Monte Carlo Solution of Boltzmann Equation for a Simple Model of Highly Nonequilibrium Diatomic Gases — Translational Rotational Energy Relaxation

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t  time
T  temperature
u  x component of velocity
v  y component of velocity
V  interaction potential, or velocity
w  z component of velocity
δ  index of power associated with point centers of repulsion model,
   (eq. (6)), \( V(R) = d/R^δ. \)
Λ  rotational level jump
Θ_r  characteristic rotational temperature
\( \Lambda \)  relaxation time, (eq. (16))
μ  reduced mass
ν  collision frequency
σ  effective collision diameter
\( \tau \)  characteristic collision time based on equilibrium translational
temperature, \( 1/(nπ\sigma^2c_o) \)
χ  deflection angle defined in equations (2) or (10).
Ω  solid angle

Subscripts:

e  equilibrium
i,j  rotational states
max  maximum value
r  rotational
t  translational
Superscript

\[ \vec{v}_j^{(k)} \text{ after } \hat{r} \text{ momentun} \]
MONTE CARLO SOLUTION OF BOLTZMANN EQUATION FOR A
SIMPLE MODEL OF HIGHLY NONEQUILIBRIUM DIATOMIC GASES

Translational Rotational Energy Relaxation

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SUMMARY

Theoretical studies of translational and rotational energy relaxation in
diatomic gases are described. The direct simulation Monte Carlo method is
employed to solve the Boltzmann equation for a rotationally excited highly
nonequilibrium gas. The gas investigated is homonuclear diatomic nitrogen,
and the semiclassical model of Itikawa is incorporated for the transition
probability that describes rotation-translation energy interchange.

The details of energy interchange between the translational motion and
the rotational energy levels of the gas are examined for spatially uniform
flow without boundary interactions (the "box" calculation) with a variety of
initial conditions. The results show:

1. The assumption that relaxation occurs via successive local Maxwellian
velocity distributions, which is a commonly used basis for finding approximate
solutions of Boltzmann equation, is not valid for gases that are initially in
highly nonequilibrium states. This is especially true for initial conditions
that involve low translational and high rotational temperatures.

2. The energy distributions for such transitions show bimodal (or double
peak) relaxation patterns; the secondary peak ("satellite peak") appears around
the Maxwellian elastic peak in the velocity distribution early during the
relaxation period. The secondary peak is due to inelastic collisions and is
analogous to the rotational Raman effect accompanying Rayleigh scattering.

3. The rotational energy distribution also shows bimodal relaxation
effects: In addition to thermal equilibrium Boltzmann peak, a weak peak also
appears at the high rotational energy levels. When the rotational energy

distribution is a delta function, however, relaxation proceeds only as a single-peak distribution. One, therefore, concludes that single- or double-peak relaxation depends on the type of initial distributions assumed.

4. Relaxation of the velocity distribution to equilibrium Maxwellian occurs relatively fast while the rotational energy relaxes more slowly. The relaxation time depends not only on equilibrium temperature, but also on initial velocity and rotational energy distributions.

Close correlation of the relaxation between the box models and fluid flows, such as, sound absorption, shock wave, and free-jet expansion experiment are described. Also presented are brief preliminary results of a shock wave showing translational and rotational energy relaxation structure.

A 16-mm movie film displays examples of the relaxation effects of the "box" model with a variety of initially specified velocity and rotational energy distributions.

INTRODUCTION

A knowledge of internal energy transfer mechanisms at the molecular level is valuable for an accurate understanding of many important nonequilibrium problems that occur in high-speed gas dynamics, acoustics, laser transmission, detonation, combustion, pollution, and atmospheric physics. For example, collision-induced rotational transitions play a major role in establishing the population inversions leading to gas-dynamic laser action, and also in evaluating the effects of highly nonequilibrium energy transfer in rarefied gas flow about spacecraft entering the planetary atmosphere.

In the present paper, several new and important results are presented on internally excited translation-rotation energy relaxation. These results are obtained by solving the Boltzmann equation by the Monte Carlo direct simulation method, which previously has been applied successfully to monatomic (without internal rotational relaxation) gas flow problems (refs. 1 to 3). An important feature of the simulation method is that it provides insight into the effects of collisional relaxation at the microscopic level. In particular, the instantaneous internal energy distributions can be continuously observed throughout the relaxation processes. To ensure that these distributions be meaningful, however, it is essential that the rotational transition probability function, used in the method, display certain features; namely, (1) probability must be conserved, (2) probabilities relating transitions to and from pairs of definite states must satisfy "detailed balancing," and (3) probabilities, when used in the simulations, must yield the correct asymptotic behavior of the distributions (refs. 4 and 5).

The Monte Carlo method itself can be described briefly as follows (see refs. 2 and 3 for details): The flow is determined by following a statistical sample comprising several thousand molecules that are allowed to collide with each other. The phase space coordinates that involve trajectory and rotational
variables are known at every instant. These coordinates are allowed to change only during a collision; the modeling of these intermolecular encounters is, of course, the essential part of an accurate simulation. To account for these sampling encounters, a molecule and a near neighbor are each selected at random as are also their impact parameter and relative orientation angles—all in a manner representative of typical molecules undergoing similar encounters. They are accepted for a collision or rejected according to a selection rule that is dependent on the collision cross section. Since the initial coordinates, that is, relative velocity, impact parameter, orientation angles, and pair of rotational quantum states, are known, the final rotational quantum states can be computed. This involves computing the transition probabilities of all quantum states that are accessible from the known initial states and then selecting randomly from this resulting distribution.

The procedure used here for the "translational" interactions parallels other investigations (refs. 1 and 3) which treat of monatomic gases only. The procedure is different, however, from those investigations that have treated of translation-rotation interactions. All investigations are easily categorized under the following descriptions: (1) semiempirical, (2) classical, (3) semiclassical, and (4) quantum mechanical. Within these categories, the semiempirical treatment includes an energy sink (ref. 6) and rough spheres and loaded spheres (ref. 7) to model the translation-rotation collision processes. While such methods do not appear to be satisfactory for highly nonequilibrium flow, they adequately describe near equilibrium steady flows. The classical models (refs. 8 and 9), although consistent with the classical direct simulation Monte Carlo procedure used here, necessarily include approximations to make the models tractable for studies of the type considered in this report. The approximations yield appropriate macroscopic behavior for a nonequilibrium example, but do not adequately provide limiting microscopic behavior. In particular, individual molecular encounters that violate energy and momentum conservation can occur.

Semiclassical methods (refs. 4, 5, and 10) appear to have physically realistic bases. The simplified model of Pearson and Hansen satisfies limiting equilibrium behavior, but, during a calculation, the model causes a drift in the answers that violates energy equipartition (ref. 4). Itikawa's model is more rigorously founded, allows for treatment of molecular collisions (ref. 5), and also satisfies conservation of probability and appropriate detailed balancing. Itikawa's model, therefore, satisfies the desirable characteristics of the ideal model that we described earlier; our investigations described in this paper depend on this model. Our intent, then, is to extend its application to even more general problems.

As regards the fourth category of the model (i.e., quantum mechanical models), the author is not aware that truly quantum mechanical results are yet viable. Such descriptions are difficult to obtain analytically, and to apply.

In this paper, we treat translation-rotation interactions for a spatially uniform gas far removed from solid boundaries. We are concerned only with a basic understanding of translation-rotation relaxation behavior in highly
nonequilibrium situations. In fact, it is our belief that the Monte Carlo method is best suited for studies of the type described in this paper.

The results presented are based on calculations involving three different basic types of initial conditions: (1) equilibrium, (2) nonequilibrium-equipartition (i.e., equipartition is satisfied, but distributions are perturbed), and (3) nonequilibrium-nonequipartition (i.e., both the equipartition and the distributions are perturbed). Also included are the results of monatomic gas simulations (rotational relaxation effects frozen) to assist comparisons with coupled translation-rotation relaxation simulations. To further assist the understanding of the Monte Carlo method, the essential mathematical relations are also given in this report.

FORMULATION AND PROCEDURE

The essence of the Monte Carlo procedure is described briefly in the Introduction. Introduced in this section are several analytical relations that assist both the understanding and use of the method. Appendices A and B provide supplementary analysis to the procedure and appendix C is a listing of the computer program that was used in the procedure.

Governing Equations

The study described in this paper concerns the temporal and spatial relaxation of the velocity and rotational energy state distributions that characterize a statistical sample representing several thousand molecules. If we assume that the molecular distributions themselves are diagonal and independent of the degenerate rotational \(m\) substates, the Boltzmann equation (or Wang, Chang and Uhlenbeck equation, see ref. 11) that relates the temporal and spatial behavior of the distribution functions \(f_i\) can be written

\[
\frac{\partial (nf_i)}{\partial t} + \nabla \cdot (\hat{v}_i f_i) = \sum_{i'j'} \int \int n^2 \left[ \frac{(2i + 1)(2j + 1)}{(2i' + 1)(2j' + 1)} f_{i'} f_{j'} - f_i f_{j'} \right] \frac{d\sigma}{d\Omega} d\Omega d^3v_2 \tag{1}
\]

where \(n\) is the number density, \(f_i = f_i(\hat{x}, \hat{v}, t)\) is the distribution function which depends on time \(t\), position \(\hat{x}\), velocity \(\hat{v}\), and rotational state \(i\), \(d\sigma/d\Omega\) is the differential cross section corresponding to solid angle \(\Omega\). External forces are assumed to be absent. The Monte Carlo procedure is used to effectively solve this equation by means of a probabilistic sampling procedure. Implicit within the equation and procedure are the conventional fluid dynamic conservation laws (i.e., conservation of mass, momentum, and energy; see, e.g., ref. 11).

Of greatest interest for the study given here is the "box" calculation wherein the gas is spatially uniform, has constant density, and is stationary; that is, the gas is entirely contained within an "imaginary box" that has unit
volume and noninteracting boundaries (sketch (a)). The procedure, in this case, concerns interactions in a closed system. Energy conservation is applied directly (i.e., exact energy conservation is imposed on the interacting pairs of molecules) and "random sampling" ensures that, over long periods of time, the number of molecules contained within the box remain constant. These concepts are treated in greater detail in the subsequent discussion.

The "box" calculation has general utility since such a calculation, when started with an appropriate set of initial conditions, provides insight into mechanisms in the more general flow situation as found in sound absorption, normal shock wave, or free jet expansion experiments (see table 1). In addition to the "box" calculation results, a simulation is also given for steady one-dimensional shock wave flow. This result is preliminary and demonstrates the capability of the method for simulating more complicated flows.

Collision Parameters

The essence of an accurate simulation is the random or probabilistic sampling used to select the interacting (colliding) molecular pairs, to determine whether a reaction occurs, to find the resulting "states," and then to advance the time interval for the next collision, and so on. To provide insight on this entire collision process and to arrive at a criterion for evaluating certain of the parameters required to define a collision, it is worthwhile to briefly review the classical representation of the equivalent process, and to observe how such relations depend on intermolecular potential.

A classical representation (ref. 7, ch. 8) is given by

\[ \chi(b, g) = \pi - 2 \int_{R_c}^{\infty} \frac{b \, dR/R^2}{\sqrt{1 - (b/R)^2 - V(R)/(1/2)\mu g^2}} \]

\[ Q^{(l)}(g) = 2\pi \int_0^{\infty} (1 - \cos \theta) \chi(b) \, db \]

where \( V(R) \) is a spherically symmetric intermolecular potential, \( \chi(b, g) \) is the encounter deflection angle, which depends on impact parameter \( b \) and on \( g \), relative velocity of approach, \( \mu \) is the reduced mass, \( R_c \) is the distance of closest approach, and \( Q^{(l)}(g) \) is the \( l \)th "momentum" transport cross section, which also depends on relative velocity (for studies reported here, \( l = 1 \)). The collision frequency, \( \nu \), is then given by
With this relation, the collision time, $\Delta t$, between encounters and the elapsed time, $t$, are given, respectively, by

$$
\Delta t = \frac{2}{N} \frac{1}{\nu}
$$

$$
t = \sum_i \Delta t_i
$$

where $N$ is the number of particles in the "box."

