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THE DYNAMICS AND CONTROL OF
LARGE FLEXIBLE SPACE STRUCTURES
PART B: DEVELOPMENT OF CONTINUUM MODEL AND
COMPUTER SIMULATION

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ABSTRACT

The equations of motion of an arbitrary flexible body in orbit are derived. The model includes the effects of gravity with all its higher harmonics. As a specific example, the motion of a long, slender, uniform beam in circular orbit is modelled. The example considers both the inplane and three dimensional motion of the beam in orbit. In the case of planar motion with only flexural vibrations, the pitch motion is not influenced by the elastic motion of the beam. For large values of the square of the ratio of the structural modal frequency to the orbital angular rate the elastic motion is decoupled from the pitch motion. However, for small values of this ratio and small amplitude pitch motion, the elastic motion is governed by a Hill's 3-term equation. Numerical simulation of this equation indicates the possibilities of instability for very low values of the square of the ratio of the modal frequency to the orbit angular rate. Also numerical simulations of the first order non-linear equations of motion for a long flexible beam in orbit have been performed. The effect of varying the initial conditions and the number of modes has been demonstrated.
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**LIST OF SYMBOLS**

- $A_n$: Modal amplitude function
- $B^0, B(s)$: Matrices defined in Appendix II
- $\bar{c}$: Resultant of external disturbances torques acting on the body
- $(C_x, C_y, C_z)$: Components of $\bar{c}$ in $\tau_3$ frame
- $\bar{D}^{(n)}, \bar{D'}^{(n)}$: Terms accounting for the center of mass shift due to elastic motion (Eqns. (30) and (48))
- $E$: External force on the elemental mass $dm$
- $E_n$: Modal components of external disturbances
- $F$: Gravity force expressed in $\tau_2(0)$ frame
- $F_0$: Gravity force at point $O$, expressed in the intrinsic frame at that point
- $(F_p, F_\eta, F_\omega)$: Gravity force components in $\tau_1$ frame
- $\bar{G}$: Gravitational torque due to rigid body motion
- $G^{(n)}$: Gravity torque due to elastic motion in the $n^{th}$ mode
- $H_{\alpha\beta}^{(n)}$: See Eqn. (III - 1) in Appendix - III
- $r^{(n)}$: See Eqn. (III - 1) in Appendix - III
- $(J_x, J_y, J_z)$: Principal moments of inertia of the body in the undeformed state
- $K_s$: See Eqn. (5)
- $K_{s0}$: Zeroth harmonic amplitude of gravity potential
- $L$: Structural operator which transforms the elastic displacement to elastic force
I

\( L^{(m_\infty)} \)

See Eqn. (III - 1) in Appendix - III

\( M, M^{(0)}, M^{(s)} \)
Matrices defined in Eqn. (6)

\( M_n \)
Generalized mass in the \( n^{th} \) mode, see Eqn. (12)

\( M^{(0)}, M^{(s)} \)
Elements of \( M^{(0)} \) and \( M^{(s)} \) respectively

\( 0' \)
Origin of \( \tau_0 \) frame (center of the earth)

\( 0 \)
Origin of the body axes

\( P^{(m)}(n) \)
\( m^{th} \) Legendre associated function of order \( s \)

\( P \)
An arbitrary point in the body

\( Q^{(n)} \)
Terms accounting for the inertia torques due to elastic motion (Eqn. (28)).

\( Q^{(n)}_x, Q^{(n)}_y, Q^{(n)}_z \)
Components of \( Q^{(n)} \)

\( \bar{R} \)
See Eqn. (27)

\( T_1, T_2, T_3 \)
Transformation matrices given in Eqns. (1), (2) and (3).

\( T \)
Orbital period

\( V \)
Gravitational potential per unit mass (see Eqn. (57)

\( O'XYZ \)
Inertial axes system (\( \tau_0 \) frame)

\( OX_0Y_0Z_0 \)
Orbit fixed reference frame (\( \tau_2 \) frame)

\( Ox_0y_0z_0 \)
Principal axes of the body (\( \tau_3 \) frame)

\( z^{(n)} \)
Mode shape of a free-free beam in its \( n^{th} \) mode
(see Eqn. (IV - 4)

\( \bar{a} \)
Inertial acceleration of \( dm \)

\( a \)
Equatorial radius of the earth

\( \bar{a}_{cm} \)
Acceleration of the center of mass
c  An arbitrary constant in Eqn. (67)
\(\bar{e}\)  External disturbance force per unit mass
\(\bar{F}\)  Gravity force per unit mass at an arbitrary point in the body, expressed in the body frame
\(\bar{F}_0\)  Gravity force at point 0 expressed in the body frame
\(g_n\)  Gravitational force acting on \(n^{th}\) mode
\(g_{mn}\)  See Eqn. (46) and Eqn. (III - 1)
\(\Omega_{i_1 i_2 i_3}\)  Local intrinsic frame (\(\tau_1\) frame)
\(\lambda\)  Length of the beam
\(m\)  Mass of the body
\(\bar{q}\)  Elastic displacement vector
\(\bar{r}\)  Instantaneous position vector of the elemental mass \(dm\) measured from 0
\(\bar{r}_0\)  Position vector of \(dm\) measured from 0 in the undeformed state
\(t\)  Time
\(z_n\)  Non-dimensionalized modal amplitude (= \(A_n/\lambda\))
\(\beta_n\)  See Eqn. (IV - 3)
\(\gamma\)  Phase angle
\(\delta_{mn}\)  Kronecker delta symbol
\(\xi\)  See Eqn. (IV - 3)
\(\eta\)  Co-latitude
\(\phi, \theta, \gamma\)  Euler angles
\( \mu \) Mass density

\( \nu \) Gravitational constant (=9.79 m/sec\(^2\) for earth)

\((\xi_x, \xi_y, \xi_z)\) Co-ordinates of an elemental mass \( dm \) in \( \tau_3 \) frame in the undeformed state

\( \rho \) Distance of \( dm \) from the inertial origin 0'

\( \tau \) Dimensionless time; \( \tau = \frac{1}{2} (\sqrt{3} \omega t + \gamma) \)

\( \phi(n) \) Mode shape of \( n \)th mode (components \( \phi_x^{(n)} \), \( \phi_y^{(n)} \), \( \phi_z^{(n)} \))

\( \phi_{sm} \) See Eqn. (5)

\( \omega \) Longitude

\( \omega_c \) orbital angular velocity

\( \omega_n \) \( n \)th mode natural frequency

\( \Omega(\eta, \omega) \) See Eqn. (5)

\( \Omega_n \) See Eqn. (IV - 3)

\( s() \) \sin ( )

\( c(\cdot) \) \cos ( )

\( (') \) \( \frac{d}{d\eta} (\cdot) \)

\( (\cdot) \) \( \frac{d}{d\eta} (\cdot) \)

\( (\cdot) \) \( \frac{d}{d\tau} (\cdot) \)
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1. INTRODUCTION

This report presents the development of the equations of motion of an arbitrary flexible body in orbit and its specialization to a long, slender, uniform beam in circular orbit. In the literature, many models of a beam satellite in orbit are presented. The model considered by Pringle consists of a massive central rigid body to which a massless elastic beam is rigidly attached. The other end of the beam carries a tip mass. For beam satellites with lengths of the order of 100 meters or more and mass distributed throughout the length this cannot be used. Ashley's model of a beam satellite is based on a continuum approach. He arrives at a set of partial differential equations for the beam model through energy methods. However, partial differential equations are not as convenient for numerical simulation. Ashley arrives at the conclusion that it is impossible to excite flexural vibrations of a beam directly through gravity gradient. The same conclusion is arrived at in section 5.1.1 of this report.

The development of the beam model presented in this report follows as a specialization of the equations of motion of an arbitrary flexible body in orbit. Also, the model presented here can be quite easily adopted for numerical simulation and also for inclusion of various control laws.

The development of equations of motion for an arbitrary flexible body presented in this report essentially follows that of Santini.
However, the distinguishing features of the present development from that of Santini are (a) extensive use of the techniques of vector calculus (b) introduction of the orbit fixed reference frame as an intermediate frame between the local intrinsic frame at the origin of the body axes system and the body axes frame. Hence, the Euler angle rotations used in the present report are different from those of Santini.\textsuperscript{4}

The equations of motion presented in this report are obtained by the Galerkin integration\textsuperscript{5,6} of the equations of motion of a generic point in the body. The motion of the generic point is assumed to be described by the superposition of rigid body motion plus a combination of the structural modes.

Section 2 of this report describes the various co-ordinate reference frames employed in the development of the equations, and the transformation relations between these reference frames. The detailed derivations of the transformation relations are presented in Appendix - I.

Section 3 presents an expression for the gravitational force on a generic element in the body. The expression also includes the higher harmonics of the earth's gravitational field. The details of this development are presented in Appendix - II.

In Section 4, the development of the equations of motion of an arbitrary flexible body in orbit is presented. The expansions of the vector expressions in equations (28), (31), (32), (43), (44), (45), and (46) are presented in Appendix - III.
Specializations of the equations of motion in section 4 to a long, slender, uniform beam are presented in section 5. Sections 5.1 and 5.2 discuss the planar, and the three dimensional motion of the beam in orbit, respectively. In Appendix - IV, the expressions for the natural mode shapes, frequencies and the modal mass for a free-free uniform beam are presented.

In Section 6, the numerical solutions to the response of the planar motion of a beam in circular orbit, undergoing only flexural vibrations, are presented. The responses indicate the possibilities of instability at very low values of \((\omega_n/\omega_c)^2\).

The final section of the report, Section 7, outlines the conclusions based on the numerical results in Section 6.
2. CO-ORDINATE FRAMES

The following co-ordinate frames are used in the development of
the equations of motion. (See Fig. 1):

$\tau_0 : 0'XYZ$ Inertial reference frame.
$O'Z$ along the earth's spin axis.
$0'X$ along the line of the ascending node.
$0'Y$ perpendicular to $0'X$ and $0'Z$.

$\tau_{1P} : Pi_1i_2i_3$ Local intrinsic frame at a generic point, $P$.
$Pi_1$ along the radius vector from $0'$ to $P$.
$Pi_2$ perpendicular to $Pi_1$ in the plane of $ZO'P$.
$Pi_3$ perpendicular to $Pi_1$ and $Pi_2$.

$\tau_{10} : 0'i_1i_2i_3$ Local intrinsic frame at $O$.

$\tau_2 : 0X_0Y_0Z_0$ Orbit fixed reference frame.
$OX_0$ along the local vertical
$OY_0$ along the orbit normal and in the negative direction
of the orbit angular momentum vector.
$OZ_0$ perpendicular to $OX_0$ and $OY_0$.

$\tau_3 : 0XYZ$ Principal axes of the body.

The above reference frames are related to each other as follows:

\[
\begin{bmatrix}
X_{10} \\
Y_{10} \\
Z_{10}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= T_1
\begin{bmatrix}
X_0 \\
Y_0 \\
Z_0
\end{bmatrix}
= T_2
\begin{bmatrix}
X_{10} \\
Y_{10} \\
Z_{10}
\end{bmatrix}
= T_3
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= T_2
\begin{bmatrix}
X_0 \\
Y_0 \\
Z_0
\end{bmatrix}
\]

where,

\[
T_1 = \begin{bmatrix}
\cos \omega & \sin \omega & 0 \\
- \sin \omega & \cos \omega & 0 \\
- \omega & 0 & 1
\end{bmatrix}
- (1)
\]
\[ T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi & s\chi \\ 0 & -s\chi & c\chi \end{bmatrix} \] (2)

\[ T_3 = \begin{bmatrix} c\phi c\theta & (s\phi c\psi + c\phi s\theta s\psi) & (s\phi s\psi - c\phi s\theta c\psi) \\ -s\phi c\theta & (c\phi c\psi - s\phi s\theta s\psi) & (c\phi s\psi + s\phi s\theta c\psi) \\ s\theta & -c\theta s\psi & c\theta c\psi \end{bmatrix} \] (3)

and \( s, c \) represent sine and cosine functions respectively. The detailed development of these transformations are presented in Appendix - I.

The body angular velocity components \( (\omega_x, \omega_y, \omega_z) \) and the Euler angular rates \( (\phi, \dot{\phi}, \psi) \) are related as follows:

\[ \omega_x = \dot{\theta} s\phi + \dot{\psi} c\phi c\theta - \omega_c \ (s\phi c\psi + c\phi s\theta s\psi) \]

\[ \omega_y = \dot{\theta} c\phi - \dot{\psi} s\phi c\theta - \omega_c \ (c\phi c\psi - s\phi s\theta s\psi) \]

\[ \omega_z = \dot{\psi} s\theta + \dot{\phi} + \omega_c c\theta s\psi \] (4)
In this section an expression for the gravitational force per unit mass, expressed in the body fixed frame ($r_3$) is presented. In deriving this expression, it is assumed that $|\vec{r}|/\rho<<1$, where $\vec{r}$ is the position vector of an arbitrary point in the body with respect to 0 and $\rho$ is the distance of the same point from 0.

We can write the gravitational potential in the most general form as:

$$V(\rho, \eta, \omega) = \frac{\nu a^2}{\rho} + \nu \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s+1} \Omega_s (\eta, \omega)$$

(5)

where,

$$K_s = K_{so} \cos \phi_{so}$$

$$\Omega (\eta, \omega) = \sum_{m=0}^{s} \left[ P_s^{(m)}(\eta) \cos (m\omega + \phi_{sm}) \right] / K_s$$

$P_s^{(m)}(\eta)$ is the $m^{th}$ associated Legendre function of order $s$.

$K_{so}$ and $\phi_{sm}$ are constants to be given experimentally through geodetic satellite techniques.

The gravitational force per unit mass at the origin of the body axes, 0, in the $r_1(0)$ frame is $\vec{F}_0 = \vec{V}|_0$.

For a point at a distance $\vec{r}$ from 0, neglecting small quantities of the order of $|\vec{r}|/\rho$, the gravity force in the $r_3$ frame is given by

$$\vec{F} = \vec{F}_0 + M \vec{r}$$

(10)
\[ \mathbf{T}_0 = \text{Gravitational force at } 0 \text{ expressed in the principal body axes frame} \]
\[ M = \left[ M^{(0)} + \sum_{s=1}^{\infty} K^s \left( \frac{a}{\rho} \right)^s M^{(s)} \right] \]
\[ M^{(0)} = \frac{\omega^2}{\rho^3} \begin{bmatrix}
3c^2 \phi c^2 \Theta - 1 & -3s \phi c \phi \Theta & 3c \phi c s \Theta \\
-3s \phi c \phi \Theta & 3s^2 \phi c^2 \Theta - 1 & -3s \phi c s \Theta \\
3c \phi c s \Theta & -3s \phi c s \Theta & 3s^2 \Theta - 1 
\end{bmatrix} \]
\[ M^{(s)} = \frac{\omega^2}{\rho^3} T_3 T_2 B^{(s)} T_2 T_3 T \]

The matrix \( B^{(s)} \) is given by eqn. (II-12) in Appendix - II. It can be observed that \( M^{(0)} \) and \( M^{(s)} \) are symmetric matrices.

