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# Equations of Motion and Two- Equation Turbulence Model for Plane or Axisymmetric Turbulent Flows in Body-Oriented Orthogonal Curvilinear Coordinates and Mass-Averaged Dependent Variables

J. P. Sislian

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Mass-Averaged Dependent Variables

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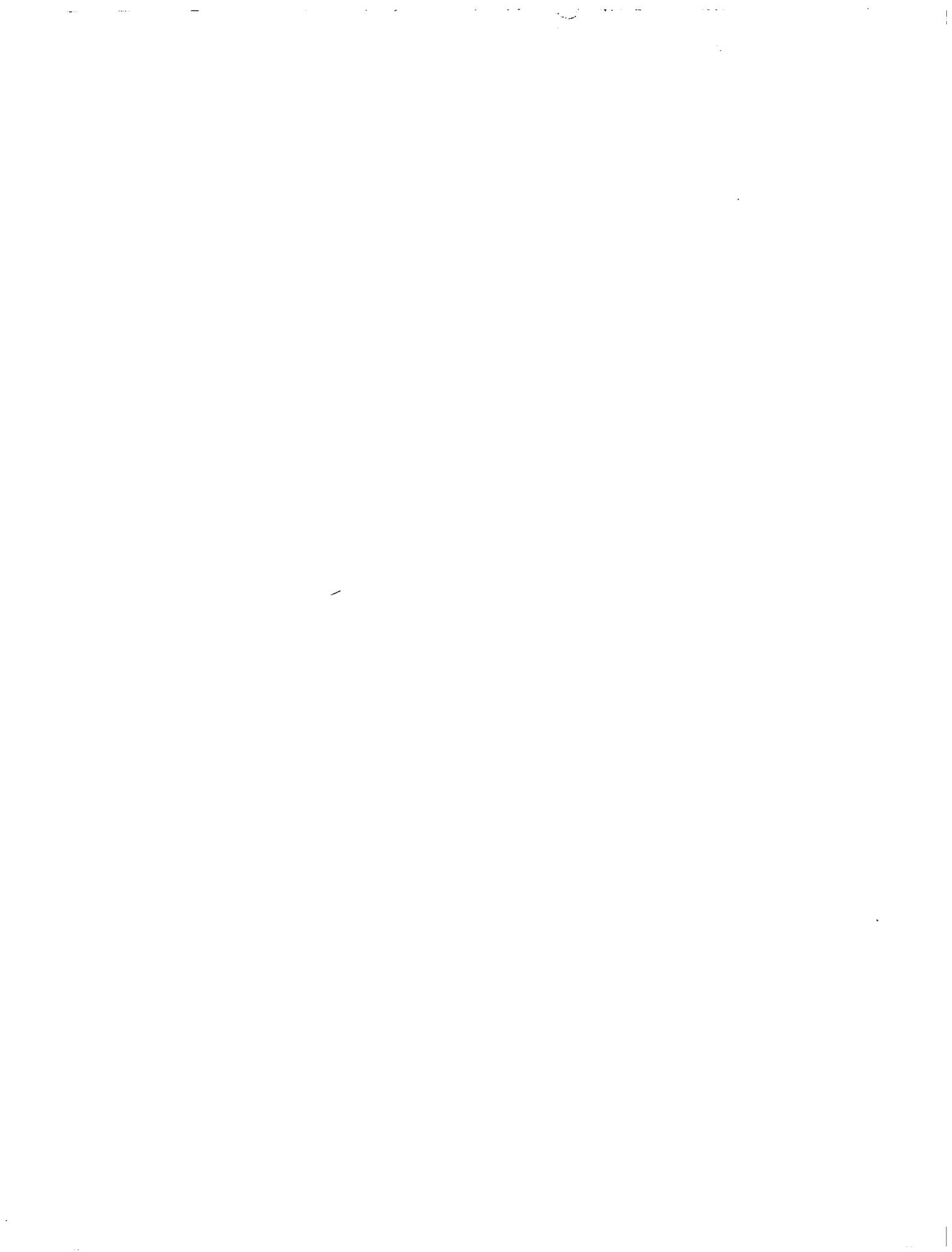
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## Summary

This report presents the full Navier-Stokes time-dependent, compressible, turbulent mean-flow equations in mass-averaged variables for plane or axisymmetric flow configurations. The equations are derived in a body-oriented orthogonal curvilinear coordinate system. Turbulence is modelled by a system of two equations for the mass-averaged turbulent kinetic energy and dissipation rate which determine the turbulent (eddy) diffusivity, recently proposed by Wilcox et al. These equations are rederived for the coordinate system and flow configuration considered, and some new features of these equations are discussed. A system of second-order boundary layer equations is then derived which includes in a consistent way the effects of longitudinal curvature and the corresponding normal pressure gradient. In this system the normal momentum equation is retained. The Wilcox and Chambers approach is used in considering effects of streamline curvature on turbulence phenomena in turbulent boundary layer type flows. Their two-equation turbulence model with curvature terms are rederived for the cases considered in the present report. The derived system of seven second-order boundary layer equations serve as a basis for an analytical-numerical investigation of a variety of boundary layer (parabolic) type problems where streamline curvature is of the order of the characteristic length in the longitudinal direction.



CONTENTS

	<u>Page</u>
Summary . . . . .	iii
Notation . . . . .	vii
1. INTRODUCTION . . . . .	1
2. MASS-AVERAGED EQUATIONS OF MOTION . . . . .	2
3. BOUNDARY-LAYER TYPE EQUATIONS OF MOTION . . . . .	14
4. TWO-EQUATION TURBULENCE MODEL FOR CURVED BOUNDARY LAYER TYPE FLOWS . . . . .	20
APPENDIX A - DERIVATION OF TWO-EQUATION TURBULENCE MODEL FOR COMPRESSIBLE PLANE OR AXISYMMETRIC TURBULENT FLOWS IN ORTHOGONAL CURVILINEAR BODY-ORIENTED COORDINATES . . . . .	26
REFERENCES . . . . .	40
FIGURES . . . . .	41



## Notation

a	speed of sound
$c_p$	specific heat at constant pressure
e	mass-averaged turbulent kinetic energy
H	total (stagnation) enthalpy
h	specific static enthalpy of the mixture, $h = \sum_i \alpha_i h_i$
$h_i$	specific static enthalpy of species i
$\vec{j}$	diffusional flux factor
k	heat conductivity
L	characteristic length of flow in x-direction
Le	Lewis number
$\ell$	dissipation length scale
M	molecular weight
Pr	Prandtl number
p	pressure
$\vec{q}$	heat conduction vector
R	universal gas constant; radius of curvature of the body surface
Re	Reynolds number
r	distance to x-axis
Sc	Schmidt number
T	temperature
t	time
u	x-component of velocity
v	y-component of velocity
$\vec{V}$	total velocity vector
$\dot{W}_i$	net mass rate of production of chemical species

$x$	longitudinal coordinate along surface of the body
$y$	coordinate normal to the body
$\alpha$	species mass fraction; angle of body with x-axis; turbulence
$\alpha^*$	closure coefficients
$\beta$	turbulence closure coefficients
$\beta^*$	
$\gamma$	specific heat ratio
$\delta_t$	characteristic length of flow in y-direction
$\epsilon$	mass-averaged turbulent dissipation rate
$\kappa$	curvature of the body
$\mu$	coefficient of viscosity
$\nu_L$	kinematic viscosity
$\nu_t$	eddy diffusivity
$\xi^*$	
$\xi_1$	turbulence closure coefficients
$\xi_2$	
$\rho$	density
$\sigma$	turbulence closure coefficients
$\sigma^*$	
$\overline{\tau}$	viscous stress tensor
$\varphi$	cylindrical or azimuthal coordinate
$\omega$	vorticity or pseudovorticity
$\Omega$	dissipation rate quantity, $\rho\omega$
$\nabla$	divergence operator

#### Subscripts

$a$	$a = 0$ plane flow, $a = 1$ axisymmetric flow
$t$	denotes turbulent quantity
$i$	denotes $i^{\text{th}}$ species

## Superscripts

~	mass-averaged value
*	nondimensional quantity
-	time-averaged value
'	time-averaged fluctuations
"	mass-averaged fluctuations

## 1. INTRODUCTION

Recently, considerable research efforts have been devoted to the understanding of the aerodynamics of combustion and the complex turbulent mixing phenomena occurring in combustion devices. Experimental investigations and theoretical modelling of such phenomena combined with sophisticated computational fluid-dynamic methods are the essential tools facilitating the economical design and operation of these combustors.

Usually all practical flames have the form of turbulent jets issuing from round orifices, the fuel gas being introduced through a central jet and the oxidizer (air) through an annulus surrounding it, so that straight concentric jets are formed. However, in order to enhance the fuel-air mixing process, the primary (fuel) jet or the secondary jet (air) or both are given a certain degree of curvature (e.g., swirl). Also, in certain types of combustors involving compressible jets, in order to achieve optimum combustion characteristics, the finite-rate chemical processes should be controlled by controlling the pressure and temperature fields in the combustor. This can be achieved by imparting to the flow various degrees of curvature.

Among the theoretical models describing curvature effects on turbulence, second-order closure two-equation turbulence models, which utilize two parameters to characterize the turbulence and to determine the eddy diffusivity, with each parameter satisfying a nonlinear transport equation, involve less empiricism than the mixing-length theories or the one equation models of turbulence. They require no advance knowledge of the flow under consideration. Yet they are simple enough to use for general engineering applications. Such models have proliferated. Recently, Wilcox and Chambers (Ref. 2) have proposed an interesting extension of their model to compressible flows with streamline curvature. The success achieved by this method in predicting the behaviour of a number of turbulent boundary-layer flows (Ref. 2) warrants its further use and testing in other compressible applications.

The present report is concerned with the analytical investigation of curvature and compressibility effects on the turbulent mixing of a compressible (fuel) jet issuing into a compressible air flow (Fig. 1). Flow variables of primary interest are the fuel/air ratio and temperature distribution. The full Navier-Stokes equations for global continuity, species continuity, momentum and energy conservation are derived for the mean mass-averaged variables of a compressible, turbulent, axisymmetric or plane flow in curvilinear orthogonal body-oriented system of coordinates (Ref. 1). The apparent (Reynolds) stresses were modelled according to the eddy viscosity concept, the turbulent viscosity being a function of two parameters, the turbulent mixing energy and the turbulent-dissipation rate determined by two nonlinear partial differential equations. The Navier-Stokes form of these equations was obtained in the coordinate system considered, with due account of compressibility and curvature effects. A compressibility term was found to exist in the equation for turbulent dissipation omitted by Wilcox and Chambers in their general equations in Cartesian coordinates. Starting from this system of seven equations, a complete set of second-order boundary-layer (parabolic) type equations was derived (terms of order unity and of order  $\delta_t/R$  are retained,  $\delta_t$  being the characteristic mixing zone thickness and  $R$  the radius of curvature, which is assumed to be of the same order of magnitude as the characteristic longitudinal length of the problem).

The continued interest shown by Professor I. I. Glass, and his helpful comments through the course of this study, is greatly appreciated. Thanks are due Dr. H. L. Beach, Jr. of NASA Langley Research Center, without whose support and encouragement this study could not have been completed. Thanks are also due to the Director, Professor J. H. de Leeuw, and staff of the Institute for Aerospace Studies, University of Toronto, for making the work possible. My thanks are extended to Mrs. Winifred Dillon for typing the manuscript.

## 2. MASS-AVERAGED EQUATIONS OF MOTION

The general equations describing flows of a reacting mixture of perfect gases in vector form are given by:

$$\text{Global Continuity:} \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} = 0 \quad (1)$$

$$\text{Species Continuity:} \quad \frac{\partial(\rho \alpha_i)}{\partial t} + \vec{\nabla} \cdot (\rho \alpha_i \vec{V} + \vec{J}_i) = \dot{W}_i \quad (2)$$

$$\text{Momentum:} \quad \frac{\partial(\rho \vec{V})}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} \vec{V} = - \vec{\nabla} p + \vec{\nabla} \cdot \hat{\Phi} \quad (3)$$

$$\text{Energy:} \quad \frac{\partial(\rho H - p)}{\partial t} + \vec{\nabla} \cdot \left[ \rho \vec{V} H + \vec{q} + \sum_i h_i \vec{J}_i - (\hat{\Phi} \cdot \vec{V}) \right] = 0 \quad (4)$$

$$\text{Equation of State:} \quad p = RT\rho \sum_i \frac{\alpha_i}{M_i} \quad (5)$$

where  $\vec{\nabla}$  is the divergence operator;  $\rho$ ,  $T$ ,  $p$  are the density, temperature and pressure of the mixture, respectively;  $\vec{V}$  the velocity vector;  $\alpha_i$  the mass fraction of the  $i^{\text{th}}$  species;  $\vec{J}_i$  the diffusional mass-flux vector,  $\vec{J}_i = \rho(\vec{V}_i - \vec{V})$ ;  $\dot{W}_i$  the net mass rate of production of species  $i$  per unit volume by chemical reaction (grams of  $i$  per  $\text{cm}^3$  per sec);  $\hat{\Phi}$  the viscous stress tensor;  $H = \sum_i \alpha_i h_i + \vec{V} \cdot \vec{V}/2$  the total (stagnation) enthalpy;  $h_i$  the specific enthalpy of the  $i^{\text{th}}$  species;  $\vec{q}$  the heat-conduction vector;  $M_i$  the molecular weight of the  $i^{\text{th}}$  species;  $R$  the universal gas constant. Constitutive relationships for the viscous stress tensor  $\hat{\Phi}$ , for the diffusional mass flux vector  $\vec{J}_i$ , and for the heat conduction vector  $\vec{q}$  must be added to these equations in order to obtain a closed system. Thus, for Newtonian fluids, i.e. fluids such that the viscous stress tensor is a linear function of the rate of strain, the viscous stress is given by:

$$\hat{\Phi} = \mu \left( \text{def } \vec{V} - \frac{2}{3} \text{div } \vec{V} \hat{I} \right) \quad (6)$$

with def  $\vec{V} = \text{grad } \vec{V} + (\text{grad } \vec{V})^T$ , the superscript T denoting the transpose of a tensor,  $\mu$  being the coefficient of viscosity and  $\underline{1}$  the unit tensor. The diffusion of each species is assumed to depend only on the gradient of the particular species mass fraction. The assumption requires that the binary diffusion coefficients  $D_{ij}$  for each species are equal and the simple Fick's law is applicable. Thus

$$\vec{J}_i = - \rho D_{ij} \text{grad } \alpha_i = - \mu \frac{Le}{Pr} \text{grad } \alpha_i \quad (7)$$

where  $Le$  and  $Pr$  are the Lewis and Prandtl numbers, respectively. Furthermore, the fluid is assumed to obey Fourier's law of heat conduction for  $q$ :

$$\vec{q} = - k \text{grad } T = - \frac{\bar{c}_p \mu}{Pr} \text{grad } T \quad (8)$$

where  $\bar{c}_p = \sum_i \alpha_i c_{pi}$ ,  $c_{pi}$  being the specific heat of species  $i$ .

For turbulent flows considered in the present investigation, it is assumed that the functional form describing the transport processes discussed above remain unchanged and the laminar coefficients are replaced by their turbulent counterparts, i.e.  $\mu \rightarrow \mu_t$ , etc. The coefficients  $\mu_t$ , ... have yet to be determined.