**Collision.**—For cases in which the intermolecular potential is inversely proportional to the power, $\delta$, of distance between colliding molecular pairs, we can readily calculate a frequency ratio (e.g., ref. 5)

$$
\left( \frac{\nu}{\nu_{\text{max}}} \right) = \left( \frac{g}{g_{\text{max}}} \right)^{(\delta-6)/\delta}
$$

where $\nu_{\text{max}}$ and $g_{\text{max}}$ are the maximum possible values in a cell. The dimensionless ratios of frequency and relative approach velocity are related through $\delta$. If the collision process can be represented by symmetric inverse power-law potentials, then equation (6) is a valid representation for all collisions and can be used as a criterion to decide whether an "encounter" is a "collision." The representation, therefore, serves as a "selection rule" for encounters.1

For the Monte Carlo results displayed in this report, we have arbitrarily picked intermolecular potentials with $\delta = 4$ (i.e., "Maxwell molecules").

Of all encounters that are collisions, we must further categorize those which are elastic from those which are inelastic (i.e., those which yield rotational transitions).

**Inelastic collision.**—Not all collisions result in a rotational transition. For example, some interacting pairs may have insufficient relative energy to

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1 Actual intermolecular potentials have a more complex behavior than the idealized $\delta$ potential upon which equation (6) is based. Relations equivalent to equation (6), but based on more accurate representations for the molecular potential, can be found in reference 12. The expressions were derived recently and, hence, were not available for the simulations described in this paper.
induce a transition. To separate such events from those which result in rotational transitions (i.e., inelastic collisions), we introduce a relation similar to that given above but with a different value for the exponent, that is,

\[
\left(\frac{v}{v_{\text{max}}}\right)_i = \left(\frac{g}{g_{\text{max}}}\right)^{C_i}
\]

where the subscript \(i\) denotes inelastic collision. The appropriate value to use for \(C_i\), however, is based on simulation results. We require that simulations, which start with Maxwell-Boltzmann distributions that satisfy equipartition, must yield nondrifting results. A value of \(C_i = 0.431\) yields this desired behavior. The procedure used to evaluate \(C_i\) is also described in reference 5, but in greater detail.

Collision Dynamics

In the previous section, the parameters required to determine the occurrence of a collision are given. In this section, we describe the procedure for finding the trajectories after a collision. A collision, of course, also can be accompanied by rotational transitions in either or both colliding pairs of molecules. These rotational transitions can also perturb the particle trajectories. In this section, we describe the relations that ensure collision symmetry (i.e., a collision is invariant with its inverse) and that enable rotational transitions to be more precisely determined.

The relative velocity and impact parameter after a collision are obtained by knowing the onset energy and momentum. The relations are given by

\[
(g')^2 = g^2 - \frac{2}{\mu} \left(\frac{E'_{r1} - E_{r1} + E'_{r2} - E_{r2}}{g'}\right)
\]

and

\[
b' = \frac{[gb - (M'_{r1} - M_{r1} + M'_{r2} - M_{r2})/\mu]}{g'}
\]

where \(E_r\) and \(M_r\) are the rotational energy and momentum before a collision and a prime distinguishes the corresponding values after a collision. We assume that rotational transitions only slightly perturb the relative velocity and impact parameter.

The deflection angle, given by equation (2), is actually dependent on the functional behavior of the intermolecular potential, but, for finding the limiting trajectories, we assume that the infinite-rise potential (i.e., a "billiard ball" collision) is adequate. The corresponding deflection angle is given by
\[\chi(b) = 2 \cos^{-1}\left(\frac{b}{\sigma}\right)\]  

(10)

where \(\sigma\) is the effective diameter of the rigid-spherical molecule. The slightly perturbing effect of inelastic collisions resulting from the rotational transitions is accounted for by the following average:

\[\bar{\chi}(b) = \frac{\chi(b) + \chi(b')}{2}\]  

(11)

To completely specify a collision, however, it is also necessary to give the orientation angle \(\varepsilon\), which references the collision plane with respect to some arbitrary coordinate plane (see, e.g., ref. 13, p. 36). The velocity components before and after the collision can be related (e.g., ref. 13)

\[g'_x = \frac{g'}{g} \left( g_x \cos \bar{\chi} - \sqrt{g^2 - g_x^2} \cos \varepsilon \sin \bar{\chi} \right) \]  

(12a)

\[g'_y = \frac{g'}{g} \left[ g_y \cos \bar{\chi} + (g_x g_y \cos \varepsilon + g g_z \sin \varepsilon) \sin \frac{\bar{\chi}}{\sqrt{g^2 - g_x^2}} \right] \]  

(12b)

\[g'_z = \frac{g'}{g} \left[ g_z \cos \bar{\chi} + (g_x g_z \cos \varepsilon - g g_y \sin \varepsilon) \sin \frac{\bar{\chi}}{\sqrt{g^2 - g_x^2}} \right] \]  

(12c)

We impose conservation of linear momentum to find the resulting velocity components after a collision. There results

\[
\begin{align*}
& u'_x = \frac{1}{2} (u_1 + u_2 + g'_x) \\
& v'_2 = \frac{1}{2} (v_1 + v_2 + g'_y) \\
& w'_2 = \frac{1}{2} (w_1 + w_2 + g'_z)
\end{align*}
\]  

(13a)

\[
\begin{align*}
& u'_1 = u'_2 - g'_x \\
& v'_1 = v'_2 - g'_y \\
& w'_1 = w'_2 - g'_z
\end{align*}
\]  

(13b)
Rotational Transition Probability

To describe inelastic collisions, also needed, in addition to the trajectory parameters introduced in the previous sections, are expressions which relate the probability of transition between initial and final rotational energy states (i.e., the rotational transition probabilities). As was pointed out in the introduction, the semiclassical probabilities derived by Itikawa (ref. 5) are used. A brief description of their properties follows.

We describe an interaction where rotational transitions occur from levels \((i,j)\) to \((i',j')\) by

\[
N_2(i) + N_2(j) = N_2(i') + N_2(j')
\]  

The collision trajectory itself, as described earlier, is determined classically: Given an analytical relation for the trajectory, the amplitude of the rotational transitions can then be determined from quantum mechanical considerations. By appropriately combining the trajectory with an expanded set of Schrödinger equations (e.g., see ref. 5), the amplitude of the rotational transitions can be obtained by solving a set of coupled differential equations. In order to reduce the rank of the system, the effective potential method of Rabitz (ref. 14) is employed. The method eliminates the dependence of the interaction matrix on the magnetic quantum number, \(m\). The resulting coupled set of ordinary differential equations are then solved by using the exponential approximation (see ref. 15). What is important is that the method treats an interaction regardless of its "strength," and, in addition, allows for the likelihood of all transitions, including those with multilevel jumps. The simultaneous transitions for both colliding molecules (i.e., rotation-rotation as well as rotation-translation) are also taken into account. The precise formulation used is given in reference 5.

Some important properties of the probabilities that pertain to the Monte Carlo simulation method are described briefly in appendix B. If a collision is inelastic as selected by equation (7), the transition of pair's molecular states are then determined by the Itikawa's rotational transition probabilities.

RESULTS AND DISCUSSION

In this section, Monte Carlo simulations are described for a stationary homogeneous molecular gas (i.e., for a "box" calculation). The simulations differ depending on the choice of the initial distribution functions (see Table 1). The initial conditions fall under three general categories: (1) complete equilibrium, (2) nonequilibrium, equipartition, and (3) nonequilibrium and nonequipartition. We use the term nonequilibrium here to denote that either the velocity distribution, \(f_v\), or the rotational energy distribution, \(f_{\gamma}\), are non-Maxwellian.
The first case, that is, complete equilibrium, tests the method, as well as the internal parameters, for self-consistency. The velocity and rotational energy distributions should remain constant for extended periods of calculation. That is, the internal energy distributions should remain Maxwellian and equipartition should be inviolate (e.g., refs. 4 and 5).

The second case, where the velocity and rotational energy distributions are specified to satisfy energy partition (i.e., the fraction of energy distributed between translation and rotation is proper, but where the distributions, themselves, are non-Maxwellian) provides a test on whether the procedure has an internal driving mechanism that will yield a relaxation to equilibrium within a physically realistic time.

The third set of initial conditions, where the initial distributions violate both equipartition and are highly nonequilibrium states, allows-even more complex investigations. For example, one can study the relaxation processes to equipartition as well as how the velocity and rotational distributions interact during the relaxation. In effect, these simulations with varied starting conditions give qualitative information on the coupling of the energy distributions and quantitative data on the rates of relaxation.

In table 1, the specific choice of initial conditions for the simulations described in this paper are listed. These results are also useful in providing qualitative information that can be used to interpret results in several equivalent experiments. The experiments are sound absorption, shock-waves, and free-jet expansions (ref. 11). Table 1 lists the simulation and the related experiment type. The simulations are described in the discussion that follows.

Initial distributions: complete equilibrium.—The first test of a good method for simulating solutions to Boltzmann's equation is that Maxwellian energy distributions, both in velocity and in rotational energy, not change for extended calculation periods. In figure 1a are given the results of such a simulation. The results show sets of paired figures for progressively increasing times corresponding to \( t/T = 0.0, 1.0, 5.0, \) and 10.0. One figure in the pair is a snapshot of the rotational energy distribution function, \( f_r \), plotted versus rotational energy level \( j \) at a definite time \( t/T \), and the other figure is the velocity distribution function, \( f_t \), plotted versus velocity \( c/c_0 \), where \( c_0 \) is the most probable molecular speed defined by \( c_0 = \sqrt{2kT_e/m} \), \( m \) is the molecular mass, and \( T_e \) is the equilibrium temperature associated with the "box" model. We observe that, although small fluctuations occur around the dotted curves (which represent the true Maxwellian distributions) during the calculation period, these fluctuations do not grow (ref. 5). In fact, figure 2b shows the results of the same calculation, but where the time average of the distributions, given by

\[
\bar{f}_r = \frac{1}{t} \int_0^t f_r dt
\]  

(15)
are plotted. We observe that the fluctuations are negligible in the second group of "snapshots." These simulations illustrate that, indeed, the procedure is stable over long calculation periods. The next case tests the capability of the procedure to drive arbitrarily specified initial distributions to the Maxwellian limit.

**Initial distributions: non-Maxwellian velocity and rotational effects frozen.**—The Monte Carlo method allows for considerable flexibility regarding the precise specification of the initial distributions. For example, one can freeze the rotational relaxation effects and investigate only the relaxation of the velocity distribution. The next example is of this type. In figure 2 are displayed the resulting time history for relaxation of the velocity distribution function, starting with two different initial distributions. In figure 2a are displayed the relaxation processes that correspond to initially letting every molecule have a speed equal to $\sqrt{3/2} c_0$. The dotted curve is a Maxwellian distribution characterized by the temperature $T_e = 320$ K. In this example, the "Dirac delta function" type of initial distribution should relax to coincidence with the dotted curve. The rotational energy, of course, is ignored. We observe that the distributions are largely equilibrated by the instant $t/\tau = 1$ (the area differences between the solid and dotted curves correspond to the number of molecules that still have initial velocity $\sqrt{3/2} c_0$ and remain to be "equilibrated"—i.e., about 10 percent of the total). At $t/\tau = 2.0$, the distribution is established and very little change occurs thereafter. One concludes from this simulation that the procedure leads to the correct Maxwellian limit, as indeed it should. In figure 2b, the initial distribution is slightly different. In this case, the energy, corresponding to $kT_e$, is distributed at two separate initial velocities: $c_0/2$ and $\sqrt{11/2} c_0$. The rotational energy is managed in the same manner as the example in figure 2a. The result for this case is nearly the same. In fact, little difference can be observed in a comparison of the relaxation history. At the instant $t/\tau = 1.0$, roughly the same fraction of molecules remain to be equilibrated as in the first example. The distribution appears to be established by the instant $t/\tau = 2.0$ and changes very little thereafter.

