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RESEARCH STUDY ON STABILIZATION AND CONTROL

MODERN SAMPLED-DATA CONTROL THEORY

SYSTEMS RESEARCH LABORATORY

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PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA
BIMONTHLY REPORT

DIGITAL CONTROLLER DESIGN

subtitle:

ANALYSIS OF THE ANNULAR SUSPENSION POINTING SYSTEM

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V. DESIGN OF THE ANALOG ASPS THROUGH DECOUPLING AND POLE PLACEMENT

5.1 Introduction

In Chapter II the analog controllers of the ASPS are designed for the control of the x, $\phi_1$, and $\phi_2$ dynamics. The eight eigenvalues of the system are assigned so that the overall system has an equivalent bandwidth of 2 Hz. We shall see later that this is not an accurate description of the bandwidth requirements of the system.

The analog controllers consist of eight state-feedback gains with no coupling between x, $\phi_1$ and $\phi_2$. In other words, the input $u_1$ of the x component is affected only by feedbacks from x and $\dot{x}$, the input $u_2$ of the $\phi_1$ component is affected only by feedbacks from $\dot{\phi}_1$, $\phi_1$ and $\dot{\phi}_1$. Similarly, the input $u_3$ of the $\phi_2$ component is realized by feedback from the states $\dot{\phi}_2$, $\phi_2$ and $\dot{\phi}_2$. Therefore, the three independent controllers are essentially PID controllers.

The values of the eight feedback gains are determined by using the Brown's method for pole-placement. The sampled-data version of the x, $\phi_1$ and $\phi_2$ dynamics of the ASPS was obtained by inserting sample-and-hold devices in the three input channels. It was shown in Chapter III that using the feedback gains designed for the analog system, the sampled-data system is stable for sampling periods less than or equal to 0.0075 seconds. However, this sampling period is still considered to be too small to be economical and practical for the ASPS.

5.2 Design of the Analog ASPS Through Decoupling And Pole Placement

As it turns out the desired bandwidths requirement of the ASPS is as follows:

- $x_1$ dynamics $\quad 0.04$ Hz
- $\phi_1$ dynamics $\quad 10$ Hz
\[ \phi_2 \text{ dynamics} \quad 1 \text{ Hz} \]

Since the \( \phi_1 \) dynamics is 250 times faster than the \( x \) dynamics, it is virtually impossible to find an equivalent bandwidth of the overall system and establish a general eigenvalue requirement for the system. Therefore, it is necessary to decouple the \( x \), \( \phi_1 \) and \( \phi_2 \) dynamics through state feedback and simultaneously place the poles to realize the desired bandwidths for all three system components. The problem is stated as:

Given the system

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

(5-1)

where

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\ddots \\
\dot{x}_8
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
x_2 \\
\ddots \\
x_8
\end{bmatrix}
\]

(5-2)

\[ u(t) = 
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{bmatrix}
\]

(5-3)

A is the 8x8 coefficient matrix and B is the 8x3 input matrix. The state feedback is defined by

\[ u(t) = -Gx(t) \]

(5-4)

where G is the 3x8 feedback matrix. The elements of G are to be selected so that the coefficient matrix of the closed-loop system, \( A - BG \), has the following form:
\[ A - BG = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ - & - & - \\ 0 & \Lambda_2 & 0 \\ - & - & - \\ 0 & 0 & \Lambda_3 \end{bmatrix} \] (5-5)

where
- \( \Lambda_1 \) is a 2x2 matrix for \( x \)
- \( \Lambda_2 \) is a 3x3 matrix for \( \phi_1 \)
- \( \Lambda_3 \) is a 3x3 matrix for \( \phi_2 \).

The eigenvalues of \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) are so selected that the bandwidths requirements are satisfied.

Referring to Fig. 2-1 the A matrix of the ASPS has the following form:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{21} & 0 & a_{24} & 0 & 0 & a_{27} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
a_{51} & 0 & 0 & a_{54} & 0 & 0 & a_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a_{81} & 0 & 0 & a_{84} & 0 & 0 & a_{87} & 0
\end{bmatrix} \] (5-6)

The B matrix is written as

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \] (5-7)
where the $a_{ij}$'s and $b_{ij}$'s in the $A$ and $B$ matrices are given as

\[
\begin{align*}
a_{21} &= -0.61207568 \\
a_{24} &= -1.4844272 \times 10^{-3} \\
a_{27} &= 1.4844272 \times 10^{-3} = -a_{24} \\
a_{51} &= -0.31202659 \\
a_{54} &= -7.589096 \times 10^{-4} \\
a_{57} &= 7.589096 \times 10^{-4} = -a_{54} \\
a_{81} &= 0.31202659 \\
a_{84} &= 7.589096 \times 10^{-4} \\
a_{87} &= -7.6884996 \times 10^{-4} \\
b_{21} &= 0.58237458 \\
b_{22} &= 0.29688544 \\
b_{23} &= -0.29688544 \\
b_{51} &= 0.29688544 \\
b_{52} &= 0.15178192 \\
b_{53} &= -0.15178192 = -b_{52} \\
b_{81} &= -0.29688544 \\
b_{82} &= -0.15178192 \\
b_{82} &= 0.15376999
\end{align*}
\]

It is simple to show that $B$ has full rank (rank = 3).

