Airplane Stability Calculations
With a Card Programmable Pocket Calculator

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ERRATA

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Please make the following corrections:

Page 15: Sentence after equation (11) should read as follows:

Equations (9) and (10) were programmed for the calculator and the program is given in appendix B.

Page 16: Equation (16) should read as follows:

\[ \text{Re}(y) = S + T - \frac{b_2}{3} \]

Page 24, Last sentence: Change step 49 to step 45.

Page 25: Step 100 should read as follows:

STO×9 \((g\sigma_T/2U_{ss}) \sin 2y_{ss}\)

Page 26, Step 105: Change - to +
Step 141: Change RCL8 to RCLB

Page 29: Delete the last sentence.

Page 49: In column headed "Output," change the values of \(a_3\), \(a_2\), \(a_1\), \(a_0\), and \(a_{12}\) to

\(a_3 = 1.3980958\)
\(a_2 = 1.1093007\)
\(a_1 = -0.0098076\)
\(a_0 = -0.021448\)
\(a_{12} = 0.0373094\)

ISSUED NOVEMBER 1978
Page 5: Equation (3) should read

\[ x_{10} = x_e \left( \frac{\ln e}{\ln 10} \right)^{1/2} = x_e \left( \frac{1}{\ln 10} \right)^{1/2} \]
Airplane Stability Calculations
With a Card Programmable
Pocket Calculator

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SUMMARY

Programs are presented for calculating airplane stability characteristics with a card programmable pocket calculator. These calculations include eigenvalues of the characteristic equations of lateral and longitudinal motion as well as stability parameters such as the time to damp to one-half amplitude or the damping ratio. The effects of wind shear are included. Background information and the equations programmed are given. The programs are written for the International System of Units, the dimensional form for the stability derivatives, and stability axes. In addition to the programs for stability calculations, an unusual and short program is included for the Euler transformation of coordinates used in airplane motions. The programs have been written for a Hewlett Packard HP-67 calculator. However, the use of this calculator does not constitute an endorsement of the product by the National Aeronautics and Space Administration.

INTRODUCTION

Over the past several years, the programmable pocket calculator has developed into a highly sophisticated device that has almost computer characteristics. Because of its sophistication, the newer models are capable of being programmed to make very complicated calculations. Since different logics are used in programmable calculators and since the available keyboard instructions vary with models of different manufacturers, it is necessary to identify the make and model of the calculator for which a program is written. The airplane stability programs presented in this paper were written for a Hewlett Packard HP-67 card programmable calculator; however, its use and identification in this report does not constitute an endorsement of the product by the National Aeronautics and Space Administration.

Programs are given for the calculation of the coefficients of the airplane lateral and longitudinal characteristic equations, the eigenvalues, and the stability parameters such as the time to damp to one-half amplitude or the damping ratio. In addition, a unique coordinate transformation program is given for transformations between inertial axes and airplane body axes. This program requires very few program steps and may be useful as part of a larger program. The equations on which the programs are based are given so that the programs can be readily adapted to other calculators that have sufficient program capacity.

The programs presented herein evolved during the study of wind shear and its effect on airplane stability and control. These programs proved useful in making stability calculations in this study and should be of use in other investigations.


SYMBOLS

A aspect ratio

\( a_0, a_1, \ldots, a_5 \) coefficients of characteristic equations

\( a_{12}, a_{13}, a_{14}, \ldots \) elements of longitudinal stability determinant

b wing span

\( b_2, b_1, b_0 \) coefficients of resolvent cubic

\( C_D \) drag coefficient, \( \frac{D}{\rho S U_2^2/2} \)

\( C_{D,0} \) drag coefficient for \( C_L = 0 \)

\( C_{D\alpha} = \frac{\partial C_D}{\partial \alpha} \)

\( C_L \) lift coefficient, \( \frac{L}{\rho S U_2^2/2} \)

\( C_{L,0} \) lift coefficient at zero angle of attack

\( C_{L\alpha} = \frac{\partial C_L}{\partial \alpha} \)

\( C_{L\alpha} = \frac{\partial C_L}{\partial \alpha} \)

\( C_{L\theta} = \frac{\partial C_L}{\partial \theta} \)

\( C_L \) rolling-moment coefficient, \( \frac{M_X}{\rho S b U_2^2/2} \)

\( C_{Lp} = \frac{\partial C_L}{\partial p} \)

\( C_{Lr} = \frac{\partial C_L}{\partial r} \)
\[ c_{l\beta} = \frac{\partial c_l}{\partial \beta} \]
\[ c_{l\phi} = \frac{\partial c_l}{\partial \phi} \]
\[ c_{m} \quad \text{pitching-moment coefficient, } \frac{M_Y}{\rho S \bar{V}_S^2 / 2} \]
\[ c_{m,0} \quad \text{total pitching-moment coefficient at zero angle of attack} \]
\[ c_{m\alpha} = \frac{\partial c_m}{\partial \alpha} \]
\[ c_{m\alpha} = \frac{\partial c_m}{\partial \alpha} \]
\[ c_{m\phi} = \frac{\partial c_m}{\partial \phi} \]
\[ c_{n} \quad \text{yawing-moment coefficient, } \frac{M_z}{\rho S \bar{V}_S^2 / 2} \]
\[ c_{n\rho} = \frac{\partial c_n}{\partial \rho} \]
\[ c_{n\alpha} = \frac{\partial c_n}{\partial \alpha} \]
\[ c_{n\beta} = \frac{\partial c_n}{\partial \beta} \]
\[ c_{n\phi} = \frac{\partial c_n}{\partial \phi} \]
\[ C_{n\phi} = \frac{\partial C_n}{\partial \phi} \]

thrust coefficient

\[ C_T = \frac{\partial C_T}{\partial u} \]

side-force coefficient, \( \frac{F_Y}{\rho S u^2_{ss}/2} \)

\[ C_{Yr} = \frac{\partial C_Y}{\partial r} \]

\[ C_{Y\beta} = \frac{\partial C_Y}{\partial \beta} \]

\[ C_{Y\delta} = \frac{\partial C_Y}{\partial \delta} \]

\( C_{11}, C_{21}, C_{30} \quad \{ b_{11}, b_{12}, b_{13}, \ldots \} \) terms in lateral stability determinant

\( \bar{c} \) mean aerodynamic chord

\( D \) drag

\( F_T \) thrust

\( F_{T, tr} \) trim thrust

\[ F_{Tu} = \frac{\partial F_T}{\partial u} \]

\( F_X, F_Y, F_Z \) forces along \( X, Y, \) and \( Z \) stability axis

\[ F_{X\delta_{e}} = \frac{\partial F_X}{\partial \delta_{e}} \]
\[ F_{Y\delta_a} = \frac{\partial F_Y}{\partial \delta_a} \]
\[ F_{Y\delta_r} = \frac{\partial F_Y}{\partial \delta_r} \]
\[ F_{Z\delta_e} = \frac{\partial F_Z}{\partial \delta_e} \]

\( g \) acceleration of gravity

\( I_x, I_y, I_z \) moments of inertia, stability axes

\( I_{xz} \) product of inertia, stability axes

\( \text{Im}( ) \) imaginary part of complex root

\( k_x, k_y, k_xz \) radii of gyration, stability axes

\( L \) lift

\( M_x, M_y, M_z \) moments about \( x, y, \) and \( z \) stability axes

\[ M_{X\delta_a} = \frac{\partial M_x}{\partial \delta_a} \]
\[ M_{X\delta_r} = \frac{\partial M_x}{\partial \delta_r} \]
\[ M_{Y\delta_e} = \frac{\partial M_y}{\partial \delta_e} \]
\[ M_{Z\delta_a} = \frac{\partial M_z}{\partial \delta_a} \]
\[ M_{Z\delta_r} = \frac{\partial M_z}{\partial \delta_r} \]

\( m \) mass

\( N_D \) number of cycles to double amplitude

\( N_{1/2} \) number of cycles to damp to one-half amplitude
p rolling velocity
$R_*$ radius
Re() real part of complex root
Re(y) real root of resolvent cubic
r yawing velocity
S wing area
t period
t$_D$ time to double amplitude
t$_{1/2}$ time to damp to one-half amplitude
U$_{SS}$ steady-state velocity
U$_w$ wind velocity
u$_{pr}$ perturbation velocity
u$_w'$ wind shear gradient
w$_w'$ updraft-downdraft gradient
X,Y,Z stability axes
X$_b$,Y$_b$,Z$_b$ airplane body axes
X$_e$,Y$_e$,Z$_e$ Earth-fixed axes
X$_{sp}$,Y$_{sp}$,Z$_{sp}$ space axes
x,y,z general variables
x$_b$,y$_b$,z$_b$ body axis coordinates
x$_{sp}$,y$_{sp}$,z$_{sp}$ space axis coordinates
$\alpha_{pr}$ perturbation angle of attack
$\alpha_{tr}$ trim angle of attack
$\alpha_1,\alpha_2$ real roots
$\alpha_3,\alpha_4$
$\beta$ sideslip angle
$\gamma_{pr}$ perturbation flight-path angle
\( \gamma_{ss} \) steady-state flight-path angle

\( \Delta \) logarithmic decrement

\( \delta_a \) aileron deflection

\( \delta_e \) elevator deflection

\( \delta_r \) rudder deflection

\( \epsilon_1, \epsilon_2 \) angles

\( \zeta \) damping ratio

\( \theta_{tr} \) trim pitch angle

\( \rho \) atmospheric density

\( \sigma_T = \sigma_u + \sigma_w \)

\( \sigma_u = \frac{U_{ss} \omega_w'}{g} \)

\( \sigma_w = \frac{U_{ss} \omega_w'}{g} \)

\( \psi, \theta, \phi \) airplane yaw (heading), pitch, and roll angles, respectively

\( \omega_n \) undamped circular frequency

Dot over a symbol indicates differentiation with respect to time.

EQUATIONS PROGRAMMED AND PROGRAM DESCRIPTIONS

Six programs are presented in this paper. The first three calculate the elements of the lateral and longitudinal stability determinants and the coefficients of the characteristic equations. In addition, program 3 extracts a real root of a fifth-order polynomial when required. Programs 4 and 5 complete the root extraction process and calculate the stability parameters. Program 6 implements the Euler angle transformation by using the polar-rectangular keys found on calculators.

Programs 1, 2, and 3 are written for the International System of Units, stability axes (fig. 1), and the dimensional form of the stability derivatives. The equations programmed are the linearized form of the equations of motion derived in appendix A of reference 1; thus, the effects of wind shear are included.
In deriving these equations, head winds and updrafts were taken as negative. Thus, a positive $u'_w$ will change a head wind into a tail wind, and a positive $w'_w$ will change an updraft into a downdraft. The signs of $u'_w$ and $w'_w$ set the signs of $\sigma_u$ and $\sigma_w$; $u'_w$ is a gradient with altitude and $w'_w$ is a gradient along the flight path.