The expression for the gravitational force in the form given in eqn. (5) is used in the development of the equations of motion presented in the next section.
4. EQUATIONS OF MOTION

In this section the equations of rotational motion and elastic motion of an arbitrary flexible body in orbit are presented. The body is assumed to be subjected to small amplitude elastic displacements, $\vec{q}$, which are transformed to elastic forces by the linear operator $L$.

Consider an elemental mass, $dm$, whose instantaneous position vector from the center of mass of the body is $\vec{r}$ (Fig. 2).

The equation of motion of $dm$ can be written as:

$$\vec{a} dm = L (\vec{q}) + \vec{f} dm + \vec{E}$$

where

- $\vec{a}$ = Inertial acceleration of $dm$
- $L(\vec{q})$ = Elastic forces acting on $dm$
- $\vec{f}$ = Gravitational force per unit mass
- $\vec{E}$ = External forces acting on $dm$
- $\vec{q}$ = Elastic displacement

The above vector equation can be written in the body fixed reference frame ($\vec{r}$) as,

$$[\ddot{\vec{a}}_{cm} + \vec{\ddot{r}} + 2 \vec{\omega} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})] dm = L (\vec{q}) + \vec{f} dm + \vec{E}$$

It is important to note that $\vec{r}$ and $\dot{\vec{r}}$ are the velocity and acceleration of $dm$ respectively as seen from the body fixed reference frame, $\vec{r}_3$. All the vectors in the above equation must be expressed in the body frame $\vec{r}_3$. 
We can write the instantaneous position vector $\mathbf{r}$ of $dm$ as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{q} \quad (9)$$

where $\mathbf{r}_0$ is the position vector of $dm$ with respect to 0 in the undeformed state; $\mathbf{q}$ is the elastic displacement of $dm$.

Hence

$$\dot{\mathbf{r}} = \dot{\mathbf{q}} \quad \text{and} \quad \ddot{\mathbf{r}} = \ddot{\mathbf{q}} \quad (10)$$

For small amplitude elastic displacements, we can write the elastic displacement, $\mathbf{q}$, as a superposition of the various modal contributions according to

$$\mathbf{q} = \sum_{n=1}^{\infty} A_n(t) \phi_n(\mathbf{r}_0) \quad (11)$$

where

$$\phi_n(\mathbf{r}_0) = \phi_x(n) \hat{i} + \phi_y(n) \hat{j} + \phi_z(n) \hat{k}$$

$A_n(t) = $ Modal amplitude

$\phi(n)(\mathbf{r}_0)$ is the mode shape associated with the natural frequency $\omega_n$ and satisfies the following orthogonality condition,

$$\int_{\text{vol}} \phi^*(m) \cdot \phi(n) \, dm = \delta_{mn} M_n \quad (12)$$

Also

$$L(\phi(n)) = -\omega_n^2 \phi(n) \quad (13)$$

Further, if the body is unconstrained, the elastic modes must be orthogonal to the rigid body modes i.e.,

$$\int_{\text{vol}} \phi^*(m) \, dm = 0 \quad (14)$$
\[
\int_{\text{vol}} \vec{r} \times \vec{\phi}'^n \, dm = 0
\]

If the body is constrained against translation and rotation at the undeformed center of mass, the corresponding modes are called "fixed modes." For fixed modes, the orthogonality conditions (14) and (15) do not hold. It should be noted that, for the case of fixed modes, the origin of the body frame, \(0\), no longer coincides with the center of mass in the deformed state. Hence, only for free modes, \( \int_{\text{vol}} \vec{r} \, dm = 0 \). However, for fixed modes \( \int_{\text{vol}} \vec{r} \, dm \neq 0 \).

4.1 Equations of rotational motion

The equations of rotational motion of the body are obtained by the following operation

\[
\int_{\text{vol}} \vec{r} \times \text{Eqn. (6)} \, dm
\]

i.e.,

\[
\int_{\text{vol}} \vec{r} \times [\vec{a}_{cm} + \vec{r} + 2 \vec{\omega} \times \vec{r} + \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})] \, dm
\]

\[
= \int_{\text{vol}} \vec{r} \times [\vec{L(q)} + \vec{F} + \vec{e}] \, dm
\]

where \( \vec{e} \) is the external force per unit mass.

The various terms in Eqn (15) can now be evaluated using the techniques of vector calculus. Assuming \( \frac{|\vec{q}|}{|\vec{F}|} \ll 1 \), only the first order terms in \( \vec{q} \) are retained in the following expansions.

\[
\int_{\text{vol}} \vec{r} \times \vec{a}_{cm} \, dm = \int_{\text{vol}} \vec{q} \, dm \times \vec{a}_{cm} \]

-10-
\[
\int_{\text{vol}} \sqrt{F} \times \dot{\sqrt{F}} \, dm = \int_{\text{vol}} (\sqrt{F_0 + q}) \times \dot{\sqrt{F_0 + q}} \, dm = \int \sqrt{F_0} \times \dot{\sqrt{F_0}} \, dm \quad (18)
\]

\[
\int_{\text{vol}} \sqrt{F} \times 2 (\dot{\sqrt{F}} \times \sqrt{F}) \, dm = 2 \int_{\text{vol}} (\sqrt{F_0 + q}) \times (\dot{\sqrt{F_0 + q}} \times \sqrt{F_0 + q}) \, dm
\]

\[
\int_{\text{vol}} \sqrt{F} \times (\dot{\sqrt{F}} \times \sqrt{F}) \, dm = \int_{\text{vol}} (\sqrt{F_0 + q}) \times \left[ \dot{\sqrt{F_0 + q}} \times \sqrt{F_0 + q} \right] \, dm
\]

\[
\int_{\text{vol}} \sqrt{F} \times \left( \dot{\sqrt{F}} \times \sqrt{F} \right) \, dm = \int_{\text{vol}} [(\sqrt{F} \cdot \dot{\sqrt{F}}) \sqrt{F} - (\sqrt{F} \cdot \dot{\sqrt{F}}) (\sqrt{F} \times \sqrt{F})] \, dm
\]

\[
\int_{\text{vol}} \sqrt{F} \times L (\dot{\sqrt{F}}) \, dm = \int_{\text{vol}} \sqrt{F_0} \times L (\dot{\sqrt{F_0}}) \, dm
\]

\[
= \sum_{n=1}^{\infty} \omega_n^2 A_n \int_{\text{vol}} \sqrt{F_0} \times \varphi^{(n)} \, dm \quad \text{(using eqn. (13))}
\]

\[
= 0 \quad \text{(for free modes. See Eqn. (15))}.
\]
\[
\iiint \mathbf{r} \times \mathbf{F} \, dm = \iiint \mathbf{r} \times (\mathbf{F}_0 + \mathbf{M} \mathbf{r}) \, dm \\
= \iiint \mathbf{q} \, dm \times \mathbf{F}_0 + \iiint (\mathbf{F}_0 \times \mathbf{q}) \times \mathbf{M} (\mathbf{F}_0 \times \mathbf{q}) \, dm \\
= \iiint \mathbf{q} \, dm \times \mathbf{F}_0 + \iiint \mathbf{F}_0 \times \mathbf{M} \mathbf{F}_0 \, dm \\
+ \iiint [\mathbf{F}_0 \times \mathbf{M} \mathbf{q} + \mathbf{q} \times \mathbf{M} \mathbf{F}_0] \, dm \quad (23)
\]

\[
\iiint \mathbf{r} \times \mathbf{e} \, dm = \mathbf{c}
\]

It can be easily shown that,
\[
\iiint \mathbf{F}_0 \times (\mathbf{w} \times \mathbf{F}_0) \, dm = J_x \dot{\omega}_x \mathbf{i} + J_y \dot{\omega}_y \mathbf{j} + J_z \dot{\omega}_z \mathbf{k} \quad (24)
\]

and
\[
\iiint (\mathbf{F}_0 \cdot \mathbf{w}) (\mathbf{w} \times \mathbf{F}_0) \, dm = (J_y - J_z) \omega_y \omega_z \mathbf{i} + (J_y - J_x) \omega_z \omega_x \mathbf{j} \\
+ (J_x - J_y) \omega_x \omega_y \mathbf{k} \quad (25)
\]

where, \( J_x, J_y, J_z \) are the principal moments of inertia of the body in the undeformed state.

After substitution of the values for the integrals in Eqn. (16) and rearrangement of terms, one can obtain the following form for the equations of rotational motion.
\[
\mathbf{R} + \sum_{n=1}^{\infty} Q^{(n)} + \mathbf{C} + \sum_{n=1}^{\infty} D^{(n)} = \mathbf{\bar{c}} + \sum_{n=1}^{\infty} \bar{g}^{(n)} \quad (26)
\]

where,
\[
\mathbf{R} = \left\{ J_x \dot{\omega}_x + (J_z - J_y) \omega_y \omega_z \right\} \mathbf{i} \\
+ \left\{ J_y \dot{\omega}_y + (J_x - J_z) \omega_z \omega_x \right\} \mathbf{j} \\
+ \left\{ J_z \dot{\omega}_z + (J_y - J_x) \omega_x \omega_y \right\} \mathbf{k} \quad (27)
\]
\[ \sum \mathcal{Q}^{(n)} = \int \left[ r_0 \times \ddot{q} + 2r_0 \times (\omega \times \dot{q}) + \ddot{r}_0 \times (\dot{\omega} \times \dot{q}) + \dot{q} \times (\dot{\omega} \times \ddot{r}_0) - (\dddot{r}_0 \cdot \ddot{w}) \right] \, dm \]

\[ \bar{C} = \int \bar{r} \times \bar{e} \, dm \]  \hspace{1cm} \text{(29)}

\[ \sum_{n=1}^{\infty} \mathcal{P}^{(n)} = \int \bar{q} \, dm \times (\dddot{\alpha}_{cm} - \dddot{r}_0) + \omega^2 \sum_{n=1}^{\infty} A_n \int \bar{r}_0 \times \mathcal{G}^{(n)} \, dm \]  \hspace{1cm} \text{(30)}

\[ \bar{G}_R = \int \bar{r}_0 \times M \bar{r}_0 \, dm \]  \hspace{1cm} \text{(31)}

\[ \sum_{n=1}^{\infty} \mathcal{G}^{(n)} = \int \left[ \bar{r}_0 \times M \bar{q} + \dot{q} \times M \ddot{r}_0 \right] \, dm \]  \hspace{1cm} \text{(32)}

The expansions of the above vector expressions are presented in Appendix - III.

The significance of the various terms in Eqn. (26) will now be briefly discussed. The terms, \( \mathcal{Q}^{(n)} \), reflect the inertia torque associated with the elastic deformations. The term, \( \bar{G}_R \), corresponds to the gravitational torque on a rigid body. The terms, \( \mathcal{G}^{(n)} \), correspond to the gravitational torque due to the elastic deformations. The terms, \( \mathcal{D}^{(n)} \), account for the difference in position between the actual center of mass and the center of mass of the undeformed body. For the case of free modes, \( \mathcal{D}^{(n)} = 0 \).
4.2. Generic mode equations

The generic mode equation is obtained by the following operation:

\[ \int_{\text{vol}} \Phi^{(n)} \cdot \Omega \, \text{dm} \]

ie

\[ \int_{\text{vol}} \Phi^{(n)} \cdot \left[ \bar{a}_{cm} + \bar{F} + 2 \bar{w} \times \bar{F} + \bar{w} \times \bar{F} + \bar{w} \times (\bar{w} \times \bar{F}) \right] \, \text{dm} \]

\[ = \int_{\text{vol}} \Phi^{(n)} \cdot \left[ \frac{L(q)}{dm} + \bar{F} + \bar{e} \right] \, \text{dm} \quad (33) \]

With the use of equations (6), (9), (10), (11), (12) and (13), the various terms appearing in Eqn. (33) can now be expanded as follows:

\[ \int_{\text{vol}} \Phi^{(n)} \cdot \bar{a}_{cm} \, \text{dm} = \int_{\text{vol}} \Phi \, \text{dm} \cdot \bar{a}_{cm} \quad (34) \]

\[ = \Sigma A_{m} \int_{\text{vol}} \Phi^{(n)} \cdot \Phi^{(m)} \, \text{dm} \]

\[ = A_{h} M_{h} \quad (35) \]

\[ \int_{\text{vol}} \Phi^{(n)} \cdot (2 \bar{w} \times \bar{F}) \, \text{dm} = 2 \int_{\text{vol}} \Phi^{(n)} \cdot \bar{w} \times \bar{F} \, \text{dm} \quad (36) \]

\[ \int_{\text{vol}} \Phi^{(n)} \cdot \left( \bar{w} \times \bar{F} \right) \, \text{dm} = \int_{\text{vol}} \left( \Phi^{(n)} \cdot \bar{w} \times \bar{F} \right) \, \text{dm} + \int_{\text{vol}} \left( \Phi^{(n)} \cdot \bar{w} \times \bar{Q} \right) \, \text{dm} \quad (37) \]

\[ \int_{\text{vol}} \Phi^{(n)} \cdot \bar{w} \times (\bar{w} \times \bar{F}) \, \text{dm} = \int_{\text{vol}} \Phi^{(n)} \cdot \bar{w} \times (\bar{w} \times \bar{F}) \, \text{dm} \]

\[ + \int_{\text{vol}} \Phi^{(n)} \cdot \bar{w} \times (\bar{w} \times \bar{Q}) \, \text{dm} \quad (38) \]
\[
\int \phi^{(n)} \cdot L(q) \, \text{vol} = -\sum_{m=1}^{N_\theta} \omega_m^2 A_m \int \phi^{(n)} \cdot \phi^{(m)} \, \text{d}m
\]

\[
= -\omega_n^2 A_n M_n
\]

\[
\int \phi^{(n)} \cdot \overline{r} \, \text{vol} = \int \phi^{(n)} \, \text{d}m \cdot \overline{r}_0 + \int \phi^{(n)} \cdot M \overline{r} \, \text{d}m
\]

\[
= \int \phi^{(n)} \, \text{d}m \cdot \overline{r}_0 + \int \phi^{(n)} \cdot M \overline{r}_0 \, \text{d}m
\]

\[
+ \int \phi^{(n)} \cdot M \overline{q} \, \text{d}m
\]

\[
\int \phi^{(n)} \cdot \overline{e} \, \text{vol} = E_n
\]

After substitution of the values for the various integrals in Eqn. (33) and rearrangement of the terms, the generic mode equation is obtained in the following form.

\[
\ddot{\phi}^{(n)} + \omega_n^2 A_n + \frac{\overline{\omega} \times \overline{r}_0}{M_n} + \sum_{m=1}^{N_\theta} \Phi_{mn} = \frac{1}{M_n} [g_n + \sum_{m=1}^{N_\theta} E_n + D_n'' - \sum_{m=1}^{N_\theta} D_{mn}]
\]

where

\[
\Phi_n = \int \phi^{(n)} \cdot \overline{w} \times \overline{r}_0 + \phi^{(n)} \cdot \overline{w} \times (\overline{w} \times \overline{r}_0) \, \text{d}m
\]

\[
\sum_{m=1}^{N_\theta} \Phi_{mn} = \int [2 \phi^{(n)} \cdot \overline{w} \times \overline{q} + \phi^{(n)} \cdot \overline{w} \times (\overline{w} \times \overline{q})] \, \text{d}m
\]

\[
g_n = \int \phi^{(n)} \cdot M \overline{r}_0 \, \text{d}m
\]

\[
\sum_{m=1}^{N_\theta} E_{mn} = \int \phi^{(n)} \cdot M \overline{q} \, \text{d}m
\]

\[
E_n = \int \phi^{(n)} \cdot \overline{e} \, \text{d}m
\]

\[
D_n'' = \int \phi^{(n)} \, \text{d}m \cdot (\overline{a}_{cm} - \overline{r}_0)
\]

\[
M_n = \int \phi^{(n)} \cdot \phi^{(m)} \, \text{d}m
\]
The expanded forms of the above expressions are presented in Appendix III.