Let the feedback matrix be represented by

\[
G = \begin{bmatrix}
g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\
g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\
g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38}
\end{bmatrix}
\]

Then $BG$ becomes

\[
(5-8)
\]
In order for $A - BG$ to be a diagonal matrix, the following relations must hold:

\[
\begin{align*}
(BG)_{23} &= a_{23} = 0 \\
(BG)_{24} &= a_{24} \\
(BG)_{25} &= a_{25} = 0 \\
(BG)_{26} &= a_{26} = 0 \\
(BG)_{27} &= a_{27} \\
(BG)_{28} &= a_{28} = 0 \\
(BG)_{51} &= a_{51} \\
(BG)_{52} &= a_{52} = 0 \\
(BG)_{56} &= a_{56} = 0 \\
(BG)_{57} &= a_{57} \\
(BG)_{58} &= a_{58} = 0 \\
(BG)_{81} &= a_{81} \\
(BG)_{82} &= a_{82} = 0 \\
(BG)_{83} &= a_{83} = 0 \\
(BG)_{84} &= a_{84}
\end{align*}
\]

where

\[
(BG)_{ij} = \sum_{k=1}^{3} b_{ik} g_{kj}, \quad i=1,2,\ldots,8, \quad j=1,2,\ldots,8
\]
These equations represent 16 equations with 24 variables in the feedback gains $g_{ij}$ ($i=1,2,3; j=1,2,...,8$).

For the present design, the $x$ dynamics is specified to have a bandwidth of 0.04 Hz. Since the $x$ dynamics are represented by a second-order system, the bandwidth of the system is given by

$$BW = \omega_n \left( 1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right)^{1/2}$$

If we choose the damping ratio $\zeta$ to be 0.707, then

$$BW = \omega_n$$

where $\omega_n$ is the natural undamped frequency.

Therefore, for the $x$ dynamics, the eigenvalues are selected to be at

$$p_1, p_2 = a_1 \pm ja_1$$

where

$$a_1 = -\frac{0.04 \times 2\pi}{\sqrt{2}} = -0.178$$

The $\phi_1$ and $\phi_2$ dynamics are of the third order. However, we can use second-order approximations by placing the real root far to the left on the real axis in the $s$-plane. For this case we let the real root of the $\phi_1$ dynamics be placed at

$$p_3 = b_2 = -200$$

Then, the complex roots are:

$$p_4, p_5 = a_2 \pm ja_2$$

where

$$a_2 = -\frac{2\pi \times 10}{\sqrt{2}} = -44.436$$

Similarly, for the $\phi_2$ dynamics, we let

$$p_6 = b_3 = -20$$
Then,
\[ p_7, p_8 = a_3 \pm ja_3 \]  
(5-19)

where
\[ a_3 = - \frac{2\pi \times 1}{\sqrt{2}} = -4.4436 \]  
(5-20)

The characteristic equations of the three decoupled subsystems are as follows:

**x** component:
\[ (s - a_1 - ja_1)(s + a_1 + ja_1) = s^2 - 2a_1s + 2a_1^2 = 0 \]  
(5-21)

**y_1** component:
\[ (s - b_2)(s^2 - a_1^2 - ja_1^2)(s - a_2 + ja_2) = (s - b_2)(s^2 - 2a_2s + 2a_2^2) \]
\[ = s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2b_2)s - 2a_2b_2 = 0 \]  
(5-22)

**y_2** component:
\[ (s - b_3)(s - a_3 - ja_3)(s - a_3 + ja_3) = (s - b_3)(s^2 - 2a_3s + 2a_3^2) \]
\[ = s^3 - (2a_3 + b_3)s^2 + (2a_3^2 + 2a_3b_3)s - 2a_3b_3 = 0 \]  
(5-23)

If there exists a set of feedback gains such that the decoupling of the subsystems \( x \), \( \phi_1 \) and \( \phi_2 \) can be accomplished, then the decoupled closed-loop system will have the independent characteristic equations of Eqs. (5-21), (5-22) and (5-23).