In writing the programs, the following conventions were used for the labels:

(1) Capital letters (A to E) are program labels

(2) Lower-case letters (a to e) are subroutine labels

(3) Numbers (0 to 9) are used for all other labels

Table I summarizes the programs presented in this paper. The key entries given in appendixes A to F are the standard HP-67 key entries given in the owner's manual. Check cases for all programs are given in appendix G.

<table>
<thead>
<tr>
<th>Program</th>
<th>Description</th>
<th>Key entries given in</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Calculates the elements of longitudinal stability determinant and normalized coefficients for characteristic equation</td>
<td>Appendix A</td>
</tr>
<tr>
<td>2</td>
<td>Calculates the elements of lateral stability determinant and starts calculating coefficients of the characteristic equation</td>
<td>Appendix B</td>
</tr>
<tr>
<td>3</td>
<td>Label A completes calculating coefficients of characteristic equations of lateral motion; label B calculates a real root of a fifth-order polynomial and reduces the fifth-order polynomial to a fourth-order one; $t_{1/2}$ or $t_D$ for the real root determined; label B can be used as a stand-alone program</td>
<td>Appendix C</td>
</tr>
<tr>
<td>4</td>
<td>Uses Ferrari's method to calculate the roots of a fourth-order polynomial and can be used as a stand-alone program; will also determine roots of cubic, quadratic, and first-order equations</td>
<td>Appendix D</td>
</tr>
<tr>
<td>5</td>
<td>Calculates stability parameters such as $t_{1/2}$, $t_D$, and $N_{1/2}$</td>
<td>Appendix E</td>
</tr>
<tr>
<td>6</td>
<td>Uses the polar-rectangular transformations of the calculator to implement the Euler transformation between space and body axes or body and space axes; this method saves about 57 program steps when compared with the more usual methods of programming</td>
<td>Appendix F</td>
</tr>
</tbody>
</table>
Programs 1 and 2 give solutions from an equilibrium flight condition. There are six parameters, \( U_{ss}, Y_{ss}, \alpha_{tr}, F_{T,tr}, \sigma_T, \) and \( \sigma_w \), that must be adjusted correctly to obtain the equilibrium flight condition. There are two equations to accomplish this adjustment. Programs 1 and 2 were set up in the following manner. The parameters \( U_{ss}, Y_{ss}, \sigma_T, \) and \( \sigma_w \) are specified by the user. The program calculates \( \alpha_{tr} \), assuming that \( F_{T,tr} = 0 \). For the flight condition \( U_{ss} = 77.12 \text{ m/sec}, Y_{ss} = -0.05236 \text{ rad}, \sigma_T = 2.0, \) and \( \sigma_w = 0.0 \), the error introduced in \( \alpha_{tr} \) by this method is 0.00081 rad, which is considered acceptable. If it is desired to monitor the calculated value of \( \alpha_{tr} \), insert a pause after step 45 of program 1.

**Program 1**

The linearized equation of longitudinal motion is in symbolic form

\[
\begin{bmatrix}
\frac{d}{dt} + a_{12} \\
a_{13} \frac{d}{dt} + a_{22} \frac{d}{dt} + a_{23} \\
\left(\frac{d}{dt}\right)^2 + a_{25} \frac{d}{dt} + a_{26} \\
\left(\frac{d}{dt}\right)^2 + a_{35} \frac{d}{dt}
\end{bmatrix}
\begin{bmatrix}
\frac{d}{dt} + a_{31} \\
a_{32} \frac{d}{dt} + a_{33} \\
\left(\frac{d}{dt}\right)^2 + a_{35} \frac{d}{dt}
\end{bmatrix}
\begin{bmatrix}
u \\
\alpha_{pr} \\
\gamma_{pr}
\end{bmatrix}
= \begin{bmatrix}
F X_{\delta e} \\
F Z_{\delta e} \\
M Y_{\delta e}
\end{bmatrix}
\]

(1)

The characteristic equation for longitudinal stability is obtained from the determinant of the \( 3 \times 3 \) matrix and has the form

\[
a_4 \left(\frac{d}{dt}\right)^4 + a_3 \left(\frac{d}{dt}\right)^3 + a_2 \left(\frac{d}{dt}\right)^2 + a_1 \frac{d}{dt} + a_0 = 0
\]

(2)

where \( a_0 \) to \( a_4 \) are given by

\[
a_4 = a_{22} - a_{32} \quad (3a)
\]

\[
a_3 = a_{12} (a_{22} - a_{32}) + (a_{23} - a_{33} - a_{25} a_{32} + a_{22} a_{35}) \quad (3b)
\]

\[
a_2 = (a_{23} a_{35} - a_{25} a_{33} - a_{26} a_{32}) + a_{12} (a_{23} - a_{33} - a_{25} a_{32} + a_{22} a_{35}) \\
+ a_{13} (a_{31} - a_{21}) \quad (3c)
\]
\[ a_1 = a_{12}(a_{23}a_{35} - a_{25}a_{33} - a_{26}a_{32}) + a_{31}(a_{13}a_{25} - a_{14}a_{22}) - a_{21}(a_{13}a_{35} - a_{14}a_{32}) - a_{26}a_{33} \]  
\[ a_0 = a_{33}(a_{21}a_{14} - a_{26}a_{12}) + a_{31}(a_{13}a_{26} - a_{14}a_{23}) \]  
and \( a_{12}, a_{13}, \text{etc.}, \) are given by

\[
a_{12} = -g \frac{\sigma_T}{2U_{SS}} \sin 2\gamma_{SS} + \left( C_{D,0} + \frac{C_L^2}{2} \right) k_1 - \frac{F_{Tu}}{m} \sin 2\gamma_{SS} \]

\[
a_{13} = C_L k_1 - g \frac{(\sigma_T \sin^2 \gamma_{SS} - \sigma_w)}{U_{SS}} \]

\[
a_{14} = -\left( C_{m_o} + C_{m_{\alpha_{tr}}} \right) k_2 \]

\[
a_{21} = C_{D_{\alpha}} k_3 \]

\[
a_{22} = \left( C_{L_{\alpha}} + C_{L_{\theta}} \right) k_3 \]

\[
a_{23} = C_{L_{\alpha}} k_3 \]

\[
a_{25} = -\left( C_{m_{\theta}} + C_{m_{\alpha}} \right) k_4 \]

\[
a_{26} = -C_{m_{\alpha}} k_4 \]

\[
a_{31} = g(\cos \gamma_{SS} - \sigma_T \cos 2\gamma_{SS}) \]

\[
a_{32} = -U_{SS} + C_{L_{\theta}} k_3 \]

\[
a_{33} = g(\sin \gamma_{SS} - \sigma_T \sin 2\gamma_{SS}) \]

\[
a_{35} = -C_{m_{\theta}} k_4 \]
where \( k_1 = \frac{\rho S u_{SS}^2}{m}, \quad k_2 = \frac{\rho S c u_{SS}}{I_Y}, \quad k_3 = \frac{\rho S u_{SS}^2}{2m}, \) and \( k_4 = \frac{\rho S c u_{SS}^2}{2I_Y}. \) In addition to the foregoing equations, the following equations are needed to calculate the values of \( C_L, \ C_D, \) and \( \alpha_{tr} \) at trim:

\[
C_L = \frac{2mg}{\rho S u_{SS}^2} (\sigma_T \sin^2 \gamma_{SS} - \sigma_w + \cos \gamma_{SS})
\]

\[
C_D = C_{D,0} + \frac{C_L^2}{\pi A}
\]

\[
\alpha_{tr} = \frac{C_L - C_{L,0}}{C_{L,0}}
\]

Because large changes in forward speed are encountered in wind shear, the effects of the \( u \) stability derivatives not normally accounted for are included in this program. This was done in the following manner:

\[
D_u = \frac{\partial D}{\partial u} = \left(C_{D,0} + \frac{C_L^2}{\pi A}\right) k_1
\]

\[
L_u = \frac{\partial L}{\partial u} = C_L k_1
\]

\[
M_y = \frac{\partial M_y}{\partial u} = \left(C_{m,0} + C_{m,0} \alpha_{tr}\right) k_2
\]

Equations (3), (4), and (5) were programmed to calculate the coefficients of the characteristic equation, which is equation (2). The key codes for program 1 are given in appendix A.

The program destroys the original input data but preserves the coefficients of the determinant in the secondary registers. The principal output is the normalized coefficients of the characteristic equation which are stored in \( R_0, \ R_1, \ R_2, \) and \( R_3. \)
Program 2

The linearized equations of lateral motion with the effects of wind shear included are, in symbolic form,

\[
\begin{bmatrix}
  c_{11} \frac{d}{dt} + b_{13} & c_{21} \frac{d}{dt} + b_{22} & c_{30} \frac{d}{dt} + b_{31} \\
  \left(\frac{d}{dt}\right)^2 + b_{14} \frac{d}{dt} + b_{15} & b_{42} \left(\frac{d}{dt}\right)^2 + b_{23} \frac{d}{dt} + b_{32} & b_{33} \frac{d}{dt} + b_{33} \\
  b_{43} \left(\frac{d}{dt}\right)^2 + b_{16} \frac{d}{dt} + b_{17} & \left(\frac{d}{dt}\right)^2 + b_{24} \frac{d}{dt} + b_{34} & b_{35} \frac{d}{dt} + b_{35}
\end{bmatrix}
\begin{bmatrix}
  \phi \\
  \psi \\
  \beta
\end{bmatrix} =
\begin{bmatrix}
  F_{\gamma_{\phi}} \\
  M_{\xi_{\phi}} \\
  M_{\xi_{\phi}}
\end{bmatrix}
\begin{bmatrix}
  \delta_a \\
  \delta_r
\end{bmatrix}
\]

(6)

and are based on equations (A5), (A8), and (A9) of reference 1. In linearizing these equations, it was assumed that no wind gradient existed in the \( Y_e \) derivative in Earth axes. If the wind gradients are zero (i.e., no wind shear), these equations reduce to the standard form of the linearized equations of lateral motion that are given in many standard works, such as reference 2. The equations are valid in the interval \(-0.17453 \leq \gamma_{ss} \leq 0.17453\).

The characteristic equation is obtained from the 3\times3 matrix on the left-hand side of equation (6) and has the form

\[
a_5 \left(\frac{d}{dt}\right)^5 + a_4 \left(\frac{d}{dt}\right)^4 + a_3 \left(\frac{d}{dt}\right)^3 + a_2 \left(\frac{d}{dt}\right)^2 + a_1 \frac{d}{dt} + a_0 = 0
\]

(7)

for \( \sigma_T \neq 0 \).

