General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)

BIMONTHLY REPORT

(NASA-CR-150807) DIGITAL CONTROLLER DESIGN: ANALYSIS OF THE ANNULAR SUSPENSION POINTING SYSTEM Bimonthly Report (Systems Research Lab., Champaign, Ill.) 26 p HC A03/MF A01 CSCL 09B G3/63 N78-31825 Unclas 30291

OCT 1978
RECEIVED
NASA STI FACILITY

RESEARCH STUDY ON STABILIZATION AND CONTROL

MODERN SAMPLED-DATA CONTROL THEORY

SYSTEMS RESEARCH LABORATORY

P.O. BOX 2277, STATION A
3206 VALLEY BROOK DRIVE
CHAMPAIGN, ILLINOIS 61820

PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

BIMONTHLY REPORT

DIGITAL CONTROLLER DESIGN

subtitle:

ANALYSIS OF THE ANNULAR SUSPENSION

POINTING SYSTEM

SEPTEMBER 1, 1978

B. C. Kuo

Contract No. NAS8-32358 1-7-ED-07418(IF)

PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

SYSTEMS RESEARCH LABORATORY

P.O. BOX 2277, STATION A

CHAMPAIGN, ILLINOIS 61820

TABLE OF CONTENTS

5.1 Introduction

5.2 Design of the Analog ASPS Through Decoupling And Pole Placement

Chapter 6. Design of the Digital ASPS Through Decoupling and Pole Placement

V. DESIGN OF THE ANALOG ASPS THROUGH DECOUPLING AND POLE PLACEMENT

5.1 Introduction

In Chapter II the analog controllers of the ASPS are designed for the control of the x, ϕ_1 and ϕ_2 dynamics. The eight eigenvalues of the system are assigned so that the overall system has an equivalent bandwidth of 2 Hz. We shall see later that this is not an accurate description of the bandwidth requirements of the system.

The analog controllers consist of eight state-feedback gains with no coupling between x, ϕ_1 and ϕ_2. In other words, the input u_1 of the x component is affected only by feedbacks from x and $\dot{x}$, the input u_2 of the ϕ_1 component is affected only by feedbacks from $\int\phi_1$, ϕ_1 and $\dot{\phi}_1$. Similarly, the input u_3 of the ϕ_2 component is realized by feedback from the states $\int\phi_2$, ϕ_2 and $\dot{\phi}_2$. Therefore, the three independent controllers are essentially PID controllers.

The values of the eight feedback gains are determined by using the Brown's method for pole-placement. The sampled-data version of the x, ϕ_1 and ϕ_2 dynamics of the ASPS was obtained by inserting sample-and-hold devices in the three input channels. It was shown in Chapter III that using the feedback gains designed for the analog system, the sampled-data system is stable for sampling periods less than or equal to 0.0075 seconds. However, this sampling period is still considered to be too small to be economical and practical for the ASPS.

5.2 Design of the Analog ASPS Through Decoupling And Pole Placement

As it turns out the desired bandwidths requirement of the ASPS is as follows:

x_1 dynamics	0.04 Hz
ϕ_1 dynamics	10 Hz

ϕ_2 dynamics 1 Hz

Since the ϕ_1 dynamics is 250 times faster than the x dynamics, it is virtually impossible to find an equivalent bandwidth of the overall system and establish a general eigenvalue requirement for the system. Therefore, it is necessary to decouple the x, ϕ_1 and ϕ_2 dynamics through state feedback and simultaneously place the poles to realize the desired bandwidths for all three system components. The problem is stated as:

Given the system

ORIGINAL PAGE IS
OF POOR QUALITY

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{5-1}$$

where

$$\underline{x}(t) = \begin{bmatrix} x \\ \dot{x} \\ \int\phi_1 \\ \phi_1 \\ \dot{\phi}_1 \\ \int\phi_2 \\ \phi_2 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \tag{5-2}$$