**Initial distributions: Maxwellian velocity and equipartition.**—Our next simulation, figure 3, illustrates the relaxation effects that occur when the initial velocity distribution is Maxwellian and the rotational distribution, which satisfies energy equipartition, approximates a delta function centered at $j_0 = 10$ (i.e., every molecule has $kT_e$ rotational energy in the 10th energy level). This rotational level also represents the probable rotational energy level for a Boltzmann distributions at temperature $kT_e$ (i.e., $j_0$ is found from $k\theta_r j_0(j_0 + 1) = kT_e$, where $\theta_r(N) = 2.9$ K and $T_e = 320$ K).

Since our investigation concerns homonuclear nitrogen, only rotational transitions that satisfy the multiples of $\Delta j = \pm 2$ are allowed. At the first instant displayed in figure 3 after relaxation begins (i.e., at $t/\tau = 0.5$), we observe a double peak appearing in the velocity distribution. This behavior is very similar to the Stokes and anti-Stokes lines that appear in Raman scattering (ref. 16). The position of these peaks can be calculated (see appendix A) and appear at the velocities $c/c_0 = 0.91$ and 1.1, respectively.
These peaks appear because the rotational energy is "dumped" into a narrow energy band. The transitions, \( \Delta j = \pm 2 \), are most probable compared to multilevel transitions, \( |\Delta j| \geq 4 \). This effect results in two perturbations appearing in the velocity distribution about the most probable velocity \( c_0 \).

**Initial distributions:** Maxwellian velocity \((T_r > T_e)\) and nonequilibrium.

The simulation displayed in figure 4 demonstrates the relaxation effects caused by violating equipartition. Rotational energy states are assumed initially to be unexcited; the energy is contained entirely within the velocity distribution which is Maxwellian \((T_r = 534 \text{ K})\). The fact that single level \((\Delta j = \pm 2 \text{ for homonuclear molecules})\) rotational transitions are the most likely compared to multilevel \((|\Delta j| \geq 4)\) transitions is also apparent here. After relaxation begins, we observe that the lowest level rotational states populate first. The gain in rotational energy appears to be at the expense of the molecules with velocities that correspond to the most probable velocity \(c_0\) or higher. As the time \(t/\tau\) increases, energy continues to be "dumped" from the translational to the rotational mode as demonstrated by the downward drifting velocity distribution and the upward drifting rotational distribution. These processes become less efficient as energy is "dumped" to higher and higher rotational energy levels. This is apparent because, as the width of the energy level increases, the energy interchange between the rotational and translational mode becomes even less efficient. We find, then, that considerable time is required to populate the uppermost rotational energy levels. In fact, the velocity distribution appears to be nearly equilibrated by the time \(t/\tau = 20.0\), while the rotational distribution is still relaxing at \(t/\tau = 50.0\).

The simulation demonstrates that the step-wise populating mechanism implicit within the Itikawa model leads to relatively slow relaxation to a Boltzmann rotational distribution. Of course, if multiple level transitions were more effective, the rate of relaxation to a Boltzmann rotational distribution would be greatly enhanced. These features are characteristic of translation-rotation transitions, and they are apparent in all simulations involving translation-rotation interactions.

The simulation displayed in the next figure, figure 5, differs from this example in that rather than follow the populating of the rotational energy levels from an initial state of "excessive" translational energy, we follow the depopulating of the rotational energy levels from an initial state of "excessive" rotational energy. Of course, "excessive" refers to the manner in which that energy is initially distributed relative to the distribution that satisfies equipartition.

**Initial distributions:** Maxwellian velocity \((T_r < T_e)\) and nonequilibrium.

In figure 5, we observe that, at the initial instant prior to relaxation, the rotational energy is stored in a Boltzmann distribution \((T_r = 793 \text{ K})\) which peaks near the most probable levels \(j = 10 \text{ or } 12\) (i.e., \(f_r(x)\) is maximum at \(x = 11.16\)). The velocity distribution, however, peaks at low velocities, \(c^*/c_0 = 0.135\).

Several interesting features can be observed during the relaxation processes. We note that a satellite peak (refs. 16 and 17) develops on the high
sides of the velocity distribution (i.e., $c/c_0 \approx 0.44$), and this peak forms at the expense of rotational energy near the most probable levels $j = 10$ or $12$ (as exemplified by the "dip" in the rotational distribution). This peak continues to grow as rotational energy is converted to thermal motion. Again, one observes the effect of the inefficient coupling of translational and rotational energy interchange with the higher rotational energy levels. This effect, of course, results in the appearance of a second peak at the higher rotational levels. This effect of double peaks is also discussed by Polanyi and Woodall (ref. 18). The peaks remain in the distribution until quite late during the relaxation process (say, $t/\tau = 50.0$). We also notice that, by this time, the velocity distribution has nearly equilibrated to Maxwellian, and this occurs before the rotational distribution becomes Boltzmann, similar to what occurred in the previous example, figure 4.

Also interesting is a comparison of how close the distributions approximate the Maxwell-Boltzmann distribution during each instant of the relaxation processes. Such comparisons are displayed in figure 5b. Here, the dotted curves are "local Maxwell-Boltzmann" distributions rather than the asymptotic limiting equilibrium distributions displayed heretofore. The dotted distributions are determined by matching the energy, in both the velocity and rotational modes, with the simulation results. These results illustrate that the actual distributions found by the simulation deviate significantly from local Maxwell-Boltzmann distributions. This demonstrates that the popular methods that rely on expansion procedures involving local Boltzmann distributions for solving the Boltzmann equation can be unreliable. In fact, the double-peaked results displayed in figure 5 illustrate that appropriate distributions can have rather complex non-Boltzmann functional behavior.

Initial distributions: Maxwellian velocity ($T_v < T_e$) and nonequipartition.—The simulation displayed in figure 6 is very similar to that displayed in figure 3; the initial rotational energy distribution approximates a Dirac delta-function, but the simulation differs in that the constant rotational energy assigned each molecule violates equipartition. Here we have an initial dumping of the rotational energy into the 16th rotational energy level. The velocity distribution, however, is Maxwellian. At the onset of relaxation, we observe the "anti-Stokes" Raman scattering effect appearing in the velocity distribution, that is, the appearance of a satellite peak (using the relations in appendix A with $j = 16$ and $\Delta j = -2$, we find the peak location to be $c/c_0 = 0.55$). The pumping mechanism by which the high rotational energy is converted into thermal motion also appears conspicuously in this example (see ref. 17). The single step transitions from the 16th to 14th level ($\Delta j = -2$) occur first with the quanta of rotational energy being preferentially absorbed by the very low-speed molecules. As this process proceeds, the number of molecules in the 14th level approaches equality with the number in the 16th level. Simultaneously, the "anti-Stokes Raman" peak broadens as the slow speed molecules are "pumped" into this region. As the relaxation progresses, the lower lying rotational levels are successively populated while the "anti-Stokes Raman" peak becomes increasingly broad.

Contrary to the previous example, figure 5a, the rotational distribution relaxes continuously with one one peak. The velocity distribution, however,
To show the manner by which the local distributions compare with local Maxwell-Boltzmann distributions during the relaxation processes, the dotted curves are again introduced in figure 6b, as was done in figure 5b. Again, we observe that the distributions do not approximate Maxwell-Boltzmann distributions during each instant of the relaxation.

Relaxation of Average Rotational Energy

The results displayed in figures 1 through 6 illustrate, in particular, the manner by which an initial energy distribution relaxes to the final Maxwellian distributions. Also interesting is the manner by which the "average energy" approaches some asymptotic constant value. Such results are given in figure 7. In this figure are displayed four curves. Three curves display the average energy relaxation associated with initial distributions, which are Dirac delta-function type, and the fourth displays results with an initial distribution that is Boltzmann with high rotational temperature. The essential feature is the comparison of results between curves that have high and low initial rotational energy (e.g., cases 4 and 5). Because the coupling between translation and rotation is efficient for the lower levels, the slope of these curves is greatest. Also, interesting is case 3 which illustrates that even though the initial distribution satisfies energy equipartition, the system is not bound to satisfy equipartition during the subsequent instants as the rotational distribution asymptotically approaches a Boltzmann distribution. Additional simulations all illustrate the downward shift (as illustrated in the figure for \( j = 10 \)) followed by the upward relaxation to "equipartition."

Figure 7 can also be used to define a useful relaxation time that characterizes the simulation results. Such a definition, of course, is not exactly clear because the relaxing curves are not exponential. One can resort to the definition given by (e.g., ref. 11)

\[
\Lambda = \frac{\left(E_r\right)_e - E_r(t)}{dE_r/dt}
\]  

This definition, however, is impractical when the energy difference in the bracket is small. One can also define the relaxation time to be that time when the bracket expression has reduced to \( 1/e \) of its initial value (e.g., ref. 13). On the basis of this latter definition, it turns out that \( \Lambda \approx 32 \) is satisfactory for both curves labeled \( j = 0 \) and \( j = 16 \). This value also seems to be consistent with the simulation displayed for all three initial functions of rotational energy distribution (i.e., \( j = 0, 10, \) and 16) in figures 3, 4, and 6. In these figures, we can see that the relaxation appears to be nearly ceased at the same instant between the two displays of the distribution function at \( t/\tau = 20 \) and 40.
The relaxation of translational energy is not shown here since, for our "box" calculation, conservation of energy $E_t$ is directly related to the average rotational energy $E_r$ via the relation

$$\frac{E_t}{(E_t)_e} = \frac{5}{3} - \frac{2}{3} \frac{E_r}{(E_r)_e} \quad (17)$$

at each instant during the relaxation processes. The subscript $e$ denotes the asymptotic limiting energies associated with equilibrium.

**Shock Wave Structure**

The previous examples, figures 1 through 6, rely on the "box" model, which is based on the assumption that the distributions have no spatial dependence. The Monte Carlo method, of course, has potential for much greater generality. We can effectively introduce a spatial dependence into the distributions and study more complicated problems. To demonstrate this effect, results of simulations for a normal shock wave structure are displayed in figure 8. For this example, the number of molecule in the sample size was not increased and, therefore, the curves are not exactly smooth.

In this figure are displayed the translational and rotational temperatures (based on average energy) and density at seven distinct instants of time. As one might expect, the translational temperature develops an overshoot. As the rotational mode is excited the high translational temperature decreases and approaches an asymptotic steady value.

This example is included to demonstrate that such simulations that involve both elastic and inelastic collisions are possible. More refined shock shapes than those displayed in figure 8 will require a considerable increase in the number of molecules within the statistical sample and in computation time, thus, no attempt has been made to check the convergence of the solution.

**CONCLUSIONS**

The Monte Carlo simulation method described in this report, including the use of the Itikawa model for representing inelastic collision processes, is a viable scheme for studying translation-rotation interactions. The method can provide very useful qualitative and quantitative information on the relaxation processes associated with at least relatively simple topological systems (i.e., one-dimensional and quasi one-dimensional systems). On the basis of experience gained here, it is not expected that the method will be considered viable at this time for more complex topological studies (i.e., three-dimensional flow simulations), because current and foreseeable computer resources appear insufficient to allow economical processing of the increased sample size that will be required in such studies.
The method, however, is very useful in its present form to visualize fundamental gas kinetic behavior as demonstrated by the results presented in this paper. A review of the simulations given in this report shows the following results:

1. Single step ($\Delta j = \pm 2$ for homonuclear molecules) transitions are the significant mechanisms of intermodal energy transfer rather than the multistep transitions (i.e., $|\Delta j| > 4$ for homonuclear molecules).