When \( \sigma_T = 0 \), the \( a_0 \) term in equation (7) becomes 0. Equation (7) now has one zero root and four finite roots and is solved as a quartic. Program 2 tests equation (7) and informs the user if a fourth- or fifth-degree polynomial is present. The coefficients \( a_0 \) to \( a_5 \) are given by

\[
a_5 = C_{30} (1 - b_{43} b_{42})
\]

(9a)

\[
a_4 = C_{11} (b_{42} b_{34} - b_{32}) - C_{21} (b_{34} - b_{43} b_{32}) + b_{31} (1 - b_{43} b_{42}) + C_{30} (b_{24} + b_{14} - b_{43} b_{23} - b_{16} b_{42})
\]

(9b)
The C and b terms that are used to generate the coefficients of the characteristic equation are given by

\[ a_3 = b_{13}(b_{42}b_{34} - b_{32}) - b_{22}(b_{34} - b_{43}b_{32}) + C_{11}(b_{42}b_{35} + b_{23}b_{34} - b_{24}b_{32} - b_{33}) - C_{21}(b_{35} + b_{14}b_{34} - b_{43}b_{33} - b_{16}b_{32}) + b_{31}(b_{24} + b_{14} - b_{43}b_{23}) - b_{16}b_{42}) + C_{30}(b_{15} + b_{14}b_{24} - b_{16}b_{23} - b_{17}b_{42}) \]  
\[ a_2 = C_{11}(b_{23}b_{35} - b_{24}b_{33}) + b_{13}(b_{42}b_{35} + b_{23}b_{34} - b_{24}b_{32} - b_{33}) - C_{21}(b_{14}b_{35} + b_{15}b_{34} - b_{16}b_{33} - b_{17}b_{32}) + C_{30}(b_{15}b_{24} - b_{17}b_{23}) - b_{22}(b_{35} + b_{14}b_{34} - b_{16}b_{33} - b_{17}b_{32}) + b_{31}(b_{15} + b_{14}b_{24} - b_{16}b_{23} - b_{17}b_{42}) \]  
\[ a_1 = b_{13}(b_{23}b_{35} - b_{24}b_{33}) - b_{22}(b_{14}b_{35} + b_{15}b_{34} - b_{16}b_{33} - b_{17}b_{32}) \]  
\[ a_0 = -b_{22}(b_{15}b_{35} - b_{17}b_{33}) \]  

The \( C \) and \( b \) terms are given by

\[ b_{11} = 0.0 \]  
\[ b_{12} = -C_{\gamma p}k_5 \]  
\[ b_{13} = -\frac{g\sigma_w}{U_{\infty}^2} - \frac{g}{U_{\infty}} \cos \gamma_{SS} \]  
\[ b_{14} = -C_{\gamma p}k_6 \]  
\[ b_{15} = u_w'(\gamma_{\gamma p}k_6) = C_{\gamma p}k_6 \]  
\[ b_{16} = -C_{\gamma p}k_7 \]  
\[ b_{17} = u_w'(\gamma_{\gamma p}k_7) = C_{\gamma p}k_7 \]  
\[ b_{21} = -C_{\gamma p}k_5 \]
\[ b_{22} = \frac{g(\sigma_u + \sigma_w)}{2u_{ss}^2} \sin 2\gamma_{ss} \]  

\[ b_{23} = -C_{I_x}k_6 \]  

\[ b_{24} = -C_{n_x}k_7 \]  

\[ b_{30} = -C_{Y_p}k_5 \]  

\[ b_{31} = -C_{Y_p}k_5 \]  

\[ b_{32} = -C_{I_{p}}k_6 \]  

\[ b_{33} = -C_{I_{p}}k_6 \]  

\[ b_{34} = -C_{n_{p}}k_7 \]  

\[ b_{35} = -C_{n_{p}}k_7 \]  

\[ b_{42} = -\frac{I_{XZ}}{I_x} \]  

\[ b_{43} = -\frac{I_{XZ}}{I_z} \]  

\[ C_{11} = b_{11} + b_{12} \]  

\[ C_{21} = 1 + b_{21} \]  

\[ C_{30} = 1 + b_{30} \]  

where \( k_5 = \frac{\rho s u_{ss}}{2m} \), \( k_6 = \frac{\rho S b u_{ss}^2}{2I_x} \), and \( k_7 = \frac{\rho S b u_{ss}^2}{2I_z} \). The trim angle of attack
was calculated from

\[ \alpha_{tr} = \left\{ \frac{2mg}{\rho S U_{ss}^2} \left[ (\sigma_T \sin^2 \gamma_{ss} - \sigma_w + \cos \gamma_{ss}) \right] - C_{L,0} \right\} (C_{L\alpha})^{-1} \]  \quad (11)

Equations (9), (10), and (11) were programmed for the calculator and the program is given in appendix B. The stability derivatives \( C_{l\phi} \), \( C_{n\phi} \), and \( C_{Y\phi} \) have been included in this program. The derivatives \( C_{l\phi} \) and \( C_{n\phi} \) are always calculated when wind shears are included. This program calculates all \( b \) and \( C \) coefficients in the determinant and starts calculating the coefficients of the characteristic equation.

Program 3

Program 3 completes the calculation of the coefficients of the characteristic equation and tests the contents of register 4 to determine if the equation is a quartic or a quintic. If it is a quartic, the number 4 is displayed and the calculator stops. If it is a quintic, the number 5 is displayed and the calculator stops. Label B of this program calculates a real root of the quintic equation by using the secant method. The initial guess for the root is obtained by dividing the coefficient \( a_4 \) by 5; this operation is done in the program.

Even with an estimate of the root, however, two estimated points are required to start the secant method. These points are obtained by either adding or subtracting 0.08 from \( a_4/5 \). The number 0.08 has proved satisfactory for several different fifth-order polynomials. However, the secant method is sensitive to this number and changes may be necessary. Subsequent estimates of the root were calculated from

\[ x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \]  \quad (12)

where \( x_i \) is always the present value and \( f(x_i) \) is the value of the function being used for \( x = x_i \). Synthetic division is used to determine when a root has been found. The fifth-order polynomial is then reduced to a quartic for processing by program 4. During the iteration process, the calculator pauses to display the value of the characteristic polynomial so that convergence can be monitored. When the display shows zero, the root has been found. When the root has been found, the calculator will stop and display 8. The root is in the Y stack register and the time to damp to one-half amplitude or time to double amplitude is in the Z register. A negative number in the Z register means that the value given is the time to double amplitude. The number of iterations required to extract the root is in the T stack register.
The test used for the determination of a root is that the polynomial must be zero to the number of digits in the calculator display; thus, the test for a root assures its accuracy.

Program 4

A quartic equation is the highest order polynomial for which an explicit analytical solution for the root exists. Ferrari's method (refs. 3 and 4) and appendix H was used to obtain the roots of the quartic from the characteristic equation. The general form of the quartic equation is

\[ a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \]  
(13)

The first step in applying Ferrari's method is to normalize equation (13) so that \( a_4 = 1 \). The determination of a real root of the following resolvent cubic is the next step:

\[ y^3 + b_2 y^2 + b_1 y + b_0 = 0 \]  
(14)

The coefficients of equation (14) are given by

\[
\begin{align*}
    b_2 &= -a_2 \\
    b_1 &= a_1 a_3 - 4a_0 \\
    b_0 &= a_0 \left( 4a_2 - a_3^2 \right) - a_1^2
\end{align*}
\]

(15)

and the root \( \text{Re}(y) \) is obtained by

\[
\text{Re}(y) = S + T = -\frac{b_2}{3} \quad (f > 0)
\]

(16)

or

\[
\text{Re}(y) = 2(R^2 + f)^{1/3} \cos \left[ \frac{1}{3} \left( \tan^{-1} \left( \frac{\sqrt{f}}{R} \right) \right) \right] - \frac{b}{2} \quad (f \leq 0)
\]

(17)

where

\[
\begin{align*}
    Q &= (3b_1 - b_2^2)/9 \\
    R &= (9b_2 b_1 - 27b_0 - 2b_2^3)/54 \\
    f &= R^2 + Q^3
\end{align*}
\]

(18a) (18b) (18c)
\[ S = \left( R + \sqrt[3]{f} \right)^{1/3} \quad (18d) \]

\[ T = \left( R - \sqrt[3]{f} \right)^{1/3} \quad (18e) \]

The root \( \text{Re}(y) \) is any root of the resolvent cubic, equation (14); this program is written to calculate the largest real root of equation (14). Once \( \text{Re}(y) \) is known, the roots of the quartic are obtained by solving the following two quadratic equations:

\[
\begin{align*}
  z^2 + (A + C)z + (B + D) &= 0 \\
  z^2 + (A - C)z + (3 - D) &= 0
\end{align*}
\]  

(19)

where

\[
A = \frac{a_3}{2} \\
B = \frac{\text{Re}(y)}{2} \\
D = \sqrt{b^2 - a_0} \\
C = \left( \frac{AB - \frac{a_1}{2}}{D} \right) \quad (D \neq 0) \\
C = \sqrt{A^2 - a_2 + \text{Re}(y)} \quad (D = 0)
\]  

(20)

Equations (15) to (20) and a quadratic solution routine were programmed to obtain the roots of a quartic equation. The key codes for program 4 are given in appendix D.

Because \( f \) and \( D \) are tested to determine program direction, special programming is required both to insure that nonsignificant digits do not influence the test and to protect against the small difference of large numbers. The expressions for \( f \) and \( D \) were written as

\[
f = R^2 \left( 1 + \frac{Q^3}{R^2} \right)
\]
\[ D = \sqrt{B^2 \left(1 - \frac{a_0}{B^2}\right)} \]

for programming. In each case, the quantity in the parenthesis was rounded to the calculator display and then tested. Special routines were added to protect against \( R \) and \( B \) being equal to 0. The introduction of rounding will introduce some error if a significant number is truncated. As the rounding is controlled by the number of decimal digits in the calculator display, there is flexibility in the amount of rounding introduced. Experience with a set of 20 test equations indicates that a display of 7 digits is satisfactory for most cases.

The roots of the quartic are stored in registers \( R_1, R_2, S_1, \) and \( S_2 \). The root indicator (-1.0 for complex roots and 0.0 for real roots) is stored in registers \( R_0 \) and \( S_0 \). If the roots are complex, the real part is stored in register 1 and the imaginary part in register 2.