The significance of the various terms in Eqn. (42) will now be briefly discussed. The term \( \phi_n \) corresponds to the forcing term due to rigid body motion. \( \Phi_{mn} \) is the forcing term due to the elastic motion in the \( m^{th} \) mode. The term, \( g_n \), represents the gravitational force acting on the \( n^{th} \) mode due to the rigid body motion. The terms \( g_{mn} \) correspond to the gravitational force acting on the \( n^{th} \) mode, due to the elastic motion in the \( m^{th} \) mode. \( E_n \) is the component of the external force acting on the \( n^{th} \) mode. \( D_n \) is the term corresponding to the displacement of the center of mass from the point \( O \).

In the next section, an application of the equations of motion developed in this section is presented.
5. APPLICATIONS

In this section, we consider the application of the equations of motion presented in the previous section, to the specific example of a beam in circular orbit shown in Fig. 3.

In section 5.1 it is assumed that the motion of the beam is restricted to the orbit plane. In section 5.2, the general case of the three dimensional beam is considered.

5.1. Planar motion of a long, slender, uniform beam in circular orbit

Since the beam motion is restricted to the orbit plane, the yaw and the roll angles vanish, i.e.

\[ \psi(t) = \phi(t) = 0 \]  \hspace{1cm} (50)

Also, the elastic deformation out of the orbit plane is assumed to be zero, i.e., \( \phi_y^{(n)} = 0 \).

For unconstrained structures, \( D^{(n)} = 0; D_n' = 0 \) and \( H_{\alpha\delta}^{(n)} = H_{\delta\alpha}^{(n)} \)

\[ \text{where} \]

\[ H_{\alpha\delta}^{(n)} = \int_{\text{vol}} \xi_\alpha \phi_\delta^{(n)} \, dm \quad (\alpha, \delta = x, y, z) \]

and \( (\xi_x, \xi_y, \xi_z) \) are coordinates of \( dm \) in the undeformed state measured in the body frame.

In the absence of out-of-plane deformations, i.e. \( \phi_y^{(n)} = 0 \), we can deduce that,

\[ H_{yy}^{(n)} = H_{xy}^{(n)} = H_{yx}^{(n)} = H_{zy}^{(n)} = H_{yz}^{(n)} = 0 \]  \hspace{1cm} (52)
Noting that, for a beam (see Fig. 3) with uniform cross section,
\[ \int \bar{e}_x \, d\bar{e}_y \, d\bar{e}_z = \int \bar{e}_y \, d\bar{e}_x \, d\bar{e}_z = \int \bar{e}_z \, d\bar{e}_x \, d\bar{e}_y = 0 \]  
(53)
we can further deduce that,
\[ H^{(n)}_{xx} = \int \frac{u}{\text{vol}} \phi_x^{(n)}(\xi) \, d\xi \, d\bar{e}_x \, d\bar{e}_z = 0 = H^{(n)}_{xz} \]  
(54)
\[ H^{(n)}_{zz} = \int \frac{u}{\text{vol}} \phi_z^{(n)}(\xi) \, d\xi \, d\bar{e}_x \, d\bar{e}_z = 0 \]  
(55)
where, \( u \) is the density of the beam material.

Since, by definition,
\[ L^{(mn)} = \int \frac{\alpha^{(m)}}{\text{vol}} \phi_s^{(n)}(\xi) \, d\xi \, d\bar{e}_s \quad (s = x, y, z) \]
in the absence of out-of-plane elastic deformation, i.e., \( \alpha^{(n)} = 0 \), it can be easily shown that
\[ L^{(mn)}_{xy} = L^{(mn)}_{yx} = L^{(mn)}_{yz} = L^{(mn)}_{zy} = 0 \]  
(56)

With the use of the above results and the results presented in Appendix - III, the pitch and the generic mode equations for the present case are obtained as:
\[ J_{yy} \dot{\gamma}_{yy} + \sum_{n=1}^{\infty} \gamma^{(n)} + C_y = \ddot{\gamma}_{yy} + \sum_{n=1}^{\infty} \gamma^{(n)} \quad \text{(pitch equation)} \]  
(57)
where,

\[ \gamma_y = \frac{\beta}{\alpha} \]
\[ \gamma^{(n)} = \frac{\alpha}{\beta} [A_n \phi - \alpha] H^{(n)} \]
\[ C_N = (C_N - C_z) M_{31} = - J_{zz} M_{31} \quad \text{(for } J_N \ll J_z) \]
\[ C_y = - \beta M_{31} H^{(n)} \]
\[ M_{31} = \frac{\beta}{\alpha} \left[ \sum_{s=1}^{\infty} \nu^{(s)} K_s \left( \frac{a}{c} \right) \nu_{31}^{(s)} \right] \]
The generic mode equations:

\[ \ddot{A}_n + \omega_n^2 A_n + \frac{\Phi_n}{M_n} + \frac{1}{M_n} \sum_{m=1}^{\infty} \phi_{mn} = \frac{1}{M_n} \left( g_n + \sum_{m=1}^{\infty} g_{nn} + \varepsilon_n \right) \]  

where,

\[ \phi_n = \left( \theta - \omega_c \right)^2 H^{(n)}_{xx} \]
\[ \phi_{mn} = (2A_m \left( \theta - \omega_c \right) + A_m \ddot{\theta}) \left( L^{(mn)}_{xx} - L^{(mn)}_{zz} \right) - A_m \left( \theta - \omega_c \right)^2 \left( L^{(mn)}_{xx} + L^{(mn)}_{zz} \right) \]
\[ g_n = M_{11} H^{(n)}_{xx} \]
\[ g_{mn} = A_m \left[ M_{11} L^{(mn)}_{xx} + M_{13} (L^{(mn)}_{xz} + L^{(mn)}_{zx}) + M_{33} L^{(mn)}_{zz} \right] \]

We will now consider several specific cases of the long slender beam assumed to be subjected to rotations and deformations only within the orbital plane.

5.1.1. The case of no longitudinal vibrations: i.e., \( \phi_x^{(n)} = 0 \).

For this case it can be seen that,

\[ H^{(n)}_{xx} = L^{(mn)}_{xz} = L^{(mn)}_{zx} = L^{(mn)}_{xx} = 0 \]  

By choosing \( \phi_z^{(n)} \) to represent the eigen-modes of bending vibrations of a free-free beam, we obtain,

\[ L^{(mn)}_{zz} = \delta_{mn} M_n \]  

where, \( \delta_{mn} \) is the kronecker delta and \( M_n \) is the generalized mass of the beam in its \( n^{th} \) mode. An expression for the value of \( M_n \) for free-free beams is presented in Appendix - IV.
Substituting the above results in equations (57) and (59), one obtains the pitch equation and generic mode equation in the following form.

\[ \ddot{\theta} + M_{31} + \frac{C_y}{y} = 0 \quad (61) \]

Generic mode equation:

\[ \ddot{A}_n + [\omega_n^2 - M_{33} - (\dot{\theta} - \omega_c)^2]A_n = \frac{E_n}{M_n} \quad (62) \]

where,

\[ M_{31} = \frac{\omega_a}{\rho^2} \left[ s \theta \cot\theta + \sum_{s=1}^{\infty} K_s \left( \frac{a_s}{\rho} \right) M_{31}^{(s)} \right] \]

\[ M_{33} = \frac{\omega_a}{\rho^2} \left[ 3 s^2 \theta^2 - 1 + \sum_{s=1}^{\infty} K_s \left( \frac{a_s}{\rho} \right) M_{33}^{(s)} \right] \]

Assumptions of a spherically symmetric gravitational field and circular orbit result in the following simplification of \( M_{31} \) and \( M_{33} \),

\[ \frac{\omega_a^2}{\rho^2} = \omega_c^2 \]

\[ M_{31} = \frac{3}{2} \omega_c^2 \sin 2\theta \]

\[ M_{33} = \omega_c^2 (\sin^2 \theta - 1) \]

Hence, equations (61) and (62) simplify to,

\[ \ddot{\theta} + \frac{3}{2} \omega_c^2 \sin 2\theta + \frac{C_y}{\gamma} = 0 \quad (63) \]

\[ \ddot{A}_n + [\omega_n^2 - \omega_c^2 (3 \sin^2 \theta - 1) - (\dot{\theta} - \omega_c)^2]A_n = \frac{E_n}{M_n} \quad (64) \]

Equation (63) is just the rigid body pitching equation. Thus from Eqn. (63), one can conclude that there is no influence of the elastic motion on the rigid body motion, under the assumptions of the present case. The same conclusions were arrived at by Ashley for the case of a thin beam undergoing flexural deformations in the orbit plane.
If the rigid body pitch oscillations are small i.e. $\theta \ll 1$, equations (63) and (64) further simplify to (assuming no external disturbances),

\[
\ddot{\theta} + 3 \omega_c^2 \dot{\theta} = 0 \tag{65}
\]
\[
\ddot{A}_n + [\omega_n^2 - \dot{\theta}^2 + 2\omega_c \dot{\theta}] A_n = 0 \tag{66}
\]

Without loss of generality, one can assume the solution of equation (65) as,

\[
\theta = c \sin (\sqrt{3} \omega_c t + \gamma) \tag{67}
\]

where, $c$ is the amplitude of the pitch motion.

Substitution of Eqn. (67) into Eqn. (66) yields,

\[
\ddot{A}_n + [\omega_n^2 - 3 \omega_c^2 c^2 \cos^2 (\sqrt{3} \omega_c t + \gamma) \\
+ 2\sqrt{3} \omega_c^2 c \cos (\sqrt{3} \omega_c t + \gamma)] A_n = 0 \tag{68}
\]

With the introduction of dimensionless variables, $\tau = \frac{1}{2} (\sqrt{3} \omega_c t + \gamma)$ and $z_n = A_n / \ell$ where $\ell$ is the length of the beam, one can write,

\[
\ddot{A}_n = \frac{\sqrt{3}}{2} \omega_c \ell \frac{dz_n}{dt} \text{ and } \dddot{A}_n = \frac{3}{4} \omega_c^2 \ell \frac{d^2z_n}{dt^2} \tag{69}
\]

After substituting Eqn. (69) into Eqn. (68), one obtains the generic mode equation in the following non-dimensional form,

\[
\frac{d^2 z_n}{dt^2} + \frac{4}{3} \left[ \left( \frac{\omega_n^2}{\omega_c^2} - 3 \frac{c^2 \cos^2 2\tau}{2} + 2 \sqrt{3} \frac{c \cos 2\tau}{2} \right) \right] z_n = 0 \tag{70}
\]

Using the trigonometric identity, $\cos^2 2\tau = \frac{1}{2} (\cos 4\tau + 1)$, Eqn. (70) can be re-written as,

\[
\frac{d^2 z_n}{dt^2} + \frac{4}{3} \left[ \left( \frac{\omega_n^2}{\omega_c^2} - \frac{3c^2}{2} \right) + 2\sqrt{3} c \cos 2\tau - \frac{3c^2}{2} \cos 4\tau \right] z_n = 0 \tag{71}
\]
Equation (71) is in the form of "Hill's 3-term equation" or "Whittaker's equation." For small pitch amplitudes i.e., \( c << 1 \), the above equation can be further approximated by the Mathieu equation

\[
\frac{d^2 z_n}{dt^2} + (\delta + \epsilon \cos 2\tau) z_n = 0
\]

(72)

where,

\[
\delta = \frac{4}{3} \left( \frac{\omega_n^2}{\omega_c^2} - \frac{3}{2} c^2 \right)
\]

\[
\epsilon = \frac{3\sqrt{3}}{3} c = 4.62 c
\]

Figure 3 shows the Mathieu stability diagram with \( \epsilon \) and \( \delta \) as parameters. It can be seen from the Mathieu chart that for the values of \( \delta \) around unity i.e., \( \frac{\omega_n^2}{\omega_c^2} = 5'(1) \) the system may enter the region of instability. However for large values of \( (\omega_n/\omega_c)^2 \) the coefficient of \( z_n \) in Eqn. (72) is dominated by \( (\omega_n/\omega_c)^2 \). Hence, in the high frequency range, the elastic modes are essentially governed by the following simple equation:

\[
\frac{d^2 z_n}{dt^2} + \frac{4}{3} \left( \frac{\omega_n}{\omega_c} \right)^2 z_n = 0 \quad (\omega_n/\omega_c >> 1)
\]

Thus, it can be concluded that for beams with \( (\omega_n/\omega_c)^2 >> 1 \) (n = 1, 2, ... \( \infty \)), the elastic motion and the rigid body pitching motion are completely decoupled from each other. However, if \( (\omega_n/\omega_c)^2 = \delta'(1) \), (highly flexible beams) one has to consider Eqn. (71) to study the elastic motion, and thus the elastic motion is coupled with the rigid body pitching motion.
In Section 6, numerical solutions of Eqn. (71) are presented, for some typical values of \( \omega_n / \omega_c \).

5.1.2. The case of no flexural vibrations: \( \phi_z^{(n)} = 0 \)

In this Section, we specialize equations (57) and (58) by restricting the beam to have no bending or flexural vibrations, but allow for the possibility of longitudinal vibrations. With the above assumption and the assumption of circular orbit in a spherically symmetric gravitational field, the following simplifications result,

\[
L_{zz}^{(mn)} = L_{zx}^{(mn)} = L_{xz}^{(mn)} = 0
\]  

\[
M_{31} = \omega_c^2 \sin^2 \theta
\]  

\[
M_{11} = \omega_c^2 (3 \sin^2 \theta - 1)
\]  

Substitution of the above results in equations (57) and (58) leads to, (assuming \( C_y = 0 \)),

\[
J_y \ddot{\theta} + 2 \sum_{n=1}^{\infty} \left[ \ddot{A}_n (\dot{\theta} - \omega_c) + A_n \ddot{\theta} \right] H^{(n)}_{xx} + \left[ J_y + \sum_{n=1}^{\infty} H^{(n)}_{xx} \right] \frac{\omega_c^2}{2} \sin 2\theta = 0
\]  

\[
\ddot{A}_n + \omega_n^2 A_n - \left[ (\dot{\theta} - \omega_c)^2 + \omega_c^2 (3 \sin^2 \theta - 1) \right] \left( \frac{H^{(n)}_{xx}}{M_n} + A_n \right) = \frac{E_n}{M_n}
\]  

Equations (76) and (77) are consistent with Ashley's equations (A-8) and (A-9) in Ref. 2, which were derived from an energy approach.