Let the matrix \( A \) of Eq. (5-6) be written as

\[ A = \begin{bmatrix} A_{11} & X & X \\ X & A_{22} & X \\ X & X & A_{33} \end{bmatrix} \]  
(5-24)

where \( A_{11} \) denotes a 2x2 matrix, \( A_{22} \) and \( A_{33} \) are 3x3 matrices. The submatrices denoted by \( X \) contain elements which are unimportant for the immediate development.

In view of Eqs. (5-5) and (5-9), the decoupled \( x \) dynamics with state feedback can be expressed as
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \Lambda_1 
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

(5-25)

where

\[
\Lambda_1 = A_{11} - \begin{bmatrix}
0 & 0 \\
(BG)_{21} & (BG)_{22}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
(a_{21} - (BG)_{21} & -(BG)_{22}
\end{bmatrix}
\]

(5-26)

The characteristic equation of \( \Lambda_1 \) is

\[
|sI - \Lambda_1| = \begin{vmatrix}
s & -1 \\
a_{21} - (BG)_{21} & s + (BG)_{22}
\end{vmatrix}
= s^2 + (BG)_{22}s - a_{21} + (BG)_{21} = 0
\]

(5-27)

Similarly, the decoupled \( \phi_1 \) dynamics with state feedback is described by

\[
\begin{bmatrix}
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{bmatrix}
= \Lambda_2 
\begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}
\]

(5-28)

where

\[
\Lambda_2 = A_{22} - \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
(BG)_{53} & (BG)_{54} & (BG)_{55}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-(BG)_{53} & a_{54} - (BG)_{54} & -(BG)_{55}
\end{bmatrix}
\]

(5-29)

The characteristic equation of \( \Lambda_2 \) is

\[
|sI - \Lambda_2| = \begin{vmatrix}
s & -1 & 0 \\
0 & s & -1 \\
(BG)_{53} & -(a_{54} + (BG)_{54} & s + (BG)_{55}
\end{vmatrix}
\]
For the $\phi_2$ component, the decoupled closed-loop state equations are

$$\begin{bmatrix} \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \Lambda_3 \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where

$$\Lambda_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (BG)_{86} & (BG)_{87} & (BG)_{88} \end{bmatrix}$$

and

$$\Lambda_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(BG)_{86} & a_{87} - (BG)_{87} & -(BG)_{88} \end{bmatrix}$$

The characteristic equation of $\Lambda_3$ is

$$|sI - \Lambda_3| = s - 1 \quad 0$$

$$0 \quad s - 1$$

$$(-a_{87} + (BG)_{87} \quad s + (BG)_{88})$$

$$= s^3 + (BG)_{88}s^2 + ((BG)_{87} - a_{87})s + (BG)_{86} = 0$$

For pole placement, in order to meet the bandwidth requirements, the corresponding coefficients of the characteristic equations in Eqs. (5-27), (5-30) and (5-33) must match those of Eqs. (5-21), (5-22) and (5-23), respectively.

Thus, for pole-placement,

$$s^2 - 2a_1s + 2a_1^2 = s^2 + (BG)_{22}s + (BG)_{21} - a_{21}$$

$$s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2b_2)s - 2a_2^2b_2 = s^3 + (BG)_{55}s^2 + ((BG)_{54} - a_{54})s + (BG)_{53}$$
Equating the like coefficients in the above three equations, we have

\begin{align}
(BG)_{22} &= b_{21}g_{12} + b_{22}g_{22} + b_{23}g_{32} = -2a_1 \\
(BG)_{21} &= b_{21}g_{11} + b_{22}g_{21} + b_{23}g_{31} = 2a_1 + a_{21} \\
(BG)_{55} &= b_{51}g_{15} + b_{52}g_{25} + b_{53}g_{35} = -(2a_2 + b_2) \\
(BG)_{54} &= b_{51}g_{14} + b_{52}g_{24} + b_{53}g_{34} = 2a_2 + 2a_2b_2 + a_{54} \\
(BG)_{53} &= b_{51}g_{13} + b_{52}g_{23} + b_{53}g_{33} = -2a_2b_2 \\
(BG)_{88} &= b_{81}g_{18} + b_{82}g_{28} + b_{83}g_{38} = -(2a_3 + b_3) \\
(BG)_{87} &= b_{81}g_{17} + b_{82}g_{27} + b_{83}g_{37} = 2a_3 + 2a_3b_3 + a_{87} \\
(BG)_{86} &= b_{81}g_{16} + b_{82}g_{26} + b_{83}g_{36} = -2a_3b_3
\end{align}

(5-37)

Notice that the 16 constraint equations on the feedback gains in Eq. (5-10) are conditions on decoupling of the three subsystems, whereas the 8 constraint equations in Eq. (5-37) are the conditions for pole placement. The 24 constraint equations contain 24 unknowns in the elements of the feedback matrix $G$, $g_{ij}$, $i=1, 2, 3, j=1, 2, \ldots, 8$.