This program is a general program for the roots of a quartic equation and may be used as a stand-alone program if the coefficients of the quartic are stored in the following locations:

- \( a_3 \) in register \( R_0 \)
- \( a_2 \) in register \( R_1 \)
- \( a_1 \) in register \( R_2 \)
- \( a_0 \) in register \( R_3 \)

In addition, this program may be used to solve for the roots of lower order equations. For the cubic where the equation has the form

\[ a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \quad a_3 = 1.0 \]

the equation is multiplied by \( x \) so that it is converted to a quartic with a zero root and the coefficients are stored as follows:

- \( a_2 \) in register \( R_0 \)
- \( a_1 \) in register \( R_1 \)
- \( a_0 \) in register \( R_2 \)
- 0.0 in register \( R_3 \)

Quadratic and first-order equations may be solved in a similar manner by multiplying through by \( x^2 \) or \( x^3 \), respectively.
Program 5

Program 5 calculates the stability parameters (ref. 5, p. 61), such as the time to damp to one-half amplitude or the damping ratio. The equations programmed are given as follows:

Time to damp to one-half amplitude $t_{1/2}$ or time to double amplitude $t_D$:

$$ t_{1/2} \text{ or } t_D = \frac{0.693}{\text{Re}(\ )} $$  \hspace{1cm} (21)

Period:

$$ t = \frac{2\pi}{\text{Im}(\ )} \quad \rho = \frac{2\pi}{\omega} $$  \hspace{1cm} (22)

Number of cycles to damp to one-half amplitude $N_{1/2}$ or time to double amplitude $N_D$:

$$ N_{1/2} \text{ or } N_D = -0.110 \frac{\text{Im}(\ )}{\text{Re}(\ )} $$  \hspace{1cm} (23)

Logarithmic decrement:

$$ \Delta = \frac{0.693}{N_{1/2} \text{ or } N_D} $$  \hspace{1cm} (24)

Undamped circular frequency:

$$ \omega_n = [\text{Re}(\ )^2 + \text{Im}(\ )^2]^{1/2} $$  \hspace{1cm} (25)

Damping ratio:

$$ \zeta = \frac{\text{Re}(\ )}{\omega_n} $$  \hspace{1cm} (26)

If $\Delta$, $t$, or $N$ is negative, unstable conditions are indicated. For instance, if $-\frac{0.693}{\text{Re}(\ )}$ is negative, the time calculated is for doubling the amplitude.

The key entries for this program are given in appendix E and the storage at the end of this program contains all the calculated information concerning airplane stability. For real roots, only the time to damp to one-half amplitude or the time to double amplitude is calculated. This program may be used as a stand-alone program.
Program 6

Program 6 uses the polar-rectangular keys of the calculator to implement the Euler transformation used in rigid-body rotation. The transformation programmed is the $\psi, \theta, \phi$ transformation that is frequently used in aeronautics (fig. 1). The use of the polar-rectangular keys permits a short program for this type of transformation.

The transformation scheme is illustrated through the use of a two-dimensional transformation. The coordinates of a point $p(x,y)$ in the $xy$ axis system are given in the $x'y'$ axis system, which is rotated through the angle $\epsilon_1$ with respect to the $xy$ axis system by

$$
\begin{align*}
  x' &= x \cos \epsilon_1 + y \sin \epsilon_1 \\
  y' &= -x \sin \epsilon_1 + y \cos \epsilon_1
\end{align*}
$$

(27)

The polar coordinates of $p(x,y)$ are $R_*, \epsilon_2$ in the $xy$ axis system, where $R_* = (x^2 + y^2)^{1/2}$ and $\epsilon_2 = \tan^{-1} \frac{y}{x}$, and are $R_*, (\epsilon_2 - \epsilon_1)$ in the $x'y'$ axis system. The $x'y'$ axis system coordinates are now given by

$$
\begin{align*}
  x' &= R_* \cos (\epsilon_2 - \epsilon_1) \\
  y' &= R_* \sin (\epsilon_2 - \epsilon_1)
\end{align*}
$$

(28)

If equation (28) is expanded ($x$ is substituted for $R_* \cos \epsilon_2$ and $y$ is substituted for $R_* \sin \epsilon_2$), equation (27) results and shows that the same transformation is taking place. This result leads to a program for a two-dimensional transformation. It is assumed that $y$ is stored in the $Y$ stack register, $x$ is stored in the $X$ stack register, and $\epsilon_1$ is stored in register $Rn$. The program is as follows:

\begin{verbatim}
  ->p
  x>y
  RCL n
  ...
  x>y
  ->R
\end{verbatim}

This program gives $x'$ and $y'$ in 6 steps instead of the usual 18 steps. This two-dimensional program is completely general. If this two-dimensional transformation program is used in conjunction with a bookkeeping program, three-dimensional transformations may be made. In reference 6 (pp. 272 to 275), a method is given that simplifies the bookkeeping problem. A program for one of the three-dimensional Euler transformations used in aeronautics is given in
appendix F. This program is for transformations between two right-hand axes systems (fig. 1) in which the Z-axis is positive downwards. The first rotation is through the angle $\psi$ about the $Z_{sp}$-axis; the second is through the angle $\theta$ about the $Y_{sp}$-axis; and the third is through the angle $\phi$ about the $X_{sp}$-axis. The angles $\psi$, $\theta$, and $\phi$ are the airplane heading, pitch, and roll angles, respectively. The program presented in appendix F is a specialized program because, in three-dimensional transformations, the order in which the rotation angles are taken and the axes about which the rotations take place vary from one transformation to another. Similar programs may be written for other three-dimensional transformations by changing the bookkeeping part of program 6. Subroutines B and C would not be changed.

The advantages of using the polar-rectangular keys in program 6 for three-dimensional transformations are not apparent unless program 6 is compared with a program that uses the traditional approach of calculating the direction cosines and then using them to make the transformation. By using direction cosines, a reasonably efficient program for the $\psi, \theta, \phi$ transformation discussed in this section takes 124 program steps and 20 storage registers, compared with 67 steps and 10 storage registers for the polar-rectangular method of this paper. The impact is even more apparent if both the polar-rectangular (P+R) and the direction-cosine (D-C) methods are considered as subprograms to a main program. Take the following example:

A vector has been computed and its components are stored in three consecutive registers. The angles $\psi$, $\theta$, and $\phi$ have also been calculated and are stored in consecutive registers. It is desired to transform the calculated vector components to a new coordinate system rotated from the original by the angles $\psi$, $\theta$, and $\phi$.

Table II summarizes the manner in which the two programs would merge with the main program. Storage for the original vector components and the angles is not counted.

<table>
<thead>
<tr>
<th>Programming considerations</th>
<th>Space to body</th>
<th>Body to space</th>
<th>Two way</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P+R method</td>
<td>D-C method</td>
<td>P+R method</td>
</tr>
<tr>
<td>Program steps in transformation</td>
<td>26</td>
<td>83</td>
<td>27</td>
</tr>
<tr>
<td>Registers used</td>
<td>----</td>
<td>13</td>
<td>----</td>
</tr>
<tr>
<td>I register</td>
<td>Used</td>
<td>Used</td>
<td>Used</td>
</tr>
<tr>
<td>Storage for new computation</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Total program steps used$^a$</td>
<td>26</td>
<td>83</td>
<td>27</td>
</tr>
<tr>
<td>Total registers used$^a$</td>
<td>3</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>Steps available for main program</td>
<td>198</td>
<td>141</td>
<td>197</td>
</tr>
<tr>
<td>Registers available for main program</td>
<td>22</td>
<td>9</td>
<td>22</td>
</tr>
</tbody>
</table>

$^a$I register is not counted.
Analysis of the data presented in table II shows the economics of using the P*R method in programs. In addition, a calculator with only 49 program steps can be programmed one way using the P*R method, and a calculator with 98 steps can handle the two-way P*R transformation. For the D-C method, the smallest programmable calculator that can handle a one-way transformation is one with 98 steps. The two-way transformation will not fit on a 98-step calculator.

USE OF PROGRAMS 1 TO 6

After a program has been keyed into the calculator, the program switch should be set to run. Set the display and trig modes, switch back to program, and record program. The display and trig mode status are now recorded on the magnetic card and the calculator will be set to the indicated status conditions whenever the program is read in. The display and trig mode status are given for each program in the appendixes.

Appendix G contains the check case for the programs in appendixes A to F. To make longitudinal stability calculations, use the following procedure:

(1) Enter program 1 (appendix A)

(2) Enter data as shown on storage map
    Push A
    At stop, coefficients of characteristics equation have been calculated

(3) Enter program 4 (appendix D)
    Push A
    At stop, roots of characteristic have been determined

(4) Enter program 5 (appendix E)
    Push A
    At stop, complete set of longitudinal stability data is stored as
    indicated on storage map

To make lateral stability calculations, use the following procedure:

(1) Enter program 2 (appendix B)

(2) Enter data as shown on storage map
    Push A

(3) At stop, enter program 3 (appendix C)
    Push A
    If 4 is displayed at stop, go to step 4
    If 5 is displayed, push B

    When 8 is displayed, the real root, the time to damp to one-half amplitude or the time to double amplitude, and the number of iterations are stored in the stacks
(4) Enter program 4 (appendix D)
    Push A
    At stop, roots of quartic have been calculated

(5) Enter program 5 (appendix E)
    Push A
    At stop, a complete set of lateral stability data has been calculated.
    Data relating to quartic is stored in calculator

Programs 4 and 5 may be used as stand-alone programs.

Program 6 may be used in several different ways. To transform from space axes \((X_{sp}, Y_{sp}, Z_{sp})\) to airplane axes \((X_b, Y_b, Z_b)\), use the following procedure:

1. Enter \(Z_{sp}, Y_{sp}, X_{sp}\) in stack in order given
   Push A

2. Enter \(\phi, \theta, \psi\) in stack in order given
   Push B
   Push C to make the transformation
   At stop, airplane axis coordinates \(x_b, y_b, z_b\) are stored in registers \(R_6, R_7, \text{and } R_8\), respectively

To transform from airplane axes \((X_b, Y_b, Z_b)\) to space axes \((X_{sp}, Y_{sp}, Z_{sp})\), use the following procedure:

1. Enter \(Z_b, Y_b, X_b\) in stack in order given
   Push A

2. Enter \(\phi, \theta, \psi\) in stack in order given
   Push B
   Push D to make the transformation
   At stop, space coordinates \(x_{sp}, y_{sp}, z_{sp}\) are stored in registers \(R_6, R_7, \text{and } R_8\), respectively

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April 28, 1978
APPENDIX A

PROGRAM 1 - LONGITUDINAL AIRPLANE STABILITY

Program 1 uses the basic physical and aerodynamic data of an airplane to calculate the coefficients of the characteristic equation of longitudinal motion. This program calculates normalized coefficients for the characteristic equation. For running this program, the calculator should be set to calculate in radians and to display seven decimal digits. A pause inserted after step 49 causes the calculated value of $\alpha_{tr}$ to be displayed.
APPENDIX A

001 LBLA
RCL9
STO+8
RCLO
X^2
CHS
RCLX (i)
RCL8
RCL7

010 SIN
X^2
×
RCL9
- \( \sigma_T \) \( \sin^2 \gamma_{ss} \) - \( \sigma_w \)
STO0
RCL7
COS
STO9
+

020 RCL5
×
g(\( \sigma_T \) \( \sin^2 \gamma_{ss} \) - \( \sigma_w \) + \( \cos \gamma_{ss} \))
RCLB
RCL2
×
RCL6
×
RCL1
÷
STOE

