>
<td>0.43878</td>
</tr>
<tr>
<td>2</td>
<td>2.5702</td>
<td>0.61711</td>
</tr>
<tr>
<td>3</td>
<td>0.8343</td>
<td>0.25676</td>
</tr>
<tr>
<td>4</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>5</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>6</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>7</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>8</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>9</td>
<td>------</td>
<td>------</td>
</tr>
</tbody>
</table>

TABLE IV. - COMPARISON OF POLE LOCATIONS FOR EXACT (16th ORDER) AND APPROXIMATE (3rd AND 9th ORDER) TRANSFER FUNCTIONS

<table>
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<th>Poles of transfer functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>16th Order</td>
</tr>
<tr>
<td>-0.648</td>
</tr>
<tr>
<td>-1.91</td>
</tr>
<tr>
<td>-2.62</td>
</tr>
<tr>
<td>-6.71±1.31j</td>
</tr>
<tr>
<td>-17.8±4.80j</td>
</tr>
<tr>
<td>-18.2</td>
</tr>
<tr>
<td>-21.6±1.55j</td>
</tr>
<tr>
<td>-38.7</td>
</tr>
<tr>
<td>-47.1</td>
</tr>
<tr>
<td>-50.6</td>
</tr>
<tr>
<td>-59.2</td>
</tr>
<tr>
<td>-175</td>
</tr>
<tr>
<td>-577</td>
</tr>
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</table>
TABLE V. - MINIMUM TRANSFER FUNCTION ORDER FOR 81-PERCENT IMPULSE RESPONSE ENERGY RATIO

<table>
<thead>
<tr>
<th>Transfer function output</th>
<th>Transfer function input</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>3</td>
</tr>
<tr>
<td>$x_3$</td>
<td>6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>11</td>
</tr>
<tr>
<td>$x_5$</td>
<td>4</td>
</tr>
<tr>
<td>$x_6$</td>
<td>3</td>
</tr>
<tr>
<td>$x_7$</td>
<td>3</td>
</tr>
<tr>
<td>$x_8$</td>
<td>4</td>
</tr>
<tr>
<td>$x_9$</td>
<td>8</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>4</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>9</td>
</tr>
<tr>
<td>$x_{12}$</td>
<td>8</td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>7</td>
</tr>
<tr>
<td>$x_{14}$</td>
<td>7</td>
</tr>
<tr>
<td>$x_{15}$</td>
<td>4</td>
</tr>
<tr>
<td>$x_{16}$</td>
<td>6</td>
</tr>
</tbody>
</table>
TABLE VI. - BOOST-PUMP PRESSURE REGULATOR TRANSFER
FUNCTION FOR PRESSURE ERROR MEASUREMENT INPUT P(s)
TO ACTUATOR PISTON-COMMANDED VELOCITY OUTPUT Q(s)

\[
Q(s) = \frac{-261.31 \, s^4 + 9821.8 \, s^4 + 139360 \, s^4 + \ldots + 0.34557 \times 10^8}{s^4 + 530.64 \, s^4 + 1239.2 \, s^4 + 17914 \, s^4 + \ldots + 0.33972 \times 10^8}
\]