2. The coupling of translation-rotation transitions is the most efficient for low lying rotationally excited molecules and is least efficient for the highly rotationally excited molecules.

3. The "relaxation time" required for molecules to reach an asymptotic steady-distribution in both the velocity and rotational states is dependent on the initial distributions.

4. Relaxation occurs via a successive set of distributions that are not Maxwell-Boltzmann (nonlocal Maxwellian).

5. Initial rotational distributions with high rotational energy and that are far removed from satisfying equipartition lead to the appearance of "satellite peak" on the velocity distribution via a mechanism that is similar to the Stokes Raman effect accompanying the Rayleigh scattering.

Subsequent studies should quantitatively compare characteristic relaxation times found by the Monte Carlo methods with similar times obtained experimentally. Of course, only qualitative comparisons are given here.

The simulations reported in this paper certainly demonstrate that the method is viable for studying translation-rotation interaction processes and that some revisions are necessary in existing analytical methods.
Certain types of rotational energy distributions can couple strongly through the collision processes to perturb the velocity distributions. In these cases, peaks occur (on the velocity distributions) that are analogous to the Stokes and anti-Stokes rotational Raman effects which accompany Rayleigh scattering. The satellite peaks have been observed experimentally (refs. 16 and 17). In one case (example 1) energy is "dumped" into a narrow band of the rotational energy levels; a pair of "satellite peaks" then appear around the maximum in the velocity distribution during subsequent relaxation. In another case (example 2) a similar effect occurs when the velocity distribution has a peak at low velocities ("cold gas") and the rotational energy is peaked at the higher levels. Here, however, only one "satellite peak" appears in the resulting velocity distribution. The nature of these peaks is such that their position can be readily estimated without solving the Boltzmann equation.

The relative velocity, \( g' \), of a pair of molecules after a collision can be found from equation (8) and is given by

\[
g' = g \sqrt{1 - \frac{\Delta E_r}{(1/2 \mu g^2)}} \tag{A1}
\]

where \( \Delta E_r \) is the change in rotational energy during a collision, and \( \mu \) is the reduced mass. We assume that only one of the pair of colliding molecules transfers rotational energy during the interaction; the rotational transitions correspond to \( j \rightarrow j \pm \Delta \), that is,

\[
\Delta E_r(j \rightarrow j \pm \Delta) = E_r(j \pm \Delta) - E_r(j) \\
= \Delta \cdot (\Delta \pm (2j + 1))k \theta_r \tag{A2}
\]

where \( k \) is Boltzmann's constant, and \( \theta_r \) is the characteristic rotational temperature which corresponds to a single transition. We introduce the following notation

velocity corresponding to the initial Maxwellian peak: \( c^* \)
velocity after rotational-translational interaction: \( c' \)
reference velocity: \( c_0 = \sqrt{2kT_0/m} \)

and, in addition, the approximations \( g \simeq 2c^* \) and \( g' \simeq 2c' \); we then obtain
\[ \frac{c'}{c_0} \approx \frac{c^*}{c_0} \sqrt{1 - \frac{\Delta E_r}{kT_e} \left[ 1 + 2 \left( \frac{\mu (c^* c_0)}{\mu c_0^2} \right) \right]} \]

\[ = \sqrt{\frac{T_1}{T_e}} \sqrt{1 - \frac{\Delta}{2} (\Delta \pm (2j + 1)) \frac{\theta_e}{T_1}} \quad (A3) \]

where we have used \( \mu = m/2 \) and \( (c^*/c_0)^2 = T_1/T_e \). We recall for homonuclear diatomic molecules, such as molecular nitrogen, that multiples of \( |\Delta| = 2 \) rather than single transitions \( |\Delta| = 1 \) are allowed. In the analogy with Raman scattering, the positive sign corresponds to "Stokes" and the negative sign to "anti-Stokes" effects. In the discussion above, example 1 displays both Stokes and anti-Stokes effects; example 2, however, shows only anti-Stokes effect.
APPENDIX B

MODIFIED ROTATIONAL TRANSITION PROBABILITY

In the Introduction, we listed several properties that the rotational transition probabilities must satisfy to ensure their proper behavior for the Monte Carlo simulation method: (1) probability must be conserved, (2) probabilities relating transitions to and from pairs of definite states must satisfy "detailed balancing," and (3) the probabilities, when used in the simulations, must yield the correct asymptotic behaviors of the distributions. The third property has been covered in the text (also, see refs. 4 and 5). The first property, that is,

\[ \sum_{i', j'} P(i, j \rightarrow i', j'; g) = 1 \]  \hspace{1cm} (B1)

is satisfied by Itikawa's relations (refs. 5 or 11) as is also collision symmetry, given by

\[ P(i, j \rightarrow i', j'; g) = P(i', j' \rightarrow i, j; g') \]  \hspace{1cm} (B2)

To ensure the satisfaction of the second property listed above, we introduce the modified transition probability, \( \hat{P} \), given by

\[ \hat{P}(i, j \rightarrow i', j'; g) = A(g) \frac{B(i, j; i', j')}{(2i + 1)(2j + 1)} P(i, j \rightarrow i', j'; g) \text{ for } (i, j) \neq (i', j') \]  \hspace{1cm} (B3)

This relation also satisfies the "principle of detailed balancing" given by

\[ (2i + 1)(2j + 1) \hat{P}(i, j \rightarrow i', j'; g) = (2i' + 1)(2j' + 1) \hat{P}(i', j' \rightarrow i, j; g') \]  \hspace{1cm} (B4)

where \( A = A(g) \) and the symmetric function \( B(i, j; i', j') \) are arbitrary relations that have functional behaviors as indicated in the parentheses. Several example relations of \( B(i, j; i', j') \) are

\[ B(i, j; i', j') = \left[ (2i + 1)(2j + 1) \right]^a \] \hspace{1cm} (B5)

or

\[ B(i, j; i', j') = \left[ (2i + 1)(2j + 1)(2i' + 1)(2j' + 1) \right]^{1/2} \] \hspace{1cm} (B6)

The notation \( i^< \) is used to designate the smaller value of either \( i \) or \( i' \), and similarly for \( j^< \).
Note that equation (B3) is obtained from equation (B4), since $g \preceq g'$ implies $A(g) \simeq A(g')$. Itikawa's expression derived in reference 5 is a specific example of the more general probability relation that is displayed here (e.g., Itikawa assumes $A(g) = 1$ and $a = 1$ in equation (B5)). It can be shown that the modified transition probability given by equation (B3) also satisfies "conservation of probability" and "detailed balancing."

It is worthwhile to comment that equation (7) in the text, which is the selection rule for inelastic encounters, was based somewhat on heuristic arguments and yielded qualitatively satisfactory results. We expect, however, that more accurate representations that will be based on more convincing physical arguments, which will involve $A(g)$ above, can be obtained for this equation.
APPENDIX C
PROGRAM LISTINGS

The entire program for the "Gas in an Imaginary Box" calculation is listed in this section.

Program listings consist of sample control cards, correction cards, main program listings, and sample input-data cards. Many unused cards are still in the listings, but are marked by a comment symbol "c," "c*," etc.
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* * *
SAMPLE CONTROL CARDS
* *
YOSHIKAWA, T500.
ACCOUNT, STGGKYY, T4606.
AUDIT, ID=YOSHIKAWA.
ATTACH, TAPE11, FILE702, ID=YOSHIKAWA, PW=STGGKYY, MR=1, CY=2.
ATTACH, OLDPL, ITILIBSOURCE, ID=YOSHIKAWA, MR=1.
UPDATE, F.
FTN, I, R=3, OPT=2, PL=100000.
ATTACH, IMSL, IMSL LIB, ID=AMESLIB.
LIBRARY (IMSL)
REQUEST, TAPE9,*PF.
REQUEST, TAPE10,*PF.
LDSET, MAP=X.
LOAD, LG0.
NOGO, MAIN.
RETURN, LG0.
MAIN.
CATALOG, TAPE9, DXX903, ID=YOSHIKAWA, PW=STGGKYY, MR=1, RP=999, CY=2.
CATALOG, TAPE10, DXX103, ID=YOSHIKAWA, PW=STGGKYY, MR=1, RP=999, CY=2.
AUDIT, ID=YOSHIKAWA.
EXIT.
CATALOG, TAPE9, DXX903, ID=YOSHIKAWA, PW=STGGKYY, MR=1, RP=999, CY=2.
CATALOG, TAPE10, DXX103, ID=YOSHIKAWA, PW=STGGKYY, MR=1, RP=999, CY=2.
AUDIT, ID=YOSHIKAWA.
"

* *
* *
* *
PROGRAM LISTING
* *
* IDENT, CORRECT
* 1 INI$$, 18
C OTHER RANDOM GENERATION
  KRN=50
  DO 50 J=1,KRN
  R=RANF(0)
  50 CONTINUE

*DECK MONT$$
PROGRAM MONTEC(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7,TAPE8,
TAPE9,TAPE10,TAPE11)
COMMON /TIME/ TO,TS,TF,TM,TDF,TDM,TN
COMMON /CV/ MAX,MAX6,CI,ROH1
COMMON /CONST/ W,A,VOM,SF,CO
COMMON /RANDOM/ R
COMMON /ANSWER/ MOM,T(4),G,DVOL,AI,POA,ROT1
COMMON /PART/ P(5001)
COMMON /CV/ ABCC(40,40,9)
DATA ABCC/14400*0./

C

MONTE CARLO PROGRAM FOR GASES IN A BOX -REVISED BY K.K. Yoshikawa

C

JUMP=0
1 CALL INPUT(JUMP)
   IF(JUMP.GT.0) GO TO 30
10 CALL INITIAL
   CALL MOMENT
   CALL OUTPUT(M)
30 IF(TN.GE.TM) GO TO 100
50 CALL JPAIRS
100 TM = TM + DTM
200 IF(TM.LT.TS) GO TO 30
209 CALL MOMENT
330 IF(TM.LT.TO) GO TO 30
   CALL OUTPUT(M)
   TO = TO + DTn
340 IF(TM.LT.TF) GO TO 30
ENDFILE 10
   CALL EXIT
   STOP
END

*DECK INP$
SUBROUTINE INPUT(JUMP)
COMMON /TIME/ TO,TS,TF,TM,TDF,TDM,TN
COMMON /CV/ MAX,MAX6,CI,ROH1
COMMON /CONST/ W,A,VOM,SF,CO
COMMON /RANDOM/ R
COMMON /ANSWER/ MOM,T(4),G,DVOL,AI,POA,ROT1
COMMON /PART/ P(5001)
DIMENSION FC(40),FW(40)
C
C DATA INPUT FOR MONTE CARLO PROGRAM ALA BIRD
DIMENSION HED(18)
READ(5,500) HED
READ(5,501) MAX, RH01, CO, W, A, VOM, MOM, DTM, TS, DTO, TF, D, R
10 MAX6=5*MAX
    AI=.25*W*D**2
120 POA=SQRT(3.14159/A)
200 C1=CO
    DVOL = 0.5*(W/(RH01*A)**2
    SF = RH01*DVOL/(MAX*W)
    WRITE(6,600) HED
    WRITE(6,601) MAX, RH01, DTM, TS, DTO, TF, CO, W, A, VOM, SF, D
500 FORMAT( 18A4)
501 FORMAT( 14/ 1OX5El0.4,13/( 4EI0.4))
600 FORMAT(IHI 18A4)
601 FORMAT( I4/ 10X5E10.4, I3/( 4E10.4))
If(MAX.LT.0) GO TO 300
RETURN
300 READ(S) MOM, T, G, DVOL, W, A, VOM, SF, CO, TO, TS, TF, TM, DTO, DTM, TN, MAX, MAX6
1, C1, RH01, P, AI, ROT1, POA, ABC, R
REWIND 8
NTM0=(TM-DTM)/DTO*0.5
100 READ(7) TAUBAR, NTM, MAX, COC2, EOE, FC, FW
WRITE(10) TAUBAR, NTM, MAX, COC2, EOE, FC, FW
If(NTM.LT.NTMO) GO TO 100
TF=TF+100.*DTM
JUMP=1
RETURN
END