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \tag{5-3}$$

A is the 8x8 coefficient matrix and B is the 8x3 input matrix. The state feedback is defined by

$$\underline{u}(t) = -G\underline{x}(t) \tag{5-4}$$

where G is the 3x8 feedback matrix. The elements of G are to be selected so that the coefficient matrix of the closed-loop system, A - BG, has the following form:

ORIGINAL PAGE IS
OF POOR QUALITY

$$A - BG = \left[\begin{array}{c|c|c} \Lambda_1 & 0 & 0 \\ \hline 0 & \Lambda_2 & 0 \\ \hline 0 & 0 & \Lambda_3 \end{array}\right] \tag{5-5}$$

where

Λ_1 is a 2x2 matrix for x

Λ_2 is a 3x3 matrix for ϕ_1

Λ_3 is a 3x3 matrix for ϕ_2.

The eigenvalues of Λ_1, Λ_2 and Λ_3 are so selected that the bandwidths requirements are satisfied.

Referring to Fig. 2-1 the A matrix of the ASPS has the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} & 0 & 0 & a_{27} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_{51} & 0 & 0 & a_{54} & 0 & 0 & a_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_{81} & 0 & 0 & a_{84} & 0 & 0 & a_{87} & 0 \end{bmatrix} \tag{5-6}$$

The B matrix is written as

$$B = \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{81} & b_{82} & b_{83} \end{bmatrix} \tag{5-7}$$

where the a_{ij}'s and b_{ij}'s in the A and B matrices are given as

$$a_{21} = -0.61207568$$
$$a_{24} = -1.4844272 \times 10^{-3}$$
$$a_{27} = 1.4844272 \times 10^{-3} = -a_{24}$$
$$a_{51} = -0.31202659$$
$$a_{54} = -7.589096 \times 10^{-4}$$
$$a_{57} = 7.589096 \times 10^{-4} = -a_{54}$$
$$a_{81} = 0.31202659$$
$$a_{84} = 7.589096 \times 10^{-4}$$
$$a_{87} = -7.6884996 \times 10^{-4}$$
$$b_{21} = 0.58237458$$
$$b_{22} = 0.29688544$$
$$b_{23} = -0.29688544$$
$$b_{51} = 0.29688544$$
$$b_{52} = 0.15178192$$
$$b_{53} = -0.15178192 = -b_{52}$$
$$b_{81} = -0.29688544$$
$$b_{82} = -0.15178192$$
$$b_{82} = 0.15376999$$

It is simple to show that B has full rank (rank = 3).

Let the feedback matrix be represented by

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \end{bmatrix} \tag{5-8}$$

Then BG becomes

$$BG = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (BG)_{21} & (BG)_{22} & (BG)_{23} & (BG)_{24} & (BG)_{25} & (BG)_{26} & (BG)_{27} & (BG)_{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (BG)_{51} & (BG)_{52} & (BG)_{53} & (BG)_{54} & (BG)_{55} & (BG)_{56} & (BG)_{57} & (BG)_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (BG)_{81} & (BG)_{82} & (BG)_{83} & (BG)_{84} & (BG)_{85} & (BG)_{86} & (BG)_{87} & (BG)_{88} \end{bmatrix} \quad (5\text{-}9)$$

where

$$(BG)_{ij} = \sum_{k=1}^{3} b_{ik} g_{kj} \qquad i=1,2,\ldots,8,\ j=1,2,\ldots,8$$

In order for A - BG to be a diagonal matrix, the following relations must hold:

$$(BG)_{23} = a_{23} = 0$$
$$(BG)_{24} = a_{24}$$
$$(BG)_{25} = a_{25} = 0$$
$$(BG)_{26} = a_{26} = 0$$
$$(BG)_{27} = a_{27}$$
$$(BG)_{28} = a_{28} = 0$$
$$(BG)_{51} = a_{51}$$
$$(BG)_{52} = a_{52} = 0$$
$$(BG)_{56} = a_{56} = 0$$
$$(BG)_{57} = a_{57}$$
$$(BG)_{58} = a_{58} = 0$$
$$(BG)_{81} = a_{81}$$
$$(BG)_{82} = a_{82} = 0$$
$$(BG)_{83} = a_{83} = 0$$
$$(BG)_{84} = a_{84}$$

$$(BG)_{85} = a_{85} = 0$$

These equations represent 16 equations with 24 variables in the feedback gains g_{ij} $(i=1,2,3;\ j=1,2,\ldots,8)$.