<table>
<thead>
<tr>
<th>Power of s</th>
<th>Denominator coefficient</th>
<th>Numerator coefficient</th>
</tr>
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<tbody>
<tr>
<td>42</td>
<td>0.5306x10^2</td>
<td>0.2613x10^3</td>
</tr>
<tr>
<td>41</td>
<td>0.1239x10^4</td>
<td>-0.9821x10^4</td>
</tr>
<tr>
<td>40</td>
<td>0.1791x10^5</td>
<td>-0.1393x10^6</td>
</tr>
<tr>
<td>39</td>
<td>9.776x10^6</td>
<td>-0.1389x10^7</td>
</tr>
<tr>
<td>38</td>
<td>1.4079x10^7</td>
<td>-0.1052x10^8</td>
</tr>
<tr>
<td>37</td>
<td>9.1649x10^7</td>
<td>-0.6817x10^8</td>
</tr>
<tr>
<td>36</td>
<td>5.219x10^8</td>
<td>-0.3706x10^9</td>
</tr>
<tr>
<td>35</td>
<td>2.5635x10^9</td>
<td>-0.1806x10^10</td>
</tr>
<tr>
<td>34</td>
<td>1.1146x10^10</td>
<td>-0.7806x10^10</td>
</tr>
<tr>
<td>33</td>
<td>4.5559x10^10</td>
<td>-0.3043x10^11</td>
</tr>
<tr>
<td>32</td>
<td>1.6623x10^11</td>
<td>-0.1091x10^12</td>
</tr>
<tr>
<td>31</td>
<td>5.5692x10^11</td>
<td>-0.3534x10^12</td>
</tr>
<tr>
<td>30</td>
<td>1.6873x10^12</td>
<td>-0.1073x10^13</td>
</tr>
<tr>
<td>29</td>
<td>4.8986x10^12</td>
<td>-0.2961x10^13</td>
</tr>
<tr>
<td>28</td>
<td>1.2415x10^13</td>
<td>-0.7672x10^13</td>
</tr>
<tr>
<td>27</td>
<td>3.1769x10^13</td>
<td>-0.1837x10^14</td>
</tr>
<tr>
<td>26</td>
<td>6.7590x10^13</td>
<td>-0.4059x10^14</td>
</tr>
<tr>
<td>25</td>
<td>1.5437x10^14</td>
<td>-0.8571x10^14</td>
</tr>
<tr>
<td>24</td>
<td>2.7519x10^14</td>
<td>-1.6048x10^15</td>
</tr>
<tr>
<td>23</td>
<td>5.6516x10^14</td>
<td>-3.0264x10^15</td>
</tr>
<tr>
<td>22</td>
<td>8.4066x10^14</td>
<td>-4.744x10^15</td>
</tr>
<tr>
<td>21</td>
<td>1.5582x10^15</td>
<td>-8.0866x10^15</td>
</tr>
<tr>
<td>20</td>
<td>1.9203x10^15</td>
<td>-1.0439x10^16</td>
</tr>
<tr>
<td>19</td>
<td>3.2126x10^15</td>
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<td>-1.6879x10^16</td>
</tr>
<tr>
<td>17</td>
<td>4.8853x10^15</td>
<td>-2.4235x10^16</td>
</tr>
<tr>
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<td>3.9945x10^15</td>
<td>-1.9629x10^16</td>
</tr>
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<td>5.3594x10^15</td>
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<tr>
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<td>3.4801x10^15</td>
<td>-1.5896x10^16</td>
</tr>
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<td>4.1055x10^15</td>
<td>-2.0056x10^16</td>
</tr>
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<td>2.0670x10^15</td>
<td>-0.8553x10^15</td>
</tr>
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TABLE VII. - POLE COMPARISON BETWEEN APPROXIMANT AND ORIGINAL SYSTEMS

(a) Approximant system

(b) Original system
TABLE VIII. - ZERO COMPARISON BETWEEN APPROXIMANT AND ORIGINAL SYSTEMS

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(a) Approximant system
(b) Original system
TABLE IX. - 26th-ORDER
ROUTH APPROXIMANT

\[
\frac{Q}{P} = \frac{\sum_{i=1}^{26} d_i s^{26-i}}{s^{26} + \sum_{i=1}^{26} c_i s^{26-i}}, \quad c_0 = 1.
\]

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### TABLE X. - ENGINE

\[ n = 16, \quad 1 = 2, \]

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<td>-1.04020</td>
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</table>

(b) \( B \) matrix

<table>
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<th>( A )</th>
<th>( B )</th>
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<tr>
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<td>-1.18466</td>
</tr>
</tbody>
</table>

\^For \( p = 16, \) \( C \) is the 16th-order identity matrix.
### MODEL DATA

\( q = 2 \)