*DECK INI**$  
SUBROUTINE INITAL
COMMON /TIME/ TO, TS, TF, TM, DTO, DTM, TN
COMMON /CV/ MAX, MAX6, C1, RH01
COMMON /CONST/ W, A, VOM, SF, CO
COMMON /RANDOM/ R
COMMON /ANSWER/ MOM, T(4), G, DVOL, AI, POA, ROT1
COMMON /PART/ P(5001)
COMMON/INI1X2/ ERO

C
C INITIAL VALUE SUBROUTINE FOR EQUILIBRIUM BOX
C
C OTHER RANDOM NUMBER GENERATION
C  KRAN=10 50 OR 100 ETC.
C  DO 50 JR=1,KRAN
C  R=RANF(0)
C 50 CONTINUE
  ERO=0.5*W*C1*C1
  JO=SQRT(1.+99401E16*ERO)/2.
C  CM=0.7071068*C1 (1/SQRT(2))*C1
C  ISEED=41111111
C  TN=0.
  P(1)=0.
100 T(IM)=0.
100 T(IM)=0.
C  CALCULATION OF INITIAL VALUES
C  I=0
C 130 P(I)=P(I+1).
C  DO 200 L=2,4
C* CALL RANDJ(Ix, IY, R)
C* IX=IY
C  IL=I+L
C*200 CALL GAUURND(IX,CM, 0. , P(II)).
C 200 P(I)=CM*GNOF(I•••SEED)
C  ROTATIONAL ENERGY AT UPSTREAM CONDITIONS
C  CALL XDIST(IROT)
C  P(I+6)=IROT
C 400 P(I+5)=SQRT((P(I+2)**2+P(I+3)**2+P(I+4)**2)
C  I=I+5
C  IF(I .GT. MAX6) GO TO 500
C  GO TO 130
C  READ INITIAL VALUES FROM THE FILE 11
C READ(11) P
500 CONTINUE
  DTO=DTO*DTM*.99999
  TS=TS*DTM*.99999
  TF=TF*DTM*.99999
  TO=TS+DTO
  TM = DTM
  RETURN
END

*DECK JPA$$
SUBROUTINE JPAIRS
COMMON /TIME/ TO,TS,TF,TM,DTO,DTM,TN
COMMON /CONST/ W,A,VM, SF, CO
COMMON /RANDOM/ R
COMMON /ANSWER/ MOM,T(4),G,DVOL, AI, POA, ROT1
COMMON /PART/ P(5001)

COLLISION PAIR SELECTION FOR THE INVERSE POWER POTENTIAL PARTICLES

VP=.0
10 VM=P(5)
NC=P(I)
20 DO 40 I=2,NC
J=5*I
IF(P(J).GT.VM) GO TO 35
IF(P(J).GT.VR) VR=P(J)
GO TO 40
35 VR=VM
VM=P(J)
40 CONTINUE
50 GM=VM+VR
60 R=RANF(0)
80 J1=P(I)*R+1
90 R=RANF(0)
110 J2=P(I)*R+1
120 IF(J1.EQ.J2) GO TO 90
J1=5*J1-3
J2=5*J2-3
150 G=SOR((P(J1)-P(J2))**2+(P(J1+1)
1-P(J2+1)**2+(P(J1+2)-P(J2+2))**2)

PAIR SELECTION RULE: IF F(G/GM) > P, TAKE A PAIR
FOR GENERAL CASE USE CC1 THROUGH CC3
F(G/GM)=(G/GM)**C
R=RANF(0)
FGRAR=(G/GM)**0.4310
KELST=0
KELST=1 FOR MONATOMIC GAS
IF(P.LT.FGRAR) GO TO 170
CC1 CELST=0.215 FOR EXAMPLE
CC2 FGEL=(G/GM)**CELST
CC* THIS PROGRAM IS SET UP FOR MAXWELLIAN MODEL CASE (FGEL=1.0)
KELST=1

END OF PAIR SELECTION RULE
170 CALL CRASH(P(J1),P(J1+1),P(J1+2),P(J1+4),P(J1+1)
1,P(J2+1),P(J2+2),P(J2+4),KGP,KELST)
IF(KGP.GE.1) GO TO 60
P(J1+3)=SOR(P(J1)**2+P(J1+1)**2+P(J1+2)**2)
P(J2+3)=SOR(P(J2)**2+P(J2+1)**2+P(J2+2)**2)
260 DTN=2.*PDVL/(P(1)**2*A*G*SF)
270  TN = TN + DTN
280  IF(TN.LT.TM) GO TO 60
290  RETURN
END

*DECK CRASS*
SUBROUTINE CRASH(U1,V1,W1, I1,U2,V2,W2, I2,KGP,KELST)
COMMON /CONST/ W,A,VOM,SO,CO
COMMON /RANDOM/ R
COMMON /ANSWER/ MOM,T(4),G,DVOL,AI,POA,ROT1
COMMON/PRBOUT/PSUM(400),RRRR,J1FIN,J2FIN
REAL I1,12

C DIATOMIC PARTICLE COLLISION TRAJECTORIES
C *ELSTIC COLLISION ; GO TO 700
IF(KELST.EQ.1) GO TO 700
G2=G*G
IF(l.GT.38..OR.I2.GT.38.) GO TO 7
ETP=0.25*W*G2
C EXCLUSION OF NO ENERGY TRANSITION AT LOW KINETIC ENERGY
GO TO 1
7 KGP=1
RETURN
1 R=RANF(O)
EPS=6.28318*R
KGP=0
ER1=4.0241E-16*I1*(I1+1)
ER2=4.0241E-16*I2*(I2+1)
RM1 =SQR(2.*AI*ER1 )
RM2 =SQR(2.*AI*ER2 )
R=RANF(O)
50 V02=(VOM**2)*R
VO=SQRT(V02)
60 CONTINUE
IF(MOM.LT.100) GO TO 100
R2=VO2*A/3.14159
GO TO 110
100 B2=VO2*A/G**(4./MCM)
C C ROTATOR MODEL
C C TRANSITION PROBABILITY FOR ROTATOR 1
C C TRANSITION PROBABILITY FOR ROTATOR 2
110 CONTINUE
BB=SORT(R2)
L10=I1+.0.1

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L20 = 12 + 0.1
120 R = RANF(0)

CALL LINKMC(L10, L20, BB, ETR)
J1 = J1FIN
J2 = J2FIN

J1 = 2 * J1 - 2 + MOD(L10, 2)
J2 = 2 * J2 - 2 + MOD(L20, 2)

******

ER1P = 4.0241E-16 * J1*(J1+1)
ER2P = 4.0241E-16 * J2*(J2+1)

GP2 = (G2 - 4. * (ER1P + ER2P - (ER1 + ER2))/W)

IF (GP2 LT 0.0) GO TO 150
GO TO 160

150 IF (KGP .GE. 1) RETURN
KGP = KGP + 1
GO TO 10

160 GP = SQRT(GP2)
RM1P = SQRT(2.*AI*ER1P)
RM2P = SQRT(2.*AI*ER2P)

BP = (G*BB + 2.*RM1P + RM1P - RM2P)/W)/GP

IF (BP .LT. 0.0) GO TO 10
GO TO 710

700 G2 = G*G
701 R = RANF(0)
EPS = 6.28318*R
710 R = RANF(0)
750 VO2 = (VOM**2)*R
760 CONTINUE

IF (MOM .LT. 100) GO TO 800
82 = VO2*A/3.14159
GO TO 810

800 B2 = VO2*A/G***(4./MOM)
810 CONTINUE
BB = SQRT(B2)

860 GP = G
905 BP = BB
J1 = 11 + .001
J2 = 12 + .001

210 IF (MOM .LT. 100) GO TO 220

VOP = BP*POA
IF(VOP.GT.99995) VOP=.99995
GO TO 230
220 VOP=BP*GP**(2./MOM)/SQRT(A)
230 XI=SCATTER(VOMOM)
250 XIB=.5*(XI+XIP)
260 GX=U2-U1
GY=V2-V1
GZ=W2-W1
300 CE=COS(EPS)
SE=SIN(EPS)
CX=COS(XIB)
SX=SIN(XIB)
GPG=GP/G
350 RTG=SORT(G2-GX**2)
GPX=GPG*(GX*CX-RTG*SX*CE)
GPY=GPG*(GY*CX+(GX*GY*CE+G*GZ*SE)/RTG*SX)
GPZ=GPG*(GZ*CX+(GX*G7*CE-G*GY*SE)/RTG*SX)
400 U2=.5*(U1+U2+GPX)
V2=.5*(V1+V2+GPY)
W2=5*(W1+W2+GPZ)
U1=U2-GPX
V1=V2-GPY
W1=W2-GPZ
460 I1=J1
I2=J2
RETURN
END

*OECK LINKS*
SUBROUTINE LINKMC(I1,I2,BB,ETR)
DIMENSION PWAVE(20,20),LABCSA(16)
COMMON /CM1/ PWAVE,FKIN,L10,L20,BIMP,NMAX,NPRINT,L1PAR,L2PAR
COMMON /CM2/ JEL1,JEL2,LABCSA,LLMAX,ISELCT
COMMON /CM3/ VA,VB,IPRT1,IPRT2,IPRT3,IPRT4
COMMON /CMVR1/ VC,VALPHA,VC6,BBIMP,EEE
C** INPUT ** SELECTION FOR OUTPUT (0 FOR PWAVE/1 FOR P)
ISELCT=1
C** INPUT ** MIN PROBABILITY
C** INPUT ** INITIAL ROTATIONAL STATES
L10=11
L20=12
C** INPUT ** RELATIVE KINETIC ENERGY (IN EV) EKIN= ETR/1.602E-12
EKIN=ETP*.6242197E+12
C** INPUT ** IMPACT PARAMETER (IN ANGSTROM)
**INPUT** ** MAX NO. OF TERMS IN EXP