Ashley has further shown in Ref. 1, that if the beam is spinning at a high enough spin rate, i.e. \( \omega_c / \Omega \ll 1 \), where \( \Omega \) here represents the spin angular velocity, the response of Eqn. (77) for only the first mode, consists of steady stretching on which can be superimposed an oscillation with the physical frequency of \( \left( \omega_{10}^2 + 2.93 \Omega^2 \right)^{1/2} \) where, \( \omega_{10} \) is the fundamental frequency of the beam in the longitudinal vibrations assuming no spin.
Ashley has also investigated another special case of equation (77) by assuming \( \theta(t) = \text{constant} \). Physically this implies that the beam is forced to maintain a constant orientation with respect to the local vertical. In this case, it can be easily seen that Eqn. (77) reduces to that of an harmonic oscillator with the physical frequency of \( \sqrt{[\omega_n^2 - \omega_c^2 + \omega_c^2(1 - 3c^2\theta)]^2} \). The third term in this frequency expression is due to the gravity-gradient effect. Thus, the gravity-gradient contribution to the frequency changes sign when \( c^2\theta = \frac{1}{3} \).

For beams with \( \omega_n = \omega_c'(\omega_c) \), the possibility of buckling instability at \( \theta = 0 \) is also evident from the frequency expression.

5.2. Three dimensional motion of a long, slender, uniform beam in circular orbit: (assuming yaw to be zero, i.e. \( \psi = 0 \)).

This section presents the equations of three dimensional motion of a beam in circular orbit. Some of the following results which were obtained in Section 5.1 also hold for the present case, i.e.

\[
D^{(n)} = 0; D'_n = 0 \quad \text{and} \quad H^{(n)}_{\alpha\beta} = H^{(n)}_{\beta\alpha} \quad (51)
\]

\[
H^{(n)}_{xz} = H^{(n)}_{zx} = H^{(n)}_{xy} = H^{(n)}_{yx} = H^{(n)}_{yy} = H^{(n)}_{y^2} = H^{(n)}_{zz} = H^{(n)}_{yz} = H^{(n)}_{zy} = 0 \quad (78)
\]

Also, \( J_y = J_z \) and \( J_y - J_x = J_x \) for long, slender beams.

Hence, the pitch equation is,

\[
J_y \ddot{\omega}_y - J_y \omega_z \omega_x + \sum_{n=1}^{\infty} Q^{(n)}_y + C_y \dot{\gamma}_y = G^{(n)}_y \quad (79)
\]
\[ W = (6 - wc) \]

\[ y = (e - wc)c \]

\[ w_0(n) = 2 \left[ A_y w_z + A_z (w_z + w_x w_z) \right] H^{(n)} \]

\[ Y' = - j M_{31} \]

\[ G(n) = - 2 M_{31} A_n H^{(n)}_{xx} \]

The roll equation is,

\[ J_z \dot{w} + J_z \omega_x \omega_y + \sum_{n=1}^{\infty} Q_z(n) + C_z = G_{Rz} + \sum_{n=1}^{\infty} C_{z(n)} \] (80)

where,

\[ Q_z(n) = 2 \left[ A_y w_z + A_z (w_z + w_x w_z) \right] H^{(n)} \]

\[ G_{Rz} = J_z M_{12} \]

\[ G_z = 2 M_{12} A_n H^{(n)}_{xx} \]

\[ M_{12} = - 3 \frac{w_x^2 \phi \sigma \phi \phi \phi}{\rho^3} + \frac{w_x^2 \phi \sigma \phi \phi \phi}{\rho^3} K_y (\frac{\phi}{\rho}) M_{12} \]

The generic mode equation is,

\[ A_n + \omega^2 A_n + \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn} = \frac{1}{M_n} \left( g_n + \sum_{m=1}^{\infty} g_{mn} + E_n \right) \] (81)

where,

\[ g_n = M_{11} H^{(n)}_{xx} \]

\[ g_{mn} = A_m \left[ L^{(mm)}_{xx} + \left( M_{22} + M_{33} + 2M_{23} \right) L^{(mm)}_{zz} + \left( L^{(mm)}_{zz} + L^{(mm)}_{xz} \right) \right] \]

\[ \phi_{mn} = 2 \left( w_y - w_z \right) \left( L^{(mm)}_{zx} - L^{(mm)}_{xz} \right) \]

\[ + A_m \left( \omega_y - \omega_z \right) \left( L^{(mm)}_{zx} - L^{(mm)}_{xz} \right) + \omega_x \left( \omega_y + \omega_z \right) \left( L^{(mm)}_{zx} + L^{(mm)}_{xz} \right) \]

\[ + \left( 2 \omega_y \omega_z - \omega_x^2 - \omega_y^2 - \omega_z^2 \right) L^{(mm)}_{zz} - \left( \omega_y^2 + \omega_z^2 \right) L^{(mm)}_{xx} \]
5.2.1 The case of no longitudinal vibrations: \( \phi_\chi = 0 \).

In this case, the following further simplifications result, i.e.

\[
H(n) = L_{xx} = L_{xz} = L_{zx} = L_{xx} = 0
\]  

(82)

Furthermore, if we choose the eigen modes of bending vibrations of a free-free beam to be \( \phi_z(n) \), then,

\[
L_{zz}(mn) = \delta_{mn} M_n.
\]  

(83)

Assumption of a spherically symmetric gravity field results in,

\[
M_{12} = -3 \omega_c^2 \sin \phi \cos \theta \cos \varphi \; M_{22} = (3 \sin^2 \theta \cos^2 \varphi - 1) \omega_c^2
\]  

\[
M_{23} = -3 \omega_c^2 \sin \phi \sin \theta \; M_{33} = (3 \sin^2 \theta - 1) \omega_c^2
\]  

\[
M_{31} = 3 \omega_c^2 \cos \phi \sin \theta
\]  

With the above results substituted in equations (79), (80) and (81), one obtains the following equations of:

Pitch:

\[
\ddot{\omega}_y - \omega_z \omega_x + M_{31} + C_{y}/J_y = 0
\]  

(85)

Roll:

\[
\dot{\omega}_z + \omega_x \omega_y - M_{12} + C_{z}/J_z = 0
\]  

(86)

Generic mode:

\[
\ddot{A}_n + (\omega_n^2 + 2 \omega_y \omega_z - 2 \omega_x^2 - \omega_y^2 - \omega_z^2 - M_{22}) A_n = \frac{E_n}{M_n}
\]  

(87)

where,

\[
\omega_x = (\dot{\theta} - \omega_c) \sin \phi
\]  

\[
\omega_y = (\dot{\theta} - \omega_c) \cos \phi
\]  

\[
\omega_z = \dot{\phi}
\]
The assumption of small amplitude pitch and roll, i.e. $\theta << 1$ and $\phi << 1$, yields the following set of linearized equations for pitch and roll, after neglecting the terms containing the products and powers of $\phi, \dot{\phi}, \theta$ and $\dot{\theta}$.

Pitch: 
\[ \ddot{\theta} + 3 \omega_c^2 \theta + \frac{C_y}{y} = 0 \] (88)

Roll: 
\[ \ddot{\phi} + 3 \omega_c^2 \phi + \frac{C_z}{z} = 0 \] (89)

The generic mode equation is,
\[ \ddot{A}_n + [\omega_n^2 + 2 \dot{\phi} (\dot{\theta} - \omega_c) - (\dot{\theta} - \omega_c)^2 - \dot{\phi}^2 + 2 \omega_c^2]A_n = \frac{E_n}{M_n} \] (90)

5.2.2 The case of no flexural vibrations ($\varphi^{(n)}_y = \psi^{(n)}_y = 0$):

In this case, the assumption of no flexural vibrations results in,
\[ L^{(mn)}_{zz} = L^{(mn)}_{zx} = L^{(mn)}_{xz} = 0 \] (91)

Also, the assumption of $\varphi^{(n)}_x$ to be the eigen-modes of longitudinal vibration of the beam leads to the result,
\[ L^{(mn)}_{xx} = \delta_{mn} M_n (m = 1, 2, \ldots \infty) \] (92)

As in Section 5.2.1, the assumption of a spherically symmetric gravity field, yeilds

\[ M_{11} = (3 s^2 \phi c^2 \theta - 1)\omega_c^2 ; \quad M_{12} = -3 \omega_c^2 \phi \phi \ c \theta \ \\
M_{22} = (3 s^2 \phi c^2 \theta - 1)\omega_c^2 ; \quad M_{23} = -3 \omega_c^2 \phi \phi \ s \theta \ \\
M_{23} = 3 \omega_c^2 \phi \phi \ c \theta \ s \theta ; \quad M_{33} = (3 s^2 \theta - 1)\omega_c^2 \] (93)
For this case, the pitch and roll equations remain the same as equations (79) and (80). The generic mode Eqn., (81), simplifies to the following form:

\[
\ddot{A}_n + \omega_n^2 A_n + \frac{\Phi_n}{M_n} + \frac{1}{M_n} \sum_{m=1}^{\infty} \Phi_{mn} = \frac{1}{M_n} (g_n + \sum_{m=1}^{\infty} g_{mn} + E_n)
\]

(94)

where,

\[\Phi_n = - (\omega_y^2 + \omega_z^2) \lambda_{nx}^{(n)} \]

\[\Phi_{mn} = - A_m [\omega_y^2 + \omega_z^2] M_n \delta_{mn} \]

\[g_n = M_{ll} \lambda_{nx}^{(n)} \]

\[g_{mn} = A_m M_{ll} M_n \delta_{mn}. \]
6. NUMERICAL RESULTS

In this section, the numerical solutions of equation (71), for a few typical values of \( (\omega_n/\omega_c)^2 \) shown in Table -1, are presented. Cases 1-5 correspond to very flexible beams. Eqn. (71) was integrated, using a Runge-Kutta fourth order method with variable step size, on a Nova 840 digital computer. As an initial guess a step size of approximately \( 1/500 \)th of the orbital period was chosen for all the computer runs. A pitch amplitude of 0.2 radians was assumed, which is the upper limit for \( \theta \) for which the approximation \( \sin \theta \approx \theta \) is still valid.

The responses shown in Figs. 5 and 6 for the value of \( (\omega_n/\omega_c)^2 = 1.0 \) indicate the instability of the system at very low natural frequencies. Point 1 corresponding to this case on the Mathieu stability diagram in Fig. 4 also indicates this instability. The other points on the Mathieu stability diagram correspond to other cases shown in Table 1 and are referred to by the appropriate case number. For the other values of \( (\omega_n/\omega_c)^2 \) considered, the beam response is sinusoidal to a very close approximation, see Figs. 7, 8, 9 and 10. Phase plane plots of cases 1, 2, 3, 4 are shown in Figs. 11, 12, 13, and 14, respectively.

It is to be noted however, that when the pitch amplitude exceeds 0.2 radians, one has to use equations (63) and (64) for simulation.

The development of a general computer program which treats the combined pitch and generic modal equations will be discussed in Section 6.1.
### Table 1: Data assumed for numerical simulation

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Assumed data</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(\omega_n/\omega_c)^2$</td>
<td>$c^{**}$</td>
</tr>
<tr>
<td>1.</td>
<td>1.0</td>
<td>0.2</td>
</tr>
<tr>
<td>2.</td>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td>3.</td>
<td>2.0</td>
<td>0.2</td>
</tr>
<tr>
<td>4.</td>
<td>5.0</td>
<td>0.2</td>
</tr>
<tr>
<td>5.</td>
<td>10.0</td>
<td>0.2</td>
</tr>
<tr>
<td>6.*</td>
<td>3200</td>
<td>0.2</td>
</tr>
</tbody>
</table>

* This case corresponds to a beam with a fundamental natural frequency of $1/100$ cps and moving in a circular orbit of 250 n. miles altitude.

** $c$ - Pitch amplitude in radians.
6.1 The Flexbeam Computer Program

A. Development of Computer Program

The first order non-linear equations of motion for the long-flexible beam in orbit as developed in section 5, Eqs. (57) and (58), were coded for numerical simulation using the Nova 840 digital computer system. The program, Flexbeam, consists of nine subprograms whose names and functions are:

FBSET - (a) set up initial state, (b) an input quantity, Precision, dependent upon the desired precision (range 1.0 to 10^-5); the size of this input parameter is inversely proportional to the desired accuracy (and also the number of iterations required per computational time step).

FBHST - set \( h_{ij}^{(n)} \) where \( n = 1, 2, \ldots, M \), \( M_{\text{max}} = 20 \) modes

FBLST - set \( l_{ij}^{(r,n)} \) where \( r = 1, 2, \ldots, M \), and \( n = 1, 2, \ldots, M \).

FRDIF - set differential equations which include primary and secondary functions

RKSCL - (a) set full scale values of the initial conditions: \( \phi_0, \dot{\phi}_0, \nabla_0, \) and \( \dot{\nabla}_0 \), (b) set the number of ordinary differential equations to be evaluated, (c) set the bounds on the maximum number of iterations (11) that the Runge-Kutta subroutine is allowed per time step in order to fulfill the desired precision specified in FBSET. If more than this number of iterations would be required the Runge-Kutta subroutine is automatically terminated.
In this case, the precision parameter or full-scale values would have to be adjusted.

RKGS - Uses the Runge-Kutta method to obtain an approximate solution of the system of first order differential equations, given initial conditions.

FBP1F - calculation of primary intermediate functions, i.e.

\[ M_{11}, M_{12}, R_2, \sin \phi \text{ and } \cos \phi, \text{ etc.} \]

FB1F - calculation of secondary intermediate functions, i.e.

\[ \phi_r, g_r, \phi_m \text{ and } g_m \]

FBOUT - sets the scale magnitudes of the output variables \( \phi, \dot{\phi}, A_n, \) and \( \dot{A}_n, \) equal to the maximum expected amplitudes for use in the plot routine.

The flow diagram, Fig. 15, depicts the various subprograms and how input information, as well as the resulting output of each subprogram, is passed through the Flexbeam main program.