\( \text{matrix} \)

| \( 0.252614 \) | 0.427346x10\(^{-1} \) | -1.19810 | 1.82058 | 0.213133 | -0.106835 | -0.997036x10\(^{-2} \) | 0.132963x10\(^{1} \) |
| \( 2.24174 \) | 0.239846 | 0.940377 | -1.855296x10\(^{-2} \) | -1.00398x10\(^{-1} \) | -1.04123x10\(^{-1} \) | -3.55279x10\(^{-1} \) |
| \(-79.0068 \) | 0.101911 | 4.41643 | 0.114667 | 0.101938 | 0.114658 | 2.24267 |
| \(-128376 \) | -12.9188 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.659112x10^{-1} \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.629665x10^{-1} \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.368312x10^{-1} \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.49.9900 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.5.99940 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.2.44677 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.12.7924 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.5.11614 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(-.375007x10^{-2} \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |
| \(.913777 \) | -1.19810 | 170.896 | 4.52342 | 4.53157 | 4.43644 | -3.61533 |

\( \text{CT} \) matrix

| 1.00000 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 1.00000 |
| 0 |
| 0 |

(c) \( \text{CT} \) matrix

| 1.00000 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 1.00000 |
| 0 |
| 0 |

33
TABLE XI. - INPUT, OUTPUT, AND STATE VECTOR DEFINITIONS

[Linear model: $\dot{x} = Ax + Bu; y = Cx$, where $x$ are state variables, $y$ are output variables, and $u$ are input variables. Since $C = I$ (identity matrix), $y = x$.]

<table>
<thead>
<tr>
<th>State variables</th>
<th>Control variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$ - Fan speed</td>
<td>$u_1$ - Main combustor fuel flow</td>
</tr>
<tr>
<td>$x_2$ - Compressor speed</td>
<td>$u_2$ - Nozzle jet area</td>
</tr>
<tr>
<td>$x_3$ - Compressor discharge pressure</td>
<td>$u_3$ - Inlet guide vane position</td>
</tr>
<tr>
<td>$x_4$ - Interturbine volume pressure</td>
<td>$u_4$ - High variable stator position</td>
</tr>
<tr>
<td>$x_5$ - Augmentor pressure</td>
<td>$u_5$ - Compressor bleed</td>
</tr>
<tr>
<td>$x_6$ - Fan inside-diameter discharge temperature</td>
<td></td>
</tr>
<tr>
<td>$x_7$ - Duct temperature</td>
<td></td>
</tr>
<tr>
<td>$x_8$ - Compressor discharge temperature</td>
<td></td>
</tr>
<tr>
<td>$x_9$ - Combustor exit temperature - fast component</td>
<td></td>
</tr>
<tr>
<td>$x_{10}$ - Combustor exit temperature - slow component</td>
<td></td>
</tr>
<tr>
<td>$x_{11}$ - Combustor exit total temperature</td>
<td></td>
</tr>
<tr>
<td>$x_{12}$ - Fan turbine inlet temperature - fast component</td>
<td></td>
</tr>
<tr>
<td>$x_{13}$ - Fan turbine inlet temperature - slow component</td>
<td></td>
</tr>
<tr>
<td>$x_{14}$ - Fan turbine exit temperature</td>
<td></td>
</tr>
<tr>
<td>$x_{15}$ - Duct exit temperature</td>
<td></td>
</tr>
<tr>
<td>$x_{16}$ - Augmentor exit temperature</td>
<td></td>
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</table>
TABLE XII. - COMPARISON OF COMPUTATIONAL ACCURACY OF FREQUENCY- AND TIME-DOMAIN FORMULATIONS

<table>
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</table>
Figure 1. - Step response comparisons for normalized compressor speed and turbine inlet temperature outputs for fuel flow input-frequency-domain approach.

Figure 2. - Frequency response comparison.
Figure 3. Step response comparisons for normalized compressor speed and turbine inlet temperature outputs for fuel flow input - time-domain approach.
The Routh approximation technique for reducing the complexity of system models was applied in the frequency domain to a 16th-order, state-variable model of the F100 engine and to a 43rd-order, transfer-function model of a launch vehicle boost-pump pressure regulator. The results motivate extending the frequency-domain formulation of the Routh method to the time domain in order to handle the state-variable formulation directly. The time-domain formulation is derived and a new characterization that specifies all possible Routh similarity transformations is given. This characterization is computed by solving two eigenvalue-eigenvector problems.

The application of the time-domain Routh technique to the state-variable engine model is described, and some results are given. Additional computational problems are discussed, including an optimization procedure that can improve the approximation accuracy by taking advantage of the transformation characterization.