030 RCL6
×
2
÷
\( \rho \) \( SU_{ss}/m \)
STO×3
STOB
÷
RCL4
P+S
STO6
C_m stored in R6

040 X+Y
STO5
C_L stored in S5
RCLD
-
RCL1
÷
\( \alpha_{tr} \)
×
STO+9
RCL5
X^2

050 RCLC

060 STO+4
RCL2
STO+3
C_L0 + C_L\( \alpha \)
RCL8
STO+7
C_m0 + C_m\( \alpha \)
RCLE

070 STO×4
STO×5
RCLA
×
STO×9
a14
RCLB
STO×(i)
STO×0
a21
STO×1
a23
STO×2

080 STO×3
a22
RCLA
STO×6
a26
STO×7
a25
STO×8
a35
P+S
RCL9
RCL7
2
×

090 COS
RCL8
STO×9
×
- 
STOA
RCL7
SIN
STO×9
RCL9

100 STO×9
\( g \sigma_T \) \( \sin^2 \gamma_{ss}/2U_{ss} \)
APPENDIX A

\[ g(\sigma_T \sin^2 \gamma_{ss} - \sigma_w)/U_{ss} \]

110

RCL0 ×
RCL9
RCL3 -
RCL6
P+S
STO-2 R+ 160
STO-4 R+ 170
STO-5 1
0 STOI
RCL3
RCL2 - 32
STOC RCL4
120
RCL1
RCLB -
RCL2
RCL7 ×
- 180
RCL3
RCL8 ×
+ STOD
+ a3 190
STO(i) ISZ
RCL1
RCL8 ×
140
RCL7
RCL8 ×
- RCL6
RCL2 ×
- 200
STOE RCLD
RCL4 ×
150
RCL8 +
- RCL9
RCL1 ×
APPENDIX A

RCLA
×
+
⇒ S
STO 3

210 RCLC
STO× 0
STO× 1
STO× 2
STO× 3
RTN
R/S
APPENDIX A

Storage Map for Program 1

<table>
<thead>
<tr>
<th>(i) Address</th>
<th>Register</th>
<th>Input storage</th>
<th>Output storage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>R0</td>
<td>kY</td>
<td>a3</td>
</tr>
<tr>
<td>1</td>
<td>R1</td>
<td>m</td>
<td>a2</td>
</tr>
<tr>
<td>2</td>
<td>R2</td>
<td>ρ</td>
<td>a1</td>
</tr>
<tr>
<td>3</td>
<td>R3</td>
<td>CTu</td>
<td>a0</td>
</tr>
<tr>
<td>4</td>
<td>R4</td>
<td>Cmα</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>R5</td>
<td>g</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>R6</td>
<td>Us</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>R7</td>
<td>γss</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>R8</td>
<td>σu</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>R9</td>
<td>σw</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>S0</td>
<td>CDα</td>
<td>a21</td>
</tr>
<tr>
<td>11</td>
<td>S1</td>
<td>CLα</td>
<td>a23</td>
</tr>
<tr>
<td>12</td>
<td>S2</td>
<td>CL0</td>
<td>a32</td>
</tr>
<tr>
<td>13</td>
<td>S3</td>
<td>CLα</td>
<td>a22</td>
</tr>
<tr>
<td>14</td>
<td>S4</td>
<td>CD,0</td>
<td>a12</td>
</tr>
<tr>
<td>15</td>
<td>S5</td>
<td>0.0</td>
<td>a13</td>
</tr>
<tr>
<td>16</td>
<td>S6</td>
<td>0.0</td>
<td>a26</td>
</tr>
<tr>
<td>17</td>
<td>S7</td>
<td>Cmδ</td>
<td>a25</td>
</tr>
<tr>
<td>18</td>
<td>S8</td>
<td>Cm0</td>
<td>a35</td>
</tr>
<tr>
<td>19</td>
<td>S9</td>
<td>Cm,0</td>
<td>a14</td>
</tr>
<tr>
<td>20</td>
<td>R_A</td>
<td>c</td>
<td>a31</td>
</tr>
<tr>
<td>21</td>
<td>R_B</td>
<td>S</td>
<td>a33</td>
</tr>
<tr>
<td>22</td>
<td>R_C</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>R_D</td>
<td>CL,0</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>R_E</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>I</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

These are the normalized coefficients of the quartic; thus, a4 = 1.00.
APPENDIX B

PROGRAM 2 - LATERAL AIRPLANE STABILITY

Program 2 uses the basic physical and aerodynamic data of an airplane to generate the coefficients of the lateral stability determinant. After completing the calculation of these coefficients, the program starts but does not finish calculating the coefficients of the characteristic equation for lateral motion. For running this program, the calculator should be set to calculate in radians and to display seven decimal digits. A pause inserted after step 44 causes the calculated value of $\alpha_{tr}$ to be displayed.
APPENDIX B

001 LBLA
RCL1
RCL2
RCL3
×
×
2
÷
RCL4
RCL5
STO+0
STO-4
RCL9
X²
STO9
STO+(i)
RCL8
X<0
SF2
F?2
CHS
CHS
ENT
ENT
RCL9
÷
STO9
X+Y
RCL5
÷
STO8
RCL0
RCL7
RCL1
÷
×
STO2
RCLC
COS

RCL0
×
\frac{g_\alpha_w}{U_{SS}^2} - \frac{g}{U_{SS}} \cos \gamma_{SS}
APPENDIX B

100 RCL0 ST0×6 ST0×7 2 3 STOI RCL8 Calculate coefficient of characteristic equation P+S RCL3 ×
110 RCL7 Secondary called - b₄₂b₃₄ - b₃₂ 160 RCL2 RCL7 × + RCL4 Primary called P+S RCL9 ×
120 RCL6 RCL3 × RCL0 RCL4 × - b₂₃b₃₅ - b₂₄b₃₃ 170 DSZ DSZ Complete calculations and store terms R+ STO-2 R+ STO-1 R+ STO-0 R+ End of program
130 P+S Primary called RCL8 ST02 R+ ST01 R+ GSBb R+ R+ 140 STO3 R+ STO+2 R+ RCL9 R+ GSBb R+ R+ 150 ST00 RCL9 P+S RCL3 + RCL1 b₃₄ - a₃₂a₃₄ 160 RCL2 RCL7 × RCL4 Primary called P+S RCL9 ×
170 DSZ DSZ Complete calculations and store terms R+ STO-2 R+ STO-1 R+ STO-0 R+ End of program
180 RTN LBLa DSZ STO×(i) RTN LBLb ENT⁺ ENT⁺ RCL(i) ×
190 X+Y DSZ RCL(i) × RTN R/S
## APPENDIX B

### Storage Map for Program 2

<table>
<thead>
<tr>
<th>Address</th>
<th>Register</th>
<th>Initial storage</th>
<th>End of program</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>R0</td>
<td>g</td>
<td>(a)</td>
</tr>
<tr>
<td>1</td>
<td>R1</td>
<td>Uss</td>
<td>(a)</td>
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<td>ρ</td>
<td>(a)</td>
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<td>S</td>
<td>(a)</td>
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<tr>
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<td>R4</td>
<td>b</td>
<td>(a)</td>
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<tr>
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<td>R6</td>
<td>σu</td>
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<td>R8</td>
<td>b&lt;sub&gt;kxz&lt;/sub&gt;</td>
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<td>kZ</td>
<td>b43</td>
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<td>C&lt;sub&gt;n&lt;/sub&gt;β</td>
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<td>C&lt;sub&gt;l&lt;/sub&gt;p</td>
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<td>b31</td>
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<td>RA</td>
<td>m</td>
<td>b22</td>
</tr>
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<td>C&lt;sub&gt;y&lt;/sub&gt;r</td>
<td>C21</td>
</tr>
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<tr>
<td>25</td>
<td>I</td>
<td>24</td>
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</tr>
</tbody>
</table>

<sup>a</sup>Registers R0 to R4 contain the partially calculated coefficients of the characteristic equation.

<sup>b</sup>If k<sub>xz</sub> is imaginary, enter k<sub>xz</sub> as a negative number.
APPENDIX C

PROGRAM 3 - LATERAL AIRPLANE STABILITY (Concluded)

Program 3 completes the calculation of the coefficients of the characteristic equation of lateral motion that was started in program 2. The program then determines if the characteristic equation is a quartic or a quintic. If it is a quartic, a 4 is displayed and the program stops. Program 4 is then used to obtain the roots of the quartic. If the characteristic equation is a quintic, a 5 is displayed and the program continues on to extract the real root of the quintic and then calculates the time to damp to one-half amplitude or the time to double amplitude. This program uses the storage that existed at the end of program 2. This program calculates normalized coefficients for the quartic and the quintic. For running this program, the calculator should be set to calculate in radians and to display seven decimal digits.
APPENDIX C

001 LBLA
RCL7
RCL6
P+S
RCL3
x
X+Y
RCL7
x

010 -
RCL2
RCL4
x

020 P+S
RCL1
Primary called
RCL6
x
X+Y
RCL7
x
-
GSBa
Complete and store calculations

030 STO4
R↓
STO-3
R↓
GSBa
STO-3
R↓
STO-2
RCL8
RCL9

040 P+S
Secondary called
RCL6
x
CHS
X+Y
RCL2
x
-
RCL5
+

050 RCL0
Secondary called
P+S
GSBa
Complete and store calculations
RCL6
x
X+Y
RCL7
x
-
RCL3
x
X>Y
RCL7
x

060 -
b24b15 - b23b17

070 b15b34 - b32b17
- b16b33 + b14b35

080 x

090 +
b15 - b17b42
- b16b23 + b14b24

Determines if equation is a quartic or a quintic
APPENDIX C

4
Indicates quartic
4

4
RTN
Stop for quartic
4

4
LBL1
GSDb

5
Indicates quintic
160

5
RTN
Stop for quintic
4

5
RCL0
Calculate real root
4

5
STOA
of quintic
4

5
RCL1
This section
4

5
STOB
positions data
4

5
RCL2
Output routine
4

5
STOC
RCL3
4

5
STOD
RCL4
170

5
STOE
0
4

5
ST07
Initialization for
4

5
FIX
secant method
4

5
RCLA
8

5
÷

5
.

5
0
180

5
ST05

5
ST06
1

5
GSBb

5
ST08

5
.