The body-oriented orthogonal curvilinear coordinate system used is shown in Fig. 1, where  $x$  is the distance along the surface of the body (measured from a certain point 0) and  $y$  the distance normal to this surface. The corresponding components of the velocity vector  $\vec{V}$  are  $u$  and  $v$ .  $R(x)$  is the radius of curvature of the body, reckoned positive for a convex body, and  $\alpha(x)$  its angle with and  $r_b(x)$  its distance from the axis. The length element is given by

$$dl^2 = h_1^2 dx^2 + h_2^2 dy^2 + h_3^2 d\phi^2 \quad (9)$$

where the metric coefficients are

$$h_1 = 1 + \kappa y, \quad h_2 = 1, \quad h_3 = r^a, \quad r = r_b + y \cos \alpha, \quad \kappa(x) = 1/R(x) \quad (10)$$

$a = 0$  for plane and  $a = 1$  for axisymmetric flows, and  $\phi$  is the lateral ( $z$ ) coordinate in plane flow and the azimuthal angle in axisymmetric flow. Then from the usual relations for vector operators in orthogonal curvilinear coordinates the continuity equation, Eq. (1), is found to be

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \rho u) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \rho v] \right\} = 0 \quad (11)$$

Using Eq. (7), the species continuity equation, Eq. (2), becomes:

$$\begin{aligned}
& \frac{\partial(\rho\alpha_i)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \rho \alpha_i u) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \rho \alpha_i v] \right\} = \\
& = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \frac{\mu}{Sc} \frac{1}{1+\kappa y} \frac{\partial \alpha_i}{\partial x} \right] + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \frac{\mu}{Sc} \frac{\partial \alpha_i}{\partial y} \right] \right\} + \dot{W}_i \quad (12)
\end{aligned}$$

where  $Sc = Pr/Le$  is the Schmidt number.

The Navier-Stokes momentum equations are (Ref. 3):

x-momentum:

$$\begin{aligned}
& \frac{\partial(\rho u)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a(\rho u^2 + p - \tau_{xx})] + \frac{\partial}{\partial y} [r^a(1+\kappa y)(\rho uv - \tau_{xy})] \right\} + \\
& + \frac{\kappa}{1+\kappa y} (\rho uv - \tau_{xy}) - \frac{a}{r} (p - \tau_{\varphi\varphi}) \sin\alpha = 0 \quad (13)
\end{aligned}$$

y-momentum:

$$\begin{aligned}
& \frac{\partial(\rho v)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a(\rho uv - \tau_{xy})] + \frac{\partial}{\partial y} [r^a(1+\kappa y)(\rho v^2 + p - \tau_{yy})] \right\} - \\
& - \frac{\kappa}{1+\kappa y} (\rho u^2 + p - \tau_{xx}) - \frac{a}{r} (p - \tau_{\varphi\varphi}) \cos\alpha = 0 \quad (14)
\end{aligned}$$

The viscous stresses  $\tau_{xx}$ , ...,  $\tau_{\varphi\varphi}$  are given by Eq. (6), and in the coordinate system considered are:

$$\begin{aligned}
\tau_{xx} &= \mu \left[ \frac{2}{1+\kappa y} \left( \frac{\partial u}{\partial x} + \kappa v \right) - \frac{2}{3} \operatorname{div} \vec{V} \right]; & \tau_{yy} &= \mu \left[ 2 \frac{\partial v}{\partial x} - \frac{2}{3} \operatorname{div} \vec{V} \right] \\
\tau_{\varphi\varphi} &= \mu \left[ \frac{2a}{r} (u \sin\alpha + v \cos\alpha) - \frac{2}{3} \operatorname{div} \vec{V} \right]; & \tau_{xy} &= \mu \left[ \frac{1}{1+\kappa y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\kappa u}{1+\kappa y} \right] \quad (15)
\end{aligned}$$

$$\operatorname{div} V = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a u) + \frac{\partial}{\partial y} [r^a(1+\kappa y) v] \right\}$$

With Eqs. (7-8) and relations:

$$\frac{\partial H}{\partial x} = \sum_i h_i(T) \frac{\partial \alpha_i}{\partial x} + \sum_i \alpha_i c_{pi} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left( \frac{v^2}{2} \right)$$

$$\frac{\partial H}{\partial y} = \sum_i h_i(T) \frac{\partial \alpha_i}{\partial y} + \sum_i \alpha_i c_{pi} \frac{\partial T}{\partial y} + \frac{\partial}{\partial y} \left( \frac{v^2}{2} \right)$$

where  $c_{pi} = dh_i/dT$ ,  $v^2 = (u^2+v^2)/2$ , and assuming  $Le = 1$ , the energy equation, Eq. (4), becomes:

$$\begin{aligned} & \frac{\partial(\rho H - p)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial(r^a \rho u H)}{\partial x} + \frac{\partial}{\partial y} [r^a(1+\kappa y) \rho v H] \right\} = \\ & = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a}{1+\kappa y} \frac{\mu}{Pr} \frac{\partial}{\partial x} \left( H - \frac{v^2}{2} \right) \right] + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \frac{\mu}{Pr} \frac{\partial}{\partial y} \left( H - \frac{v^2}{2} \right) \right] \right\} + \\ & + \frac{\partial}{\partial x} [r^a(\tau_{xx} u + \tau_{xy} v)] + \frac{\partial}{\partial y} [r^a(1+\kappa y)(\tau_{xy} u + \tau_{yy} v)] \end{aligned} \quad (16)$$

An alternate form of the energy equation may be obtained if we make use of the equation for the kinetic energy of the flow. Multiplying Eq. (13) by  $u$  and Eq. (14) by  $v$  and adding, we obtain, after some calculation:

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \rho \frac{v^2}{2} \right) + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \rho u \frac{v^2}{2}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \rho v \frac{v^2}{2}] \right\} = \\ & = - \frac{u}{1+\kappa y} \frac{\partial p}{\partial x} - v \frac{\partial p}{\partial y} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a(\tau_{xx} u + \tau_{xy} v)] + \frac{\partial}{\partial y} [r^a(1+\kappa y)(\tau_{xy} u + \tau_{yy} v)] \right\} - \\ & - \frac{\tau_{xx}}{1+\kappa y} \left( \frac{\partial u}{\partial x} + \kappa v \right) - \tau_{xy} \left( \frac{1}{1+\kappa y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\kappa u}{1+\kappa y} \right) - \tau_{yy} \frac{\partial v}{\partial y} - \frac{a}{r} \tau_{\varphi\varphi} (u \sin \alpha + v \cos \alpha) \end{aligned} \quad (17)$$

Noting that  $H - (v^2/2) = h$  and using Eq. (17), we derive from Eq. (16) the energy equation in the following second form:

$$\begin{aligned} & \frac{\partial(\rho h)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \rho u h) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \rho v h] \right\} = \\ & = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left( \frac{r^a}{1+\kappa y} \frac{\mu}{Pr} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \frac{\mu}{Pr} \frac{\partial h}{\partial y} \right] \right\} + \frac{\partial p}{\partial t} + \frac{u}{1+\kappa y} \frac{\partial p}{\partial x} + \end{aligned}$$

(18)  
Contd...

$$+ v \frac{\partial p}{\partial y} + \mu \left[ 2 \left( \frac{1}{1+\kappa y} \frac{\partial u}{\partial x} + \frac{\kappa v}{1+\kappa y} \right)^2 + \left( \frac{1}{1+\kappa y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{u\kappa}{1+\kappa y} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + \right. \\ \left. + \frac{2a}{r^2} (u \sin \alpha + v \cos \alpha)^2 - \frac{2}{3} \left( \frac{1}{1+\kappa y} \frac{\partial u}{\partial x} + \frac{\kappa v}{1+\kappa y} + \frac{\partial v}{\partial y} + a \frac{u \sin \alpha + v \cos \alpha}{r} \right)^2 \right] \quad (18)$$

In order to obtain the governing conservation equations for turbulent flows, it is convenient to replace the instantaneous quantities in the above equations by their mean and their fluctuating quantities. In this treatment, the mass-weighted-averaging procedure is used. With this procedure the resulting equations for the average turbulent flow quantities will have a form very similar to the equations for laminar flow.

If  $f$  is any flow quantity, the conventional time average of this quantity is denoted by  $\bar{f}$ . Then, the mass average of any quantity except density, pressure and viscous stresses (fluctuations of transport coefficients are neglected) is given by

$$\tilde{f} = \frac{\bar{\rho f}}{\bar{\rho}}$$

The quantity  $f$  may then be written as

$$f = \tilde{f} + f''$$

where  $f''$  is the departure or fluctuation from the mass-averaged value  $\tilde{f}$ . It should be noted that  $\bar{f}'' \neq 0$ ;  $f'' = -\rho' f' / \bar{\rho}$ . Also, it is easy to show that  $\bar{\rho f''} = 0$ . Representing the density, pressure and viscous stresses by the sum of their time average and its departure or fluctuation from the time average, namely

$$\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p', \quad \tau_{ij} = \bar{\tau}_{ij} + \tau'_{ij} \quad (19)$$

and the velocities, the total enthalpy, static enthalpy, temperature and mass fractions by their mass average and its departure from the mass average, namely

$$u = \tilde{u} + u'', \quad v = \tilde{v} + v'', \quad H = \tilde{H} + H'', \quad h = \tilde{h} + h''$$

$$T = \tilde{T} + T'', \quad \alpha_i = \tilde{\alpha}_i + \alpha_i'' \quad \left( \tilde{\alpha}_i = \frac{\overline{\rho \alpha_i}}{\bar{\rho}} = \frac{\overline{\rho_i}}{\bar{\rho}} \right)$$

substituting these quantities into Eqs. (11)-(18), and applying the Reynolds rules of averaging (Ref. 4) yields the following conservation equations for plane or axisymmetric turbulent flows:

Global Continuity:

$$\frac{\partial \bar{p}}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{v}] \right\} = 0 \quad (21)$$

Species Continuity:

$$\begin{aligned} & \frac{\partial (\bar{\rho} \tilde{\alpha}_i)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{\alpha}_i \tilde{u}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{\alpha}_i \tilde{v}] \right\} = \\ & = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \frac{\mu}{Sc} \frac{1}{1+\kappa y} \frac{\partial \tilde{\alpha}_i}{\partial x} - \overline{\rho \alpha_i'' u''} \right) \right] + \right. \\ & \quad \left. + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \left( \frac{\mu}{Sc} \frac{\partial \tilde{\alpha}_i}{\partial y} - \overline{\rho \alpha_i'' v''} \right) \right] \right\} + \tilde{W}_i \end{aligned} \quad (22)$$

x-momentum:

$$\begin{aligned} & \frac{\partial (\bar{\rho} \tilde{u})}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u}^2) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{u} \tilde{v}] \right\} + \frac{\kappa \bar{\rho} \tilde{u} \tilde{v}}{1+\kappa y} = - \frac{1}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} + \\ & + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a (\bar{\tau}_{xx} - \overline{\rho u'' u''})] + \frac{\partial}{\partial y} [r^a(1+\kappa y) (\bar{\tau}_{xy} - \overline{\rho u'' v''})] \right\} + \\ & + \frac{\kappa}{1+\kappa y} (\bar{\tau}_{xy} - \overline{\rho u'' v''}) - \frac{a}{r} (\bar{\tau}_{\varphi\varphi} - \overline{\rho w'' w''}) \sin \alpha \end{aligned} \quad (23)$$

y-momentum:

$$\begin{aligned} & \frac{\partial (\bar{\rho} \tilde{v})}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \tilde{v}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{v}^2] \right\} - \frac{\kappa}{1+\kappa y} \bar{\rho} \tilde{u}^2 = - \frac{\partial \bar{p}}{\partial y} + \\ & + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a (\bar{\tau}_{xy} - \overline{\rho u'' v''})] + \frac{\partial}{\partial y} [r^a(1+\kappa y) (\bar{\tau}_{yy} - \overline{\rho v'' v''})] \right\} - \\ & - \frac{\kappa}{1+\kappa y} (\bar{\tau}_{xx} - \overline{\rho u'' u''}) - \frac{a}{r} (\bar{\tau}_{\varphi\varphi} - \overline{\rho w'' w''}) \cos \alpha \end{aligned} \quad (24)$$

Energy:

$$\frac{\partial}{\partial t} [r^a(1+\kappa y) (\bar{\rho} \tilde{H} - \bar{p})] + \frac{\partial}{\partial x} [r^a (\bar{\rho} \tilde{u} \tilde{H} + \overline{\rho u'' H''})] + \frac{\partial}{\partial y} [r^a(1+\kappa y) (\bar{\rho} \tilde{v} \tilde{H} + \overline{\rho v'' H''})] =$$

(25)  
Contd...

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left\{ r^a \left[ \frac{\mu}{Pr} \frac{1}{1+\kappa y} \frac{\partial \tilde{h}}{\partial x} + \overline{(\bar{\tau}_{xx} + \tau'_{xx})(\tilde{u} + u'')} + \overline{(\bar{\tau}_{xy} + \tau'_{xy})(\tilde{v} + v'')} \right] \right\} + \\
&+ \frac{\partial}{\partial y} \left\{ r^a(1+\kappa y) \left[ \frac{\mu}{Pr} \frac{\partial \tilde{h}}{\partial y} + \overline{(\bar{\tau}_{xy} + \tau'_{xy})(\tilde{u} + u'')} + \overline{(\bar{\tau}_{yy} + \tau'_{yy})(\tilde{v} + v'')} \right] \right\} \quad (25)
\end{aligned}$$

But it can be shown that

$$\begin{aligned}
\overline{\rho u'' \tilde{H}''} &= \rho u'' \left( h'' + \frac{u'' u'' + v'' v''}{2} \right) + \tilde{u} \overline{\rho u'' u''} + \tilde{v} \overline{\rho u'' v''} \\
\overline{\rho v'' \tilde{H}''} &= \rho v'' \left( h'' + \frac{u'' u'' + v'' v''}{2} \right) + \tilde{u} \overline{\rho u'' v''} + \tilde{v} \overline{\rho v'' v''} \\
\bar{\rho} \tilde{u} \tilde{H} &= \bar{\rho} \tilde{u} \left( \tilde{h} + \frac{\tilde{u}^2 + \tilde{v}^2}{2} + e \right) \\
\bar{\rho} \tilde{v} \tilde{H} &= \bar{\rho} \tilde{v} \left( \tilde{h} + \frac{\tilde{u}^2 + \tilde{v}^2}{2} + e \right)
\end{aligned}$$

where

$$e = \frac{\overline{\rho u'' u'' + \rho v'' v''}}{2\bar{\rho}} \quad (26)$$

is the mass averaged specific turbulent kinetic energy. Hence Eq. (25) can be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t} [r^a(1 + \kappa y)(\bar{\rho} \tilde{H} - \bar{p})] + \frac{\partial}{\partial x} \left[ r^a \bar{\rho} \tilde{u} \left( \tilde{h} + \frac{\tilde{u}^2 + \tilde{v}^2}{2} + e \right) \right] + \\
&+ \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \bar{\rho} \tilde{v} \left( \tilde{h} + \frac{\tilde{u}^2 + \tilde{v}^2}{2} + e \right) \right] = \\
&= \frac{\partial}{\partial x} \left\{ r^a \left[ \frac{\mu}{Pr} \frac{1}{1+\kappa y} \frac{\partial \tilde{h}}{\partial x} + \overline{\tau_{xx} u'' + \tau_{xy} v''} - \rho u'' \left( h'' + \frac{u'' u'' + v'' v''}{2} \right) \right] \right\} + \\
&+ \frac{\partial}{\partial y} \left\{ r^a(1 + \kappa y) \left[ \frac{\mu}{Pr} \frac{\partial \tilde{h}}{\partial y} + \overline{\tau_{xy} u'' + \tau_{yy} v''} - \rho v'' \left( h'' + \frac{u'' u'' + v'' v''}{2} \right) \right] \right\} + \quad (27)
\end{aligned}$$

Contd...

$$\begin{aligned}
& + \frac{\partial}{\partial x} \left\{ r^a [\tilde{u}(\bar{\tau}_{xx} - \overline{\rho u''u''}) + \tilde{v}(\bar{\tau}_{xy} - \overline{\rho u''v''})] \right\} + \\
& + \frac{\partial}{\partial y} \left\{ r^a(1 + \kappa y) [\tilde{u}(\bar{\tau}_{xy} - \overline{\rho u''v''}) + \tilde{v}(\bar{\tau}_{yy} - \overline{\rho v''v''})] \right\} \quad (27)
\end{aligned}$$