B. Flexbeam Usage

Flexbeam is a computer simulation of the dynamics of a flexible beam in a circular orbit about the earth. The programming language used in the main program is Fortran Five. This language allows the results to be obtained faster than the same results obtained, using Fortran Four. (The Flexbeam output program is coded in Fortran Four.) Flexbeam, contains three principal components: Flexbeam Main Program, (refer to Fig. 15), Flexbeam Input Data and Flexbeam Output Program.
Inside the main program the subroutine FBINP is associated with Flexbeam Input Data. The purpose of this subroutine is to specify the order and format with which the input data is read into the computer. Block data, also inside the main program, sets the constant parameters, i.e. gravitational constant, NU, orbit frequency, WD, external forces, QCP, C2, and terms involving the higher harmonics of the earth's gravitational field, J2 and J3, to their full-scale values; also constant parameters associated with each mode are set, i.e. the modal frequency, DW. The number of modes (M) and the number of differential equations being considered, NORD, are also specified here.

Flexbeam Input Data is the component program that allows the main program to be executed. The first input of the first data card denotes the total time interval over which the numerical integration is to be performed, e.g. 0.0 to 300.0 sec. The second input on this card is the computational time step, 0.5 sec. The third input is the precision, 1.0. The second data card contains the initial conditions: \( \phi_0 \), \( \dot{\phi}_0 \) and their scale values. The remaining data cards contain \( A_n_0 \), \( \dot{A}_n_0 \), their scale values and the modal frequency, in that order for each mode being considered. Sample data cards are shown in Fig. 16 and input data, for a specific case, is shown on page 12 of the program listing (Appendix V - 12).

The Flexbeam Output Program prints and/or plots any and/or all of the following: pitch angle (and pitch rate), modal amplitudes (and rates), and deflections taken at particular points along the undeformed beam, all as functions of time.
(The designated points at which the deflections are to be calculated
are specified according to the position of the particular point from
the left end of the beam, non-dimensionalized by the total beam length.)
The desired outputs can be implemented by placing the "amplitude vs. time"
deck of cards, (page V - 17 of the program listing), "print out-put"
deck of cards, (page V - 16) of the program listing), or the "call
to deflection plot (DFPLO)"
subroutine, (page V -13 of the program listing), into the main part of this program which is also on page V - 13.
6.2 Numerical results of Flexbeam Program

The results obtained from the computer simulation of the dynamics of a flexible beam in orbit using the Flexbeam Program will be considered. The beam is assumed to be a long slender hollow tube made of wrought aluminum (2014T6). The length of the beam is taken as 100 meters, its outside diameter and thickness are: 0.05 meters and 0.005 meters, respectively. The structural rigidity (EI) of the beam is $7.707 \times 10^3$ n·m² and the mass per unit length is $9.906 \times 10^4$ kg/m. It is further assumed that the c.m. of the system follows a 250 n.mi (463.31 km) altitude circular orbit.

The initial conditions which remained constant throughout all simulation runs are: $\phi(0) = 0.2$ rad., $\dot{\phi}(0) = 0.0$ rad./sec., $A_n(0) = 0.5$ meters (maximum value) and $\dot{A}_n(0) = 0.0$ meters/sec. Other parameters, i.e. the time interval of the numerical simulation and the generic modal frequencies, were varied. Variations in these parameters and their effect upon the deflection of the beam throughout its entire length will be discussed.

The first two cases of the simulation were constructed to test the program. The first, depicted in Fig. 17, is a simulation of pitch motion with a superposition of the first and second generic modes, (refer to Table 2). Each mode is assumed to have an initial amplitude coefficient of 0.5 meters. The deflection is calculated using Eq. (11), where $\hat{\phi}^{(n)}(r_0)$ has only one component, $\phi_z^{(n)}$. 
The deflected beam was examined at three locations, the left and right nodal points of the first mode and the central nodal point of the second mode, (Table 2). In Figs. 17a and c, we observe the predominant effect of the frequency response of the second mode, whereas in Fig. 17b, that of the first mode. Discrepancies between the responses shown in Fig. 17 and purely simple harmonic motion at a particular modal frequency, are attributed to: (1) small numerical errors in calculating the exact location of the nodal points and (2) nonlinearities associated with pitch-rate coupling in the generic modal equations. In the second case, Fig. 18, the frequency of the first mode is set equal to the orbital frequency. The frequencies of modes two and three are calculated based on criterion consistent with free-free beams, (Table 2). Figs. 18a and b. depict the responses of the first generic mode and then all three generic modes, respectively. In each figure, we note the growth of the amplitude of the first mode due to orbital resonance as simulated. Fig. 19 is associated with Fig. 18b. This figure illustrates the deflection of the entire beam for a one hundred second time interval. This interval was chosen to show the dominating effect of the first mode during this part of the response. It should be noted that the case shown in Fig. 18 for the response of the first generic mode, duplicates the earlier result described in Fig. 5, after accounting for the nondimensionalization of $A_1$ and noting that the initial conditions in $\phi$ and $\dot{\phi}$ shown in Fig. 18a, will result in a pitch amplitude of 0.2 radians.
The third and remaining cases of the simulation shown in Figs. 20 - 29, all depict the following plots; modal amplitude and deflection vs. non-dimensional beam length as functions of time. The modal amplitude responses are used to study the interaction of the modes superimposed upon one another; first with initial values of the mode amplitudes equally weighted, i.e. $A_n(0) = 0.5$ meters (Fig. 20), then unequally weighted, i.e. Figs. 22, 25a, and 27. Figs. 22 and 25a can be compared since the initial conditions are similar. It can be observed that with the frequencies chosen for the first generic mode, $\omega_1 = 0.0628$ in Fig. 22 and $\omega_1 = 0.628$ in Fig. 25a, which represents $\frac{1}{100}$ and $\frac{1}{10}$ cycles per second, respectively, the responses shown, illustrate the effect of the greater rigidity in the latter case. The deflection plots are used to study the deflection of the entire beam during specific time intervals. As a particular example, we consider Figs. 20 and 21 jointly, to study the effect of the superimposition of the different modes at different intervals. We note that at the beginning of the simulation the signs of all three modal amplitudes ($A_1 - A_3$) are positive. Figures 21a and b. show the deflection of the beam throughout the length of the entire beam during the first 21 seconds. From Table 2, it is observed that the deflections at the left end of the beam due to the first three (or more) generic modes, will be additive providing the modal amplitude factors have the same sign. This phenomena is apparent in Fig. 21a, where the initial larger deflection at the left end of the beam should be noted.
The first and second modal amplitudes have negative values between 42.5 and 48 secs. whereas the third modal amplitude has a positive value, (Fig. 20). It should also be noted that the first modal amplitude reaches a maximum negative value at 45 secs. During this time period, the dominating influence of the first mode in the deflection response (Fig. 21c), is apparent. The contributions of the second and third mode tend to compensate each other.

The remaining amplitude and deflection responses showing the effect of different initial conditions and the numbers of modes in the model can be examined in a similar manner and will be useful in the forthcoming simulation of the free-free beam under the action of various control devices.

Figure 30 shows a typical response of the pitch motion of the beam for a simulation which involved only one generic mode. Since the pitch motion is not coupled to first order with the generic modal motion [Eqns. (88) and (90)], this type of response is representative of all pitch motions simulated for small pitch amplitudes.

For all numerical cases reported here, an average of 10-12 minutes computational time was required to simulate the dynamics over an 80 minute interval of real time using the NOVA 840 computer system.
This report presents the development of the equations of motion of an arbitrary flexible body in orbit. In the case of planar motion of a long, slender beam in a circular orbit, undergoing small pitch oscillations and flexural vibrations, the pitch motion completely decouples from the elastic motion and the elastic motion is governed by a Hill's 3-term equation. For large values of $(\omega_n/\omega_c)^2$ the elastic motion completely decouples from the pitch motion and the elastic motion closely approximates that of an harmonic oscillator. However, the numerical results indicate the possibilities of instabilities at very low values of $(\omega_n/\omega_c)^2$. A general computer program which treats both the pitching motion and the first-order flexural vibrations of a thin beam in orbit is developed; this program can be modified to simulate the effects of both external environmental torques and control torques that may be provided by actuators located at various positions along the beam.
REFERENCES


0' - Center of the earth

0 - Center of mass of the body

P - An arbitrary point in the body

Fig. 1: Co-ordinate Frames
$\mathbf{r}_0$ - Position vector of $P$ before deformation

$\mathbf{r}$ - Position vector of $P$ after deformation

$\mathbf{q}$ - Elastic displacement

Fig. 2: Deformed and Undeformed Configurations of the Body.
Fig. 3: Beam Satellite
Fig. 4: Mathieu Stability Diagram
(Shaded regions indicate stability)
T = orbital period

\[ \left( \frac{\omega_n}{\omega_c} \right)^2 = 1.0 \]

\[ c = 0.2 \]

Fig. 5: Modal Amplitude Response - Very Flexible Beam
Fig. 6: Modal Amplitude Response - Very Flexible Beam

\[ \frac{\omega_n}{\omega_c} = 1.0 \]
\[ \varepsilon = 0.05 \]

\( T = \) orbital period

\( z_n \times 10^5 \)

\( t/T \)
Fig. 7: Modal Amplitude Response - Effect of Increased Beam Stiffness

\[ T = \text{orbital period} \]

\[ \left(\frac{\omega_n}{\omega_c}\right)^2 = 2.0 \]

\[ c = 0.2 \]
Fig. 8: Modal Amplitude Response - Effect of Increased Beam Stiffness
Fig. 9: Modal Amplitude Response - Effect of Increased Beam Stiffness

\[ T = \text{Orbital period} \]

\[ \left( \frac{\omega_n}{\omega_c} \right)^2 = 10.0 \]

\[ \psi = 0.2 \]

\[ 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ t/T \]
Fig. 10: Modal Amplitude Response - Effect of Increased Beam Stiffness.
Fig. 11: Phase Plane Plot (case 1)

\[
\left(\frac{\omega_n}{\omega_c}\right)^2 = 1
\]

\[
c = 0.2 \text{ rad.}
\]

\[
\tau = 11.0
\]
Fig. 12: Phase Plane Plot (case 2)
Fig. 13: Phase Plane Plot (case 3)

\[ \frac{n}{\omega_c} = 2 \]
\[ c = 0.2 \text{ rad.} \]

\[ \tau = 11.0 \]
\[ \tau = 0 \]
\[ z_n \times 10^5 \]
Fig. 14: Phase Plane Plot (case 4)
Fig. 15: Flow Diagram of Flexbeam Program
Howard University NOVA - 840 Digital Computer
Fig. 16. Sample Data Input Cards
Deflection at the Left Nodal Point of First Mode, (meters)

$$\delta (\xi, t)|_{\xi = 0.224}$$
Fig. 17b: Beam Deflection Time Response
Fig. 17c: Beam Deflection Time Response

\[ \delta (x, t) \big|_{t} = 0.776 \]

\[ \omega_2 = 0.173317 \text{ r/s} \]
Fig. 18a: Time Response of First Generic Mode
(Orbital Frequency Equals Mode Frequency)
Fig. 18b: Time Response of Three Modes Equally Weighted in $A_i(0)$
Fig. 19: Deflection vs. Non-dimensional Beam Length for 6400 ≤ T ≤ 6500 (secs)

\[ \omega_1 = 0.001115 \text{ r/s} \]
\[ \omega_2 = 0.003076 \text{ r/s} \]
\[ \omega_3 = 0.006013 \text{ r/s} \]
Fig. 20: Time Response of Three Modes Equally Weighted in $A_n(0)$

- $\omega_1 = 0.0628 \text{ r/s}$
- $\omega_2 = 0.1733 \text{ r/s}$
- $\omega_3 = 0.3374 \text{ r/s}$
Fig. 21a: Deflection vs. Non-dimensional Beam Length for $0 \leq T \leq 10$ (secs)

- $\omega_1 = 0.0628 \text{ r/s}$
- $\omega_2 = 0.1733 \text{ r/s}$
- $\omega_3 = 0.3347 \text{ r/s}$
Fig. 21b: Deflection vs. Non-dimensional Beam Length for $11 \leq T \leq 21$ (secs)

$\omega_1 = 0.0628 \text{ r/s}$

$\omega_2 = 0.1733 \text{ r/s}$

$\omega_3 = 0.3374 \text{ r/s}$
Fig. 21c: Deflection vs. Non-dimensional Beam Length for $42.5 \leq T \leq 52.5$ (secs)

- $T = 42.5$ sec.
- $T = 43.5$ sec.
- $T = 44.5$ sec.
- $T = 45.5$ sec.
- $T = 46.5$ sec.
- $T = 47.5$ sec.
- $T = 48.5$ sec.
- $T = 49.5$ sec.
- $T = 50.5$ sec.
- $T = 51.5$ sec.
- $T = 52.5$ sec.

Deflection (meters)

Time (seconds)

Non-dimensional Beam Length

$w_1 = 0.0628 \text{ r/s}$

$w_2 = 0.1733 \text{ r/s}$

$w_3 = 0.3374 \text{ r/s}$
Mode Amplitude (meters)

Fig. 22: Time Response of Three Modes Unequally Weighed

I.C.: $A_1 = 0.5 \text{ m}, A_2 = 0.2 \text{ m}$

$A_3 = 0.2m$

$\omega_1 = 0.0628 \text{ rad/s}$

$\omega_2 = 0.1733 \text{ rad/s}$

$\omega_3 = 0.3371 \text{ rad/s}$

ORIGINANL PAGE IS OF POOR QUALITY
$\omega_1 = 0.0628 \text{ r/s}$

$\omega_2 = 0.1733 \text{ r/s}$

$\omega_3 = 0.3374 \text{ r/s}$

I.C.: $A_1 = 0.5 \text{ m}$

$A_2 = 0.3 \text{ m}$

$A_3 = 0.2 \text{ m}$

Fig. 23: Deflection vs. Non-dimensional Beam Length for 0.1 T = 1.0 sec.
Fig. 24: Deflection vs. Non-dimensional Beam Length

I.C.: $A_1 = 0.5m$, $A_2 = 0.3m$, $A_3 = 0.2m$

- $w_0 = 0.00628 \text{ r/s}$
- $w_0 = 0.1733 \text{ r/s}$
- $w_0 = 0.3374 \text{ r/s}$

Time (seconds)  Deflection (meters)

For $47.5 < T < 57.5$ (secs)
Fig. 25: Time Response of Three Modes Unequally Weighted in $A_n(0)$

- $\omega_1 = 0.6283 \text{ r/s}$
- $\omega_2 = 1.7332 \text{ r/s}$
- $\omega_3 = 3.3741 \text{ r/s}$

I.C.: $A_1 = 0.5a$
- $A_2 = 0.3m$
- $A_3 = 0.2m$
Fig. 26: Deflection vs. Non-dimensional Beam Length for 0 < T < 10 (secs)

- \( \omega_1 = 0.628 \text{ r/s} \)
- \( \omega_2 = 1.733 \text{ r/s} \)
- \( \omega_3 = 3.374 \text{ r/s} \)

I.C.: 
- \( A_1 = 0.5 \text{ m} \)
- \( A_2 = 0.3 \text{ m} \)
- \( A_3 = 0.0 \text{ cm} \)
Fig. 27: Time Response of Five Modes Unequally Weighted in $A_n(0)$