NMAX = 80

**INPUT ** INDEX FOR PRINT OUT

IPRT1 = 1
IPRT2 = 1
IPRT3 = NMAX + 1
IPRT4 = 0

** POTENTIAL PARAMETERS FOR SPHERICAL PART

V(R) = VC*EXP(-VALPHA*R)-VC6/R**6

VC = 3440.
VALPHA = 3.160
VC6 = 73.40

** POTENTIAL PARAMETERS FOR NON-SPHERICAL PART

VA = .2
VB = .2
JMAX = 20

5010 CONTINUE
DO 5016 I = 1, JMAX
DO 5016 J = I, JMAX
PWAVE(I, J) = 0.0

5016 CONTINUE

** NPRINT

NPRINT = NMAX
IF (Ekin .LE. 0.001) Ekin = 0.001
IF (Ekin .GE. 0.5) Ekin = 0.5

CALL PROB

5999 CONTINUE
RETURN
END

*DECk PROB*

SUBROUTINE PROB
COMMON /PRBOUT/PSUM(400), PRRR, J1FIN, J2FIN
COMMON /MV1/ AMATRX(20, 20, 9)
COMMON /MV2/ VBB, VAA, PROT, EROT, BRC, VVALP
COMMON /CMV1/ VC, VALPHA, VC6, BIMP, EEEE
COMMON /MV3/ NCOUNT
COMMON /CM1/ PWAVE, Ekin, L110, L20, BBBB, NPRINT, L1PAR, L2PAR
COMMON /CM2/ JEL1, JEL2, LABCSA, LLMAX, ISELCT
COMMON /CM3/ VA, VB, IPRT1, IPRT2, IPRT3, IPRT4
DIMENSION PWAVEC(20, 20)
DIMENSION P00D(20, 20), PEVN(20, 20), PWAVE(20, 20)
DIMENSION AKSUMO(20, 20), AKSUM1(20, 20)
DIMENSION LABCSA(16)
TCLOCK = 0
BIMP = BRRB
JMAX = 20
LMAX = 2*JMAX-2
C************* REDUCED MASS
RMASS = 14.02
C************* ROTATIONAL CONSTANT
BROT = 0.2512E-3
VVALP = 0.045723*VALPHA/SQRT(RMASS)
VAA = 0.4472136*VA
VBB = 0.2*VR*0.6298283
ETOT = EKIN + FLOAT(L10*(L10+1)+L20*(L20+1))*BROT
EB = ETOT/BROT
FB1 = SQRT(EB)
LMAX = INT(FB1)+1
C***** PRINT 901-1
11 CONTINUE
TF(LLMAX.GT.LMAX) LLMAX=LMAX
DO 18 I=1,JMAX
DO 18 J=1,JMAX
PENV(I,J) = 0.0
PDD(I,J) = 0.0
PWAVE(I,J) = 0.0
PWAVEC(I,J) = 0.0
AKSUMO(I,J) = 0.0
AKSUM1(I,J) = 0.0
DO 16 M=1,9
AMATRX(I,J,M) = 0.0
16 CONTINUE
18 CONTINUE
LL10 = L10+1
LL20 = L20+1
JEL1 = (LL10+1)/2
JEL2 = (LL20+1)/2
L1PAR = MOD(L10,2)+1
L2PAR = MOD(L20,2)+1
CLO = FLOAT((2*L10+1)*(2*L20+1))
PEO = 0.0
C************ N= 1
N=1
LL1I = LL10
LL2I = LL20
NCOUNT = 0
DO 57 K=1,9
CALL VMATRX(LL1T,LL2T,K)  
50 CONTINUE
LL1FMN= MAX0(LL10-2,LL1PAR)
LL1FMX= LL10+2
LL2FMN= MAX0(LL20-2,LL2PAR)
LL2FMX= LL20+2
K= 1
JJTI= (LL11+1)/2
JJT2= (LL2I+1)/2
DO 64 I=1,3
LL2F= LL20-4+2*I
DO 63 J=1,3
LL1F= LL10-4+2*J
IF(LL2F.LT.1.OR.LL1F.LT.1) GO TO 62
JJF1= (LL1F+1)/2
JJF2= (LL2F+1)/2
AKSUMO(JJF1,JJF2)= AMATRX(JJTI,JJT2,K)
62 K=K+1
63 CONTINUE
64 CONTINUE
TOTPW= 0.0
DO 74 LL1F=LL1FMN,LL1FMX,2
DO 73 LL2F=LL2FMN,LL2FMX,2
JJF1= (LL1F+1)/2
JJF2= (LL2F+1)/2
PODN(JJF1,JJF2)= AKSUMO(JJF1,JJF2)
PEVN(JJF1,JJF2)= 0.0
IF(LL1F.EQ.LL11.AND.LL2F.EQ.LL21) PEVN(JJF1,JJF2)=1.0
PWAVE(JJF1,JJF2)= PODN(JJF1,JJF2)**2*4.0+PEVN(JJF1,JJF2)**2
P=PWAVE(JJF1,JJF2)
L01=LL1F-1
L02=LL2F-1
C***** PRINT 902-1
72 CONTINUE
TOTPW=TOTPW+P
73 CONTINUE
74 CONTINUE
C***** PRINT 904-1
75 CONTINUE
IF(NMAX.EQ.1) GO TO 300
N= ?
C*********100
100 CONTINUE
NCOUNT= 0
N2= 2*N-2

32
LL21 MN = MAX(0, LL20 - N2, L2 PAR)
LL21 MX = LL20 + N2
IF(LL21 MX, GE, LL MAX) LL21 MX = LL MAX
DO 149 JJI1 = LL21 MN, LL21 MX, 1
JJII = (LL2I + 1)/2
E82 = SQR(E82 - FLAT(LL2I - 1)**2)
LL MAX1 = INT(E82) + 1
LL11 = LL10 + N2
JJ11 = (LL1I + 1)/2
IF(LL MAX1, GE, LL MAX) LL MAX1 = LL MAX
IF(LL11 .GT. LL MAX1) GO TO 130
DO 114 K = 5, 8, 3
CALL VMATRX(LL1I, LL2I, K)
114 CONTINUE
DO 119 K = 3, 9, 3
CALL VMATRX(LL1I, LL2I, K)
119 CONTINUE
IF(LL11 - 2 .LT. LL I) GO TO 125
AMATRX(JJI1, JJII2, 4) = AMATRX(JJI1 - 1, JJII2, 6)
K = 4
IF(LL2I .EQ. LL20 - N2, OR, LL2I .EQ. LL20 + N2) CALL VMATRX(LL1I, LL2I, K)
AMATRX(JJI1, JJII2, 7) = AMATRX(JJI1 - 1, JJII2 + 1, 3)
K = 7
IF(LL2I .EQ. LL20 + N2 - 2, OR, LL2I .EQ. LL20 + N2) CALL VMATRX(LL1I, LL2I, K)
IF(LL2I - 2 .LT. LL I) GO TO 130
AMATRX(JJI1, JJII2, 1) = AMATRX(JJI1 - 1, JJII2 - 1, 9)
K = 1
IF(LL2I .EQ. LL20 - N2 + 2, OR, LL2I .EQ. LL20 - N2) CALL VMATRX(LL1I, LL2I, K)
125 CONTINUE
IF(LL2I - 2 .LT. LL I) GO TO 130
AMATRX(JJI1, JJII2, 2) = AMATRX(JJI1, JJII2 - 1, 8)
K = 2
IF(LL2I .EQ. LL20 - N2) CALL VMATRX(LL1I, LL2I, K)
130 CONTINUE
LL11 = LL10 - N2
JJII1 = (LL11 + 1)/2
IF(LL1I1 .LT. LL1I PAR) GO TO 149
DO 134 K = 5, 8, 3
CALL VMATRX(LL1I, LL2I, K)
134 CONTINUE
DO 139 K = 1, 7, 3
CALL VMATRX(LL1I, LL2I, K)
139 CONTINUE
AMATRX(JJI1, JJII7, 6) = AMATRX(JJI1 + 1, JJII2, 4)
K = 6
IF(LL1.EQ.LL20-N2) CALL VMATRX(LL1,LL21,K)
AMATRX(JJI1,JJI2,9) = AMATRX(JJI1+1,JJI2+1,1)
K=9
IF(LL2.EQ.LL20-N2) CALL VMATRX(LL1,LL21,K)
AMATRX(JJI1,JJI2,2) = AMATRX(JJI1,JJI2-1,8)
K=7
IF(LL1.EQ.LL20-N2) CALL VMATRX(LL1,LL21,K)
AMATRX(JJI1,JJI2,3) = AMATRX(JJI1+1,JJI2-1,7)
K=7
149 CONTINUE
C************* 150
LL1MN= MAX0(LL10-N2+2,LLPAR)
LL1MX= LL10+N2-2
IF(LL1MX.GE.LLMAX) LL1IMX= LLMAX
DO 199 LL1I=LL1IMN,LL1IMX,2
JJI1=(LL1I+1)/2
EB2= SQRT(EB- FLOAT(LL1I-1)**2)
LLMAX1= INT(F82)+1
LL2I= LL20+N2
JJI2=(LL2I+1)/2
IF(LLMAX1.GE.LLMAX) LLMAX1= LLMAX
IF(LL2I.GT.LLMAX1) GO TO 180
DO 164 K=5,9
IF(LL1.EQ.LL10+N2-2.AND.K.EQ.6) GO TO 163
CALL VMATRX(LL1I,LL2I,K)
GO TO 164
163 CONTINUE
AMATRX(JJI1,JJI2,6) = AMATRX(JJI1+1,JJI2,4)
164 CONTINUE
IF(LL2I-2.LT.1) GO TO 175
AMATRX(JJI1,JJI2,2) = AMATRX(JJI1,JJI2-1,8)
AMATRX(JJI1,JJI2,3) = AMATRX(JJI1+1,JJI2-1,7)
IF(LL1I-2.LT.1) GO TO 180
AMATRX(JJI1,JJI2,1) = AMATRX(JJI1-1,JJI2-1,9)
175 CONTINUE
IF(LL1I-2.LT.1) GO TO 180
AMATRX(JJI1,JJI2,4) = AMATRX(JJI1-1,JJI2,6)
180 CONTINUE
LL21=LL20-N2
JJI2=(LL2I+1)/2
IF(LL2I.LT.LLPAR) GO TO 199
DO 184 K=5,6
IF(LL1.EQ.LL10+N2-2.AND.K.EQ.6) GO TO 183
CALL VMATRX(LL11,LL21,K)
GO TO 184
183 CONTINUE
AMATRX(JJI1,JJI2,6) = AMATRX(JJI1+1,JJI2,4)
184 CONTINUE
DO 189 K=1,3
CALL VMATRX(LL11,LL21,K)
189 CONTINUE
AMATRX(JJI1,JJI2,9) = AMATRX(JJI1,JJI2+1,2)
AMATRX(JJI1+1,JJI2,9) = AMATRX(JJI1+1,JJI2+1,1)
IF(LL11-2.LT.1) GO TO 189
AMATRX(JJI1,JJI2,4) = AMATRX(JJI1-1,JJI2,6)
AMATRX(JJI1,JJI2,7) = AMATRX(JJI1-1,JJI2+1,3)

C********** 199
199 CONTINUE
LL1FMX=LL10+2*N
LL2FMX= LL20+2*N
LL1FMN= LL10-2*N
IF(LL1FMN.LT.L1PAR) GO TO 201
LL11=LL1FMN+2
200 CONTINUE
LL2FMN=LL20-2*N
IF(LL2FMN.LT.L2PAR) GO TO 202
LL22= LL2FMN+2
GO TO 205
201 CONTINUE
LL1FMN= L1PAR
LL11= L1PAR
GO TO 200
202 CONTINUE
LL2FMN=L2PAR
LL22= L2PAR
205 CONTINUE
LL22F= LL2FMX-2
IF(LL2FMX.LE.LLMAX) GO TO 206
LL2FMX=LLMAX
LL22F= LLMAX
206 CONTINUE
DO 249 L2=LL22,LL22F,2
LL1F= LL1FMX-2
EB2 = SQRT(EB2-FLOA-(L2-1)**2)
LLMAX1= INT(FR2)+1
IF(LLMAX1.GE.LLMAX) LLMAX1= LLMAX
IF(LL1FMX.GE.LLMAX1) LL11F= LLMAX1
DO 248 L1= LL11,LL11F,2
K1 = 1
K2 = 0
L22 = L2 - 2
L11 = L1 - 2
IF(L22 .LE. L1) L22 = L1
IF(L121 .LE. L111) L11 = L1
IF(L22 .LE. L111) K1 = 4
IF(L121 .LE. L111) K2 = 1
K = K1
J1 = (L1+1)/2
J2 = (L2+1)/2
AK = AKSUM0(J1, J2)
L2P2 = L2 + 2
DO 219 LL2 = L22, L2P2, 2
   K = K + K
   L1P2 = L1 + 2
   DO 218 LL1 = L11, L1P2, 2
   JJ1 = (LL1+1)/2
   JJ2 = (LL1+1)/2
   AKSUM1(JJ1, JJ2) = AK*AMATRX(J1, J2, K) + AKSUM1(JJ1, JJ2)
   K = K + 1
   CONTINUE