In Eqs. (22), (23), (24) and (27) the Reynolds (apparent) stresses and the turbulent mass and energy transports are calculated by applying an eddy transport concept which is postulated as: (Ref. 6)

$$\begin{aligned}
\tau_{xx}^t &= \overline{-\rho u''u''} = \bar{\rho} \nu_t \left[ \frac{2}{1+\kappa y} \left( \frac{\partial \tilde{u}}{\partial x} + \kappa \tilde{v} \right) - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right] - \frac{2}{3} \bar{\rho} e \\
\tau_{yy}^t &= \overline{-\rho v''v''} = \bar{\rho} \nu_t \left[ 2 \frac{\partial \tilde{v}}{\partial y} - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right] - \frac{2}{3} \bar{\rho} e \\
\tau_{\varphi\varphi}^t &= \overline{-\rho w''w''} = \bar{\rho} \nu_t \left[ \frac{2a}{r} (\tilde{u} \sin \alpha + \tilde{v} \cos \alpha) - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right] - \frac{2}{3} \bar{\rho} e
\end{aligned} \quad (28)$$

$$\begin{aligned}
\tau_{xy}^t &= \tau_{yx}^t = \overline{-\rho u''v''} = \bar{\rho} \nu_t \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa}{1+\kappa y} \tilde{u} \right) \\
\overline{-\rho \alpha_i''u''} &= \frac{\bar{\rho} \nu_t}{Sc_t} \frac{1}{1+\kappa y} \frac{\partial \tilde{\alpha}_i}{\partial x} ; \quad \overline{-\rho \alpha_i''v''} = \frac{\bar{\rho} \nu_t}{Sc_t} \frac{\partial \tilde{\alpha}_i}{\partial y} \\
\overline{-\rho u''h''} &= \frac{\bar{\rho} \nu_t}{Pr_t} \frac{1}{1+\kappa y} \frac{\partial \tilde{h}}{\partial x} , \quad \overline{-\rho v''h''} = \frac{\bar{\rho} \nu_t}{Pr_t} \frac{\partial \tilde{h}}{\partial y}
\end{aligned}$$

where  $\nu_t$  is the turbulent (eddy) diffusivity, and  $Pr_t$  and  $Sc_t$  the turbulent Prandtl and Schmidt numbers, respectively. It should be noted that unlike  $\mu$ , the molecular viscosity,  $\nu_t$  is not a property of the fluid. Its value varies from point to point in the flow and is determined by the structure of turbulence at the point considered. Substituting Eq. (28) into the conservation equations, Eqs. (22)-(24) and (27) we get

Species Continuity:

$$\begin{aligned}
\frac{\partial(\bar{\rho} \tilde{\alpha}_i)}{\partial t} &+ \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \tilde{\alpha}_i \tilde{u}) + \frac{\partial}{\partial y} [r^a(1 + \kappa y) \tilde{\alpha}_i \tilde{v}] \right\} = \\
&= \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \frac{\mu}{Sc} + \frac{\bar{\rho} \nu_t}{Sc_t} \right) \frac{1}{1+\kappa y} \frac{\partial \tilde{\alpha}_i}{\partial x} \right] + \right. \\
&\quad \left. + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \left( \frac{\mu}{Sc} + \frac{\bar{\rho} \nu_t}{Sc_t} \right) \frac{\partial \tilde{\alpha}_i}{\partial y} \right] \right\} + \tilde{w}_i \quad (29)
\end{aligned}$$

x-momentum:

$$\begin{aligned}
& \frac{\partial(\bar{\rho} \tilde{u})}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u}^2) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{u} \tilde{v}] \right\} + \frac{\kappa \bar{\rho} \tilde{u} \tilde{v}}{1+\kappa y} = \\
& - \frac{1}{1+\kappa y} \frac{\partial \bar{P}}{\partial x} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a(\mu + \bar{\rho} v_t) \left( \frac{2}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{2\kappa v}{1+\kappa y} - \frac{2}{3} \operatorname{div} \vec{V} \right) \right] + \right. \\
& \quad \left. + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y)(\mu + \bar{\rho} v_t) \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) \right] \right\} + \\
& \quad + \frac{\kappa}{1+\kappa y} (\mu + \bar{\rho} v_t) \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) - \\
& \quad - (\mu + \bar{\rho} v_t) \frac{a}{r} \left[ \frac{2a}{r} (\tilde{u} \sin \alpha + \tilde{v} \cos \alpha) - \frac{2}{3} \operatorname{div} \vec{V} \right] \sin \alpha \quad (30)
\end{aligned}$$

y-momentum:

$$\begin{aligned}
& \frac{\partial(\bar{\rho} \tilde{v})}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \tilde{v}) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v}^2] \right\} - \frac{\kappa}{1+\kappa y} \bar{\rho} \tilde{u}^2 = \\
& - \frac{\partial \bar{P}}{\partial y} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a(\mu + \bar{\rho} v_t) \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa}{1+\kappa y} \tilde{u} \right) \right] + \right. \\
& \quad \left. + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y)(\mu + \bar{\rho} v_t) \left( 2 \frac{\partial \tilde{v}}{\partial y} - \frac{2}{3} \operatorname{div} \vec{V} \right) \right] \right\} - \\
& \quad - \frac{\kappa}{1+\kappa y} (\mu + \bar{\rho} v_t) \left( \frac{2}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{2\kappa v}{1+\kappa y} - \frac{2}{3} \operatorname{div} \vec{V} \right) - \\
& \quad - \frac{a}{r} (\mu + \bar{\rho} v_t) \left[ \frac{2a}{r} (\tilde{u} \sin \alpha + \tilde{v} \cos \alpha) - \frac{2}{3} \operatorname{div} \vec{V} \right] \cos \alpha \quad (31)
\end{aligned}$$

Energy:

$$\begin{aligned}
& \frac{\partial}{\partial t} [r^a(1+\kappa y)(\bar{\rho} \tilde{H} - \bar{p})] + \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \tilde{H}) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v} \tilde{H}] = \\
& = \frac{\partial}{\partial x} \left[ r^a \left( \frac{\mu}{\operatorname{Pr}} + \frac{\bar{\rho} v_t}{\operatorname{Pr}_t} \right) \frac{1}{1+\kappa y} \frac{\partial}{\partial x} \left( \tilde{H} - \frac{\tilde{u}^2}{2} + \frac{\tilde{v}^2}{2} \right) \right] + \quad (32)
\end{aligned}$$

Contd...

$$\begin{aligned}
& + \frac{\partial}{\partial y} \left[ r^a (1 + \kappa y) \left( \frac{\mu}{Pr} + \frac{\bar{\rho} v_t}{Pr_t} \right) \frac{\partial}{\partial y} \left( \tilde{H} - \frac{\tilde{u}^2 + \tilde{v}^2}{2} \right) \right] + \\
& + \frac{\partial}{\partial x} \left\{ r^a \left[ (\mu + \bar{\rho} v_t) \tilde{u} \left( \frac{2}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{2\kappa \tilde{v}}{1+\kappa y} - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right) + \right. \right. \\
& \quad \left. \left. + (\mu + \bar{\rho} v_t) \tilde{v} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) - \frac{2}{3} \bar{\rho} \tilde{u} e \right] \right\} + \\
& + \frac{\partial}{\partial y} \left\{ r^a (1 + \kappa y) \left[ (\mu + \bar{\rho} v_t) \tilde{u} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) + \right. \right. \\
& \quad \left. \left. + (\mu + \bar{\rho} v_t) \tilde{v} \left( 2 \frac{\partial \tilde{v}}{\partial y} - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right) - \frac{2}{3} \bar{\rho} \tilde{v} e \right] \right\} \quad (32)
\end{aligned}$$

where  $\bar{P} = \bar{p} + 2\bar{\rho}e/3$ ,  $\tilde{H} = \tilde{h} + (\tilde{u}^2 + \tilde{v}^2)/2$  and the turbulent kinetic energy has been neglected compared to the mean kinetic energy for a high-speed flow.

Using the equation for the mean kinetic energy of the turbulent flow,  $\bar{\rho}\tilde{v}^2/2 = \rho(\tilde{u}^2 + \tilde{v}^2)/2$ , obtained by multiplying Eq. (30) by  $\tilde{u}$  and Eq. (31) by  $\tilde{v}$  and adding

$$\begin{aligned}
& \frac{\partial(\bar{\rho} \tilde{v}^2/2)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left( r^a \bar{\rho} \tilde{u} \frac{\tilde{v}^2}{2} \right) + \frac{\partial}{\partial y} \left[ r^a (1 + \kappa y) \bar{\rho} v \frac{\tilde{v}^2}{2} \right] \right\} = \\
& - \frac{\tilde{u}}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} - \tilde{v} \frac{\partial \bar{p}}{\partial y} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a (\tau_{xx}^{\text{eff}} \tilde{u} + \tau_{xy}^{\text{eff}} \tilde{v})] + \right. \\
& \quad \left. + \frac{\partial}{\partial y} [r^a (1 + \kappa y) (\tau_{xy}^{\text{eff}} \tilde{u} + \tau_{yy}^{\text{eff}} \tilde{v})] \right\} - \tau_{xx}^{\text{eff}} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{\kappa \tilde{v}}{1+\kappa y} \right) - \\
& - \tau_{xy}^{\text{eff}} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) - \tau_{yy}^{\text{eff}} \frac{\partial \tilde{v}}{\partial y} - \frac{a}{r} \tau_{\varphi\varphi}^{\text{eff}} (\tilde{u} \sin \alpha + \tilde{v} \cos \alpha) \quad (33)
\end{aligned}$$

where

$$\begin{aligned}
\tau_{xx}^{\text{eff}} &= (\mu + \bar{\rho} v_t) \left( \frac{2}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{2\kappa \tilde{v}}{1+\kappa y} - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right) - \frac{2}{3} \bar{\rho} e \\
\tau_{yy}^{\text{eff}} &= (\mu + \bar{\rho} v_t) \left( 2 \frac{\partial \tilde{v}}{\partial y} - \frac{2}{3} \operatorname{div} \vec{\tilde{v}} \right) - \frac{2}{3} \bar{\rho} e
\end{aligned}$$

$$\tau_{\phi\phi}^{\text{eff}} = (\mu + \bar{\rho}v_t) \left( 2a \frac{\tilde{u}\sin\alpha + \tilde{v}\cos\alpha}{r} - \frac{2}{3} \text{div } \vec{V} \right) - \frac{2}{3} \bar{\rho}e$$

$$\tau_{xy}^{\text{eff}} = (\mu + \bar{\rho}v_t) \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa}{1+\kappa y} \tilde{u} \right)$$

Substituting the above into Eq. (32) we obtain the following alternate form of the energy equation:

$$\begin{aligned} & \frac{\partial(\bar{\rho} \tilde{h})}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \tilde{h}) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v} \tilde{h}] \right\} = \\ & = \frac{1}{r^a(1+y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \frac{\mu}{\text{Pr}} + \frac{\bar{\rho}v_t}{\text{Pr}_t} \right) \frac{1}{1+\kappa y} \frac{\partial \tilde{h}}{\partial x} \right] + \right. \\ & + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \left( \frac{\mu}{\text{Pr}} + \frac{\bar{\rho}v_t}{\text{Pr}_t} \right) \frac{\partial \tilde{h}}{\partial y} \right] \left. \right\} + \frac{\partial \bar{\rho}}{\partial t} + \frac{\tilde{u}}{1+\kappa y} \frac{\partial \bar{\rho}}{\partial x} + \tilde{v} \frac{\partial \bar{\rho}}{\partial y} + \\ & + (\mu + \bar{\rho}v_t) \left\{ 2 \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{\kappa \tilde{v}}{1+\kappa y} \right)^2 + \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right)^2 + \right. \\ & + 2 \left( \frac{\partial \tilde{v}}{\partial y} \right)^2 + \frac{2a}{r^a} (\tilde{u}\sin\alpha + \tilde{v}\cos\alpha)^2 - \frac{2}{3} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{u}}{\partial x} + \frac{\kappa \tilde{v}}{1+\kappa y} + \right. \\ & \left. \left. + \frac{\partial \tilde{v}}{\partial y} + a \frac{\tilde{u}\sin\alpha + \tilde{v}\cos\alpha}{r} \right)^2 \right\} - \frac{2}{3} \bar{\rho}e \text{div } \vec{V} \end{aligned} \quad (34)$$

Equations (21), (29), (30), (31), (32) or (34) and (5) form the complete set of Navier-Stokes equations describing plane or axisymmetric laminar and turbulent reacting flows containing i distinct species. The turbulent diffusivity  $v_t$  is yet to be determined in terms of known or calculable quantities. Following Wilcox and Traci (Ref. 5) it is assumed that the eddy diffusivity is expressed by the equation

$$v_t = \frac{\bar{\rho}e}{\epsilon} \quad (35)$$

where  $\epsilon$  is the mass-averaged dissipation rate and is defined as  $\epsilon^2 = \overline{\rho\Omega''\Omega''}/\rho$  where  $\Omega = \tilde{\Omega} + \Omega'' = \rho\omega$ ,  $\omega$  being the vorticity or pseudo-vorticity. Both  $e$  and  $\epsilon$  are determined from the following two partial differential equations (for a detailed derivation of the equations for the particular coordinate system considered here see Appendix A).

$$\begin{aligned}
& \frac{\partial(\bar{\rho}e)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} e) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v} e] \right\} = \\
& \quad = (\alpha^* \bar{\rho} \tilde{S} - \beta^* \epsilon) e - \xi^* e \bar{\rho} \operatorname{div} \vec{V} + \\
& + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma^* \bar{\rho} v_t)}{1 + \kappa y} \frac{\partial e}{\partial x} \right] + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y)(\mu + \sigma^* \bar{\rho} v_t) \frac{\partial e}{\partial y} \right] \right\}
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
& \frac{\partial(\bar{\rho} \epsilon^2)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \epsilon^2) + \frac{\partial}{\partial y} [r^a(1 + \kappa y)\bar{\rho} \tilde{v} \epsilon^2] \right\} = \\
& \quad = \alpha \bar{\rho} \tilde{S} \epsilon^2 - \left\{ \beta + 2\sigma \left[ \left( \frac{1}{1+\kappa y} \frac{\partial l}{\partial x} \right)^2 + \left( \frac{\partial l}{\partial y} \right)^2 \right] \right\} \epsilon^3 + \\
& + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma \bar{\rho} v_t)}{1 + \kappa y} \frac{\partial \epsilon^2}{\partial x} \right] + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y)(\mu + \sigma \bar{\rho} v_t) \frac{\partial \epsilon^2}{\partial y} \right] \right\} - \\
& \quad \quad \quad \underbrace{\xi_2 \bar{\rho} \epsilon^2 \operatorname{div} \vec{V}}_1 \quad - \quad \underbrace{\xi_3 \frac{\bar{\rho} e}{a} \frac{\operatorname{div} \vec{V}}{1+\kappa y} \left( \frac{\partial \bar{\rho}}{\partial x} \frac{\partial \bar{\rho}}{\partial y} - \frac{\partial \bar{\rho}}{\partial y} \frac{\partial \bar{\rho}}{\partial x} \right)}_2
\end{aligned} \tag{37}$$

The term in box 1 in Eq. (37) is absent in the corresponding equation given in Ref. 5 by Wilcox and Traci for general three-dimensional Cartesian coordinates. The term in box 2 is neglected in Ref. 5; here it is kept and modelled (see Appendix A for details). The importance of these terms, as well as the values of the closure constants  $\xi_2$  and  $\xi_3$  can be assessed by comparison with appropriate and reliable experimental results. The turbulent Prandtl and Schmidt numbers  $Pr_t$  and  $Sc_t$  in Eqs. (29) and (32) or (34) and the closure coefficients  $\alpha$ ,  $\alpha^*$ ,  $\beta$ ,  $\beta^*$ ,  $\sigma$ ,  $\sigma^*$ ,  $\xi^*$  appearing in Eqs. (35) and (36) are (see Ref. 5 for a detailed discussion):

$$\beta = \frac{3}{20}, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{1}{2}, \quad \xi^* = \frac{5}{2} \tag{38}$$

Cont'd...

$$\alpha = \frac{1}{3} \left[ 1 - \frac{10}{11} \exp(-Re_t/2) \right], \quad \alpha^* = \frac{3}{10} \left[ 1 - \frac{10}{11} \exp(-2Re_t) \right]$$

$$l = \bar{\rho} e^{1/2} / \epsilon, \quad Re_t = \bar{\rho} e^{1/2} \frac{l}{\mu}, \quad Pr_t = Sc_t = \frac{8}{9} \quad (Le = 1) \quad (38)$$

$l$  being defined as the length scale of turbulence

$$\tilde{S} = \sqrt{2(e_{xx}e_{xx} + e_{yy}e_{yy} + 2e_{xy}e_{yx})} \quad (39)$$

$$e_{xx} = \frac{1}{1+\kappa y} \left( \frac{\partial \tilde{u}}{\partial x} + \kappa \tilde{v} \right); \quad e_{yy} = \frac{\partial \tilde{v}}{\partial y};$$

$$e_{xy} = \frac{1}{2} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) \quad (40)$$

To complete the formulation of the set of equations an appropriate set of boundary conditions must be specified. Generally, for nonturbulent regions  $e = \epsilon = v_t = 0$ , while for solid boundaries the usual no-slip boundary condition on  $u$  and  $v$  applies. Moreover, either the temperature or heat flux must be specified. The turbulent energy and length scale must satisfy the condition  $e = l = 0$  (see Ref. 5).