Mode Amplitude (meters)

Time (seconds)

$\omega_1 = 0.0628 \, \text{r/s}$
$\omega_2 = 0.1733 \, \text{r/s}$
$\omega_3 = 0.3374 \, \text{r/s}$
$\omega_4 = 0.5579 \, \text{r/s}$
$\omega_5 = 0.8337 \, \text{r/s}$

I.C.: $A_1 = 0.5m$
$A_2 = 0.4m$
$A_3 = 0.3m$
$A_4 = 0.2m$
$A_5 = 0.1m$
Fig. 28: Deflection vs. Non-dimensional Beam length for 0 sec to 10 sec (sec)

Deflection (meters)

Time (seconds)

Non-dimensional Beam Length 1.0

\( \omega_1 = 0.628 \text{ r/s} \)
\( \omega_2 = 0.1733 \text{ r/s} \)
\( \omega_3 = 0.3374 \text{ r/s} \)
\( \omega_5 = 0.8337 \text{ r/s} \)

I.C.: \( A_1 = 0.5 \text{m} \)
\( A_2 = 0.4 \text{m} \)
\( A_3 = 0.3 \text{m} \)
\( A_4 = 0.2 \text{m} \)
\( A_5 = 0.1 \text{m} \)
$\omega_1 = 0.0628 \text{ r/s}$

$\omega_2 = 0.1733 \text{ r/s}$

$\omega_3 = 0.3374 \text{ r/s}$

$\omega_4 = 0.5579 \text{ r/s}$

$\omega_5 = 0.8337 \text{ r/s}$

I.C.: $A_1 = 0.5m$

$A_2 = 0.4m$

$A_3 = 0.3m$

$A_4 = 0.2m$

$A_5 = 0.1m$

Fig. 29: Deflection vs. Non-dimensional Beam Length for $47.5 \leq T \leq 57.5$ (sec)
Fig. 30: Time Response of the Pitch Motion

\[ T_{\text{orbit}} = 5632.28 \text{ sec} \quad \omega_1 = 0.0051 \text{ rad/sec} \quad \text{I.C.: } \phi(0) = 0.2 \text{ rad} \]

\[ \phi(0) = 0.0 \text{ rad/sec} \]
Appendix I

Transformation Relations:

1. Transformation from $\tau_0$ to $\tau_1$

Fig. (A-1): Inertial and intrinsic frames

From Fig. (A-1) it is evident that,

$$
\begin{align*}
   i_1 & = \begin{bmatrix} s \nu c \omega & s \nu s \omega & c \nu \end{bmatrix} \\
   i_2 & = \begin{bmatrix} c \nu c \omega & c \nu s \omega & s \nu \end{bmatrix} \\
   i_3 & = \begin{bmatrix} -s \omega & c \omega & 0 \end{bmatrix}
\end{align*}
$$

\begin{pmatrix}
   X \\
   Y \\
   Z
\end{pmatrix}

\begin{pmatrix}
   T_1 \\
   \tau_0
\end{pmatrix}

(I-1)
2. Transformation from the intrinsic frame ($\tau_1$) to the orbit fixed frame ($\tau_2$):

From Fig. (A-2),

$$
\begin{bmatrix}
X_0 \\
Y_0 \\
Z_0 \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & c\chi & s\chi \\
0 & -s\chi & c\chi \\
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
\end{bmatrix}
$$

(I-2)

3. Transformation from orbit frame ($\tau_2$) to the body frame ($\tau_3$):

From Fig. (A-3),

$$
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
c\phi & s\phi & 0 \\
-s\phi & c\phi & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0 \\
z_0 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
c\phi c\theta & s\phi c\psi + c\phi s\theta s\psi & s\phi s\psi - c\phi s\theta c\psi \\
-s\phi c\theta & c\phi c\psi - s\phi s\theta s\psi & c\phi s\psi + s\phi s\theta c\psi \\
s\theta & -c\theta s\psi & c\theta c\psi \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
$$

i.e.

$$
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
x_0 \\
y_0 \\
z_0 \\
\end{bmatrix}
$$

$T_3 (\phi, \theta, \psi)$

(I-3)
The transformation matrix connecting $\tau_1 (P)$ and $\tau_3$, where $P$ is a general point in the body, is developed as below,

(i) Transform a vector in $\tau_1 (P)$ to a vector in $\tau_0$ using the transformation $T_1^{-1} (P)$.

(ii) Transform the vector in $\tau_0$ to a vector in $\tau_1 (O)$ by the transformation $T_1 (O)$

we can write,

$$T_1^{-1} (P) = T_1^{-1} (O) + AT_1^{-1}$$

$$= T_1^{-1} (O) + \frac{\partial T_1}{\partial \eta} \Delta \eta + \frac{\partial T_1}{\partial \omega} \Delta \omega + \text{H.O.T.}$$

Hence,

$$T_1 (O) T_1^{-1} (P) = I + \Delta$$  \hspace{1cm} (II - 1)

where,

$$\Delta = T_1 (O) \left[ \frac{\partial T_1}{\partial \eta} \Delta \eta + \frac{\partial T_1}{\partial \omega} \Delta \omega \right]$$

and from Eqn. (I - 1)

$$\Delta = \Delta_n \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta_\omega \begin{bmatrix} 0 & 0 & -\sin \theta \\ 0 & 0 & -\cos \theta \\ \sin \theta & \cos \theta & 0 \end{bmatrix}$$  \hspace{1cm} (II - 2)

Thus, the transformation $(I + \Delta)$ transforms a vector in $\tau_1 (P)$ to a vector in $\tau_1 (O)$, i.e.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}_{\tau_1 (O)} = (I + \Delta) \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}_{\tau_1 (P)}$$  \hspace{1cm} (II - 3)
The gravity force per unit mass at $0$, expressed in the intrinsic frame at that point, $\tau_1(0)$ is given by,

$$
\mathbf{F}_0(\rho, \eta, \omega) = \nabla V|_0 = \begin{bmatrix}
\frac{\partial V}{\partial \rho} \\
\frac{1}{\rho} \frac{\partial V}{\partial \eta} \\
\frac{1}{\rho \sin \omega} \frac{\partial V}{\partial \omega}
\end{bmatrix}
$$

Where, $V$ is the gravitational potential given by Eqn (5).

The gravity force at $P$ referred to the intrinsic frame at $P$ can be written as:

$$
\mathbf{F}_P = \mathbf{F}_0(\rho, \eta, \omega) + \frac{\partial \mathbf{F}}{\partial (\rho, \eta, \omega)} \begin{bmatrix}
\Delta \rho \\
\Delta \eta \\
\Delta \omega
\end{bmatrix} + \text{H.O.T}
$$

Hence, the gravity force at $P$ can be referred to the intrinsic frame at $0$, $\tau_1(0)$, through the use of transformation Eqn. (II-3), by:

$$
\mathbf{F} = (I + \Delta) \mathbf{F}_P
$$

i.e.,

$$
\mathbf{F} = (I + \Delta) \left[ \mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial (\rho, \eta, \omega)} \begin{bmatrix}
\Delta \rho \\
\Delta \eta \\
\Delta \omega
\end{bmatrix} \right]
$$

(II-6)

To a first order approximation in $\Delta \rho, \Delta \eta, \Delta \omega$,

$$
\mathbf{F} \approx \mathbf{F}_0 + \Delta \mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial (\rho, \eta, \omega)} \begin{bmatrix}
\Delta \rho \\
\Delta \eta \\
\Delta \omega
\end{bmatrix}
$$

(II-7)

From Eqns. (II - 2) & (II - 3), it is easy to show that,

$$
\Delta \mathbf{F}_0 = \begin{bmatrix}
0 & -F_\eta & -F_\omega s\eta \\
0 & F_\rho & -F_\omega c\eta \\
0 & 0 & F_\rho s\eta + F_\omega c\eta
\end{bmatrix} \begin{bmatrix}
\Delta \rho \\
\Delta \eta \\
\Delta \omega
\end{bmatrix}
$$

(II-8)
where,
\[
F_\rho = \frac{\partial V}{\partial \rho} = -\frac{v^2}{\rho} + v \sum_{s=1}^{\infty} K_s \left(\frac{\Delta \rho}{\rho}\right)^{s+1},
\]
\[
F_\eta = \frac{1}{\rho} \frac{\partial V}{\partial \eta} = \frac{v}{\rho} \sum_{s=1}^{\infty} K_s \left(\frac{\Delta \rho}{\rho}\right)^{s+1},
\]
\[
F_\omega = \frac{1}{\rho \omega} \frac{\partial V}{\partial \omega} = \frac{v}{\rho \omega} \sum_{s=1}^{\infty} K_s \left(\frac{\Delta \rho}{\rho}\right)^{s+1}.
\]

By some matrix manipulations, it can be shown that,
\[
\Delta F_0 + \frac{\partial F}{\partial (\rho, \eta, \omega)} \begin{bmatrix} \Delta \rho \\ \Delta \eta \\ \Delta \omega \end{bmatrix} = \mathbf{B}^* \begin{bmatrix} \Delta \rho \\ \rho \Delta \eta \\ \rho \eta \Delta \omega \end{bmatrix}
\]

where,
\[
\mathbf{B}^* = \begin{bmatrix}
\frac{\partial F_\rho}{\partial \rho} & \frac{1}{\rho} & \left(\frac{\partial F_\rho}{\partial \eta} - F_\eta\right) & \frac{1}{\rho \omega} & \left(\frac{\partial F_\rho}{\partial \omega} - F_\omega \eta\right) \\
\frac{\partial F_\eta}{\partial \rho} & \frac{1}{\rho} & \left(\frac{\partial F_\eta}{\partial \eta} + F_\rho\right) & \frac{1}{\rho \omega} & \left(\frac{\partial F_\eta}{\partial \omega} - F_\omega \eta\right) \\
\frac{\partial F_\omega}{\partial \rho} & \frac{1}{\rho} & \left(\frac{\partial F_\omega}{\partial \eta}\right) & \frac{1}{\rho \omega} & \left(\frac{\partial F_\omega}{\partial \omega} + F_\rho \eta + F_\eta \omega\right)
\end{bmatrix}
\]

Evaluating the partial derivatives in \(\mathbf{B}^*\), using Eqn. (II-9), we arrive at the following expression.
\[
\mathbf{B}^* = \frac{v a^2}{\rho^3} \left[\mathbf{B}^*\right] + \sum_{s=1}^{\infty} K_s \left(\frac{\Delta \rho}{\rho}\right)^s \mathbf{B}(s)
\]

where,
\[
\mathbf{B}(0) = \begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

(II-3)
and
\[
B(s) = \begin{bmatrix}
(s+1) (s+2) \Omega_s' & -(s+2) \Omega_s' & -s(s+2) \frac{\Omega_s}{s\eta} \\
-(s+2) \Omega_s' & [\Omega_s'' - (s+1) \Omega_s] & \left( \frac{\Omega_s}{s\eta} \right)' \\
-(s+2) \frac{\Omega_s}{s\eta} & \left( \frac{\Omega_s}{s\eta} \right)' & [\Omega_s' \cot \eta - \Omega_s (s+1 + \frac{m^2}{s^2 \eta})]
\end{bmatrix}
\]

Also, to a first order approximation,
\[
\begin{bmatrix}
\Delta \rho \\
\rho \Delta \eta \\
\rho \omega \Delta \omega
\end{bmatrix} = (T_3 T_2)^{-1} \bar{r}
\]

where \( \bar{r} \) is the position vector expressed in the body axes frame, \( \tau_3 \).

After substitution of Eqns. (II - 10) and (II - 14) into Eqn. (II - 7), there results,
\[
\bar{F} = \bar{F}_0 + B^s (T_3 T_2)^{-1} \bar{r}
\]

Reprojecting on the body fixed frame, we get
\[
\bar{F} = \bar{F}_0 + M \bar{r}
\]

where,
\[
\bar{F}_0 = T_3 T_2 \bar{F}_0
\]

\[
M = T_3 T_2 B^s (T_3 T_2)^{-1}
\]

If we write, \( M = M(0) + \sum K_s (\frac{\rho}{\rho_0})^s M(s) \)

then,
\[
M(0) = T_3 T_2 B(0) (T_3 T_2)^{-1} \frac{v_s^2}{\rho^3}
\]

\[
M(s) = T_3 T_2 B(s) (T_3 T_2)^{-1} \frac{v_i^2}{\rho^3}
\]

The matrix \( M(0) \) is given in Eqn. (6).
Appendix - III

This appendix presents the expanded forms of the vector expressions in equations (28), (31), (32), (43), (44), (45), and (46).

Let us introduce the following notations,

\[ H_{\alpha \beta}^{(n)} = \int_{\text{vol}} \xi_\alpha \phi_\beta \, dm \]

\[ L_{\alpha \beta}^{(mn)} = \int_{\text{vol}} \phi_\alpha (m) \phi_\beta (n) \, dm \]  \hspace{1cm} (III - 1)

\[ I_{\alpha}^{(n)} = \frac{1}{m} \int_{\text{vol}} \phi_{\alpha} (n) \, dm \]

By the assumption \( \vec{q} = \sum_{n=1}^{\infty} A_n (t) \overline{\phi} (n) (\overline{r_0}) \), we have

\[ \dot{\vec{q}} = \sum_{n=1}^{\infty} A_n (t) \overline{\dot{\phi}} (n) (\overline{r_0}) \]

and \[ \ddot{\vec{q}} = \sum_{n=1}^{\infty} A_n (t) \overline{\ddot{\phi}} (n) (\overline{r_0}) \]  \hspace{1cm} (III - 2)

Consider Eqn. (28),

\[ \sum_{n=1}^{\infty} \overline{Q}^{(n)} = \int_{\text{vol}} [\overline{r_0} \times \overline{q} + 2 \overline{r_0} \times (\overline{\omega} \times \overline{q}) + \overline{r_0} \times (\overline{\omega} \times \overline{q})] \, dm \]

\[ + \overline{q} \times (\overline{\omega} \times \overline{r_0}) - (\overline{r_0} \cdot \overline{\omega}) (\overline{\omega} \times \overline{q}) - (\overline{q} \cdot \overline{\omega}) (\overline{\omega} \times \overline{r_0}) \]  \hspace{1cm} (III - 3)
Using the methods of vector algebra, the above expression can be expanded in the component form to obtain

\[ Q_x^{(n)} = \hat{A}_n [ (H_{yx}^{(n)} + H_{zy}^{(n)}) \omega_x - H_{yx}^{(n)} \omega_y - H_{zx}^{(n)} \omega_z ] + A_n [2(H_{yy}^{(n)} + H_{zz}^{(n)}) \omega_x - (H_{xy}^{(n)} + H_{yx}^{(n)}) \omega_y - (H_{xz}^{(n)} + H_{zx}^{(n)}) \omega_z ] \\
- 2 \omega_x \omega_z (H_{zz}^{(n)} - H_{yy}^{(n)}) - \omega_x \omega_y (H_{xz}^{(n)} + H_{zx}^{(n)}) + \omega_x \omega_y (H_{xy}^{(n)} + H_{yx}^{(n)}) + (\omega_x^2 - \omega_y^2) (H_{yz}^{(n)} + H_{zy}^{(n)}) \]

(III - 4)

\[ Q_y^{(n)} \] and \[ Q_z^{(n)} \] are obtained by the cyclic permutation of \( x, y, z \) in the expression for \( Q_x^{(n)} \).