C************ PRINT 3F AMATRX
219 CONTINUE

C************ 250
LL1FMX = LL10 + 2*N
TOTPC = 0.0
TOTPW = 0.0
DO 299 LL2F = LL2FMN, LL2FMX, 2
   LL1F = LL1FMX
   F2 = SQRT(FB - FLOAT(LL2F-1)**2)
   LLMAX1 = INT(F2)+1
   IF(LLMAX1 .GE. LMAX) LLMAX1 = LMAX
   IF(LL1FMX .GE. LLMAX1) LL1F = LLMAX1
   DO 298 LL1F = LL1FMN, LL1FMX
   JJF1 = (LL1F+1)/2
   JJF2 = (LL2F+1)/2
   AKSUM1(JJF1, JJF2) = AKSUM1(JJF1, JJF2)/FLOAT(N)
   IF(MOFN, 2) .NE. 0.1 GO TO 294
   AKSUM1(JJF1, JJF2) = -4.0*AKSUM1(JJF1, JJF2)
   PEVN(JJF1, JJF2) = PEVN(JJF1, JJF2) + AKSUM1(JJF1, JJF2)
   PWAVF(JJF1, JJF2) = PDDO(JJF1, JJF2)**2*4.0*PEVN(JJF1, JJF2)**2

36
GO TO 295
294 CONTINUE
   DPON(JLF1,JLF2)=DPON(JLF1,JLF2)+AKSUM1(JLF1,JLF2)
   PWAVE(JLF1,JLF2)=DPON(JLF1,JLF2)**2*4.0+PEVN(JLF1,JLF2)**2
295 CONTINUE
   P=PWAVE(JLF1,JLF2)
   L01=LL1F-1
   L02=LL2F-1
C********** PRINT OF PWAVE
296 CONTINUE
C***** PRINT 902-2
297 CONTINUE
   TOTPW=TOTPW+P
   AKSUM0(JLF1,JLF2)=AKSUM1(JLF1,JLF2)
   AKSUM1(JLF1,JLF2)=0.0
298 CONTINUE
299 CONTINUE
   PE1=PWAVE(JEL1,JEL2)
   PE10=ARSI((PE1-PE0)/PE1)
C***** PRINT 904-2
300 CONTINUE
301 CONTINUE
C********** ELASTIC
C********** FINAL PRINT
   IF(IND4.EQ.1) GO TO 325
   MPRINT=0
   PMAX=0.1E-5
301 CONTINUE
C***** PPINT 901-2
C***** PPINT 910-1
   IMN=LL2FMN
   IMX=LL2FMX
   IF(MOD(LL2FMX-L2PAR,2).NE.0) IMX=LL2FMX-1
   Z19=IMN,IMX,2
   LLF=IMX+IMN-1
   J2=(LLF+1)/2
   LL1F=LL1FMX
   E92=SQRT(EB-FLOAT(LL2F-1)**2)
   LLMAX1=INT(E92)+1
   IF(LLMAX1.GE.LMAX) LLMAX1=LMAX
IF(LL1FMX.GE.LLMAX1) LL1F=LLMAX1
J1MX=(LL1F+1)/2
IF(J1MX.GT.16) J1MX=16
L02=L12F=1

C**** PRINT 911-1
IF(MPRINT.EQ.1) GO TO 319
LMT2=MNO(L01,L02)
CL10=FLOAT(2*LMT2+1)
J1MN=(L1PAR+1)/2
DO 318 J1=J1MN,J1MX
L01=2*J1-3*L1PAR
LMN1=MNO(L01,L01)
CL1=CL10=FLOAT(2*LMT2+1)
P=PWAVE(J1,J2)
PC_CORCT=CL1/CL0*P
TOTPC=TOTPC+PC_CORCT
PWAVEC(J1,J2)=PC_CORCT
IF(J1.EQ.JEL1.AND.J2.EQ.JEL2) GO TO 318
IF(ISELCT.EQ.0) POUT=P
IF(ISELCT.EQ.1) POUT=PC_CORCT
318 CONTINUE
319 CONTINUE
LABCSAT1=L1PAR-1
DO 320 I=2,16
LABCSA(I)=LABCSA(I-1)+(I-1)*2
320 CONTINUE

C**** PRINT 912-1
C IF(MPRINT.EQ.1) GO TO 9906
C************ PRINT OF PCORCT WITH PELASTIC MODIFIED
325 CONTINUE
PC=1.0-TOTPC+PWAVEC(JEL1,JEL2)
PWAVEC(JEL1,JEL2)=PC
IF(ISELCT.EQ.0) PSUM1=PWAVEC(JEL1,JEL2)
IF(ISELCT.EQ.1) PSUM1=PWAVEC(JEL1,JEL2)
C STATISTIC MODIFICATION OF ELASTIC COLLISION
C RPRP: P(N)=P(N)-PSUM1/(1.-PSUM1) PBAR VS RPRP
RPRP=(1.0-PSUM1)*RPRP
C IF(RPRP.GT.PSUM1) GO TO 1905
C J1FIN=JEL1
C J2FIN=JEL2
C GO TO 2001
1905 CONTINUE
NN=1
N=1
1910 CONTINUE

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR
J1MN = JFL1-N

IF(J1MN.LT.1) J1MN=1

J1MX = JFL1+N

IF(J1MX.GT.JMAX) J1MX = JMAX

K = 0

J2 = JEL2-N

IF(J2.LT.1) GO TO 1942

1920 CONTINUE

DO 1940 J1 = J1MN, J1MX

NN = NN + 1

IF(ISLECTION.EQ.1) PWAVE(J1,J2) = PWAVEC(J1,J2)

CALL SUMP(J1,J2,NN)

C IF(RRRR.GT.PSUM(1)) GO TO 1940

IF((RRRR.GT.(PSUM(1)-PSUM(1)))) GO TO 1940

J1FIN = J1

J2FIN = J2

GO TO 2001

1940 CONTINUE

IF(K.EQ.1) GO TO 1950

1942 K = 1

J2 = JEL2+N

IF(J2.GT.JMAX) GO TO 1950

GO TO 1920

1950 CONTINUE

J2MN = JEL2-N+1

IF(J2MN.LT.1) J2MN = 1

J2MX = JFL2+N-1

IF(J2MX.GT.JMAX) J2MX = JMAX

K = 0

J1 = JEL1-N

IF(J1.LT.1) GO TO 1992

1970 CONTINUE

DO 1990 J2 = J2MN, J2MX

NN = NN + 1

IF(ISLECTION.EQ.1) PWAVE(J1,J2) = PWAVEC(J1,J2)

CALL SUMP(J1,J2,NN)

C IF(RRRR.LT.PSUM(1)) GO TO 1985

IF((RRRR.LT.(PSUM(1)-PSUM(1)))) GO TO 1985

IF(ABS(1.0-PSUM(1)).LT.1.0E-4) GO TO 1985

IF(NN.GE.400) GO TO 1985

GO TO 1990

1985 CONTINUE

J1FIN = J1

J2FIN = J2

GO TO 2001
CONTINUE
IF(K.EQ.1) GO TO 2000
K=1
J1= JEL1+N
IF(J1.GT.JMAX) GO TO 2000
GO TO 1970
CONTINUE
N=N+1
GO TO 1910
CONTINUE
RETURN
END

*DECK SUMP$
SUBROUTINE SUMP(J1,J2,NNN)
COMMON/PRBOUT/PSIJM(400),RRRR,J1FIN,J2FIN
COMMON/CMI/ PAVE,EKIN,L10,L20,RRBBB,NMAX,NPRINT,L1PAR,L2PAR
DIMENSION PAVE(20,20)
N=NNN
PSUM(N)= PSUM(N-1)+PAVE(J1,J2)
RETURN
END

*DECK VMAT$*
SUBROUTINE VMATRX(LL1,LL2,K)
C**** REVISED FOR ROOT
COMMON /MV1/ AMATRX(20,20,9)
COMMON /MV2/ VRB,VAR,ETOT,BRC,VVALP
COMMON /CMVR1/ VC,VALPHA,VC6,BIMP,EIZ
COMMON /MV3/ NCOUNT
COMMON/C1/ABCC(40,40,9)
L1J= LL1-1
L2J= LL2-1
IF(K.GT.3) GO TO 1011
L2J= L2J-2
IF(L2J.LT.0) GO TO 1510
GO TO 1100
1011 IF(K.GT.6) GO TO 1012
L2J= L2J+2
GO TO 1100
1012 L2J= L2J+2
1100 CONTINUE
IF(MOD(K,3).NE.1) GO TO 1111
L1J= L1J-2
IF(L1I.LT.0) GO TO 1510
GO TO 1200
1111 IF(MOD(K,3).NE.2) GO TO 1112
LII= L1J
GO TO 1200
1112 LII= L1J+2
1200 CONTINUE
ABC= ABCC(LL1,LL2,K)
I=(ABC.NE.0.0) GO TO 1299
CALCULATION OF VEFF(L1I,L2I / L1J,L2J)
CC1= CS20(L1I,L1J)
CC2= CG20(L2I,L2J)
IF(K.LE.5) C$IGN= (-1.0)**((L1J+L2J)
IF(K.GE.6) CSIGN= (-1.0)**((L1I+L2I)
C= FLOAT((2*L1I+1)*(2*L2I+1)*(2*L1J+1)*(2*L2J+1))**0.25*CSIGN
B= VBB*CC1*CC2
A=0.0
IF(L2I.NE.L2J) GO TO 1215
A= VAA*CC1/ SQRT(FLOAT(2*L2I+1))
IF(MOD(L2I,2).NE.0) A=-A
1215 IF(L1I.NE.LIJ) GO TO 1219
AA= VAA*CC2/ SQRT(FLOAT(2*L1I+1))
IF(MOD(L1I,2).NE.0) AA=-AA
A= A+AA
1219 CONTINUE
ABC= C*B+A
ABC(LL1,LL2,K)= ABC
1299 CONTINUE
WI= BROOT* FLOAT(L1I*(L1I+1)+L2I*(L2I+1))
WJ=BROOT* FLOAT(L1J*(L1J+1)+L2J*(L2J+1))
WIJ= ABS(WI-WJ)
EI= EROT-WI
EJ= EROT-WJ
IF(EI.LE.0.0.OR.EJ.LE.0.0) GO TO 1510
EIJ= 0.5*(EI+EJ)
CALL POOT(RC)
BRC= 1.0-(ESMP/RC)**2+VC6/EIJ/RC**6
EIJ= EIJ*BRC
IF(K.EQ.5) GO TO 1500
IF(L1I.EQ.L2J.AND.K.EQ.3) GO TO 1500
IF(L1I.EQ.L2J.AND.K.EQ.7) GO TO 1500
AIJ= -VVALP* SQRT(EIJ)/WIJ
EW= EIJ/WIJ
DBALPH= EW/AIJ
APA1= 1.570796327/AIJ
F = EXP(-APAI)
FAIJ = 2.0*APAIT*F/(1.0-F*F)
AAA = ABC*DBALPH*FAIJ
1498 CONTINUE
C*** 999 CHECK PRINT
990 CONTINUE
NCOUNT = NCOUNT+1
1499 CONTINUE
JJ1 = (LL1+1)/2
JJ2 = (LL2+1)/2
AMATRX(JJ1, JJ2, K) = AAA
RETURN
1500 CONTINUE
AAA = ARC/VVALP* SQRT(EIJ1)
GO TO 1498
1510 CONTINUE
AAA = 0.0
GO TO 1499
END
*DECK .CS$*
FUNCTION CG20(J1, J2)
C DOUBLE PRECISION FUNCTION CG20(J1, J2)
IF(J2.EQ.J1+2) GO TO 8001
IF(J2.EQ.J1-2) GO TO 8002
C = 0.0
GO TO 8100
8001 CONTINUE
IF(J2.EQ.J1+2) J=J1.
IF(J2.EQ.J1-2) J=J2.
X1 = FLOAT(J+2)/ FLOAT(2*J+5)
X2 = FLOAT(J+1)/ FLOAT(2*J+3)
X3 = 1.0/ FLOAT(2*J+1)
C = SQRT(1.0*X1*X2*X3)
GO TO 8099
8002 CONTINUE
J = J1
X1 = FLOAT(J+1)/ FLOAT(2*J+3)
X2 = FLOAT(J)/ FLOAT(2*J+1)
X3 = 1.0/ FLOAT(2*J-1)
C = - SQRT(X1*X2*X3)
8099 IF(MOP(J1, J2).NE.0) C = -C
8100 CG20=C
RETURN
END