### 3. BOUNDARY-LAYER TYPE EQUATIONS OF MOTION

To compare the relative magnitude of the terms in the equations of motion for the mean turbulent flow given in the previous section in situations where the characteristic length of the flow in the  $x$  direction is  $L$  and in the  $y$  direction is  $\delta_t \ll L$ , we introduce the following new dimensionless variables that are of order unity in the flow field considered:

$$x^* = \frac{x}{L}, \quad R^* = \frac{R}{L}, \quad r^* = \frac{r}{L}, \quad y^* = \frac{y}{\delta_t}, \quad t^* = \frac{U_e t}{L}$$

$$\tilde{u}^* = \frac{\tilde{u}}{U_e}, \quad \tilde{v}^* = \frac{\tilde{v}}{V_e}, \quad \bar{\rho}^* = \frac{\bar{\rho}}{\rho_e}, \quad p^* = \frac{\bar{p}}{\rho_e U_e^2}, \quad \tilde{H}^* = \frac{\tilde{H}}{U_e^2} \quad (41)$$

$$v_t^* = \frac{v_t}{V_t \delta_t}, \quad e^* = \frac{e}{V_t^2}, \quad \epsilon^* = \frac{\epsilon \delta_t}{\rho_e V_t}, \quad l^* = \frac{l}{\delta_t} = \frac{\bar{\rho}^* e^{*1/2}}{\epsilon^*}$$

where  $U_e$  is a characteristic velocity in x-direction,  $\rho_e$  a characteristic density and  $V_t$  can be any velocity characteristic of the turbulent field. The characteristic velocity  $V_e$  is yet to be determined.

In nondimensional form the continuity, Eq (21), is:

$$\begin{aligned} \frac{\partial \bar{\rho}^*}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^*) + \right. \\ \left. + \frac{L V_e}{\delta_t U_e} \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \bar{\rho}^* \tilde{v}^* \right] \right\} = 0 \end{aligned} \quad (42)$$

If all the terms of this equation are to be of the same order of magnitude then

$$\frac{L V_e}{\delta_t U_e} = 1 \quad \text{and} \quad V_e = \frac{U_e \delta_t}{L} \quad (43)$$

Thus the continuity equation, Eq. (42), becomes

$$\frac{\partial \bar{\rho}^*}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^*) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \bar{\rho}^* \tilde{v}^* \right] \right\} = 0 \quad (44)$$

The species continuity equation, Eq. (29), in nondimensional form, is:

$$\begin{aligned} \frac{\partial (\bar{\rho}^* \tilde{\alpha}_i)}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{\alpha}_i) + \right. \\ \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \bar{\rho}^* \tilde{v}^* \tilde{\alpha}_i \right] \right\} = \\ = \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \left( \frac{\delta_t}{L} \right)^2 \frac{\partial}{\partial x^*} \left[ r^{*a} \left( \frac{\bar{\rho}^* \hat{e}}{Sc} + \frac{\bar{\rho}^* v_t^*}{Sc_t} \right) \frac{1}{\left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \frac{\partial \tilde{\alpha}_i}{\partial x^*} \right] + \right. \\ \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \left( \frac{\bar{\rho}^* \hat{e}}{Sc} + \frac{\bar{\rho}^* v_t^*}{Sc_t} \right) \frac{\partial \tilde{\alpha}_i}{\partial y^*} \right] \right\} + \tilde{W}_i^* \end{aligned} \quad (45)$$

In the above equation  $\tilde{W}_i^* = \rho_e \tilde{W}_i$  and  $\tilde{W}_i^*$  is the nondimensional mean rate of production of species  $i$ . The relation  $V_t/U_e = \delta_t/L$  is used, and the quantity  $\hat{e} = \nu_L/V_t \delta_t$ , where  $\nu_L$  is the laminar kinematic viscosity, can be interpreted as the ratio of the viscous layer length scale  $\delta_\nu$  and the defect layer length scale  $\delta_t$ :

$$\hat{\epsilon} = \frac{v_L}{V_t \delta_t} = \frac{\delta_v}{\delta_t} \quad (46)$$

Thus, in turbulent boundary layer type flows, as a result of making the equations nondimensional by using the variables in Eqs. (41) and (43), two small parameters,  $\delta_t/L = V_t/U_e$  and  $\hat{\epsilon} = \delta_v/\delta_t$  appear which correspond to three regions of flow field: the inviscid region where the length scale is  $L$ , the defect-layer region where the characteristic length is  $\delta_t$  and the viscous layer region where the characteristic length is  $\delta_v$ . Mellor (Ref. 10; see also Ref. 11) has shown that expansion in one parameter  $\delta_t/L$  is sufficient as  $\hat{\epsilon} = o[(\delta_t/L)^n]$  for arbitrary  $n$ . From Eqs. (35) and (36) the nondimensional eddy diffusivity,  $v_t^*$ , is:

$$v_t^* = e^{*1/2} l^* \quad (47)$$

In nondimensional variables, Eqs. (30)-(32) and (34) are:

x-momentum:

$$\begin{aligned} & \frac{\partial(\bar{\rho}^* \tilde{u}^*)}{\partial t^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^{*2}) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) \bar{\rho}^* \tilde{u}^* \tilde{v}^* \right] \right\} + \\ & + \frac{\delta_t}{L} \frac{\bar{\rho}^* \tilde{u}^* \tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} = - \frac{1}{\left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \frac{\partial \bar{P}^*}{\partial x^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \left\{ \left(\frac{\delta_t}{L}\right)^2 \frac{\partial}{\partial x^*} x \right. \\ & \times \left[ r^{*a} (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \left( \frac{2}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \tilde{u}^*}{\partial x^*} + 2 \left(\frac{\delta_t}{L}\right) \frac{\tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} - \frac{2}{3} \text{div } \vec{\tilde{v}}^* \right) \right] + \\ & + \left(\frac{\delta_t}{L}\right) \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \left( \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \right. \right. \\ & \left. \left. + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right) \right] \right\} + \left(\frac{\delta_t}{L}\right)^2 \frac{(\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*)}{\left(R^* + \frac{\delta_t}{L} y^*\right)} \left( \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \right. \\ & \left. + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right) - \left(\frac{\delta_t}{L}\right)^2 (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \frac{a}{r^*} \left[ \frac{2a}{r^*} \left( \tilde{u}^* \sin \alpha + \right. \right. \\ & \left. \left. + \frac{\delta_t}{L} \tilde{v}^* \cos \alpha \right) - \frac{2}{3} \text{div } \vec{\tilde{v}}^* \right] \sin \alpha \quad (48) \end{aligned}$$

y-momentum:

$$\begin{aligned}
& \frac{\delta_t}{L} \frac{\partial(\bar{\rho}^* \tilde{v}^*)}{\partial t^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{v}^*) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) x \right. \right. \\
& \left. \left. x \bar{\rho}^* \tilde{v}^{*2} \right] \right\} - \frac{\bar{\rho}^* \tilde{u}^{*2}}{R^* + \frac{\delta_t}{L} y^*} = - \frac{L}{\delta_t} \frac{\partial \bar{P}^*}{\partial y^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \left\{ \left(\frac{\delta_t}{L}\right)^2 \frac{\partial}{\partial x^*} x \right. \\
& \left. x \left[ r^{*a} (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left( \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right) \right] + \right. \\
& \left. + \frac{\delta_t}{L} \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left( 2 \frac{\partial \tilde{v}^*}{\partial y^*} - \frac{2}{3} \text{div } \vec{V}^* \right) \right] \right\} - \\
& - \left(\frac{\delta_t}{L}\right)^2 \frac{(\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*)}{R^* + \frac{\delta_t}{L} y^*} \left( \frac{2}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \tilde{u}^*}{\partial x^*} + 2 \frac{\delta_t}{L} \frac{\tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} - \frac{2}{3} \text{div } \vec{V}^* \right) - \\
& - \left(\frac{\delta_t}{L}\right)^2 \frac{a}{r^*} (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left[ \frac{2a}{r^*} (\tilde{u}^* \sin \alpha + \frac{\delta_t}{L} \tilde{v}^* \cos \alpha) - \frac{2}{3} \text{div } \vec{V}^* \right] \cos \alpha \quad (49)
\end{aligned}$$

Energy:

$$\begin{aligned}
& \frac{\partial}{\partial t^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) (\bar{\rho}^* \tilde{H}^* - \bar{p}^*) \right] + \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{H}^*) + \frac{\partial}{\partial y^*} x \\
& x \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) \bar{\rho}^* \tilde{v}^* \tilde{H}^* \right] = \left(\frac{\delta_t}{L}\right)^2 \frac{\partial}{\partial x^*} \left[ r^{*a} \left( \frac{\bar{\rho}^* \hat{e}}{\text{Pr}} + \frac{\bar{\rho}^* v_t^*}{\text{Pr}_t} \right) x \right. \\
& x \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial}{\partial x^*} \left( \tilde{H}^* - \frac{\tilde{u}^{*2} + (\delta_t \tilde{v}^*/L)^2}{2} \right) \left. \right] + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) x \right. \\
& x \left( \frac{\bar{\rho}^* \hat{e}}{\text{Pr}} + \frac{\bar{\rho}^* v_t^*}{\text{Pr}_t} \right) \frac{\partial}{\partial y^*} \left( \tilde{H}^* - \frac{\tilde{u}^{*2} + (\delta_t \tilde{v}^*/L)^2}{2} \right) \left. \right] + \left(\frac{\delta_t}{L}\right)^2 \frac{\partial}{\partial x^*} x
\end{aligned}$$

(50)  
Contd...

$$\begin{aligned}
& \times \left\{ r^{*a} \left[ (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \tilde{u}^* \left( \frac{2}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\partial \tilde{u}^*}{\partial x^*} + 2 \frac{\delta_t}{L} \frac{\tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} - \frac{2}{3} \operatorname{div} \vec{V}^* \right) + \right. \right. \\
& + (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \frac{\delta_t}{L} \tilde{v}^* \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right) - \\
& \left. \left. - \frac{2}{3} \bar{\rho}^* \tilde{u}^* e^* \right] \right\} + \frac{\delta_t}{L} \frac{\partial}{\partial y^*} \left\{ r^{*a} \left( 1 + \frac{\delta_t y^*}{L R^*} \right) \left[ (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \tilde{u}^* \times \right. \right. \\
& \times \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right) + \frac{\delta_t}{L} (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \tilde{v}^* \times \\
& \left. \left. \times \left( 2 \frac{\partial \tilde{v}^*}{\partial y^*} - \frac{2}{3} \operatorname{div} \vec{V}^* \right) - \frac{2}{3} \frac{\delta_t}{L} \bar{\rho}^* \tilde{v}^* e^* \right] \right\} \quad (50)
\end{aligned}$$

or in the alternative form:

$$\begin{aligned}
& \frac{\partial(\bar{\rho}^* \tilde{h}^*)}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t y^*}{L R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{h}^*) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t y^*}{L R^*} \right) \times \right. \right. \\
& \left. \left. \times \bar{\rho}^* \tilde{v}^* \tilde{h}^* \right] \right\} = \frac{1}{r^{*a} \left( 1 + \frac{\delta_t y^*}{L R^*} \right)} \left\{ \left( \frac{\delta_t}{L} \right)^2 \frac{\partial}{\partial x^*} \left[ r^{*a} \left( \frac{\bar{\rho}^* \hat{\epsilon}}{\operatorname{Pr}} + \frac{\bar{\rho}^* v_t^*}{\operatorname{Pr}_t} \right) \times \right. \right. \\
& \left. \left. \times \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\partial \tilde{h}^*}{\partial x^*} \right] + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t y^*}{L R^*} \right) \left( \frac{\bar{\rho}^* \hat{\epsilon}}{\operatorname{Pr}} + \frac{\bar{\rho}^* v_t^*}{\operatorname{Pr}_t} \right) \frac{\partial \tilde{h}^*}{\partial y^*} \right] \right\} + \\
& + \frac{\partial \bar{p}^*}{\partial t^*} + \frac{\tilde{u}^*}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\partial \bar{p}^*}{\partial x^*} + \tilde{v}^* \frac{\partial \bar{p}^*}{\partial y^*} + \left( \frac{\delta_t}{L} \right)^2 (\bar{\rho}^* \hat{\epsilon} + \bar{\rho}^* v_t^*) \left[ 2 \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \times \right. \right. \\
& \left. \left. \times \frac{\partial \tilde{u}^*}{\partial x^*} + \frac{\delta_t}{L} \frac{\tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} \right)^2 + \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \right)^2 + \right. \\
& \left. + 2 \left( \frac{\partial \tilde{v}^*}{\partial y^*} \right)^2 + \frac{2a}{r^{*2}} \left( \tilde{u}^* \sin \alpha + \frac{\delta_t}{L} \tilde{v}^* \cos \alpha \right)^2 - \frac{2}{3} \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\partial \tilde{u}^*}{\partial x^*} + \right. \quad (51) \\
& \left. \operatorname{Contd} \dots \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\delta_t}{L} \frac{\tilde{v}^*}{R^* + \frac{\delta_t}{L} y^*} + \frac{\partial \tilde{v}^*}{\partial y^*} + a \frac{\tilde{u}^* \sin \alpha + \delta_t \tilde{v}^* \cos \alpha L}{r^*} \Big)^2 \Big] - \frac{2}{3} \left( \frac{\delta_t}{L} \right)^2 x \\
& x \bar{\rho}^* e^* \operatorname{div} \vec{V}^* \tag{51}
\end{aligned}$$

The nondimensional equations of motion for mean turbulent boundary-layer type flows obtained by retaining terms of order unity and terms of order  $(\delta_t/L)$  in Eqs. (44), (45), (48)-(51), which include in a consistent way longitudinal curvature and normal pressure gradient, are:

Continuity:

$$\frac{\partial \bar{\rho}^*}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^*) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \bar{\rho}^* \tilde{v}^* \right] \right\} = 0 \tag{52}$$