The 24 constraint equations in Eqs. (5-10) and (5-37) can be written as

\[ WG = C \]

(5-38)

where

\[ W = \begin{bmatrix}
  b_{21} & b_{22} & b_{23} \\
  b_{51} & b_{52} & b_{53} \\
  b_{81} & b_{82} & b_{83}
\end{bmatrix} \]

$G$ is the feedback matrix given in Eq. (5-8), and $C$ is a 3x8 matrix whose elements are composed of $a_1, a_2, a_3, b_1, b_2, b_3$, and $a_{ij}$, $i=1, 2, \ldots, 8, j=1, 2, \ldots, 8$.

The feedback gain matrix $G$ can be solved form Eq. (5-38) if $W$ is nonsingular.
Then, \[ G = W^{-1}C \] (5-39)

For \( W \) to be nonsingular, it is necessary and sufficient for \( B \) to have full rank.

For the system parameters given for the ASPS, the feedback matrix is solved from Eq. (5-39), and the results are given as follows:

\begin{align*}
  g_{11} &= 36.449 \\
  g_{12} &= 212.4 \\
  g_{13} &= -9.256981248 \times 10^8 \\
  g_{14} &= -2.547162624 \times 10^7 \\
  g_{15} &= -3.3892568 \times 10^5 \\
  g_{16} &= 0 \\
  g_{17} &= 0 \\
  g_{18} &= 0 \\
  g_{21} &= -73.35 \\
  g_{22} &= -415.4544 \\
  g_{23} &= 2.212612552 \times 10^9 \\
  g_{24} &= 6.088252671 \times 10^7 \\
  g_{25} &= 8.1017273 \times 10^5 \\
  g_{26} &= 3.96750304 \times 10^5 \\
  g_{27} &= 1.091703152 \times 10^5 \\
  g_{28} &= 1.452664 \times 10^4 \\
  g_{31} &= 0 \\
  g_{32} &= 0 \\
  g_{33} &= 3.96750304 \times 10^8 \\
  g_{34} &= 1.091703152 \times 10^7 \\
  g_{35} &= 1.452664 \times 10^5 \\
  g_{36} &= 3.96750304 \times 10^5
\end{align*}
\[ g_{37} = 1.091703102 \times 10^5 \]
\[ g_{38} = 1.452664 \times 10^4 \]

Substituting \( G \) into Eq. (5-5), the elements of the closed-loop \( A - BG \) matrix are obtained as follows:

\[
\Lambda_1 = \begin{bmatrix}
0 & 1 \\
-6.25 \times 10^{-2} & -3.5 \times 10^{-1}
\end{bmatrix}
\]

\[
\Lambda_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-7.88768 \times 10^5 & -2.170384 \times 10^4 & -2.888 \times 10^2
\end{bmatrix}
\]

\[
\Lambda_3 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Thus, the three components of the \( x, \phi_1 \) and \( \phi_2 \) dynamics are completely decoupled, and they have bandwidths of 0.04, 10 and 1 Hz, respectively.
VI DESIGN OF THE DIGITAL ASPS THROUGH DECOUPLING
AND POLE PLACEMENT

In this chapter the digital ASPS is to be designed through decoupling and pole placement. In Chapter IV the digital ASPS was realized by placing sample-and-hold units at the three inputs to the $x, \phi_1$, and $\phi_2$ components. The feedback gains were the same as those of the analog ASPS.

The dynamics of the system are still described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $A$, $B$, $x(t)$ are as defined in Chapter V. The control vector $u(t)$ is given by

$$u(t) = -Gx(kT) \quad kT < t < (k+1)T$$

The solution of Eq. (6-1) over the interval $kT < t < (k+1)T$ is

$$x(t) = \phi(t-kT)x(kT) + \int_{kT}^{t} \phi(t-\tau)Bu(\tau)d\tau$$

where

$$\phi(t) = e^{At}$$

Substituting $t = (k+1)T$ into Eq. (6-3), we get

$$x((k+1)T) = (\phi(T) - \int_{0}^{T} \phi(T-\tau)d\tau B)Gx(kT)$$

Let

$$\theta(T) = \int_{0}^{T} \phi(T-\tau)d\tau B$$

Then Eq. (6-5) is simplified to

$$x((k+1)T) = (\phi(T) - \theta(T))Gx(kT)$$

Equation (6-7) now represents the difference state equations of the closed-loop state-feedback ASPS.
The design objectives are to select the elements of the 3x8 feedback matrix \( G \) so that the coefficient matrix of the closed-loop digital system, \( \phi(T) - \theta(T)G \), has the following form:

\[
\begin{bmatrix}
A_1 & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & A_3
\end{bmatrix}
\]