Now consider equation (31), which is given by,

\[ G_R = \int_{\text{vol}} (\mathbf{r}_0 \times M \mathbf{r}_0) \, dm \]

On expanding in the component form,

\[ G_R = (J_z - J_y) M_{23} \hat{i} + (J_x - J_z) M_{31} \hat{j} + (J_y - J_x) M_{12} \hat{k} \]

(III - 5)

where, \( M_{ij} \) is an element of \( M \) matrix.

\( J_x, J_y, J_z \) are the principal moments of inertia of the body in the undeformed state.

Considering Eqn. (32) for \( G^{(n)} \), we have,

\[ \sum_{n=1}^{\infty} G^{(n)} = \int_{\text{vol}} (\mathbf{r}_0 \times M \mathbf{r}_0 + \mathbf{q} \times M \mathbf{r}_0) \, dm \]

\[ = \sum_{n=1}^{\infty} A_n \int_{\text{vol}} (\mathbf{r}_0 \times M \phi^{(n)} + \mathbf{\Phi}^{(n)} \times M \mathbf{r}_0) \, dm \]

(III - 6)
Hence, the components of (III - 6) are,

\[ G^{(n)} = A_n \left[ (M_{33} - M_{22}) (H_{yz}^{(n)} + H_{zy}^{(n)}) - M_{21} (H_{xz}^{(n)} + H_{zx}^{(n)}) + M_{31} (H_{xy}^{(n)} + H_{yx}^{(n)}) + 2 M_{23} (H_{yy}^{(n)} - H_{zz}^{(n)}) \right] \]

\[ G_y^{(n)} \text{ and } G_z^{(n)} \] components are obtained by the cyclic permutation

do \text{ of } x, y, z \text{ in (III - 7).}

Now consider the following scalar quantities.

Eqn. (43):

\[ \phi_n = \int_{\text{vol}} [\overline{\phi}^{(n)} \cdot \overline{\omega} \times \overline{r} + \phi^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{r})] \, \text{d}m \]

\[ = \omega_x (H_{yz}^{(n)} - H_{zy}^{(n)}) + \omega_y (H_{xz}^{(n)} - H_{zx}^{(n)}) + \omega_z (H_{xy}^{(n)} - H_{yx}^{(n)}) + \omega \omega_y (H_{zy}^{(n)} + H_{yz}^{(n)}) + \omega \omega_z (H_{yz}^{(n)} + H_{zy}^{(n)}) \]

\[ - \omega_x^2 (H_{yy}^{(n)} + H_{zz}^{(n)}) - \omega_y^2 (H_{zz}^{(n)} + H_{xx}^{(n)}) - \omega_z^2 (H_{xx}^{(n)} + H_{yy}^{(n)}) \quad (\text{III - 8}) \]

Eqn. (44):

\[ \sum_{m=1}^{\infty} \phi_{mn} = \int_{\text{vol}} \left[ 2 \overline{\phi}^{(n)} \cdot \overline{\omega} \times \overline{r} + \phi^{(n)} \cdot \overline{\omega} \times \overline{q} + \phi^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{q}) \right] \, \text{d}m \]

\[ = \sum_{m=1}^{\infty} \int_{\text{vol}} \left[ 2 A_m \left( \overline{\phi}^{(n)} \cdot \overline{\omega} \times \overline{\phi}^{(m)} \right) \right. \]

\[ + \left. A_m \left( \phi^{(n)} \cdot \frac{\overline{\omega}}{\overline{\phi}} \right) + \phi^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{\phi}^{(m)}) \right] \, \text{d}m \]

Hence,

\[ \phi_{mn} = 2 A_m \left[ \omega_x (L_{yz}^{(mn)} - L_{zy}^{(mn)}) + \omega_y (L_{xz}^{(mn)} - L_{zx}^{(mn)}) + \omega_z (L_{xy}^{(mn)} - L_{yx}^{(mn)}) \right] \]

\[ + A_m \left[ \omega_x (L_{yz}^{(mn)} - L_{zy}^{(mn)}) + \omega_y (L_{xz}^{(mn)} - L_{zx}^{(mn)}) + \omega_z (L_{xy}^{(mn)} - L_{yx}^{(mn)}) \right] \]

\[ + \omega_x \omega_y (L_{xy}^{(mn)} + L_{yx}^{(mn)}) + \omega_y \omega_z (L_{yz}^{(mn)} + L_{zy}^{(mn)}) + \omega_z \omega_x (L_{xz}^{(mn)} + L_{zx}^{(mn)}) \]

\[ - \omega_x^2 (L_{yy}^{(mn)} + L_{zz}^{(mn)}) - \omega_y^2 (L_{zz}^{(mn)} + L_{xx}^{(mn)}) - \omega_z^2 (L_{xx}^{(mn)} + L_{yy}^{(mn)}) \quad (\text{III - 9}) \]
Eqn. (45): $g_n = \int \varphi^{(n)} \cdot M \bar{r}_0 \, dm = \sum \sum H^{(n)}_{\alpha \beta} \, M_{\alpha \beta}$  

Eqn. (46): $\sum_{m=1}^\infty g_{mn} = \int \varphi^{(n)} \cdot M \bar{q} \, dm = \sum_{m=1}^\infty A_m \int \varphi^{(n)} \cdot M \bar{\varphi}^{(m)} \, dm$

Hence,

$$g_{mn} = A_m \sum_{\alpha \beta} L^{(mn)}_{\alpha \beta} \, M_{\alpha \beta}$$  

where, $\alpha \beta = x, y, z$ or 1, 2, 3. For example, when $\alpha$ is $x$ in $H^{(n)}_{\alpha \beta}$, the corresponding value of $\alpha$ in $M_{\alpha \beta}$ is 1. In a similar way when $\alpha$ is $y$ in $H^{(n)}_{\alpha \beta}$, $\alpha$ is 2 in $M_{\alpha \beta}$ and when $\alpha$ is $z$ in $H^{(n)}_{\alpha \beta}$, $\alpha$ is 3 in $M_{\alpha \beta}$. Same reasoning holds for $\beta$ also.
Appendix IV

Natural frequencies, mode shapes and modal mass for free-free uniform beams:

The natural frequencies of a free-free beam can be obtained by solving the following frequency equation.9

\[ \cos \beta l \cosh \beta l = 1 \]  \hspace{1cm} (IV - 1)

where,

- \( \lambda \) - length of the beam
- \( \beta^n = \frac{\omega^2 m'}{EI} \)
- \( m' \) - mass per unit length of the beam
- \( EI \) - bending modulus
- \( \omega \) - natural frequency

The mode shapes corresponding to the frequencies obtained by Eqn. (IV - 1) are given by,

\[
Z_n(x') = D_n \left[ \frac{\cos \beta_n \lambda - \cosh \beta_n \lambda}{\sinh \beta_n \lambda - \sin \beta_n \lambda} \right] (\sin \beta_n x' + \sinh \beta_n x') \\
+ (\cos \beta_n x' + \cosh \beta_n x') 
\]  \hspace{1cm} (IV - 2)

Fig. (D-1): Free-free beam
where, \( \beta_n \) (\( n = 1, 2, \ldots \)) are roots of Eqn. (IV - 1).

It should be noted that \( x' \) is measured from one end of the beam, as shown in the Fig (D-1). Hence, \( x' = x + \lambda/2 \)

Define, \( \frac{x'}{\lambda} = \zeta \) \hspace{1cm} (IV - 3)

\( \beta_n \lambda = \Omega_n \)

Hence, the non-dimensionalized form of the mode shape in Eq. (IV - 2) is,

\[
Z(n)(\zeta) = \left[ \frac{\cos \Omega_n \lambda - \cosh \Omega_n \lambda}{\sin \Omega_n \lambda - \sinh \Omega_n \lambda} \right] \left( \sin \Omega_n \zeta + \sinh \Omega_n \zeta \right)
+ \left( \cos \Omega_n \zeta + \cosh \Omega_n \zeta \right) \hspace{1cm} (IV - 4)
\]

The following Table - 1 gives the first five natural modes and the approximate values of the corresponding natural frequencies of a free-free beam.\(^{10}\)

The modal mass (generalized mass) in the \( n^{th} \) mode is given by,

\[
M_n = \int \phi(n)^2(x) \mu \, dv \hspace{1cm} (IV - 5)
\]

Since the beam is assumed to be uniform, \( \mu = \text{constant} \),

Hence,

\[
M_n = m' \int_{x=-\lambda/2}^{x=\lambda/2} \phi(n)^2(x) \, dx \hspace{1cm} (IV - 6)
\]

Using the change of variable \( \zeta = \frac{x}{\lambda} + \frac{1}{2} \), we get

\[
M_n = m' \lambda \int_0^1 [Z(n)(\zeta)]^2 \, d\zeta \hspace{1cm} (IV - 7)
\]

Substituting (IV - 4) in (IV - 7),

\[
M_n = D_n^2 m' \lambda \int_0^1 \left[ K_n (\sin \Omega_n \zeta + \sinh \Omega_n \zeta) \right. \\
+ \left. (\cos \Omega_n \zeta + \cosh \Omega_n \zeta) \right]^2 \, d\zeta \hspace{1cm} (IV - 8)
\]
Table 2: Mode shapes and frequencies of free-free beams

<table>
<thead>
<tr>
<th>Mode shape</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta = 1$</td>
<td>$\omega_1 = 3.56 \frac{EI}{m^2l^3}$</td>
</tr>
<tr>
<td>0.224 0.776</td>
<td></td>
</tr>
<tr>
<td>0.132 0.5 0.868</td>
<td>$\omega_2 = 9.82 \frac{EI}{m^2l^3}$</td>
</tr>
<tr>
<td>0.094 0.356 0.644 0.906</td>
<td>$\omega_3 = 19.2 \frac{EI}{m^2l^3}$</td>
</tr>
<tr>
<td>0.0734 0.277 0.5 0.723 0.927</td>
<td>$\omega_4 = 31.8 \frac{EI}{m^2l^3}$</td>
</tr>
<tr>
<td>0.06 0.227 0.409 0.591 0.776 0.96</td>
<td>$\omega_5 = 47.5 \frac{EI}{m^2l^3}$</td>
</tr>
</tbody>
</table>

Note: The frequency value of zero corresponds to the rigid body modes.
where $K_n = (\cos \Omega_n - \cosh \Omega_n)/(\sinh \Omega_n - \sin \Omega_n)$

The various definite integrals appearing in Eqn. (II - 8) are evaluated in Table - 3.

Table - 3: Useful definite integrals

\[
\int_0^1 \sin^2 \Omega_n \, d\xi = \frac{1}{2} - \frac{\sin 2\Omega_n}{4\Omega_n}
\]

\[
\int_0^1 \sin \Omega_n \cos \Omega_n \, d\xi = \frac{\sin^2 \Omega_n}{2\Omega_n}
\]

\[
\int_0^1 \sin \Omega_n \sinh \Omega_n \, d\xi = \frac{\cosh \Omega_n \sin \Omega_n - \sinh \Omega_n \cos \Omega_n}{2\Omega_n}
\]

\[
\int_0^1 \sin \Omega_n \cosh \Omega_n \, d\xi = \frac{\sinh \Omega_n \sin \Omega_n - \cosh \Omega_n \cos \Omega_n}{2\Omega_n} + \frac{1}{2\Omega_n}
\]

\[
\int_0^1 \cos^2 \Omega_n \, d\xi = \frac{1}{2} + \frac{\sin 2\Omega_n}{4}
\]

\[
\int_0^1 \cos \Omega_n \sinh \Omega_n \, d\xi = \frac{\cosh \Omega_n \cos \Omega_n + \sinh \Omega_n \sin \Omega_n}{2\Omega_n}
\]

\[
\int_0^1 \cos \Omega_n \cosh \Omega_n \, d\xi = \frac{\sinh \Omega_n \cos \Omega_n + \cosh \Omega_n \sin \Omega_n}{2\Omega_n}
\]

\[
\int_0^1 \sinh^2 \Omega_n \, d\xi = \frac{\sinh 2\Omega_n}{4\Omega_n} - \frac{1}{2}
\]
\[
\int_0^1 \sinh \Omega_n \cosh \Omega_n \, d\zeta = \frac{\sinh^2 \Omega_n}{2 \Omega_n}
\]

\[
\int_0^1 \cosh^2 \Omega_n \, d\zeta = \frac{\sinh 2\Omega_n}{4 \Omega_n} + \frac{1}{2}.
\]

Substituting the values of various definite integrals in Eqn. (IV - 8), after simplification we get

\[
M_n = \frac{D_n^2 \Omega_n^2}{4 \Omega_n} \left[ (K_n^2 + 1) \left( \frac{\sinh 2\Omega_n}{4} + \cosh \Omega_n \sin \Omega_n \right) \right.
\]

\[
- (K_n^2 - 1) \left( \frac{\sin 2\Omega_n}{4} + \sinh \Omega_n \cos \Omega_n \right)
\]

\[
+ K_n \left( \sin \Omega_n + \sinh \Omega_n \right)^2 + \Omega_n \left] \quad \text{(IV - 9)}\right.
\]
**HOWARD UNIVERSITY -- SCHOOL OF ENGINEERING -- NOVA 840 COMPUTER**

**FORTRAN/R/E/P_T01, RR/B_T08, LS/L**

**COMPILER DOUBLE PRECISION**

**FLX BEAM MAIN PROGRAM**

```fortran
REAL L11, L13, L31, L33, M11, M13, M31, M33, MS, J2, J3, LS, LD, NU

COMMON /BEAM/ PARM(5), Y(42), DY(42),
2  H11(20), H33(20), H13(20),
3  L11(20,20), L33(20,20), L13(20,20), L31(20,20),
4  M11, M33, M13, M31, MS, ALPHA,
5  P(20), PP(20,20), B(20), GG(20,20),
6  DW(20), HS(20), HO(20), T, NUS, A, RD, J2, J3,
7  GC(20), QCP, NO, E(20), P2, LS(20,20), LD(20,20), C2, M, NORD

COMMON /LOG/ SZNOM(42), SIZMAX(42), LINES, LSTEP

DIMENSION AUX(12,42)

OPEN 5, "SYSOUT", ATT="AP"
DELETE "FLEXBEAM"
OPEN 3, "FLEXBEAM"
OPEN 2, "TMP", ATT="I"

CALL FRINP(PRCSN)
CALL FBSET(PRCSN)

LINES = 0

LSTEP = 0.20 * (PARM(2) - PARM(1))/PARM(3) + 0.5

CALL PKGS(PAP4, Y, DY, NORD, IHLF, AUX)

DO 7 I = 1, NORD
7  CONTINUE

PATIO = (SIZMAX(I)/SZNOM(I)) * 100

WRITE (5, 91) I, PATIO

WRITE (5, 9) IHLF

CALL EXIT

9 FORMAT ('OFINAL VALUE OF IHLF:', T3)
91 FORMAT ('DY(*, I2,*') PEAK:', F6, 1)

END

FINAL PRINT

PRINT 1