Species Continuity:

$$\begin{aligned}
& \frac{\partial (\bar{\rho}^* \tilde{\alpha}_i)}{\partial t^*} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{\alpha}_i) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) x \right. \right. \\
& \left. \left. x \bar{\rho}^* \tilde{v}^* \tilde{\alpha}_i \right] \right\} = \frac{1}{r^{*a}} \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \left( \frac{\bar{\rho}^* \hat{e}}{Sc} + \frac{\bar{\rho}^* v_t^*}{Sc_t} \right) \frac{\partial \tilde{\alpha}_i}{\partial y^*} \right] + \tilde{W}_i^* \tag{53}
\end{aligned}$$

x-momentum:

$$\begin{aligned}
& \bar{\rho}^* \left( \frac{\partial \tilde{u}^*}{\partial t^*} + \frac{\tilde{u}^*}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \tilde{u}^*}{\partial x^*} + \tilde{v}^* \frac{\partial \tilde{u}^*}{\partial y^*} + \frac{\delta_t}{L} \frac{\tilde{u}^* \tilde{v}^*}{R^*} \right) = - \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \bar{p}^*}{\partial x^*} + \\
& + \frac{1}{r^{*a}} \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left( \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\delta_t}{L} \frac{\tilde{u}^*}{R^*} \right) \right] + \\
& + \frac{\delta_t}{L} \frac{(\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*)}{R^*} \frac{\partial \tilde{u}^*}{\partial y^*} \tag{54}
\end{aligned}$$

y-momentum:

$$\begin{aligned} \frac{\delta_t}{L} \bar{\rho}^* \left( \frac{\partial \tilde{v}^*}{\partial t^*} + \tilde{u}^* \frac{\partial \tilde{v}^*}{\partial x^*} + \tilde{v}^* \frac{\partial \tilde{v}^*}{\partial y^*} \right) - \frac{\bar{\rho}^* \tilde{u}^{*2}}{R^*} = - \frac{L}{\delta_t} \frac{\partial \bar{p}^*}{\partial y^*} + \frac{\delta_t}{L} \frac{1}{r^{*a}} \frac{\partial}{\partial x^*} \times \\ \times \left[ r^{*a} (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \frac{\partial \tilde{u}^*}{\partial y^*} \right] + \frac{\delta_t}{L} \frac{1}{r^{*a}} \frac{\partial}{\partial y^*} \left[ r^{*a} (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left( 2 \frac{\partial \tilde{v}^*}{\partial y^*} - \right. \right. \\ \left. \left. - \frac{2}{3} \operatorname{div} \vec{V}^* \right) \right] \end{aligned} \quad (55)$$

Energy:

$$\begin{aligned} \frac{\partial}{\partial t^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) (\bar{\rho}^* \tilde{H}^* - \bar{p}^*) \right] + \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \tilde{H}^*) + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \times \right. \\ \left. \times \bar{\rho}^* \tilde{v}^* \tilde{H}^* \right] = \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \left( \frac{\bar{\rho}^* \hat{e}}{\operatorname{Pr}} + \frac{\bar{\rho}^* v_t^*}{\operatorname{Pr}_t} \right) \frac{\partial}{\partial y^*} \left( \tilde{H}^* - \frac{\tilde{u}^{*2}}{2} \right) \right] + \\ + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \left( \frac{\partial (\tilde{u}^{*2}/2)}{\partial y^*} - \frac{\delta_t}{L} \frac{\tilde{u}^{*2}}{R^*} \right) \right] \end{aligned} \quad (56)$$

or

$$\begin{aligned} \bar{\rho}^* \left( \frac{\partial \tilde{h}^*}{\partial t^*} + \frac{\tilde{u}^*}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \tilde{h}^*}{\partial x^*} + \tilde{v}^* \frac{\partial \tilde{h}^*}{\partial y^*} \right) = \frac{1}{r^{*a}} \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \left( \frac{\bar{\rho}^* \hat{e}}{\operatorname{Pr}} + \right. \right. \\ \left. \left. + \frac{\bar{\rho}^* v_t^*}{\operatorname{Pr}_t} \right) \frac{\partial \tilde{h}^*}{\partial y^*} \right] + \frac{\partial \bar{p}^*}{\partial t^*} + \frac{\tilde{u}^*}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial \bar{p}^*}{\partial x^*} + \tilde{v}^* \frac{\partial \bar{p}^*}{\partial y^*} + (\bar{\rho}^* \hat{e} + \bar{\rho}^* v_t^*) \times \\ \times \left[ \left( \frac{\partial \tilde{u}^*}{\partial y^*} \right)^2 - 2 \frac{\delta_t}{L} \frac{\partial \tilde{u}^*}{\partial y^*} \frac{\tilde{u}^*}{R^*} \right] \end{aligned} \quad (57)$$

#### 4. TWO-EQUATION TURBULENCE MODEL FOR CURVED BOUNDARY LAYER TYPE FLOWS

The turbulence quantity defined in Eq. (26) and satisfying the model equation, Eq. (36), is regarded as the total kinetic energy of turbulence, with  $u''$  and  $v''$  denoting the fluctuating velocity components in the streamwise and in the normal to the shear plane directions, respectively (in this section we

consider thin shear layers parallel to y-constant lines). However, Wilcox and Chambers (Ref. 2) argue that this definition of e is physically realistic only if the kinetic energy of turbulence is equipartitioned, i.e., if the turbulence is isotropic. Stating that turbulence in boundary layer type flows is anisotropic, they postulate that, more appropriately, e is proportional to  $v''^2$ . The proportionality coefficient is then established by requiring that e be numerically equal to the kinetic energy of turbulence in the law-of-the-wall region (Ref. 5) of a flat-plate boundary layer, i.e.,  $e = 9\overline{v''^2}/4$ . Furthermore, by proper consideration of the physics of turbulent flows with streamline curvature based on the classical stability arguments for flow over a curved wall (see Ref. 12), Wilcox and Chambers show (Ref. 2) that the stability of such a flow depends mainly on the behaviour of vertically moving fluid particles and not on any attendant nonvertical fluctuations. Hence they conclude that in a turbulent boundary layer, velocity fluctuations normal to the wall,  $v''^2$ , play an important role in curved streamline turbulent flows.

The model equation for the "mixing" energy,  $e = 9\overline{v''^2}/4$ , may be obtained from the exact normal Reynolds stress equation (A33) derived in Appendix A, rewritten in the following form:

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \bar{\rho} \frac{\overline{\rho v'' v''}}{\bar{\rho}} \right) + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left( r^a \bar{\rho} \tilde{u} \frac{\overline{\rho v'' v''}}{\bar{\rho}} \right) + \right. \\ & \left. + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \bar{\rho} \tilde{v} \frac{\overline{\rho v'' v''}}{\bar{\rho}} \right] \right\} - \frac{2\kappa}{1+\kappa y} \tilde{u} \overline{\rho u'' v''} = \\ & = - 2\overline{\rho u'' v''} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) - \overline{\rho v'' v''} 2 \frac{\partial \tilde{v}}{\partial y} - \\ & \text{----- Production -----} \\ & - \frac{2\tau_{xy}}{1+\kappa y} \frac{\partial v''}{\partial x} - 2\tau_{yy} \frac{\partial v''}{\partial y} + \\ & \text{----- Dissipation -----} \\ & + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} [r^a(2\tau_{xy} v'' - u'' \overline{\rho v'' v''})] + \right. \\ & \left. + \frac{\partial}{\partial y} [r^a(1 + \kappa y)(2\tau_{yy} v'' - v'' \overline{\rho v'' v''} - 2p' v'')] \right\} + \end{aligned}$$

— Laminar, Turbulent and Pressure Diffusion —

(Contd...)

$$+ \frac{2\kappa}{1+\kappa y} \overline{u''\rho u''v''} + \frac{2\kappa}{1+\kappa y} \overline{v''(p' - \tau_{xx})} + \frac{2a}{r} \overline{v''(p' - \tau_{\varphi\varphi})} \cos\alpha + p' \frac{\overline{\delta v''}}{\delta y}$$

----- Turbulent Work -----

$$- \overline{v''} \frac{\partial \bar{p}}{\partial y} \quad (58)$$

Compressibility

Multiplying Eq. (58) by  $9/4$ , modelling the groups of terms for the corresponding physical quantities as for Eq. (A34) and taking into account Eq. (28), we obtain the following model equation for the mass-averaged "mixing" energy  $e = 9\overline{v''v''}/4\bar{\rho}$ :

$$\begin{aligned} & \frac{\partial(\bar{\rho} e)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial(r^a \bar{\rho} \tilde{u} e)}{\partial x} + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v} e] \right\} + \\ & + \frac{9}{2} \frac{\kappa \tilde{u}}{1+\kappa y} \bar{\rho} v_t \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right) = \\ & = (\alpha^* \bar{\rho} S - \beta^* \epsilon) e - \xi^* e \bar{\rho} \operatorname{div} \vec{V} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma^* \bar{\rho} v_t)}{1+\kappa y} \frac{\partial e}{\partial x} \right] + \right. \\ & \left. + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y)(\mu + \sigma^* \bar{\rho} v_t) \frac{\partial e}{\partial y} \right] \right\} \quad (59) \end{aligned}$$

Written in nondimensional variables of Eq. (41), Eq. (50) becomes:

$$\begin{aligned} & \frac{\partial(\bar{\rho}^* e^*)}{\partial t^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \left\{ \frac{\partial(r^{*a} \bar{\rho}^* \tilde{u}^* e^*)}{\partial x^*} + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right) \bar{\rho}^* \tilde{v}^* e^* \right] \right\} + \\ & + \frac{9}{2} \frac{\tilde{u}^*}{R^* + \frac{\delta_t}{L} y^*} \bar{\rho}^* v_t^* \left( \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^*} \right) = \\ & = (\alpha^* \bar{\rho}^* \tilde{S}^* - \beta^* \epsilon^*) e^* - \xi^* e^* \bar{\rho}^* \operatorname{div} \vec{V}^* + \frac{1}{r^{*a} \left(1 + \frac{\delta_t}{L} \frac{y^*}{R^*}\right)} \times \quad (60) \\ & \hspace{15em} \text{Contd...} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \left( \frac{\delta_t}{L} \right)^2 \frac{\partial}{\partial x^*} \left[ \frac{r^* (\bar{\rho}^* \hat{\epsilon} + \sigma^* \bar{\rho}^* v_t^*)}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\partial e^*}{\partial x^*} \right] + \right. \\
& \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) (\bar{\rho}^* \hat{\epsilon} + \sigma^* \bar{\rho}^* v_t^*) \frac{\partial e^*}{\partial y^*} \right] \right\} \quad (60)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{S}^* = & \sqrt{2 \left\{ \left[ \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \left( \frac{\partial \tilde{u}^*}{\partial x^*} + \frac{\delta_t}{L} \frac{\tilde{v}^*}{R^*} \right) \right]^2 + \left( \frac{\partial \tilde{v}^*}{\partial y^*} \right)^2 \right\} +} \\
& \sqrt{+ \left[ \frac{1}{1 + \frac{\delta_t}{L} \frac{y^*}{R^*}} \frac{\delta_t}{L} \frac{\partial \tilde{v}^*}{\partial x^*} + \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^*} \right]^2} \quad (61)
\end{aligned}$$

Retaining terms of order not higher than  $\delta_t/L$  in Eqs. (60)-(61) we get:

$$\begin{aligned}
& \frac{\partial(\bar{\rho}^* e^*)}{\partial t} + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \left\{ \frac{\partial(r^{*a} \bar{\rho}^* \tilde{u}^* e^*)}{\partial x^*} + \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \bar{\rho}^* \tilde{v}^* e^* \right] \right\} + \\
& + \frac{9}{2} \frac{\tilde{u}^*}{R^*} \bar{\rho}^* v_t^* \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} = \left[ \alpha^* \bar{\rho}^* \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^*} - \beta^* e^* \right] e^* - \\
& - \xi^* e^* \bar{\rho}^* \operatorname{div} \vec{V}^* + \frac{1}{r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right)} \frac{\partial}{\partial y^*} \left[ r^{*a} \left( 1 + \frac{\delta_t}{L} \frac{y^*}{R^*} \right) \times \right. \\
& \left. \times (\bar{\rho}^* \hat{\epsilon} + \sigma^* \bar{\rho}^* v_t^*) \frac{\partial e^*}{\partial y^*} \right] \quad (62)
\end{aligned}$$

Except for the compressibility term, which is retained here in its general form, Eq. (62) reduces to the corresponding equation in Ref. 2 for  $a = \kappa = 0$ .

Writing the dissipation rate equation, Eq. (37), in nondimensional variables, Eq. (41), we have:

$$\begin{aligned}
& \frac{\partial(\bar{\rho}^* \epsilon^{*2})}{\partial t^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right)} \left\{ \frac{\partial}{\partial x^*} (r^{*a} \bar{\rho}^* \tilde{u}^* \epsilon^{*2}) + \right. \\
& \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right) \bar{\rho}^* \tilde{v}^* \epsilon^{*2} \right] \right\} = \alpha \bar{\rho}^* \tilde{S}^* \epsilon^{*2} - \\
& - \left\{ \beta + 2\sigma \left[ \left( \frac{1}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\delta_t}{L} \frac{\partial l^*}{\partial x^*} \right)^2 + \left( \frac{\partial l^*}{\partial y^*} \right)^2 \right] \right\} \epsilon^{*3} + \\
& + \frac{1}{r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right)} \left\{ \left( \frac{\delta_t}{L} \right)^2 \frac{\partial}{\partial x^*} \left[ \frac{r^{*a} (\bar{\rho}^* \hat{\epsilon} + \sigma \bar{\rho}^* v_t^*)}{1 + \frac{\delta_t y^*}{L R^*}} \frac{\partial \epsilon^{*2}}{\partial x^*} \right] + \right. \\
& \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right) (\bar{\rho}^* \hat{\epsilon} + \sigma \bar{\rho}^* v_t^*) \frac{\partial \epsilon^{*2}}{\partial y^*} \right] \right\} - \\
& - \xi_2 \bar{\rho}^* \epsilon^{*2} \operatorname{div} \vec{V}^* - \xi_3 \frac{\bar{\rho}^* e^*}{\tilde{a}^{*2}} \frac{\delta_t}{L} \frac{\operatorname{div} \vec{V}^*}{1 + \frac{\delta_t y^*}{L R^*}} \left( \frac{\partial \bar{\rho}^*}{\partial x^*} \frac{\partial \bar{p}^*}{\partial y^*} - \frac{\partial \bar{\rho}^*}{\partial y^*} \frac{\partial \bar{p}^*}{\partial x^*} \right) \quad (63)
\end{aligned}$$

Retaining terms of order not higher than  $\delta_t/L$  in Eqs. (63) we get:

$$\begin{aligned}
& \frac{\partial(\bar{\rho}^* \epsilon^{*2})}{\partial t^*} + \frac{1}{r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right)} \left\{ \frac{\partial(r^{*a} \bar{\rho}^* \tilde{u}^* \epsilon^{*2})}{\partial x^*} + \right. \\
& \left. + \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right) \bar{\rho}^* \tilde{v}^* \epsilon^{*2} \right] \right\} = \\
& = \left\{ \alpha \bar{\rho}^* \left| \frac{L}{\delta_t} \frac{\partial \tilde{u}^*}{\partial y^*} - \frac{\tilde{u}^*}{R^*} \right| - \left[ \beta + 2\sigma \left( \frac{\partial l^*}{\partial y^*} \right)^2 \right] \right\} \epsilon^{*2} + \\
& + \frac{1}{r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right)} \frac{\partial}{\partial y^*} \left[ r^{*a} \left(1 + \frac{\delta_t y^*}{L R^*}\right) (\bar{\rho}^* \epsilon + \sigma \bar{\rho}^* v_t^*) \frac{\partial \epsilon^{*2}}{\partial y^*} \right] - \\
& - \xi_2 \bar{\rho}^* \epsilon^{*2} \operatorname{div} \vec{V}^* - \xi_3 \frac{\bar{\rho}^* e^*}{\tilde{a}^{*2}} \frac{\delta_t}{L} \frac{\operatorname{div} \vec{V}^*}{1 + \frac{\delta_t y^*}{L R^*}} \left( \frac{\partial \bar{\rho}^*}{\partial x^*} \frac{\partial \bar{p}^*}{\partial y^*} - \frac{\partial \bar{\rho}^*}{\partial y^*} \frac{\partial \bar{p}^*}{\partial x^*} \right) \quad (64)
\end{aligned}$$

Equations (62) and (64) represent the model equations for the quantities  $e^*$  and  $\epsilon^*$  for boundary-layer type flows ( $\delta_t/L \ll 1$ ) needed to determine the turbulent (eddy) diffusivity  $\nu_t^*$ . The values of the closure constants,  $\alpha$ ,  $\alpha^*$ ,  $\beta$ ,  $\beta^*$ ,  $\sigma$  and  $\sigma^*$  and  $\xi^*$ , are the same as in Eq. (28). The magnitudes of the coefficients  $\xi_2$  and  $\xi_3$  and hence the importance of the terms which contain these coefficients may be determined by comparing theoretical predictions with reliable experimental results.