5
1

5
ST0+6
6

5
STBL0
Evaluates polynomial
190

5
STO+7
and tests for
0

5
GSDb
solution
1

5
ST09

5
RND

5
Pause

5
X=0
Displays value of

5
GOTO1
polynomial

5
RCL6
polynomial

5
RCL5
Calculates and stores
200

5
X+Y
new value of X

5
RCL6
ST05
150

5
X+Y
ST04
RTN

5
-

5
LBLc

5
35

Indicates quartic
Stop for quartic
Indicates quintic
Stop for quintic
Calculate real root of quintic
This section positions data
Output routine
Initialization for secant method
RCL(i) • RCL(i) = characteristic equation
Evaluates polynomial and tests for solution
Displays value of polynomial
Calculates and stores new value of X
Synthetic division subroutine
End of program Subroutine for calculating coefficient of characteristic equation
Polynomial evaluation subroutine

APPENDIX C

RCL6
×
RCL(i)
+
210 ISZ
RTN
LBLd Normalization subroutine
RCLE
STO÷0
STO÷1
STO÷2
STO÷3
STO÷4
RTN

220 R/S
APPENDIX C

Storage Map for Program 3

<table>
<thead>
<tr>
<th>(i) Address</th>
<th>Register</th>
<th>Initial storage&lt;sup&gt;a&lt;/sup&gt;</th>
<th>End of program&lt;sup&gt;b&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>R0</td>
<td></td>
<td>a3</td>
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<td>R3</td>
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<td>a0</td>
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<tr>
<td>4</td>
<td>R4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>R5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>R6</td>
<td>b15</td>
<td></td>
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<tr>
<td>7</td>
<td>R7</td>
<td>b17</td>
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<tr>
<td>8</td>
<td>R8</td>
<td>b42</td>
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</tr>
<tr>
<td>9</td>
<td>R9</td>
<td>b43</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>S0</td>
<td>b24</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>S1</td>
<td>b35</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>S2</td>
<td>b16</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>S3</td>
<td>b34</td>
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<tr>
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<td>S4</td>
<td>b33</td>
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<tr>
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<td>S6</td>
<td>b23</td>
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<td>b32</td>
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<td>18</td>
<td>S8</td>
<td>b31</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>S9</td>
<td>C30</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>RA</td>
<td>b22</td>
<td>a4</td>
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<td>21</td>
<td>RB</td>
<td>C21</td>
<td>a3</td>
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<tr>
<td>25</td>
<td>I</td>
<td>in use</td>
<td></td>
</tr>
</tbody>
</table>

The initial storage is the same as that at end of program 2. The partially calculated coefficients of the characteristic equation are stored in R0 to R4.

The end storage is the same for display signals 4 and 8; the normalized coefficients of the quartic are in registers R0 to R3. The real root of the quintic is in the Y register and the time to damp to one-half amplitude or the time to double amplitude is in the Z register when 8 is displayed. Pressing R+ moves the real root of the quintic to the X register; pressing R+ again moves the time to damp to one-half amplitude or the time to double amplitude to the X register. The number of iterations required to obtain the root is in the stack T register and may be obtained by pressing R+.
APPENDIX D

PROGRAM 4 - ROOTS OF A QUARTIC EQUATION

Program 4 applies Ferrari's method for the roots of a quartic equation to the output of either program 1 or program 3 to determine the remaining eigenvalues of the characteristic equation of longitudinal or lateral motion. Normalized coefficients must be used for this program. For running this program, the calculator should be set to calculate in radians and to display seven decimal digits.
APPENDIX D

001 LBLA Calculate coefficients of resolvent cubic
4 STO6
RCL1
STOC
STO×6
CHS
STO4 b_2 in R_4
RCL0
STOB
STO5
x^2
STO-6
RCL2
STOD
STO×5
x^2
RCL3 070 x+y
STOE
STO×6
4 x
STO-5 b_1 in R_5
x+y
STO-6 b_0 in R_6
3 STO÷4 Calculate Q, R, Q^3, R^2, and f
STO+5
RCL5
RCL4 x^2
x+y
STO×5
y^x Q^3
RCL4
RCL5 x
RCL6
- 090
2 STO÷0
RCL4 ÷
3 STO:0 8
y^x R
STO7 R^2
x^2
STOA X≠0
GOTO0
GSBa

GOTO1 060 LBL1 ABS
Calculate largest real root of resolvent cubic
F72
GOTO2
GOTO2
RCL7
RCL7
+P
GSbb
GSbb
080 +
GOTO3
GOTO3
LCL3
LCL3
RND
GOTO4
Pause
Display Re(y)
STO8
Calculate A, B, C, D
STO9
A ± C, B ± D
2
2
C, D
GOTO00
1
GOTO2
RCL3

39
APPENDIX D

CHS
RCL8
X²
X≠0
GOTO4
110
X+Y
GOTO5
GSBa
LBL4
DIV
GSBa
RCL8
X²
×
LBL5
170
X+Y
LBL8
120
√
D
STO+7
B + D
STO-8
B - D
F?2
GOTO6
RCL0
X²
RCL1
-180
X<0
GOTO0
130
√
C, D = 0
GOTO7
LBL6
DIV
LBL7
STO+0
A + C
STO-6
A - C
+190
RCL0
Solve for roots
of quartic
RCL7
Solve for roots
190
STO÷5
RCL5
140
GSBc
RCL8
RCL6
GOTO1
0
P+S
LBR0
ABS
RCL4
√
1
RND
200
CHS
Pause
Displays quantity
tested
XLB9
1
ENT†
SF2
ENT†
RTN
LBLb
Subroutine for cube
root of positive or
negative number
40
X<0
-X-
SF2
R†

Quadradic solution
GOTO0
X≠0
X+Y
X=0
GOTO9

Protects against case
A ± C = B ± D = 0

Calculates -(b/2) and (b/2)² - C

Determines if roots are real or complex

Solves for real roots

Solves for complex roots

Enters zero roots

Stores and displays roots
APPENDIX D

STO1
\!10 -X-
R↓
STO2
-X-
RTN
### APPENDIX D

#### Storage Map for Program 4

<table>
<thead>
<tr>
<th>Address</th>
<th>Register</th>
<th>Initial storage$^a$</th>
<th>End of program$^b$</th>
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<tbody>
<tr>
<td>0</td>
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<td>a3</td>
<td>Root-type indicator</td>
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<td>R1</td>
<td>a2</td>
<td>Re( ) or $\alpha_1$</td>
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<td>R2</td>
<td>a1</td>
<td>Im( ) or $\alpha_2$</td>
</tr>
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<td>R3</td>
<td>a0</td>
<td></td>
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<tr>
<td>25</td>
<td>I</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^a$Initial storage is provided by output of program 1 or program 3.

$^b$The root-type indicator is 0 for real roots and -1 for complex roots. The real part of the complex root is stored in R1 or S1 and the imaginary part in R2 or S2.
APPENDIX E

PROGRAM 5 - STABILITY PARAMETERS

Program 5 utilizes the eigenvalues computed by program 4 to calculate stability parameters, such as the time to damp to one-half amplitude or the damping ratio. For running this program, the calculator should be set to calculate in radians and to display seven decimal digits.
APPENDIX E

001 LBLA   Clear registers and protects roots and root indicators
RCL0
RCL1
RCL2
CLRREG
STO2
R↓
STO1
R↓

010 STO0 060 RCL1
P+S
RCL0
RCL1
RCL2
CLRREG
STO2
R↓
STO1
R↓

020 STO0 070 X+Y
P+S

Stores constants and initializes I register
6
9
3
CHS
STOA
1

030 1 080 RCL2
CHS
STOB
π
2
STO1
X
STOC
LBLB
RCL0
X≠0
SF2
RCL1
ABS
RCL2
ABS
+X=0
GOTO1
F?2
GOTO2
RCLA

040 X≠0 090 GOTOB
DSZ
R/S

050 GOTO2 Switch for complex roots
STO3 Calculates t1/2 or tD for real roots

Calculate stability parameters for complex roots
Protects against zero real part of complex root

t1/2 or tD

N1/2 or Nd

Switch for second set of roots and program stop

\( t \)
\( \Delta \)
\( \omega_n \)
\( \zeta \)

\( P+S \)
RCL1
→P
STO7
CHS
STO8
LBL1

Switch for complex roots
APPENDIX E

Storage Map for Program 5

<table>
<thead>
<tr>
<th>(i)Address</th>
<th>Register</th>
<th>Initial storage&lt;sup&gt;a&lt;/sup&gt;</th>
<th>End of program</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( R_0 )</td>
<td>Root-type indicator</td>
<td>Root-type indicator</td>
</tr>
<tr>
<td>1</td>
<td>( R_1 )</td>
<td>( \text{Re}( ) ) or ( \alpha_1 )</td>
<td>( \text{Re}( ) ) or ( \alpha_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( R_2 )</td>
<td>( \text{Im}( ) ) or ( \alpha_2 )</td>
<td>( \text{Im}( ) ) or ( \alpha_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( R_3 )</td>
<td></td>
<td>( t_{1/2} ) or ( t_D )</td>
</tr>
<tr>
<td>4</td>
<td>( R_4 )</td>
<td></td>
<td>( \Delta )</td>
</tr>
<tr>
<td>5</td>
<td>( R_5 )</td>
<td></td>
<td>( N_{1/2} ) or ( N_D )</td>
</tr>
<tr>
<td>6</td>
<td>( R_6 )</td>
<td></td>
<td>( t )</td>
</tr>
<tr>
<td>7</td>
<td>( R_7 )</td>
<td></td>
<td>( \omega_n )</td>
</tr>
<tr>
<td>8</td>
<td>( R_8 )</td>
<td></td>
<td>( \zeta )</td>
</tr>
<tr>
<td>9</td>
<td>( R_9 )</td>
<td></td>
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</tr>
<tr>
<td>10</td>
<td>( S_0 )</td>
<td>Root-type indicator</td>
<td>Root-type indicator</td>
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<tr>
<td>11</td>
<td>( S_1 )</td>
<td>( \text{Re}_1( ) ) or ( \alpha_3 )</td>
<td>( \text{Re}_1( ) ) or ( \alpha_3 )</td>
</tr>
<tr>
<td>12</td>
<td>( S_2 )</td>
<td>( \text{Im}_1( ) ) or ( \alpha_4 )</td>
<td>( \text{Im}_1( ) ) or ( \alpha_4 )</td>
</tr>
<tr>
<td>13</td>
<td>( S_3 )</td>
<td></td>
<td>( t_{1/2} ) or ( t_D )</td>
</tr>
<tr>
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<td>( S_4 )</td>
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<td>( \Delta )</td>
</tr>
<tr>
<td>15</td>
<td>( S_5 )</td>
<td></td>
<td>( N_{1/2} ) or ( N_D )</td>
</tr>
<tr>
<td>16</td>
<td>( S_6 )</td>
<td></td>
<td>( t )</td>
</tr>
<tr>
<td>17</td>
<td>( S_7 )</td>
<td></td>
<td>( \omega_n )</td>
</tr>
<tr>
<td>18</td>
<td>( S_8 )</td>
<td></td>
<td>( \zeta )</td>
</tr>
<tr>
<td>19</td>
<td>( S_9 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>( R_A )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>( R_B )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>( R_C )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>( R_D )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>( R_E )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>( I )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup>The initial storage is the same as the storage at the end of program 4.