(6-8)

where \( A_1 \) is 2x2, \( A_2 \) and \( A_3 \) are 3x3 matrices, and the eigenvalues of \( A_1 \), \( A_2 \) and \( A_3 \) are to be placed so that the bandwidth requirements described earlier are satisfied.

Before the design can be carried out, one difficulty remains. It is necessary to first calculate the matrices \( \phi(T) \) and \( \theta(T) \), knowing \( A \), \( B \) and the sampling period \( T \). Since \( A \) is 8x8 in the present case, the Laplace transform method becomes a tedious task. The method chosen here is that of approximating \( \phi(T) \) by a truncated power series; i.e.,

\[
\phi(T) = \sum_{n=0}^{N} A^n T^n / n! 
\]

(6-9)

where \( N \) is some positive integer.

The matrix \( \theta(T) \) is written

\[
\theta(T) = \int_{0}^{T} \phi(T - \tau) d\tau B \\
= \int_{0}^{T} \sum_{n=0}^{\infty} \frac{A^n (t - \tau)^n}{n!} d\tau B 
\]

(6-10)

If the function \( \theta(T) \) is uniformly convergent, the integral and the summation in the last equation can be interchanged. Also, truncating the infinite series at \( n=N \), we have

\[
\theta(T) = \left[ \sum_{n=0}^{N} A^n T^n / (n+1)! \right] B 
\]

(6-11)

For \( T = 0.02 \) sec, the results of \( \phi(T) \) and \( \theta(T) \) are given as follows:
For the same bandwidth requirements described in Chapter 5, the following

eigenvalues are selected in the s-plane:

\[ \phi(0.02) = \begin{bmatrix}
9.998 \times 10^{-1} & 1.999 \times 10^{-2} & 0 & -2.968 \times 10^{-7} & -1.979 \times 10^{-9} & 0 & 2.968 \times 10^{-7} & 1.979 \times 10^{-9} \\
-1.224 \times 10^{-2} & 9.998 \times 10^{-1} & 0 & -2.968 \times 10^{-5} & -2.968 \times 10^{-7} & 0 & 2.968 \times 10^{-5} & 2.968 \times 10^{-7} \\
-4.160 \times 10^{-7} & -2.080 \times 10^{-9} & 1 & 1.999 \times 10^{-2} & 1.999 \times 10^{-4} & 0 & 1.011 \times 10^{-9} & 5.059 \times 10^{-12} \\
-6.240 \times 10^{-5} & -4.160 \times 10^{-7} & 0 & 9.999 \times 10^{-1} & 1.999 \times 10^{-2} & 0 & 1.517 \times 10^{-7} & 1.011 \times 10^{-9} \\
-6.240 \times 10^{-3} & -6.240 \times 10^{-5} & 0 & -1.517 \times 10^{-5} & 1.999 \times 10^{-1} & 0 & 1.517 \times 10^{-5} & 1.517 \times 10^{-7} \\
4.160 \times 10^{-7} & 2.080 \times 10^{-9} & 0 & 1.011 \times 10^{-9} & 1.999 \times 10^{-2} & 0 & 5.059 \times 10^{-12} & 1.999 \times 10^{-4} \\
6.240 \times 10^{-5} & 4.160 \times 10^{-7} & 0 & 1.517 \times 10^{-7} & 1.011 \times 10^{-9} & 0 & 9.999 \times 10^{-1} & 1.999 \times 10^{-2} \\
6.240 \times 10^{-3} & 6.240 \times 10^{-5} & 0 & 1.517 \times 10^{-5} & 1.517 \times 10^{-7} & 0 & -1.537 \times 10^{-5} & 9.999 \times 10^{-1} 
\end{bmatrix} \]