APPENDIX A

DERIVATION OF TWO-EQUATION TURBULENCE MODEL

FOR COMPRESSIBLE PLANE OR AXISYMMETRIC TURBULENT FLOWS

IN ORTHOGONAL CURVILINEAR BODY-ORIENTED COORDINATES

The mass-averaged turbulent kinetic (mixing) energy  $e$ , and the mass-averaged turbulent dissipation rate  $\epsilon$ , are needed to define the eddy diffusivity  $\nu_t$  given by Eq. (35) on p.12. The following detailed derivation of the model partial differential transport equations in the coordinate system considered which  $\epsilon$  and  $e$  satisfy follows the Wilcox and Traci approach (Ref. 5) of turbulence modelling.

1. The Turbulent Dissipation Rate Equation

The dissipation rate quantity is chosen as  $\epsilon^2 = \overline{\rho \Omega_i'' \Omega_i''} / \rho$  where  $\Omega_i = \tilde{\Omega}_i + \Omega_i$ ,  $\Omega_i = \rho \omega_i$  and the vector  $\vec{\omega} = \text{rot } \vec{v}$  is the vorticity. In vector form the momentum equation may be written as (see, for example, Ref. 7, Ch. 1):

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \left( \frac{\vec{v}^2}{2} \right) + \vec{\omega} \times \vec{v} = \frac{\vec{\nabla}}{\rho} \cdot (-p \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau}) \quad (A1)$$

where  $\overset{\leftrightarrow}{I}$  and  $\overset{\leftrightarrow}{\tau}$  are the unit and viscous stress tensors respectively. Taking the rot of Eq. (A1) we get, using the vector relationship  $\text{rot}(\vec{\omega} \times \vec{v}) = (\vec{v} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \vec{\omega}(\vec{\nabla} \cdot \vec{v})$ ,

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \vec{\omega}(\vec{\nabla} \cdot \vec{v}) = \text{rot} \left[ \frac{\vec{\nabla} \cdot (-p \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau})}{\rho} \right] \quad (A2)$$

From the continuity equation (1) we have  $\vec{\nabla} \cdot \vec{v} = - (1/\rho)(d\rho/dt)$  and  $\partial \vec{\omega} / \partial t + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = d\vec{\omega} / dt$ . Hence Eq. (A2) becomes

$$\rho \frac{d\vec{\omega}}{dt} - (\rho \vec{\omega} \cdot \vec{\nabla})\vec{v} - \vec{\omega} \frac{d\rho}{dt} = \rho \text{rot} \left[ \frac{\vec{\nabla} \cdot (-p \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau})}{\rho} \right]$$

or

$$\frac{d\vec{\Omega}}{dt} - (\vec{\Omega} \cdot \vec{\nabla})\vec{v} + 2\vec{\Omega}(\vec{\nabla} \cdot \vec{v}) = \rho \text{rot} \left[ \frac{\vec{\nabla} \cdot (-p \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau})}{\rho} \right] \quad (A3)$$

The right-hand side of Eq. (A3) may be written as

$$\frac{\vec{\nabla} \cdot (-p \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau})}{\rho} = - \frac{\text{grad} p}{\rho} + \frac{\vec{\nabla} \cdot \overset{\leftrightarrow}{\tau}}{\rho}$$

The components of the vector  $\vec{\nabla} \cdot \hat{\mathbf{e}}_{\tau}$  are, in the orthogonal curvilinear body-oriented coordinate system, considered:

x-component:

$$\theta_x = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \tau_{xx}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \tau_{xy}] \right\} + \frac{\kappa \tau_{xy}}{1+\kappa y} - \frac{a}{r} \tau_{\varphi\varphi} \sin\alpha$$

y-component:

$$\theta_y = \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \tau_{xy}) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \tau_{yy}] \right\} - \frac{\kappa \tau_{xx}}{1+\kappa y} - \frac{a}{r} \tau_{\varphi\varphi} \cos\alpha$$

and

$$\frac{\vec{\nabla} \cdot \hat{\mathbf{e}}_{\tau}}{\rho} = \frac{\theta_x}{\rho} \vec{\alpha} + \frac{\theta_y}{\rho} \vec{\beta} + 0 \cdot \vec{\gamma} = \Pi_x \vec{\alpha} + \Pi_y \vec{\beta} + 0 \cdot \vec{\gamma}$$

where  $\vec{\alpha}$ ,  $\vec{\beta}$  and  $\vec{\gamma}$  are the unit coordinate vectors in the x, y and  $\varphi$  directions respectively. Hence

$$\begin{aligned} \text{rot} \left( -\frac{\text{grad} p}{\rho} \right) &= \frac{1}{\rho^2} \text{grad} p \times \text{grad} p = \frac{1}{\rho^2} \begin{array}{ccc} \vec{\alpha} & \vec{\beta} & \vec{\gamma} \\ \frac{1}{h_1} \frac{\partial \rho}{\partial x} & \frac{1}{h_2} \frac{\partial \rho}{\partial y} & 0 \\ \frac{1}{h_1} \frac{\partial p}{\partial x} & \frac{1}{h_2} \frac{\partial p}{\partial y} & 0 \end{array} = \\ &= \frac{1}{(1+\kappa y)\rho^2} \left( \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right) \vec{\gamma} \end{aligned}$$

and similarly

$$\text{rot} \left( \frac{\vec{\nabla} \cdot \hat{\mathbf{e}}_{\tau}}{\rho} \right) = \frac{1}{1+\kappa y} \left\{ \frac{\partial \Pi_y}{\partial x} - \frac{\partial [(1+\kappa y)\Pi_x]}{\partial y} \right\} \vec{\gamma}$$

Equation (A3) may thus be written as

$$\begin{aligned} \rho \frac{d\vec{\Omega}}{dt} &= \rho(\vec{\Omega} \cdot \vec{\nabla})\vec{v} - 2\rho\vec{\Omega}(\vec{\nabla} \cdot \vec{v}) + \frac{1}{1+\kappa y} \left( \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right) \vec{\gamma} + \frac{\rho^2}{1+\kappa y} \left\{ \frac{\partial \Pi_y}{\partial x} - \right. \\ &\quad \left. - \frac{\partial [(1+\kappa y)\Pi_x]}{\partial y} \right\} \vec{\gamma} \end{aligned} \quad (\text{A4})$$

For the only component  $\Omega_3 = \Omega$  of the vector  $\vec{\Omega}$  we get from Eq. (A4), in the coordinate system considered,

$$\begin{aligned} \rho \frac{\partial \Omega}{\partial t} + \frac{\rho u}{1+\kappa y} \frac{\partial \Omega}{\partial x} + \rho v \frac{\partial \Omega}{\partial y} = & - 2\rho\Omega \operatorname{div} \vec{V} + \frac{1}{1+\kappa y} \left( \frac{\partial \rho}{\partial x} \frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial \rho}{\partial x} \right) + \\ & + \frac{\rho^2}{1+\kappa y} \left\{ \frac{\partial \Pi_y}{\partial x} - \frac{\partial[(1+\kappa y)\Pi_x]}{\partial y} \right\} \end{aligned} \quad (\text{A5})$$

Decomposing  $\Omega = \tilde{\Omega} + \Omega''$ , using Eqs. (19) and (20), and multiplying Eq. (A5) by  $r^a(1+\kappa y)\Omega''$  and taking the average, we obtain, term by term:

$$\begin{aligned} \overline{\rho\Omega'' \frac{\partial}{\partial t} [r^a(1+\kappa y)(\tilde{\Omega} + \Omega'')] } = \frac{1}{2} \frac{\partial}{\partial t} \left[ r^a(1+\kappa y) \overline{\bar{\rho} \frac{\rho\Omega''\Omega''}{\bar{\rho}}} \right] - \\ - r^a(1+\kappa y) \frac{\overline{\Omega''\Omega''}}{2} \frac{\partial \bar{\rho}}{\partial t} \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \overline{r^a \rho \Omega'' (\tilde{u} + u'') \frac{\partial (\tilde{\Omega} + \Omega'')}{\partial x}} = \frac{1}{2} \frac{\partial}{\partial x} \left\{ r^a \left[ \bar{\rho} \tilde{u} \frac{\overline{\rho\Omega''\Omega''}}{\bar{\rho}} + \overline{u''\rho\Omega''\Omega''} \right] \right\} - \\ - \frac{\overline{\Omega''\Omega''}}{2} \frac{\partial}{\partial x} (r^a \rho u) + r^a \overline{\rho u''\Omega''} \frac{\partial \tilde{\Omega}}{\partial x} \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \overline{r^a(1+\kappa y)\rho\Omega''(\tilde{v} + v'') \frac{\partial (\tilde{\Omega} + \Omega'')}{\partial y}} = \frac{1}{2} \frac{\partial}{\partial y} \left\{ r^a(1+\kappa y) \left[ \bar{\rho} \tilde{v} \frac{\overline{\rho\Omega''\Omega''}}{\bar{\rho}} + \overline{v''\rho\Omega''\Omega''} \right] \right\} - \\ - \frac{\overline{\Omega''\Omega''}}{2} \frac{\partial}{\partial y} [r^a(1+\kappa y)\rho v] + r^a(1+\kappa y) \overline{\rho v''\Omega''} \frac{\partial \tilde{\Omega}}{\partial y} \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} - 2r^a(1+\kappa y)\rho\Omega''(\tilde{\Omega} + \Omega'') \operatorname{div}(\vec{V} + \vec{V}'') = - 2r^a(1+\kappa y) \overline{\bar{\rho} \frac{\rho\Omega''\Omega''}{\bar{\rho}}} \operatorname{div} \vec{V} - \\ - 2r^a(1+\kappa y) [\overline{\tilde{\Omega} \rho\Omega'' \operatorname{div} \vec{V}''} + \overline{\rho\Omega''\Omega'' \operatorname{div} \vec{V}''}] \end{aligned} \quad (\text{A9})$$

$$\begin{aligned}
& r^a \overline{\Omega''} \left[ \frac{\partial(\bar{\rho} + \rho')}{\partial x} \cdot \frac{\partial(\bar{p} + p')}{\partial y} - \frac{\partial(\bar{\rho} + \rho')}{\partial y} \cdot \frac{\partial(\bar{p} + p')}{\partial x} \right] = \\
& = r^a \left[ \overline{\Omega''} \left( \frac{\partial \bar{\rho}}{\partial x} \frac{\partial \bar{p}}{\partial y} - \frac{\partial \bar{\rho}}{\partial y} \frac{\partial \bar{p}}{\partial x} \right) - \overline{\Omega''} \frac{\partial \rho'}{\partial y} \cdot \frac{\partial \bar{p}}{\partial x} - \overline{\Omega''} \frac{\partial \rho'}{\partial x} \frac{\partial \bar{p}}{\partial y} - \right. \\
& \left. - \overline{\Omega''} \frac{\partial \rho'}{\partial y} \frac{\partial \rho'}{\partial x} + \overline{\Omega''} \frac{\partial \rho'}{\partial x} \cdot \frac{\partial \bar{p}}{\partial y} + \overline{\Omega''} \frac{\partial \rho'}{\partial y} \cdot \frac{\partial \bar{p}}{\partial x} + \overline{\Omega''} \frac{\partial \rho'}{\partial x} \frac{\partial \rho'}{\partial y} \right] \quad (A10)
\end{aligned}$$

and the last term in Eq. (A5) may be written as

$$r^a \overline{\Omega''} \rho^2 \left\{ \frac{\partial \bar{\Pi}_y}{\partial x} - \frac{\partial}{\partial y} [(1+\kappa y) \bar{\Pi}_x] \right\} \quad (A11)$$

Adding Eqs. (A6)-(A11), taking into account the continuity equation, Eq. (11), and regrouping the terms we get the following equation for the turbulent dissipation rate quantity  $\epsilon^2 = \overline{\rho \Omega'' \Omega''} / \rho$ :

$$\begin{aligned}
& \frac{\partial}{\partial t} (\bar{\rho} \epsilon^2) + \frac{1}{r^a (1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \epsilon^2) + \frac{\partial}{\partial y} [r^a (1 + \kappa y) \bar{\rho} \tilde{v} \epsilon^2] \right\} = \\
& = - 2 \left[ \frac{\overline{\rho u'' \Omega''}}{1+\kappa y} \frac{\partial \tilde{\Omega}}{\partial x} + \frac{\overline{\rho v'' \Omega''}}{\partial y} \frac{\partial \tilde{\Omega}}{\partial y} \right] - \\
& \quad \text{----- Production -----} \\
& - \frac{1}{r^a (1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \overline{u'' \rho \Omega'' \Omega''}) + \frac{\partial}{\partial y} [r^a (1 + \kappa y) \overline{v'' \rho \Omega'' \Omega''}] \right\} + \\
& \quad \text{----- Turbulent Diffusion -----} \\
& + \frac{2 \overline{\Omega''^2} \rho^2}{1+\kappa y} \left\{ \frac{\partial \bar{\Pi}_y}{\partial x} - \frac{\partial [(1+\kappa y) \bar{\Pi}_x]}{\partial y} \right\} - \\
& \quad \text{----- Laminar Diffusion and Dissipation -----}
\end{aligned}$$

$$\begin{aligned}
& - 4\bar{\rho} \epsilon^2 \operatorname{div} \vec{V} - 4\tilde{\Omega} \overline{\rho\Omega''} \operatorname{div} \vec{V} - 4\overline{\rho\Omega''\Omega''} \operatorname{div} \vec{V} + \frac{2}{1+\kappa\gamma} \left[ \overline{\Omega''} \left( \frac{\partial\bar{\rho}}{\partial x} \frac{\partial\bar{p}}{\partial y} - \frac{\partial\bar{\rho}}{\partial y} \frac{\partial\bar{p}}{\partial x} \right) + \right. \\
& \left. + \overline{\Omega''} \left( \frac{\partial\rho'}{\partial x} \frac{\partial p'}{\partial y} - \frac{\partial\rho'}{\partial y} \frac{\partial p'}{\partial x} \right) + \overline{\Omega''} \frac{\partial\rho'}{\partial y} \frac{\partial\bar{p}}{\partial x} + \overline{\Omega''} \frac{\partial\rho'}{\partial x} \frac{\partial\bar{p}}{\partial y} - \overline{\Omega''} \frac{\partial\rho'}{\partial y} \frac{\partial\bar{p}}{\partial x} - \overline{\Omega''} \frac{\partial\rho'}{\partial x} \frac{\partial\bar{p}}{\partial y} \right]
\end{aligned}$$


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Compressibility

(A12)

The physical meaning of each group of terms is indicated in Eq. (A12). Each group of terms should be "modelled" to express the physical quantities actually present (rates of turbulence production, dissipation and diffusion of the quantity  $\epsilon^2$  as well as the effect of compressibility on the rate of change of  $\epsilon^2$ ) in functional forms containing a closed set of dependent variables. These functional forms should be dimensionally consistent and may contain empirical constants that are expected to be insensitive to the character of individual flow fields. The following groups of terms are modelled after Wilcox and Traci (Ref. 5):

Production:

Using the eddy diffusivity approximation,

$$- \overline{\rho u''\Omega''} = \frac{\bar{\rho}v_t}{1+\kappa\gamma} \frac{\partial\tilde{\Omega}}{\partial x}, \quad - \overline{\rho v''\Omega''} = \bar{\rho}v_t \frac{\partial\tilde{\Omega}}{\partial y}$$

this group of terms may be written as

$$2\bar{\rho}v_t \left[ \left( \frac{1}{1+\kappa\gamma} \frac{\partial\tilde{\Omega}}{\partial x} \right)^2 + \left( \frac{\partial\tilde{\Omega}}{\partial y} \right)^2 \right] \tag{A13}$$

and is modelled by Wilcox and Traci by the dimensionally consistent expression

$$\alpha \bar{\rho} \tilde{S} \epsilon^2 \tag{A14}$$

where

$$\tilde{S} = \sqrt{2(e_{xx}e_{xx} + e_{yy}e_{yy} + 2e_{xy}e_{yx})} \tag{A15}$$

$$e_{xx} = \frac{1}{1+\kappa y} \left( \frac{\partial \tilde{u}}{\partial x} + \kappa \tilde{v} \right), \quad e_{yy} = \frac{\partial \tilde{v}}{\partial y}, \quad (A16)$$

$$e_{xy} = e_{yx} = \frac{1}{2} \left( \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} \right)$$

and  $\alpha$  is a closure coefficient.