For real roots, only the time to damp to one-half amplitude or the time to double amplitude is calculated. This quantity is stored in register 3 for the root in register 1 and in register 4 for the root in register 2.
APPENDIX F

PROGRAM 6 - EULER TRANSFORMATION FOR AERONAUTICS

Program 6 is for the standard Euler transformation that is used in aeronautics between inertial axes and airplane axes. The trigonometric mode and the number of decimal digits in the display are assigned by the user.
APPENDIX F

001 LBLA Stores \( X_{SP}, Y_{SP}, Z_{SP} \) or \( X_b, Y_b, Z_b \) GSBC
STO0 R↓ STO6 
STO1 R↓ STO7 
STO2 RTN LBLC Transformation subroutine \( X_b, Y_b, Z_b \)
RTN LBLB Stores \( \psi, \theta, \phi \) 060 X+Y 
STO3 RCL(i) RCL2 +P Y_b, Z_b to \( X_{SP}, Y_{SP}, Z_{SP} \)
RTN STO4 CHS GSBB 
RTN STO5 X+Y GSBC 
RTN STO6 +R GSBB 
RTN STO7 DSZ GSBB 
RTN STO8 R/S GSBB 
RTN STO8 R/S GSBB 
RTN STO9 R/S GSBB 
RTN STO9 R/S GSBB 

030 LBLD Transformation subroutine \( X_{SP}, Y_{SP}, Z_{SP} \)
R↓ X+Y \( X_b, Y_b, Z_b \) to \( X_{SP}, Y_{SP}, Z_{SP} \)
R↓ +R ISZ

040 RTN LBLD 
R↓ LBLD 
R↓ LBLD 
R↓ LBLD 
R↓ LBLD 

050 GSBC STO8 GSBC 
RTN RTN RTN RTN RTN
This appendix gives check cases for each program given in appendixes A to F. Each check case is complete in itself and does not depend on the output of a previous program. For program 3, two check cases are given - one for label A and one for label B. There is no check case given for programs 1, 2, and 3 for $\sigma_u = \sigma_w = 0.0$. All check cases are independent of previous results.
## APPENDIX G

### Check Case for Program 1

<table>
<thead>
<tr>
<th>Register</th>
<th>Input storage</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₀</td>
<td>( k_Y = 10.463784 )</td>
<td>( a_3 = 1.3924836 )</td>
</tr>
<tr>
<td>R₁</td>
<td>( m = 90909.1 )</td>
<td>( a_2 = 1.1016636 )</td>
</tr>
<tr>
<td>R₂</td>
<td>( \rho = 1.2929 )</td>
<td>( a_1 = -0.0160353 )</td>
</tr>
<tr>
<td>R₃</td>
<td>( C_{Tu} = -0.000248411 )</td>
<td>( a_0 = -0.0210558 )</td>
</tr>
<tr>
<td>R₄</td>
<td>( C_{m\alpha} = -1.115 )</td>
<td>( a_{21} = 5.9757330 )</td>
</tr>
<tr>
<td>R₅</td>
<td>( g = 9.80665 )</td>
<td>( a_{23} = 55.0128915 )</td>
</tr>
<tr>
<td>R₆</td>
<td>( U_{ss} = 77.12 )</td>
<td>( a_{32} = -73.9231523 )</td>
</tr>
<tr>
<td>R₇</td>
<td>( \gamma_{ss} = -0.052359878 )</td>
<td>( a_{22} = 4.2010871 )</td>
</tr>
<tr>
<td>R₈</td>
<td>( \sigma_u = 2.0 )</td>
<td>( a_{12} = 0.0316972 )</td>
</tr>
<tr>
<td>R₉</td>
<td>( \sigma_w = 0.0 )</td>
<td>( a_{13} = 0.2546699 )</td>
</tr>
<tr>
<td>S₀</td>
<td>( C_{D\alpha} = 0.529 )</td>
<td>( a_{26} = 0.8064467 )</td>
</tr>
<tr>
<td>S₁</td>
<td>( C_{L\alpha} = 4.87 )</td>
<td>( a_{25} = 0.6856605 )</td>
</tr>
<tr>
<td>S₂</td>
<td>( C_{L\theta} = 0.283 )</td>
<td>( a_{35} = 0.5113523 )</td>
</tr>
<tr>
<td>S₃</td>
<td>( C_{L\alpha} = 0.0889 )</td>
<td>( a_{14} = 0.0007159 )</td>
</tr>
<tr>
<td>S₄</td>
<td>( C_{D,\theta} = 0.038 )</td>
<td>( a_{31} = -9.7126460 )</td>
</tr>
<tr>
<td>S₅</td>
<td>( 0.0 )</td>
<td>( a_{33} = 1.5369077 )</td>
</tr>
<tr>
<td>S₆</td>
<td>( 0.0 )</td>
<td>( a_{4} = 78.1242394 )</td>
</tr>
<tr>
<td>S₇</td>
<td>( C_{m\theta} = -0.241 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>S₈</td>
<td>( C_{m\theta} = -0.707 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>S₉</td>
<td>( C_{m,\theta} = 0.0 )</td>
<td>( 20 )</td>
</tr>
<tr>
<td>Rₐ</td>
<td>( \overline{C} = 7.0104 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>R₉</td>
<td>( S = 267.1 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>R₉</td>
<td>( A = 7.03 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>R₉</td>
<td>( C_{L,\phi} = 0.705 )</td>
<td>( 0.0 )</td>
</tr>
</tbody>
</table>
# APPENDIX G

## Check Case for Program 2

<table>
<thead>
<tr>
<th>Register</th>
<th>Input storage</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0</td>
<td>( g = 9.80665 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>R1</td>
<td>( U_{ss} = 77.12 )</td>
<td>( 0.6027688 )</td>
</tr>
<tr>
<td>R2</td>
<td>( \rho = 1.2929 )</td>
<td>( 0.3089144 )</td>
</tr>
<tr>
<td>R3</td>
<td>( S = 267.1 )</td>
<td>( 0.0064130 )</td>
</tr>
<tr>
<td>R4</td>
<td>( b = 43.4 )</td>
<td>( -11.394069 )</td>
</tr>
<tr>
<td>R5</td>
<td>( k_X = 6.559296 )</td>
<td>( b_{15} = -0.1779356 )</td>
</tr>
<tr>
<td>R6</td>
<td>( \sigma_u = 2.0 )</td>
<td>( b_{16} = 0.0594608 )</td>
</tr>
<tr>
<td>R7</td>
<td>( \sigma_w = -0.5 )</td>
<td>( b_{17} = 0.0473607 )</td>
</tr>
<tr>
<td>R8</td>
<td>( k_{xz} = -1.28016 )</td>
<td>( b_{17} = 0.0380903 )</td>
</tr>
<tr>
<td>R9</td>
<td>( k_Z = 12.49912 )</td>
<td>( b_{18} = 0.0109210 )</td>
</tr>
<tr>
<td>S0</td>
<td>( C_{n_r} = -0.057 )</td>
<td>( b_{24} = 0.1862234 )</td>
</tr>
<tr>
<td>S1</td>
<td>( C_{n_{\beta}} = 0.173 )</td>
<td>( b_{35} = -0.5652042 )</td>
</tr>
<tr>
<td>S2</td>
<td>( C_{n_p} = -0.0182 )</td>
<td>( b_{16} = 0.0594608 )</td>
</tr>
<tr>
<td>S3</td>
<td>( C_{n_{\beta}} = 0.0 )</td>
<td>( b_{34} = 0.0 )</td>
</tr>
<tr>
<td>S4</td>
<td>( C_{l_{\beta}} = -0.21 )</td>
<td>( b_{33} = 2.3929305 )</td>
</tr>
<tr>
<td>S5</td>
<td>( C_{l_p} = -0.111 )</td>
<td>( b_{14} = 1.2648347 )</td>
</tr>
<tr>
<td>S6</td>
<td>( C_{l_r} = 0.0614 )</td>
<td>( b_{23} = -0.6996473 )</td>
</tr>
<tr>
<td>S7</td>
<td>( C_{l_{\beta}} = 0.0 )</td>
<td>( b_{32} = 0.0 )</td>
</tr>
<tr>
<td>S8</td>
<td>( C_{Y_{\beta}} = -0.866 )</td>
<td>( b_{31} = 0.1268488 )</td>
</tr>
<tr>
<td>S9</td>
<td>( C_{Y_{\beta}} = 0.0 )</td>
<td>( C_{30} = 1.0 )</td>
</tr>
<tr>
<td>R_A</td>
<td>( m = 90909.1 )</td>
<td>( B_{22} = -0.0001293 )</td>
</tr>
<tr>
<td>R_B</td>
<td>( C_{Y_r} = 0.0881 )</td>
<td>( C_{21} = 0.9870954 )</td>
</tr>
<tr>
<td>R_C</td>
<td>( Y_{ss} = -0.052359878 )</td>
<td>( b_{13} = -0.1278111 )</td>
</tr>
<tr>
<td>R_D</td>
<td>( C_{Y_p} = 0.0539 )</td>
<td>( C_{11} = -0.0078951 )</td>
</tr>
<tr>
<td>R_E</td>
<td>( 0.0 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>I</td>
<td>( 24 )</td>
<td>( 21 )</td>
</tr>
</tbody>
</table>
APPENDIX G

Check Case for Program 3 - Label A

<table>
<thead>
<tr>
<th>Register</th>
<th>Input storage</th>
<th>Output&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0</td>
<td>0.0</td>
<td>( a_4 = 1.5838890 )</td>
</tr>
<tr>
<td>R1</td>
<td>0.6854141</td>
<td>( a_3 = 0.9679675 )</td>
</tr>
<tr>
<td>R2</td>
<td>0.3031611</td>
<td>( a_2 = 1.1621140 )</td>
</tr>
<tr>
<td>R3</td>
<td>0.0063915</td>
<td>( a_1 = 0.0095553 )</td>
</tr>
<tr>
<td>R4</td>
<td>-490.2586228</td>
<td>( a_0 = -0.0001405 )</td>
</tr>
<tr>
<td>R5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R6</td>
<td>( b_{15} = -0.1779356 )</td>
<td></td>
</tr>
<tr>
<td>R7</td>
<td>( b_{17} = 0.0473607 )</td>
<td></td>
</tr>
<tr>
<td>R8</td>
<td>( b_{42} = 0.0380903 )</td>
<td></td>
</tr>
<tr>
<td>R9</td>
<td>( b_{43} = 0.0109210 )</td>
<td></td>
</tr>
<tr>
<td>S0</td>
<td>( b_{24} = 0.1862234 )</td>
<td></td>
</tr>
<tr>
<td>S1</td>
<td>( b_{35} = -0.5652042 )</td>
<td></td>
</tr>
<tr>
<td>S2</td>
<td>( b_{16} = 0.0594608 )</td>
<td></td>
</tr>
<tr>
<td>S3</td>
<td>( b_{34} = 0.0 )</td>
<td></td>
</tr>
<tr>
<td>S4</td>
<td>( b_{33} = 2.3929305 )</td>
<td></td>
</tr>
<tr>
<td>S5</td>
<td>( b_{14} = 1.2648347 )</td>
<td></td>
</tr>
<tr>
<td>S6</td>
<td>( b_{23} = -0.6996473 )</td>
<td></td>
</tr>
<tr>
<td>S7</td>
<td>( b_{32} = 0.0 )</td>
<td></td>
</tr>
<tr>
<td>S8</td>
<td>( b_{31} = 0.1268488 )</td>
<td></td>
</tr>
<tr>
<td>S9</td>
<td>( C_{30} = 1.0 )</td>
<td></td>
</tr>
<tr>
<td>RA</td>
<td>( b_{22} = -0.0110087 )</td>
<td></td>
</tr>
<tr>
<td>RB</td>
<td>( C_{21} = 0.9870954 )</td>
<td></td>
</tr>
<tr>
<td>RC</td>
<td>( b_{13} = -0.1273814 )</td>
<td></td>
</tr>
<tr>
<td>RD</td>
<td>( C_{11} = -0.0421244 )</td>
<td></td>
</tr>
<tr>
<td>RE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup>They are normalized coefficients for the quintic; thus, \( a_5 = 1.0 \).
Check Case for Program 3 - Label B