```

**ORIGINAL PAGE IS OF POOR QUALITY**
DELETE TMP, LS

COMPILED DOUBLE PRECISION

SUBROUTINE FCT(T,AY,DY)

REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU

COMMON /BEAM/ PARM(5),Y(42),DY(42),

2 M11(20),M33(20),M13(20),
3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
4 M11,M33,M13,M31,MS,ALPHA,
5 P(20),PP(20,20),G(20),GG(20,20),
6 Dw(20),HS(20),HD(20),Z,NU,AR,J2,J3,
7 QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD

FBP1F—DIFF EQN'S

CALL FRP1F
CALL FASIF
DY(1)=ALPHA
DY(2)=Y(1)
DO 2 I=1,M
     DY(2*I+1)=FASFR(I)
     DY(2*I+2)=Y(2*I+1)
2 CONTINUE
RETURN

END
CENTER

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ORIGINAL PAGE IS OF POOR QUALITY.
FORTRAN R/E/P TMP TMP, LS/L FALST, PR/R

SUBROUTINE FALST
REAL L11, L13, L31, L33, M11, M13, M31, M33, MS, JP, J3, LS, LO, N11
COMMON /REAL/ PAR,Y(42), NY(42),
2 H11(20), H33(20), H13(20),
3 L11(20,20), L33(20,20), L13(20,20), L31(20,20),
4 M11, M33, M13, M31, MS, ALPH,A,
5 P(20), PP(20,20), GG(20,20),
6 DW(20), HS(20), HD(20,17), NU, A, R0, JP, J3,
7 QC(20), QCP, ND, E(20), R2, LS(20,20), LD(20,20), C2, M, NORD

FALST=SET 'L1J'

DO 20 I=1,M
DO 20 J=1,M

20 L11(I,J)=0.0
DO 30 I=1,M
DO 30 J=1,M

30 L13(I,J)=0.0
DO 40 I=1,M
DO 40 J=1,M

40 L31(I,J)=0.0
DO 50 I=1,M
DO 50 J=1,M

50 L33(I,J)=0.0
50 L33(I,J)=1.0
DO 60 I=1,M
DO 60 J=1,M

60 LD(I,J)=L31(I,J)-L13(I,J)
LS(I,J)=L11(I,J)+L33(I,J)
61 CONTINUE
RETURN

END
LIST

1

DTRAN/R/E/P TMPLNPS/L FPPIF.RR/R

COMPILER DOUBLE PRECISION

SUBROUTINE FPPIF

RFAI L11, L13, L31, L33, M11, M13, M31, M33, MS, JP, J3, LS, LD, NU

COMMON /REAM/ PARM(5), Y(42), DY(42),

2 H11(20), H33(20), H13(20),

3 L11(20, 20), L33(20, 20), L13(20, 20), L31(20, 20),

4 M11, M33, M13, M31, MS, ALPHA,

5 P(20), PP(20, 20), G(20), GG(20, 20),

6 DW(20), HS(20), HD(20), 7, NU, A, RO, JP, J3,

7 QC(20), QCP, WD, F(20), R2, LS(20, 20), LD(20, 20), CP, M, NORD

FPPIF=PRIMARY INTERMEDIATE FUNCTIONS

SPH=STN(Y(2))

CPH=COS(Y(2))

M11=3*CPH**2-1

M13=-3*SPH

M31=-3*SPH*CPH

M33=3*SPH**2-1

MS=M13+M31

R2=Y(1)+WD

RETURN

END
SUBROUTINE FRSIF
INTEGER R
REAL L11, L13, L31, L33, M11, M13, M31, M33, MS, J2, J3, LS, LD, NU
COMMON /RFA/, /PARM(5), Y(42), DY(42),
2 H11(20), H33(20), H13(20),
3 L11(20,20), L33(20,20), L13(20,20), L31(20,20),
4 M11, M33, M13, M31, MS, ALPHA,
5 P(20), PP(20,20), R(20), GG(20,20),
6 D(20), HS(20), HP(20), 7, NU, A, RO, J2, J3,
7 QC(20), QCP, JD, E(20), Q2, LS(20,20), LD(20,20), C2, M, NORD

FRSIF—SECONDARY INTERMEDIATE FUNCTIONS

Y(2*Y2)
TY2=2.0*Y2
CTY2=CS(TY2)
STY2=SN(TY2)
SUM1 = 0.0
SUM2 = 0.0
DO 1 R = 1, M
1 SUM1 = SUM1 + 3.0*(H13(R)*CTY2 - 0.5*HD(R)*STY2)*Y(2*R+2)
  SUM2 = SUM2 + HS(R)*Y(2*R+2)
CONTINUE
ALPHA = Z*(-1.5*J3*STY2 + 2.0*SUM1 - C2 - QCP)/(J2 + 2.0*SUM2)
DO 70 R=1, M
2 P(R)=R2**2*HS(R)
G(R)=Z*(H11(R)*M11+H33(R)*M33+P*H13(R)*MS)
DO 80 N=1, M
2 PP(R,N)=(Z*Y(2*N+1)*R2*Y(2*N+2)*ALPHA)*LD(R,N)
2 Y(2*N+2)*R2**2*LS(R,N)
2*L33(R,N)*M33
60 CONTINUE
70 CONTINUE
RETURN
END
FUNCTION FBA2P(R)
INTEGER R
REAL L11, L13, L31, L33, M11, M13, M31, M33, MS, J2, J3, LS, LD, NU
COMMON /REAL/ PARM(5), Y(42), DY(42),
2 H11(20), H33(20), H13(20),
3 L11(20,20), L33(20,20), L13(20,20), L31(20,20),
4 M11, M33, M13, M31, MS, ALPHA,
5 P(20), PP(20,20), G(20), GG(20,20),
6 DW(20), HS(20), HD(20), Z, NU, A, RO, J2, J3,
7 OC(20,, OCP, WD, E(20), R2, LS(20,20), LD(20,20), C2, M, NORD

FBA2P---EVALUATE 2-ND DER OF A(R)
SUM=0.0
DO 90 N=1,M
SUM=SUM+PP(R,N)+GG(R,N)
90 CONTINUE
FBA2P=-DW(R)**2*Y.(2*R+2)-P(R)+SUM+G(R)+E(R)+OC(R)
RETURN

END
SUBROUTINE OUTP(T,AY,AY,THLF,N,APARM)

COMMON /RFM4/ PARM(5),Y(42),DY(42),
2 H11(20),H33(20),H13(20),
3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
4 M1, M3, M13, M31, MS, ALPHA,
5 P(20),PP(20,20),G(20,20),GG(20,20),
6 DW(20),HS(20),HD(20),Z,NU,A,PO,J2,J3,
7 RC(20),OCP,NO,ET(20),L2,LS(20,20),LD(20,20),C2,M,NOPD
COMMON /LOG/ SIZMOD(42), SIZMAX(42), LINES, LSTEP

LOGICAL PKNXT

FRONT -- TAKE OUTPUT VALUES

DO 2 I = 1, NORD
   YMAX = ABS(Y(I))
   IF(YMAX .GT. SIZMAX(I)) SIZMAX(I) = YMAX
2 CONTINUE

IF (.NOT. PKNXT(THLF)) GO TO 8
   WRITE BINARY(3),T,THLF,(Y(I),I=1,NORD)
   IT = ((T - PARM(1))/PARM(2))*100.0 + 0.5
   IF (MOD(LINES,LSTEP) .EQ. 0 ) TYPE TT,THLF
   LINES = LINES + 1
8 CONTINUE

RETURN

END
FLEXBEAM -- INPUT

MMP1 = 2*M + 1
MMP2 = 2*M + 2

READ (2, A1) (PARM(I), I = 1, 3), PRCMN
WRITE (5, 91) (PARM(I), I = 1, 3), PRCMN
READ (2, A2) Y(2), Y(1), DY(2), DY(1)
WRITE (5, 92) Y(2), Y(1), DY(2), DY(1)
DO 2 L = 1, M
   I1 = 2*L + 2
   IP = P*L + 1
   READ (2, B2) Y(I1), Y(I2), DY(I1), DY(IP), DW(L)
   WRITE (5,93) L, Y(I1), Y(I2), DY(I1), DY(IP), DW(L)
2 CONTINUE
DO 4 I = 1, NORD
   SIZNOM(I) = DY(I)
   SIZMAX(I) = 0.0
4 CONTINUE
RETURN

91 FORMAT ('NR FROM', F7.1, ' TO', F7.1, ' RY', F7.1, 8X)
   ' PRECISION = ', F9.6)
92 FORMAT (1X, 'INIT VAL INIT DER SCALE VAL SCALE DER',
   2 6X, 'OMEGA ', ' THETA ', PF11.6, F13.6, F12.6)
93 FORMAT (' MODE', IP, 2F11.6, F13.6, F12.6, F13.6)
FORTRAN/VI/P TMP TOP.RA/R TMP.LS/L

COMPIlER DOUBLE PRECISION

FILE DATA

REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,L0,NU

COMMON /REAL/ PARM(S),Y(L),DY(LP),
2 H11(20),H33(20),H13(20),
3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
4 M11,M33,M13,M31,MS,ALPHA,
5 P(20),PP(20,20),S(20),CS(20,20),
6 DQ(20),DS(20),HD(20),Z,NU,Ä,RO,J2,J3,
7 QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NOR

COMMON /RKNX0/ IHOLD, NEXT

DATA M/5/, MORD/12/

DATA NU/9.797/, A/6378388/, RO/6841388/, J2/R25500,/

DATA J3/825500/, QCP/0.0/, WD/.001115/, C2/0.0/

DATA Y/42*0.0/

DATA DY/42*0.0/

DATA DW/20*0.0/

DATA QC/0.0, 19*0.0/

DATA E/0.0, 19*0.0/

DATA PARM/S*0.0/

DATA IHOLD/0/, NEXT/1/

END

PRINT 1
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<td>0.0</td>
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</tbody>
</table>

DELETE DPO: FLEXBEAM
XFER FLEXREAM DPO: FLEXBEAM
FLFXREAM OUTPUT PROGRAM

// FORTRAN
PARAMETER NORD = 12, POINTS = 601, MODLES = 5
DOUBLE PRECISION TT, Y(NORD)
DIMENSION T(POINTS)
DIMENSION PHI(POINTS)
DIMENSION A(POINTS, MODLES)

CALL INOUT (2, 5)
CALL FOPEN (3, "FLFXREAM")
DO 2 I = 1, POINTS
    READ BINARY (3) TT, IHLE, Y
    T(I) = SNGL(TT)
    PHI(I) = SNGL(Y(2))
    DO 1 J = 1, MODLES
        A(I, J) = SNGL(Y(2*J+2))
    1 CONTINUE
2 CONTINUE
CALL DFPLO(POINTS, MODLES, A)
CALL EXIT

END
PARAMETER MAXM = 5, NX = 101
PARAMETER IT1 = 95, IT2 = 116, DT = 2
PARAMETER RMAX = 3.0

SUBROUTINE DFPLS(NPTS, MODES, A)
DIMENSION A(NPTS, MODES), SHAPE(NX, MAXM), DEF(NX)
DOUBLE PRECISION OMEGA
COMMON /DFPLD/ OMEGA(MAXM)
DATA OMEGA / 4.7303404D0, 7.8532046D0, 10.9956078D0, 
14.1371655D0, 17.2787596D0/

CALL DFPLS (OMEGA, MODES, NX, SHAPE)
CALL PSIZE (5.0, 11.0)
CALL PGRID (1, 11)
CALL PLOGO (0.0, 11.0)
CALL PSIZE (5.0, 1.0)
DO 8 IT = IT1, IT2, DT
    CALL PLOGO (0.0, -1.0)
    DO 5 TX = 1, NX
        G = 0.0
        DO 4 IM = 1, MODES
            Q = Q + A(IT, IM)*SHAPE(TX, IM)
        4 CONTINUE
        DEF(TX) = Q
    5 CONTINUE
    CALL PLOG (-DMAX, DEF, DMAX, NX)
8 CONTINUE
RETURN
END
SUBROUTINE DFPLS(OMEGA, MODES, NX, SHAPE)
DOUBLE PRECISION OMEGA (MODES)
DIMENSION SHAPE (NX, MODES)
DOUBLE PRECISION DZ, W, EW, SHW, CHW, COEF
DOUBLE PRECISION ZETA, WZ, EWZ, SHWZ, CHWZ, DSHAPE

DZ = 1.000/DFLOAT(NX - 1)
DO 8 IM = 1, MODES
   W = OMEGA(IM)
   EW = DEXP(W)
   SHW = (EW - 1.000/EW)/2.000
   CHW = (EW + 1.000/EW)/2.000
   COEF = (DCOS(W) - CHW)/(SHW - DSIN(W))
   DO 4 IX = 1, NX
      ZETA = DFLOAT(IX - 1)*DZ
      WZ = W*ZETA
      EWZ = DEXP(WZ)
      SHWZ = (EWZ - 1.000/EWZ)/2.000
      CHWZ = (EWZ + 1.000/EWZ)/2.000
      DSHAPE = COEF*(DSIN(WZ) + SHWZ) + (DCOS(WZ) + CHWZ)
      SHAPE(IX, IM) = SNGL(DSHAPE)
   4 CONTINUE
8 CONTINUE
RETURN
END

// LOAD PLOT/L
CALL DP01FLEXBEAM FLEXBEAM
C PRINT OUTPUT
ON & L = 1,41
IPHI = 10*L - 9
IA1A = L
IA1B = L + 960
4 CONTINUE
DO 6 L = 02, 101
IPHI = 10*L - 9
6 CONTINUE
WRITE (5,91) T(IPHI), PHI(IPHI), T(IA1A), AT(IA1A,1), A(IA1A,2)
WRITE (5,91) T(IA1B), A(IA1B,1), A(IA1B,2).
91 FORMAT (IX, FR.1, F4.9, 2(F13.1, E14.9))
PLOT AMPLITUDE VS. TIME
CALL PSIZE (6.0, 5.0)
CALL PROX
CALL PAXES
CALL PLOX (-0.5, A(1,1), 0.5, POINTS)
CALL PLOX (-0.5, A(1,2), 0.5, POINTS)
CALL PLOX (-0.5, A(1,3), 0.5, POINTS)
CALL PLOX (-0.5, A(1,4), 0.5, POINTS)
CALL PLOX (-0.5, A(1,5), 0.5, POINTS)