\[ \theta(0.02) = \begin{bmatrix}
1.164 \times 10^{-4} & 5.937 \times 10^{-5} & -5.937 \times 10^{-5} \\
1.64 \times 10^{-2} & 5.937 \times 10^{-3} & -5.937 \times 10^{-3} \\
3.958 \times 10^{-7} & 2.023 \times 10^{-7} & -2.023 \times 10^{-7} \\
5.937 \times 10^{-5} & 3.035 \times 10^{-5} & -3.035 \times 10^{-5} \\
5.937 \times 10^{-3} & 3.035 \times 10^{-3} & -3.035 \times 10^{-3} \\
-3.958 \times 10^{-7} & -2.023 \times 10^{-7} & 2.050 \times 10^{-7} \\
-5.937 \times 10^{-5} & -3.035 \times 10^{-5} & 3.075 \times 10^{-5} \\
-5.937 \times 10^{-3} & -3.035 \times 10^{-3} & 3.075 \times 10^{-3} 
\end{bmatrix} \]

\[ (6-12) \]

\[ (6-13) \]

x component: (0.04 Hz bandwidth)

\[ P_1, P_2 = -0.178 \pm j0.178 \]

\[ \phi_1 \] component: (10 Hz bandwidth)

\[ P_3 = -200 \]
\[ p_4, p_5 = -44.436 \pm 144.436 \]

\[ \phi_2 \text{ component: (1 Hz bandwidth)} \]

\[ p_6 = -20 \]

\[ p_7, p_8 = -4.4436 \pm 14.4436 \]

Using the transformation \( z = e^{Ts} \), these eigenvalues are transformed into the z-plane. The corresponding z-plane eigenvalues are:

\[ x: \ z_1, z_2 = 0.9964457 \pm j0.0035417 \]

\[ \phi_1: \ z_3 = 0.01831564 \]

\[ z_4, z_5 = 0.2592945 \pm j0.3191948 \]

\[ \phi_2: \ z_6 = 0.67032 \]

\[ z_7, z_8 = 0.911366 \pm j0.096987 \]

The condition of decoupling of the closed-loop digital system given in Eq. (6-8) results in 42 constraint equations. This is due to the fact that Eq. (6-8) represents 64 scalar equations, but there are 42 zero elements in the matrix. The pole placements of the eighth-order system would produce 8 more constraint equations, so there are only 24 unknowns in the feedback matrix \( G \). Therefore, in general, there would not be a set of solutions with more equations than unknowns. However, a closer look at the elements of \( \Phi(T) \) and \( \Theta(T) \) reveals that some of the elements are extremely small so that a good approximation can be obtained by assuming that these small elements are zeros. In other words, the state transition matrix \( \Phi(0.02) \) given in Eq. (6-12) can be approximated by the following matrix.
Similarly, we can approximate $\Theta(0.02)$ by the following matrix:

\[
\Theta(0.02) = \begin{bmatrix}
0 & 0 & 0 \\
1.164 \times 10^{-2} & 5.937 \times 10^{-3} & -5.937 \times 10^{-3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-5.937 \times 10^{-3} & -3.035 \times 10^{-3} & 3.075 \times 10^{-3}
\end{bmatrix}
\]

Then, the matrix $\Theta(0.02)G$ becomes
Based on Eqs. (6-14), (6-15) and (6-16), the following 16 equations are obtained for decoupling the $\phi - \theta G$ matrix:

\[
\begin{align*}
(\theta G)_{23} &= \theta_{21}g_{13} + \theta_{22}g_{23} + \theta_{23}g_{33} = \phi_{23} = 0 \\
(\theta G)_{24} &= \theta_{21}g_{14} + \theta_{22}g_{24} + \theta_{23}g_{34} = \phi_{24} = -2.968 \times 10^{-5} \\
(\theta G)_{25} &= \theta_{21}g_{15} + \theta_{22}g_{25} + \theta_{23}g_{35} = \phi_{25} = -2.968 \times 10^{-7} \\
(\theta G)_{26} &= \theta_{21}g_{16} + \theta_{22}g_{26} + \theta_{23}g_{36} = \phi_{26} = 0 \\
(\theta G)_{27} &= \theta_{21}g_{17} + \theta_{22}g_{27} + \theta_{23}g_{37} = \phi_{27} = 2.968 \times 10^{-5} \\
(\theta G)_{28} &= \theta_{21}g_{18} + \theta_{22}g_{28} + \theta_{23}g_{38} = \phi_{28} = 2.968 \times 10^{-7} \\
(\theta G)_{51} &= \theta_{51}g_{11} + \theta_{52}g_{21} + \theta_{53}g_{31} = \phi_{51} = -6.240 \times 10^{-5} \\
(\theta G)_{52} &= \theta_{51}g_{12} + \theta_{52}g_{22} + \theta_{53}g_{32} = \phi_{52} = -4.160 \times 10^{-7} \\
(\theta G)_{56} &= \theta_{51}g_{16} + \theta_{52}g_{26} + \theta_{53}g_{36} = \phi_{56} = 0 \\
(\theta G)_{57} &= \theta_{51}g_{17} + \theta_{52}g_{27} + \theta_{53}g_{37} = \phi_{57} = 1.517 \times 10^{-5} \\
(\theta G)_{58} &= \theta_{51}g_{18} + \theta_{52}g_{28} + \theta_{53}g_{38} = \phi_{58} = 1.517 \times 10^{-7} \\
(\theta G)_{81} &= \theta_{81}g_{11} + \theta_{82}g_{21} + \theta_{83}g_{32} = \phi_{81} = 6.240 \times 10^{-3}
\end{align*}
\]
\[
(\Theta G)_{82} = \theta_{81g_{11}} + \theta_{82g_{22}} + \theta_{83g_{32}} = \phi_{82} = 6.240 \times 10^{-5}
\]