<table>
<thead>
<tr>
<th>Register</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0</td>
<td>$a_4 = 1.583889$</td>
<td>$a_3 = 1.5915011$</td>
</tr>
<tr>
<td>R1</td>
<td>$a_3 = 0.9679675$</td>
<td>$a_2 = 0.9800821$</td>
</tr>
<tr>
<td>R2</td>
<td>$a_2 = 1.1621140$</td>
<td>$a_1 = 1.1695744$</td>
</tr>
<tr>
<td>R3</td>
<td>$a_1 = 0.0095552$</td>
<td>$a_0 = 0.0184581$</td>
</tr>
<tr>
<td>R4</td>
<td>$a_0 = -0.0001405$</td>
<td></td>
</tr>
</tbody>
</table>

The stack contains the quintic data as follows:

- Stack register T: Number of iterations = 12
- Stack register Z: $t_D$ or $t_{1/2}$ = $t_D = 91.0398496$ (displayed as negative number)
- Stack register Y: Root = 0.0076121
- Stack register X: 8.0 Indicates root has been found

Use the R4 to move data into the X register for recording.
APPENDIX G

Check Case for Program 4

Store:
\[
\begin{align*}
a_3 &= 1.4007102 \text{ in } R_0 \\
a_2 &= 1.1058038 \text{ in } R_1 \\
a_1 &= -0.0158317 \text{ in } R_2 \\
a_0 &= -0.0227494 \text{ in } R_3 \\
\end{align*}
\]

Results:

<table>
<thead>
<tr>
<th></th>
<th>Root indicator</th>
<th>Real part</th>
<th>Imaginary part</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_0)</td>
<td>-1.00 (indicates complex roots)</td>
<td>-0.6946683</td>
<td>0.7924165</td>
</tr>
<tr>
<td>(R_1)</td>
<td>First real root</td>
<td>-0.1489289</td>
<td></td>
</tr>
<tr>
<td>(R_2)</td>
<td>Second real root</td>
<td>0.1375553</td>
<td></td>
</tr>
</tbody>
</table>

\(53\)
APPENDIX G

Check Case for Program 5

<table>
<thead>
<tr>
<th>Register</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₀</td>
<td>-1.0</td>
<td>-1.0 Root</td>
</tr>
<tr>
<td>R₁</td>
<td>-0.6946683</td>
<td>-0.6946683 indicator</td>
</tr>
<tr>
<td>R₂</td>
<td>0.7924165</td>
<td>0.7924165 and roots</td>
</tr>
<tr>
<td>R₃</td>
<td></td>
<td>( t_{1/2} = 0.9975984 )</td>
</tr>
<tr>
<td>R₄</td>
<td></td>
<td>( \Delta = 5.5228662 )</td>
</tr>
<tr>
<td>R₅</td>
<td></td>
<td>( N_{1/2} = 0.1254783 )</td>
</tr>
<tr>
<td>R₆</td>
<td></td>
<td>( t = 7.9291450 )</td>
</tr>
<tr>
<td>R₇</td>
<td></td>
<td>( \omega_n = 1.0537969 )</td>
</tr>
<tr>
<td>R₈</td>
<td></td>
<td>( \zeta = 0.6592051 )</td>
</tr>
<tr>
<td>S₀</td>
<td>0.0</td>
<td>0.0 Root</td>
</tr>
<tr>
<td>S₁</td>
<td>-0.1489288</td>
<td>-0.1489288 indicator</td>
</tr>
<tr>
<td>S₂</td>
<td>0.1375553</td>
<td>0.1375553 and roots</td>
</tr>
<tr>
<td>S₃</td>
<td></td>
<td>( t_{1/2} = 4.6532303 ) First root</td>
</tr>
<tr>
<td>S₄</td>
<td></td>
<td>( t_D = -5.0379738 ) Second root</td>
</tr>
</tbody>
</table>

I
APPENDIX G

Check Case for Program 6

Space axes \((X_{sp}, Y_{sp}, Z_{sp})\) to body axes \((X_b, Y_b, Z_b)\):
\[X_{sp} = Y_{sp} = Z_{sp} = 1.0\]
\[\psi = 25^\circ; \quad \theta = 10^\circ; \quad \phi = 30^\circ\]

Results:
\[x_b = 1.1351 \text{ in R6}\]
\[y_b = 1.0267 \text{ in R7}\]
\[z_b = 0.8109 \text{ in R8}\]

Body axes \((X_b, Y_b, Z_b)\) to space axes \((X_{sp}, Y_{sp}, Z_{sp})\):
\[x_b = 1.1351 \quad y_b = 1.0267 \quad z_b = 0.8109\]
\[\psi = 25^\circ; \quad \theta = 10^\circ; \quad \phi = 30^\circ\]

Results:
\[x_{sp} = 1.0000 \text{ in R6}\]
\[y_{sp} = 1.0000 \text{ in R7}\]
\[z_{sp} = 1.0000 \text{ in R8}\]
APPENDIX H

A DISCUSSION OF FERRARI'S METHOD FOR THE SOLUTION OF A QUARTIC EQUATION

Ferrari (1522-1575), an Italian mathematician, obtained the solution of a quartic by reducing the problem to the solution of two quadratic equations. As the details of obtaining the quadratic equations are not consistent among authors, the details of obtaining the quadratics used for the solution in this paper are presented.

The general quartic equation is

\[ x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \]  \hspace{1cm} (H1)

Rewrite this equation as

\[ x^4 + a_3x^3 = -a_2x^2 - a_1x - a_0 \]  \hspace{1cm} (H2)

and complete the square

\[ \left( \frac{x^2 + \frac{a_3}{2}x}{\frac{a_2}{2}} \right)^2 = \left( \frac{a_3^2}{4} - a_2 \right)x^2 - a_1x - a_0 \]  \hspace{1cm} (H3)

Now, add \( \left( \frac{x^2 + \frac{a_3}{2}x}{\frac{a_2}{2}} \right)y + \frac{y^2}{4} \) to each side of equation (H3), \( y \) being a dummy variable

\[ \left( x^2 + \frac{a_3}{2}x + \frac{y}{2} \right)^2 = \left( \frac{a_3^2}{4} - a_2 + y \right)x^2 + \left( \frac{a_3}{2}y - a_1 \right)x + \left( \frac{y^2}{4} - a_0 \right) \]  \hspace{1cm} (H4)

The left-hand side of equation (H4) is a perfect square. If the right-hand side is also a perfect square, it can be written as the square of a linear function of \( x \), say \( Cx + D \). Thus, the pair of quadratics that must be solved for the roots of the quartic are

\[ x^2 + \frac{a_3}{2}x + \frac{y}{2} = \pm(Cx + D) \]  \hspace{1cm} (H5)

The right-hand side of equation (H4) is a perfect square if, and only if, its discriminant is 0.
In this equation \( y \) has not been defined, and if equation (H6) is written as a function of \( y \), it becomes

\[
y^3 - a_2y^2 + (a_3a_1 - 4a_0)y + \left[ a_0(4a_2 - a_3^2) - a_1^2 \right] = 0 \quad \text{(H7)}
\]

This equation is called the resolvent cubic and any root \( y_i \) of equation (H7) insures that equation (H6) is 0.

All that remains is the determination of the coefficients \( C \) and \( D \). The discriminant equation (H6)

\[
\left( \frac{a_3y}{4} - \frac{a_1}{2} \right)^2 \left( \frac{a_3^2}{4} - a_2 + y \right) \left( \frac{y^2}{4} - a_0 \right) = 0
\]

permits the right-hand side of equation (H4) to be written as

\[
\left( \frac{a_3^2}{4} - a_2 + y \right) = \left( \frac{a_3y}{4} - \frac{a_1}{2} \right)^2 \left( \frac{y^2}{4} - a_0 \right)
\]

which is a perfect square, and the coefficients \( C \) and \( D \) are

\[
C = \left( \frac{a_3y}{4} - \frac{a_1}{2} \right) \sqrt{\frac{y^2}{4} - a_0} \quad \text{(H8)}
\]

\[
D = \sqrt{\frac{y^2}{4} - a_0} \quad \text{(H9)}
\]
only if $D \neq 0$. The right-hand side of equation (H4) as written is a perfect square because

$$\frac{a_3 y}{2} - a_1 = 2 \sqrt{\left(\frac{a_3^2}{4} - a_2 + y\right)\left(\frac{y^2}{4} - a_0\right)}$$

so that

$$C = \sqrt{\frac{a_3^2}{4} - a_2 + y} \quad (H10)$$

and are used in place of equation (H8) if $D = 0$. 
REFERENCES


Figure 1.— Coordinate systems and Euler angles. Order of rotation for Euler angles is \( \psi, \theta, \) and \( \phi \). Moving axes translate with airplane and remain parallel to Earth fixed axes. Positive directions are shown.
Programs are presented for calculating airplane stability characteristics with a card programmable pocket calculator. These calculations include eigenvalues of the characteristic equations of lateral and longitudinal motion as well as stability parameters such as the time to damp to one-half amplitude or the damping ratio. The effects of wind shear are included. Background information and the equations programmed are given. The programs are written for the International System of Units, the dimensional form of the stability derivatives, and stability axes. In addition to the programs for stability calculations, an unusual and short program is included for the Euler transformation of coordinates used in airplane motions. The programs have been written for a Hewlett Packard HP-67 calculator. However, the use of this calculator does not constitute an endorsement of the product by the National Aeronautics and Space Administration.