Dissipation:

The group of terms describing the dissipation is modelled as follows: from dimensional consideration the rate of dissipation may be taken as  $\epsilon/\bar{\rho}$ [1/sec] and the dissipation rate of the quantity  $\bar{\rho} \epsilon^2$  in Eq. (A13) may be taken as

$$- \beta \bar{\rho} \epsilon^2 \cdot \frac{\epsilon}{\bar{\rho}} = - \beta \epsilon^3 \quad (A17)$$

where  $\beta$  is the proportionality coefficient.

Laminar and Turbulent Diffusion:

The form of this group of terms and the eddy diffusivity approximation for the quantities  $u''\rho\Omega''\Omega''$  and  $v''\rho\Omega''\Omega''$  suggest the following modelling of these terms:

$$\frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma \bar{\rho} v_t)}{1+\kappa y} \frac{\partial \epsilon^2}{\partial x} \right] + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y)(\mu + \sigma \rho v_t) \frac{\partial \epsilon^2}{\partial y} \right] \right\}$$

where  $\sigma$  is a closure coefficient.

Compressibility:

The first group of terms

$$- 4(\bar{\rho} \epsilon^2 \operatorname{div} \vec{V} + \tilde{\Omega} \overline{\rho \Omega'' \operatorname{div} V''} + \overline{\rho \Omega'' \Omega'' \operatorname{div} V''})$$

is missing from the corresponding equation for  $\bar{\rho} \epsilon^2$  written in general three-dimensional Cartesian coordinates in Ref. 5. This term is modelled here simply as

$$- \xi_2 \bar{\rho} \epsilon^2 \operatorname{div} \vec{V} \quad (A19)$$

where  $\xi_2$  is a proportionality (closure coefficient). It is worth noting that this term is similar in form to the compressibility term for the turbulent kinetic energy in Ref. 5 or in Eq. (A40).

The second group of terms in Eq. (A12) labelled "compressibility" is neglected in Ref. 5. However, this term may be retained and modelled in the following way: from mass-averaging relationships we have

$$\Omega'' = - \frac{\Omega' \rho'}{\bar{\rho}} \quad (\text{A20})$$

Consider now the idealized situation of one-dimensional high-speed flow for which the total temperature is approximately constant, the fluid is a perfect gas and pressure and total temperature fluctuations can be neglected. Then from the energy equation

$$\frac{\gamma-1}{2} u^2 + a^2 = \frac{\gamma+1}{2} a^{*2}$$

where  $a$  is the speed of sound and  $a^*$  the critical speed of sound. With  $u = \bar{u} + u'$  and  $a = \bar{a} + a'$  we get

$$\frac{\gamma-1}{2} (\bar{u}^2 + 2\bar{u}u' + u'^2) + \bar{a}^2 + 2\bar{a}a' + a'^2 = \frac{\gamma+1}{2} a^{*2}$$

(constant total temperature)

Since  $(\gamma-1)\bar{u}^2/2 + \bar{a}^2 = (\gamma+1)a^{*2}/2$  we have

$$(\gamma-1)\bar{u}u' + \frac{\gamma-1}{2} u'^2 + 2\bar{a}a' + a'^2 = 0 \quad (\text{A21})$$

On the other hand  $p a^2 = \gamma p$ ; with  $\rho = \bar{\rho} + \rho'$  and  $a = \bar{a} + a'$  (pressure fluctuations are neglected) we get

$$\frac{\rho'}{\bar{\rho}} (\bar{a}^2 + 2\bar{a}a') = - 2\bar{a}a' + a'^2 \quad \text{or} \quad \frac{\rho'}{\bar{\rho}} = - \frac{2a'}{\bar{a}} + \frac{3a'^2}{\bar{a}^2} \dots \quad (\text{A22})$$

Equation (A21) gives

$$(\gamma-1) \frac{\bar{u}u'}{\bar{a}^2} = - 2 \frac{a'}{\bar{a}} - \frac{\gamma-1}{2} \frac{u'^2}{\bar{a}^2} - \frac{a'^2}{\bar{a}^2}$$

Substituting the value of  $-2a'/\bar{a}$  from Eq. (A22) into the above equation, we get

$$(\gamma-1) \frac{uu'}{\bar{a}^2} = \frac{\rho'}{\bar{\rho}} - 3 \left( \frac{a'}{\bar{a}} \right)^2 \dots$$

or approximately

$$\frac{\rho'}{\bar{\rho}} \approx (\gamma-1) \frac{\bar{u}u'}{a^2} \quad (\text{A23})$$

For two-dimensional flows we can generalize this equation to

$$\frac{\rho'}{\bar{\rho}} \approx \frac{\gamma-1}{a^2} (\bar{u}u' + \bar{v}v') \quad (\text{A24})$$

Using Eq. (A24), Eq. (A20) may be written as

$$\bar{\Omega}'' \approx - \frac{\gamma-1}{a^2} (\bar{u} \overline{u'\Omega'} + \bar{v} \overline{v'\Omega'})$$

or using again the eddy diffusivity approximation

$$- \overline{u'\Omega'} \approx \frac{\nu_t}{1+\kappa\gamma} \frac{\partial \epsilon}{\partial x}, \quad - \overline{v'\Omega'} \approx \nu_t \frac{\partial \epsilon}{\partial y}$$

we obtain

$$\bar{\Omega}'' \approx \frac{\nu_t}{a^2} \left( \frac{\bar{u}}{1+\kappa\gamma} \frac{\partial \epsilon}{\partial x} + \bar{v} \frac{\partial \epsilon}{\partial y} \right) \quad (\text{A25})$$

However, from Eq. (A12) for  $\epsilon^2$ , in the approximation considered, we can write,

$$\frac{\bar{u}}{1+\kappa\gamma} \frac{\partial \epsilon}{\partial x} + \bar{v} \frac{\partial \epsilon}{\partial y} \sim - \epsilon \operatorname{div} \vec{V}$$

Taking into account Eq. (35), Eq. (A25) becomes

$$\bar{\Omega}'' \approx - \xi_3 \frac{\nu_t}{a^2} \epsilon \operatorname{div} \vec{V} = - \xi_3 \frac{\bar{\rho}}{a^2} \frac{e}{\epsilon} \epsilon \operatorname{div} \vec{V} = - \xi_3 \frac{\bar{\rho}}{a^2} e \operatorname{div} \vec{V} \quad (\text{A26})$$

where  $\xi_3$  is a closure coefficient. Hence the second group of compressibility terms in Eq. (A13) is modelled as

$$- \xi_3 \frac{\bar{\rho}}{a^2} e \operatorname{div} \vec{V} \left( \frac{\partial \bar{\rho}}{\partial x} \frac{\partial \bar{\rho}}{\partial y} - \frac{\partial \bar{\rho}}{\partial y} \frac{\partial \bar{\rho}}{\partial x} \right) \quad (\text{A27})$$

which is dimensionally sound and similar in form to Eq. (A19). From Eqs. (A14), (A17)-(A19) and (A20), the resulting model equation (A12) for the rate of change of the quantity  $\epsilon^2$ , is then

$$\begin{aligned}
& \frac{\partial(\bar{\rho} \epsilon^2)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} \epsilon^2) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{v} \epsilon^2] \right\} = \\
& = \alpha \bar{\rho} \tilde{S} \epsilon^2 - \left\{ \beta + 2\sigma \left[ \left( \frac{1}{1+\kappa y} \frac{\partial \ell}{\partial x} \right)^2 + \right. \right. \\
& + \left. \left. \left( \frac{\partial \ell}{\partial y} \right)^2 \right] \right\} \epsilon^3 + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma \bar{\rho} v_t)}{1+\kappa y} \frac{\partial \epsilon^2}{\partial x} \right] + \right. \\
& + \left. \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y)(\mu + \sigma \bar{\rho} v_t) \frac{\partial \epsilon^2}{\partial y} \right] \right\} - \\
& - \xi_2 \bar{\rho} \epsilon^2 \operatorname{div} \vec{V} - \xi_3 \frac{\bar{\rho} e}{a^2} \frac{\operatorname{div} \vec{V}}{1+\kappa y} \left( \frac{\partial \bar{\rho}}{\partial x} \frac{\partial \bar{\rho}}{\partial y} - \frac{\partial \bar{\rho}}{\partial y} \frac{\partial \bar{\rho}}{\partial x} \right) \quad (A28)
\end{aligned}$$

The dissipation-rate equation (A28) contains a term proportional to  $(1/1+\kappa y)^2 x (\partial \ell / \partial x)^2 + (\partial \ell / \partial y)^2$  referred to as the gradient-dissipation term in Ref. 5. Its introduction was motivated by comparison of the Wilcox-Traci model with other two-equation turbulence models, particularly those developed by Ng and Spalding (Ref. 8) and by Jones and Launder (Ref. 9). It enables one to predict more realistic values for the turbulence length scale  $\ell$ , especially near solid boundaries. The values of the closure coefficients  $\alpha$ ,  $\beta$  and  $\sigma$  are given by Wilcox and Traci (Ref. 5) as:

$$\alpha = \frac{1}{3} \left[ 1 - \frac{10}{11} \exp \left( - \frac{Re_t}{2} \right) \right]; \quad \beta = \frac{3}{20}; \quad \sigma = \frac{1}{2}; \quad Re_t = \frac{\bar{\rho} e^{1/2} \ell}{\mu}; \quad \ell = \frac{\bar{\rho} e^{1/2}}{\epsilon} \quad (A29)$$

The magnitudes of the coefficients  $\xi_2$  and  $\xi_3$  may be assessed by comparing theoretical predictions with reliable experimental results.

## 2. The Turbulent Kinetic Energy Equation

The x-momentum equation (13) may be written as

$$\begin{aligned}
& r^a(1 + \kappa y) \bar{\rho} \frac{\partial u}{\partial t} + r^a \bar{\rho} u \frac{\partial u}{\partial x} + r^a(1 + \kappa y) \bar{\rho} v \frac{\partial u}{\partial y} + r^a \kappa \bar{\rho} v = - r^a \frac{\partial p}{\partial x} + \\
& + \frac{\partial(r^a \tau_{xx})}{\partial x} + \frac{\partial}{\partial y} [r^a(1 + \kappa y) \tau_{xy}] + r^a \kappa \tau_{xy} - \tau_{\varphi\varphi} \frac{\partial r^a}{\partial x} \quad (A30)
\end{aligned}$$

Let

$$u = \tilde{u} + u'', \quad v = \tilde{v} + v'', \quad \rho = \bar{\rho} + \rho', \quad p = \bar{p} + p'$$

and

$$\tau_{ij} = \bar{\tau}_{ij} + \tau'_{ij} \quad (A31)$$

Multiply Eq. (A30) by  $u''$  and take the average; after some calculation we get

$$\begin{aligned} & r^a(1 + \kappa y) \frac{\partial}{\partial t} \left( \bar{\rho} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} \right) + \frac{\partial}{\partial x} \left[ r^a \left( \bar{\rho} \tilde{u} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} + \frac{\overline{u'' \rho u'' u''}}{2} \right) \right] + r^a \overline{\rho u'' u''} \frac{\partial \tilde{u}}{\partial x} + \\ & + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \left( \bar{\rho} \tilde{v} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} + v'' \frac{\overline{\rho u'' u''}}{2} \right) \right] + r^a(1 + \kappa y) \overline{\rho u'' v''} \frac{\partial \tilde{u}}{\partial y} + \\ & + r^a \kappa (\tilde{v} \overline{\rho u'' u''} + \tilde{u} \overline{\rho u'' v''} + \overline{u'' \rho u'' v''}) = - r^a \left( u'' \frac{\partial \bar{p}}{\partial x} + u'' \frac{\partial p'}{\partial x} \right) + \\ & + \frac{\partial}{\partial x} (r^a \overline{u'' \tau_{xx}}) - r^a \overline{\tau_{xx} \frac{\partial u''}{\partial x}} + \frac{\partial}{\partial y} [r^a(1 + \kappa y) \overline{u'' \tau_{xy}}] - \\ & - r^a(1 + \kappa y) \overline{\tau_{xy} \frac{\partial u''}{\partial y}} + r^a \kappa \overline{u'' \tau_{xy}} - \overline{u'' \tau_{\varphi\varphi}} \frac{\partial r^a}{\partial x} \end{aligned}$$

or rearranging the terms,

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \bar{\rho} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} \right) + \frac{1}{r^a(1 + \kappa y)} \left\{ \frac{\partial}{\partial x} \left( r^a \bar{\rho} \tilde{u} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} \right) + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \bar{\rho} \tilde{v} \frac{\overline{\rho u'' u''}}{2\bar{\rho}} \right] \right\} + \\ & + \frac{\kappa}{1 + \kappa y} \tilde{u} \overline{\rho u'' v''} = - \frac{\overline{\rho u'' u''}}{1 + \kappa y} \frac{\partial \tilde{u}}{\partial x} - \overline{\rho u'' v''} \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{v}}{1 + \kappa y} \overline{\rho u'' u''} - \frac{\kappa}{1 + \kappa y} \overline{u'' \rho u'' v''} - \\ & - \frac{\overline{u''}}{1 + \kappa y} \frac{\partial \bar{p}}{\partial x} + \frac{1}{r^a(1 + \kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \overline{u'' \tau_{xx}} - \overline{u'' \frac{\rho u'' u''}{2}} - \overline{p' u''} \right) \right] + \right. \\ & \left. + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \left( \overline{u'' \tau_{xy}} - \overline{v'' \frac{\rho u'' u''}{2}} \right) \right] \right\} - \frac{\overline{\tau_{xx}}}{1 + \kappa y} \frac{\partial u''}{\partial x} - \overline{\tau_{xy}} \frac{\partial u''}{\partial y} + \\ & + \frac{\kappa}{1 + \kappa y} \overline{u'' \tau_{xy}} - \frac{a}{r} \overline{u'' (\tau_{\varphi\varphi} - p')} \sin \alpha + \frac{\overline{p'}}{1 + \kappa y} \frac{\partial u''}{\partial x} \quad (A32) \end{aligned}$$