\[
(\Theta G)_{83} = \theta_{81g_{13}} + \theta_{82g_{23}} + \theta_{83g_{33}} = \phi_{83} = 0
\]

\[
(\Theta G)_{84} = \theta_{81g_{14}} + \theta_{82g_{24}} + \theta_{83g_{34}} = \phi_{84} = 1.517 \times 10^{-5}
\]

\[
(\Theta G)_{85} = \theta_{81g_{15}} + \theta_{82g_{25}} + \theta_{83g_{35}} = \phi_{85} = 1.517 \times 10^{-7}
\]

The eight additional constraint equations for the solution of the 24 elements of the feedback gain matrix \( G \) come from the pole placement requirements on the submatrices, \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \).

After the matrix \( \phi - \Theta G \) has been decoupled, it can be written as

\[
\phi(0.02) - \Theta(0.02)G = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]

(6-18)

where

\[
\Lambda_1 = \begin{bmatrix}
\phi_{11} & \phi_{12} \\
\phi_{21} & \phi_{22}
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
(\Theta G)_{21} & (\Theta G)_{22}
\end{bmatrix}
\]

(6-19)

\[
\Lambda_2 = \begin{bmatrix}
\phi_{33} & \phi_{34} & \phi_{35} \\
\phi_{43} & \phi_{44} & \phi_{45} \\
\phi_{53} & \phi_{54} & \phi_{55}
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 \\
(\Theta G)_{53} & (\Theta G)_{54} & (\Theta G)_{55}
\end{bmatrix}
\]

(6-20)
The characteristic equation of the decoupled $x$ dynamics is

$$|zI - \Lambda_1| = \begin{vmatrix} z - \phi_{11} & -\phi_{12} \\ -\phi_{21} + (\theta G)_{21} & z - \phi_{22} + (\theta G)_{22} \end{vmatrix}$$

$$= z^2 + ((\theta G)_{22} - \phi_{11} - \phi_{22})z + \phi_{11}\phi_{22} - \phi_{11}(\theta G)_{22} + \phi_{12}(\theta G)_{21} - \phi_{21})$$

$$= (z + a_1 + jb_1)(z + a_1 - jb_1)$$

where $a_1 = -0.9964457$ and $b_1 = -0.0035417$.

Now solving for the unknowns in Eq. (6-22), we have

$$(\theta G)_{21} = \theta_{21}\theta_{11} + \theta_{22}\theta_{21} + \theta_{23}\theta_{31}$$

$$= \frac{1}{\phi_{12}}(a_1^2 + b_1^2 + \phi_{11}^2 + 2a_1\phi_{11} + \phi_{12}\phi_{21})$$

$$(\theta G)_{22} = \theta_{21}\theta_{12} + \theta_{22}\theta_{22} + \theta_{23}\theta_{32} = \phi_{11} + \phi_{12} + 2a_1$$

For the $\phi_1$ dynamics, the characteristic equation of $\Lambda_2$ is

$$|zI - \Lambda_2| = \begin{vmatrix} z - \phi_{33} & -\phi_{34} & -\phi_{35} \\ -\phi_{43} & z - \phi_{44} & -\phi_{45} \\ -\phi_{53} + (\theta G)_{53} & -\phi_{54} + (\theta G)_{54} & z - \phi_{55} + (\theta G)_{55} \end{vmatrix}$$

$$= z^3 + ((\theta G)_{55} - \phi_{55} - \phi_{44} - \phi_{33})z^2 + ((\phi_{44} + \phi_{33})(\theta G)_{55} + \phi_{45}(\theta G)_{54}$$
The desired characteristic equation is written as

\[(z + a_2 + jb_2)(z + a_2 - jb_2)(z + c_2) = 0 \tag{6-26}\]

where
\[c_2 = -0.01831564\]
\[a_2 = -0.2592945\]
\[b_2 = -0.3191948\]