42
SUBROUTINE RNOT(P,C)
C***** REVISED 8/26/74
COMMON /CMVRV/ VC,VALPHA,VC6,BIMP,EKIN
RMIN= 1.12
RRO= ALOG(VC/EKIN)/VALPHA
IF(RPO.GE.4.1) RRO= 4.1
N=1
3099 CONTINUE
RR=P
3100 CONTINUE
RL= EKTN*BIMP**2/RR**2
V= VC* EXP(-VALPHA*RR)
V1= -VALPHA*V
IF(VC6.EQ.0.0) GO TO 3101
VP= VC6/PR**6
V= V-VR
V1= V1+6.0*VR/RR
3101 CONTINUE
F= (V+RL-EKIN)/(2.0*RL/RR-V1)
IF(ABS(F/RR).LT.0.1E-5) GO TO 3199
IF(N.GE.100) GO TO 3299
RR= RR+F
IF(PR.LT.RMIN) GO TO 3900
N=N+1
GO TO 3100
3199 CONTINUE
RC= RP
RETURN
3299 CONTINUE
WRITE(6,998) RR,F,RRO
998 FORMAT(1HO//5X,14HERROR N GT 100,3X,3HRR=E13.5,3X,2E13.5//)
RR=RPO
GO TO 3199
3900 CONTINUE
RRO= 0.5*(PRO+RMIN)
GO TO 3099
END

SUBROUTINE OUTPIJT(M)
COMMON /TIME/ T0,T5,TF,TM,T0G,DTM,TN
COMMON /CV/ MAX,MAX6,C1,RHO1
COMMON /CONST/ W,A,VOM,SF,CD
COMMON /RANDOM/ R
COMMON /ANSWER/ M(4),T(4),G,DVOL,AT,POA,ROT1
COMMON /PART/ *(5001)
COMMON /CV1/ABCC(40,40,9)
DIMENSION FC(40),FW(40)

C
NTM= (TW-NTM)/DTO+.5
TARBAR= 0.1*NTM/DTD/TM
COC= T(2)/(T(1)*C1)*.886227
COC2= T(3)/(T(1)*1.5*C1*C1)
EOE= .80482E-15*(4)/(T(1)*W*C1*C1)
WRITE (6,600) COC,COC2,EOE,TBAR
DO 100 I=1,40
FC(I)=0
100 FC(I)=FC(I)+1
DO 200 I=5,MAX6,5
JC= P(I)*COE*10. +1
JW= P(I+1)*1.01
IF (JC .GT. 40) JC=40
IF (JW .GT. 40) JW=40
FC(JC)=FC(JC)+1.
FW(JW)=FW(JW)+1.
200 CONTINUE
DO 300 I=1,40
FC(I)=FC(I)/MAX
300 FW(I)=FW(I)/MAX
600 FORMAT (5HCOC=G13.6,1X6HCOC2 = G13.6,1X6HEOE = G13.6,
11X7HTAUAR= F5.1)

C THIS IS TO PLOT FC AND FW
WRITE (10) TBAR,NTM,MAX,COC2,EOE,FC,FW

C THESE CARDS ARE FOR RESTART PROGRAM TIL NEXT *** MARKS
WRITE (9) M(4),T(4),G,DVOL,W,A,VOM,SF,CO,TO,TS,TF,TM,DTO,DTM,TN,MAX,
1MAX6,C1,RH01,P,AT,ROT1,POA,ABCC,R
END FILE 9
REWIND 9

C *** END OF THE RESTART PROGRAM
RETURN
END

*DECK X2**
SUBROUTINE X2DIST(TROT)
COMMON /RANDOM/ R
COMMON /INTX2/ ER0

44
TO DETERMINE THE ROTATIONAL FREQUENCY (OM) FROM THE
X2-DISTRIBUTION (KAI-SQUARE)

IROT = 1.0 * SQRT (.25 - 2.48503 * RHO * ALG (1 - R)) -.5 + .5 FOR 1 JUMP
IROT = .50 * SQRT (.25 - 2.48503 * RHO * ALG (1 - R)) -.25 + .5 FOR EVEN JUMP

R = RANF (0)
IROT = 1.0 * SQRT (.25 - 2.48503 * RHO * ALG (1 - R)) -.1666
RETURN
END

*DECK SCA$$
FUNCTION SCATER (VO, MOM)
IF (VO .GE. 0.0) SCATER = 2.0 * ACOS (VO)
IF (VO .LT. 0.0) SCATER = 2.0 * (3.1415926 + ACOS (-VO))
RETURN
END

*DECK MOM$$
SUBROUTINE MOMENT
COMMON /CV/ MAX, MAX6, C1, RHO1
COMMON /ANSWER/ MOM, T(4), G, DVL, AI, POA, R0T1
COMMON /PART/ P(5001)

COMPUTES DESIRED MOMENTS

DO 200 T = 1, MAX
LU = 5 * T - 3
T(1) = T(1) + 1
T(2) = T(2) + P(LU + 3)
T(3) = T(3) + P(LU + 3) ** 2
T(4) = T(4) + P(LU + 4) ** 2 + P(LU + 4)
200 CONTINUE
RETURN
END

SAMPLE INPUT DATA

F (G/CM) = SELECT, ISELECT = 1, C = 0.4310, PC = 0
1000 1.2647E-243.632933 .46519E-22.44272E-141.0 100.0
3.0 E-110.0 1.0 1.0
1.09 E-83.33 E-1
"
"
REFERENCES


### Table I. Initial Distributions and Flow Correlation

<table>
<thead>
<tr>
<th>Figure</th>
<th>Velocity(^a)</th>
<th>Rotational(^a)</th>
<th>Energy partition</th>
<th>Temperature</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a &amp; b</td>
<td>( f_t = f_{tm} (T) )</td>
<td>( f_r = f_{rm} (T) )</td>
<td>( E_t / E_r = 3/2 )</td>
<td>( T = T_e )</td>
<td>Uniform flow</td>
</tr>
<tr>
<td>2a &amp; b</td>
<td>( f_t \neq f_{tm} )</td>
<td>( f_r = \text{frozen} )</td>
<td>( - - - )</td>
<td>( - - - )</td>
<td>- - -</td>
</tr>
<tr>
<td>3</td>
<td>( f_t = f_{tm} (T_t) )</td>
<td>( f_r \neq f_{rm} )</td>
<td>( E_t / E_r = 3/2 )</td>
<td>( T = T_e )</td>
<td>Sound absorption experiment</td>
</tr>
<tr>
<td>4</td>
<td>( f_t = f_{tm} (T_t) )</td>
<td>( f_r = f_{rm} (T_r) )</td>
<td>( E_t / E_r &gt;&gt; 3/2 )</td>
<td>( T &gt;&gt; T_t &gt;&gt; T_r )</td>
<td>Shock wave</td>
</tr>
<tr>
<td>5a &amp; b</td>
<td>( f_t = f_{tm} (T_t) )</td>
<td>( f_r = f_{rm} (T_r) )</td>
<td>( E_t / E_r &lt;&lt; 3/2 )</td>
<td>( T_t &lt;&lt; T_r )</td>
<td>Free-jet expansion</td>
</tr>
<tr>
<td>6a &amp; b</td>
<td>( f_t = f_{tm} (T_t) )</td>
<td>( f_r \neq f_{rm} )</td>
<td>( E_t / E_r &lt;&lt; 3/2 )</td>
<td>( T_t &lt;&lt; T_e )</td>
<td>Chemical-fluorescence experiment</td>
</tr>
</tbody>
</table>

\(^a f_{tm}(T) \) and \( f_{rm}(T) \) denote translational and rotational Maxwell-Boltzmann distributions with temperature \( T \).
(a) Monte Carlo Results: \( f_T(0, x) = \text{Maxwellian where } x = c/c_0; \)
\( f_r(0, j) = \text{Boltzmann; } T_T = T_r = 320 \text{ K}. \)

Figure 1.- Distribution functions for complete equilibrium.
(b) Time average of Monte Carlo results.

Figure 1.—Continued.
Figure 1.- Concluded.

(b) Time average of Monte Carlo results - Concluded.
Figure 2.- Monatomic gas simulation (rotational effect frozen).

(a) Delta function initial velocity distribution: $f_{\xi}(0,x) = 1$ at $x = \sqrt{3}/2$. 

REPRODUCIBILITY OF THE ORIGINAL PAGE IS POOR
(a) Delta function initial velocity distribution - Concluded.

Figure 2.- Continued.
(b) Double delta function initial velocity distributions;
\[ f_t(0, x_1) = \frac{1}{2} \text{ at } x_1 = \frac{1}{2} \]
\[ f_t(0, x_2) = \frac{1}{2} \text{ at } x_2 = \frac{\sqrt{11}}{2} \]

Figure 2.- Continued.
(b) Double delta function initial velocity distributions – Concluded.

Figure 2.— Concluded.
Figure 3.- Maxwellian initial velocity distribution; delta function rotational energy distribution ($f_r(0,x) = 1$ at $j = 10$): Equipartition satisfied.
Figure 3.— Continued.
Figure 3.- Concluded.
Figure 4. - Maxwellian initial velocity distribution; delta function rotational distribution \( f_r(0, x) = 1 \) at \( j = 0 \):

Equi-partition not satisfied.
Figure 4.— Continued.
Figure 4.- Concluded.
(a) Comparison with equilibrium distributions.

Figure 5.- Maxwellian initial velocity distribution; Boltzmann rotational energy distribution \( (T_c = 6 \text{ K}, T_r = 793 \text{ K}) \): Equipartition not satisfied.
(a) Comparison with equilibrium distributions - Concluded.

Figure 5.- Continued.
(b) Comparison with local Maxwell-Boltzmann distributions.

Figure 5.— Continued.
(b) Comparison with local Maxwell-Boltzmann distributions - Continued.

Figure 5.—Continued.
(b) Comparison with local Maxwell-Boltzmann distributions - Concluded.

Figure 5.- Concluded.
(a) Comparison with equilibrium distributions.

Figure 6. Maxwellian initial velocity distribution; delta function rotational distribution \( f_r(0,x) = 1 \) at \( j = 16 \); Equipartition not satisfied.
(a) Comparison with equilibrium distributions - Concluded.

Figure 6.- Continued.
(b) Comparison with local Maxwell-Boltzmann distributions.

Figure 6.— Continued.
(b) Comparison with local Maxwell-Boltzmann distributions - Concluded.

Figure 6.- Concluded.
Figure 7.- Relaxation behavior of average rotational energy for a variety of initial distributions.
Figure 8.—Preliminary study of shock wave structure.
The direct simulation Monte Carlo method is applied to solve the Boltzmann equation for collisions between internally excited diatomic gases in highly non-equilibrium states. The semiclassical transition probability is incorporated in the simulation for energy exchange between rotational and translational energy.

The results provide details on the fundamental mechanisms of gas kinetics where analytical methods are impractical. The validity of the local Maxwellian assumption and relaxation time, rotational-translational energy transition, and a velocity analysis of the inelastic collision are discussed in detail.