Using Eq. (A31) and multiplying the y-momentum equation, Eq. (14), written as

$$r^a(1 + \kappa y)\rho \frac{\partial v}{\partial t} + r^a \rho u \frac{\partial v}{\partial x} + r^a(1 + \kappa y)\rho v \frac{\partial v}{\partial y} - r^a \kappa \rho u^2 = \frac{\partial(r^a \tau_{xy})}{\partial x} - r^a(1 + \kappa y) \frac{\partial p}{\partial y} +$$

$$+ \frac{\partial}{\partial y} [r^a(1 + \kappa y)\tau_{yy}] - r^a \kappa \tau_{xx} - (1 + \kappa y)\tau_{\varphi\varphi} \frac{\partial r^a}{\partial y}$$

by  $v''$  and taking the average we get

$$r^a(1 + \kappa y) \frac{\partial}{\partial t} \left( \bar{\rho} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} \right) + \frac{\partial}{\partial x} \left[ r^a \left( \bar{\rho} \tilde{u} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} + \overline{u'' \frac{\rho v'' v''}{2}} \right) \right] + r^a \overline{\rho u'' v''} \frac{\partial \tilde{v}}{\partial x} +$$

$$+ \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \left( \bar{\rho} \tilde{v} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} + \overline{v'' \frac{\rho v'' v''}{2}} \right) \right] + r^a(1 + \kappa y) \overline{\rho v'' v''} \frac{\partial \tilde{v}}{\partial y} -$$

$$- 2r^a \kappa \tilde{u} \overline{\rho u'' v''} - r^a \kappa \overline{u'' \rho u'' v''} = v'' \frac{\partial(r^a \tau_{xy})}{\partial x} - r^a(1 + \kappa y) v'' \frac{\partial \bar{p}}{\partial y} -$$

$$- r^a(1 + \kappa y) v'' \frac{\partial \bar{p}'}{\partial y} + v'' \frac{\partial}{\partial y} [r^a(1 + \kappa y)\tau_{yy}] - r^a \overline{\kappa v'' \tau_{xx}} - (1 + \kappa y) \overline{v'' \tau_{\varphi\varphi}} \frac{\partial r^a}{\partial y}$$

or rearranging the terms

$$\frac{\partial}{\partial t} \left( \bar{\rho} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} \right) + \frac{1}{r^a(1 + \kappa y)} \left\{ \frac{\partial}{\partial x} \left( r^a \bar{\rho} \tilde{u} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} \right) + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \bar{\rho} \tilde{v} \frac{\overline{\rho v'' v''}}{2\bar{\rho}} \right] \right\} -$$

$$- \frac{2\kappa}{1 + \kappa y} \tilde{u} \overline{\rho u'' v''} = - \frac{\overline{\rho u'' v''}}{1 + \kappa y} \frac{\partial \tilde{v}}{\partial x} - \overline{\rho v'' v''} \frac{\partial \tilde{v}}{\partial y} + \frac{\kappa}{1 + \kappa y} \overline{u'' \rho u'' v''} - \frac{\tau_{xy}}{1 + \kappa y} \frac{\partial v''}{\partial x} -$$

$$- \tau_{yy} \frac{\partial v''}{\partial y} + \frac{1}{r^a(1 + \kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \frac{\tau_{xy} v''}{2} - \overline{u'' \frac{\rho v'' v''}{2}} \right) \right] + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y) \right. \right.$$

$$\left. \left. x \left( \frac{\tau_{yy} v''}{2} - \overline{v'' \frac{\rho v'' v''}{2}} - \overline{p' v''} \right) \right] \right\} + \frac{\kappa}{1 + \kappa y} \overline{v'' (p' - \tau_{xx})} + \frac{a}{r} \overline{v'' (p' - \tau_{\varphi\varphi})} \cos \alpha -$$

$$- \overline{v''} \frac{\partial \bar{p}}{\partial y} + \overline{p'} \frac{\partial v''}{\partial y} \quad (A33)$$

Combining Eqs. (A32) and (A33) we obtain for the mass-averaged turbulent kinetic energy, Eq. (26), the following equation:

$$\begin{aligned}
 & \frac{\partial(\bar{\rho}e)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} e) + \frac{\partial}{\partial y} [r^a(1+\kappa y)\bar{\rho} \tilde{v} e] \right\} = \\
 & \left[ - \frac{\overline{\rho u'' u''}}{1+\kappa y} \left( \frac{\partial \tilde{u}}{\partial x} + \kappa \tilde{v} \right) - \overline{\rho u'' v''} \left( \frac{\partial \tilde{u}}{\partial y} - \frac{\kappa \tilde{u}}{1+\kappa y} + \frac{1}{1+\kappa y} \frac{\partial \tilde{v}}{\partial x} \right) - \overline{\rho v'' v''} \frac{\partial \tilde{v}}{\partial y} \right] \\
 & \qquad \qquad \qquad \text{Production} \\
 & - \frac{\overline{\tau_{xx} u''}}{1+\kappa y} - \overline{\tau_{xy} u''} - \frac{\overline{\tau_{xy} v''}}{1+\kappa y} - \overline{\tau_{yy} v''} + \\
 & \qquad \qquad \qquad \text{Dissipation} \\
 & \left[ + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ r^a \left( \overline{u'' \tau_{xx} + \tau_{xy} v''} - u'' \frac{\overline{\rho u'' u'' + \rho v'' v''}}{2} - \right. \right. \right. \right. \\
 & \left. \left. \left. - \overline{p' u''} \right) \right] + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y) \left( \overline{u'' \tau_{xy} + v'' \tau_{yy}} - v'' \frac{\overline{\rho u'' u'' + \rho v'' v''}}{2} - \overline{p' v''} \right) \right] \right\} + \\
 & \qquad \qquad \qquad + \frac{\kappa}{1+\kappa y} \overline{v'' p'} + \frac{a}{r} \overline{p' (u'' \sin \alpha + v'' \cos \alpha)} + \\
 & \qquad \qquad \qquad \text{Laminar, Turbulent and Pressure Diffusion} \\
 & \left[ + \frac{\kappa}{1+\kappa y} \overline{(u'' \tau_{xy} - v'' \tau_{xx})} - \frac{a}{r} \overline{\tau_{\varphi\varphi} (u'' \sin \alpha + v'' \cos \alpha)} \right] \\
 & \qquad \qquad \qquad \text{Turbulent Work} \\
 & \left[ - \frac{\overline{u''}}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} - \overline{v''} \frac{\partial \bar{p}}{\partial y} + \right. \\
 & \qquad \qquad \qquad \text{Compressibility} \\
 & \left. + \frac{\overline{p'}}{1+\kappa y} \frac{\partial u''}{\partial x} + \overline{p'} \frac{\partial v''}{\partial y} \right] \\
 & \qquad \qquad \qquad \text{Turbulent Pressure Work}
 \end{aligned} \tag{A34}$$

Each group of terms expressing the physical quantity present is "modelled" in the following way (Ref. 5):

### Production

Equation (28) suggests a modelling of this quantity in the form

$$\alpha^* \bar{\rho} \tilde{S} e \quad (\text{A35})$$

where

$$\tilde{S} = \sqrt{2(e_{xx}e_{xx} + e_{yy}e_{yy} + 2e_{xy}e_{yx})} \quad (\text{A36})$$

and  $\alpha^*$  is a closure coefficient. The expressions for  $e_{xx}$ ,  $e_{yy}$ ,  $e_{xy}$  are given in Eq. (A16).

### Dissipation

The rate of dissipation may be taken as  $\epsilon/\bar{\rho}$  [1/sec], hence the dissipation of the quantity  $\bar{\rho}e$  may be modelled as

$$- \beta^* \bar{\rho} e \frac{\epsilon}{\bar{\rho}} = - \beta^* e \epsilon \quad (\text{A37})$$

where  $\beta^*$  is a proportionality (closure) coefficient.

### Laminar and Turbulent Diffusion

The form of this group of terms suggests the following modelling:

$$\frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma^* \bar{\rho} v_t)}{1+\kappa y} \frac{\partial e}{\partial x} \right] + \frac{\partial}{\partial y} \left[ r^a(1 + \kappa y)(\mu + \sigma^* \bar{\rho} v_t) \frac{\partial e}{\partial y} \right] \right\} \quad (\text{A38})$$

with  $\sigma^*$  as a closure coefficient.

### Compressibility

Using the relationships

$$u'' = - \frac{\rho' u'}{\bar{\rho}}, \quad v'' = - \frac{\rho' v'}{\bar{\rho}}$$

and Eq. (A24), we can write this group of terms as

$$- \frac{u''}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} - v'' \frac{\partial \bar{p}}{\partial y} = \frac{\rho'}{\bar{\rho}} \frac{u'}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} + \frac{\rho' v'}{\bar{\rho}} \frac{\partial \bar{p}}{\partial y} \approx \frac{e}{a} \left( \frac{\tilde{u}}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} + \tilde{v} \frac{\partial \bar{p}}{\partial y} \right)$$

and

$$- \frac{u''}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} - v'' \frac{\partial \bar{p}}{\partial y} \approx e \left( \frac{\tilde{u}}{1+\kappa y} \frac{\partial \bar{p}}{\partial x} + \tilde{v} \frac{\partial \bar{p}}{\partial y} \right)$$

From the continuity equation  $\text{div}(\bar{\rho} \vec{V}) = 0$  we have

$$\bar{\rho} \text{div} \vec{V} + \frac{\tilde{u}}{1+\kappa y} \frac{\partial \bar{\rho}}{\partial x} + \tilde{v} \frac{\partial \bar{\rho}}{\partial y} = 0$$

Hence the compressibility term is modelled as:

$$-\frac{\overline{u''}}{1+y} \frac{\partial \bar{\rho}}{\partial x} - \overline{v''} \frac{\partial \bar{\rho}}{\partial y} = -\xi^* \epsilon \bar{\rho} \text{div} \vec{V} \quad (\text{A39})$$

where  $\xi^*$  is a closure coefficient.

The resulting model equation for the mass-averaged turbulent kinetic energy  $e$  is then

$$\begin{aligned} \frac{\partial(\bar{\rho} e)}{\partial t} + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} (r^a \bar{\rho} \tilde{u} e) + \frac{\partial}{\partial y} [r^a(1+\kappa y) \bar{\rho} \tilde{v} e] \right\} = \\ (\alpha^* \bar{\rho} \tilde{S} - \beta^* \epsilon) e + \frac{1}{r^a(1+\kappa y)} \left\{ \frac{\partial}{\partial x} \left[ \frac{r^a(\mu + \sigma^* \bar{\rho} v_t)}{1+\kappa y} \frac{\partial e}{\partial x} \right] + \right. \\ \left. + \frac{\partial}{\partial y} \left[ r^a(1+\kappa y)(\mu + \sigma^* \bar{\rho} v_t) \frac{\partial e}{\partial y} \right] \right\} - \xi^* \epsilon \bar{\rho} \text{div} \vec{V} \quad (\text{A40}) \end{aligned}$$

The values of closure coefficients (see Ref. 5) are:

$$\alpha^* = \frac{3}{10} \left[ 1 - \frac{10}{11} \exp(-2\text{Re}_t) \right], \quad \beta^* = \frac{9}{100}, \quad \sigma^* = \frac{1}{2}, \quad \xi^* = \frac{5}{2} \quad (\text{A41})$$

The turbulent Reynolds number  $\text{Re}_t$  is given in Eq. (A29). The turbulent work terms in Eq. (A34) are neglected.

Equations (A40) and (A28) represent a two-equation turbulence model for compressible plane or axisymmetric Navier-Stokes flows in the coordinate system considered.

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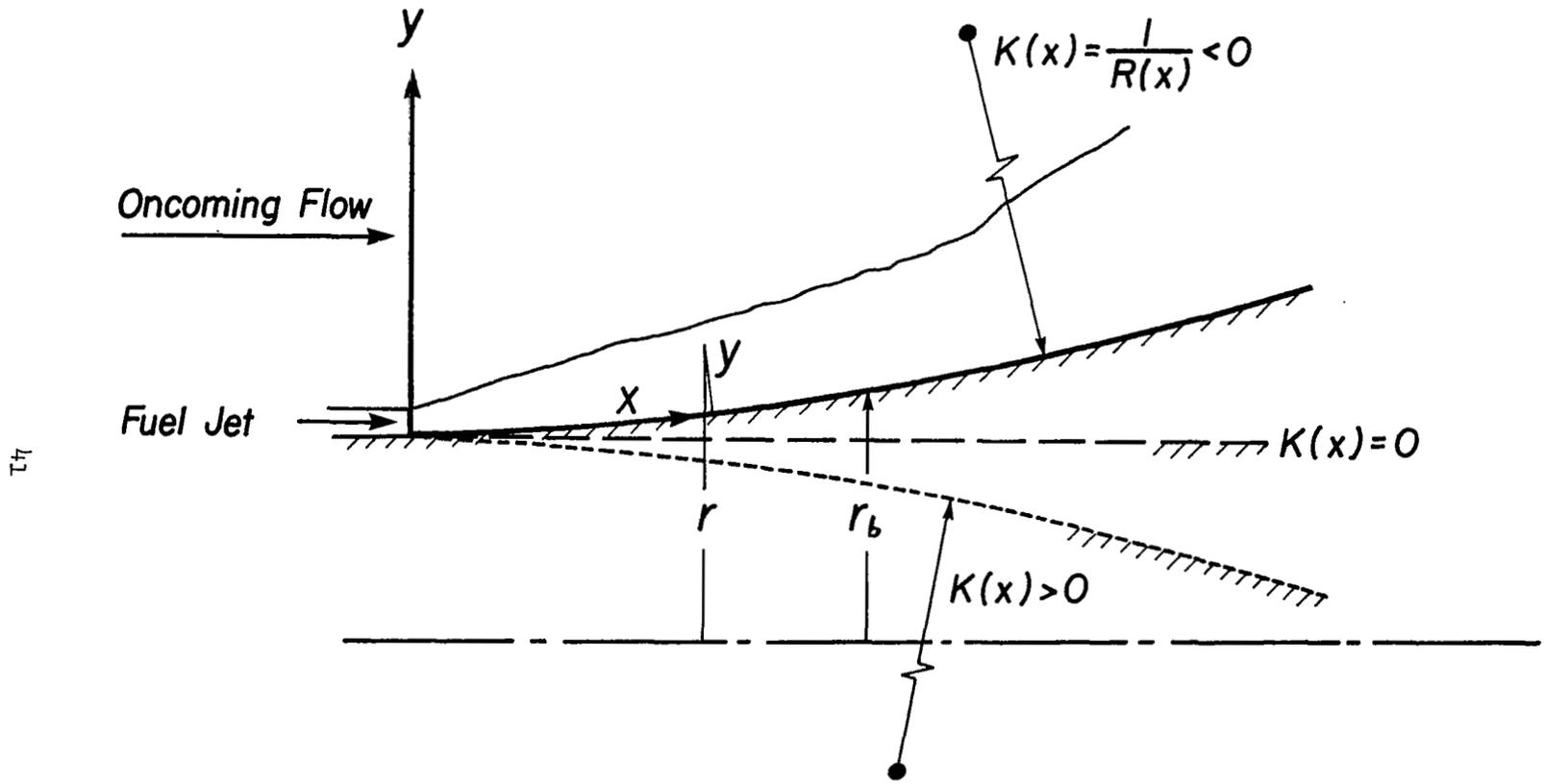


FIG. 1 FLOW CONFIGURATION AND COORDINATE SYSTEM

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16. Abstract This report presents the full Navier-Stokes time-dependent, compressible, turbulent mean-flow equations in mass-averaged variables for plane or axisymmetric flow. The equations are derived in a body-oriented, orthogonal, curvilinear coordinate system. Turbulence is modelled by a system of two equations for mass-averaged turbulent kinetic energy and dissipation rate proposed by Wilcox, et.al. These equations are rederived and some new features are discussed. A system of second-order boundary layer equations is then derived which includes the effects of longitudinal curvature and the normal pressure gradient. The Wilcox and Chambers approach is used in considering effects of streamline curvature on turbulence phenomena in turbulent boundary layer type flows. Their two-equation turbulence model with curvature terms are rederived for the cases considered in the present report. The derived system equations serves as a basis for an investigation of problems where streamline curvature is of the order of the characteristic length in the longitudinal direction.					
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