Equating Eqs. (6-25) and (6-26), we get

\[
(\Theta G)_{55} = \Theta_{51}g_{15} + \Theta_{52}g_{25} + \Theta_{53}g_{35} = \phi_{55} + \phi_{44} + 1 + 2a_2 + c_2 \tag{6-27}\]

\[
(\Theta G)_{54} = \Theta_{51}g_{14} + \Theta_{52}g_{24} + \Theta_{53}g_{34} = \frac{-\phi_{35}(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{45}(\phi_{35} + \phi_{34}g_{45} - \phi_{35}g_{44})} + \frac{1}{\phi_{45}}((\phi_{44} + 1)(\phi_{55} + \phi_{44} + 1 + 2a_2 + c_2) - \phi_{44}g_{45} + \phi_{45}g_{54} - \phi_{55} - \phi_{44} + 2a_2c_2 + a_2^2 + b_2^2) \tag{6-28}\]

\[
(\Theta G)_{53} = \Theta_{51}g_{13} + \Theta_{52}g_{23} + \Theta_{53}g_{33} = \frac{(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{35} + \phi_{34}g_{45} - \phi_{35}g_{44}} \tag{6-29}\]

For the \(\phi_2\) dynamics, the characteristic equation of \(\Lambda_3\) is

\[
|zI - \Lambda_3| = \begin{vmatrix}
  z - \phi_{66} & -\phi_{67} & -\phi_{68} \\
  -\phi_{76} & -\phi_{77} & -\phi_{78} \\
  -\phi_{86} + (\Theta G)_{86} & -\phi_{87} + (\Theta G)_{87} & z - \phi_{88} + (\Theta G)_{88}
\end{vmatrix}
\]
The desired characteristic equation is written as

$$(a + a_3 + jb_3)(z + a_3 - jb_3)(z + c_3) = 0$$

where

- $c_3 = -0.67032$
- $a_3 = -0.91136$
- $b_3 = -0.096987$

Equating Eqs. (6-30) and (6-31), we get

$$\begin{align*}
(\theta G)_{86} &= \theta_{81}^{g}16 + \theta_{82}^{g}26 + \theta_{83}^{g}36 \\
&= \frac{(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{68} + \phi_{67}^\theta 78 - \phi_{68}^\theta 77} \\
(\theta G)_{87} &= \theta_{81}^{g}17 + \theta_{82}^{g}27 + \theta_{83}^{g}37 \\
&= \frac{- \phi_{68}(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{78}(\phi_{68} + \phi_{67}^\theta 78 - \phi_{68}^\theta 77)} + \frac{1}{\phi_{78}}((\phi_{77} + 1)(\phi_{88} + \phi_{77} + 1 + 2a_3 + c_3) \\
(6G)_{88} &= \theta_{81}^{g}18 + \theta_{82}^{g}28 + \theta_{83}^{g}38 \\
&= \phi_{88} + \phi_{77} + 1 + 2a_3 + c_3
\end{align*}$$

Equations (6-23), (6-24), (6-27), (6-28), (6-29), (6-32), (6-33) and (6-34) form the additional eight constraint equations, together with the 16 equations in Eq. (6-17) are adequate for the solution of the 24 unknown feedback gains in G.
The solution of the 24 constraint equations gives the following results:

\[
\begin{align*}
9_{11} &= 35.43251395904985 \\
9_{12} &= 209.57461847387285 \\
9_{13} &= -9.368484250209332 \times 10^7 \\
9_{14} &= -8.11397701621271032 \times 10^6 \\
9_{15} &= -1.4453443216549823 \times 10^5 \\
9_{16} &= 9_{17} = 9_{18} = 0 \\
9_{21} &= -71.361744259888901 \\
9_{22} &= -409.94846384061196 \\
9_{23} &= 2.2392641102247482 \times 10^8 \\
9_{24} &= 1.9394105849068884 \times 10^7 \\
9_{25} &= 3.454675889387745 \times 10^5 \\
9_{26} &= 3.578272520216161 \times 10^5 \\
9_{27} &= 1.3239693135355619 \times 10^6 \\
9_{28} &= 1.2749737170875715 \times 10^4 \\
9_{31} &= 9_{32} = 0 \\
9_{33} &= 4.0152927555000438 \times 10^7 \\
9_{34} &= 3.4776162570406019 \times 10^6 \\
9_{35} &= 6.194684677849835 \times 10^4 \\
9_{36} &= 3.578272527392904 \times 10^5 \\
9_{37} &= 1.3239693094129444 \times 10^6 \\
9_{38} &= 1.2749737129324844 \times 10^4 
\end{align*}
\]