For the present design, the x dynamics is specified to have a bandwidth of 0.04 Hz. Since the x dynamics are represented by a second-order system, the bandwidth of the system is given by

$$BW = \omega_n\left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{\frac{1}{2}} \tag{5-11}$$

If we choose the damping ratio ζ to be 0.707, then

$$BW = \omega_n \tag{5-12}$$

where ω_n is the natural undamped frequency.

Therefore, for the x dynamics, the eigenvalues are selected to be at

$$p_1,\ p_2 = a_1 \pm ja_1 \tag{5-13}$$

where

$$a_1 = -\frac{0.04 \times 2\pi}{\sqrt{2}} = -0.178 \tag{5-14}$$

The ϕ_1 and ϕ_2 dynamics are of the third order. However, we can use second-order approximations by placing the real root far to the left on the real axis in the s-plane. For this case we let the real root of the ϕ_1 dynamics be placed at

$$p_3 = b_2 = -200 \tag{5-15}$$

Then, the complex roots are:

$$p_4,\ p_5 = a_2 \pm ja_2 \tag{5-16}$$

where

$$a_2 = -\frac{2\pi \times 10}{\sqrt{2}} = -44.436 \tag{5-17}$$

Similarly, for the ϕ_2 dynamics, we let

$$p_6 = b_3 = -20 \tag{5-18}$$

Then,

$$p_7, p_8 = a_3 \pm ja_3 \tag{5-19}$$

where

$$a_3 = -\frac{2\pi \times 1}{\sqrt{2}} = -4.4436 \tag{5-20}$$

The characteristic equations of the three decoupled subsystems are as follows:

x component:
$$\begin{aligned}(s - a_1 - ja_1)(s + a_1 + ja_1) \\ = s^2 - 2a_1 s + 2a_1^2 = 0\end{aligned} \tag{5-21}$$

ϕ_1 component:
$$\begin{aligned}&(s - b_2)(s - a_2 - ja_2)(s - a_2 + ja_2) \\ &= (s - b_2)(s^2 - 2a_2 s + 2a_2^2) \\ &= s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2 b_2)s - 2a_2^2 b_2 = 0\end{aligned} \tag{5-22}$$

ϕ_2 componwnt:
$$\begin{aligned}&(s - b_3)(s - a_3 - ja_3)(s - a_3 + ja_3) \\ &= (s - b_3)(s^2 - 2a_3 s + 2a_3^2) \\ &= s^3 - (2a_3 + b_3)s^2 + (2a_3^2 + 2a_3 b_3)s - 2a_3^2 b_3 = 0\end{aligned} \tag{5-23}$$

If there exists a set of feedback gains such that the decoupling of the subsystems x, ϕ_1 and ϕ_2 can be accomplished, then the decoupled closed-loop system will have the independent characteristic equations of Eqs. (5-21), (5-22) and (5-23).

Let the matrix A of Eq. (5-6) be written as

$$A = \left[\begin{array}{c|c|c} A_{11} & X & X \\ \hline X & A_{22} & X \\ \hline X & X & A_{33} \end{array}\right] \tag{5-24}$$

where A_{11} denotes a 2x2 matrix, A_{22} and A_{33} are 3x3 matrices. The submatrices denoted by X contain elements which are unimportant for the immediate develcpment.

In view of Eqs. (5-5) and (5-9), the decoupled x dynamics with state feedback can be expressed as

ORIGINAL PAGE IS
OF POOR QUALITY

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \Lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5-25}$$

where

$$\Lambda_1 = A_{11} - \begin{bmatrix} 0 & 0 \\ (BG)_{21} & (BG)_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21} - (BG)_{21} & -(BG)_{22} \end{bmatrix} \tag{5-26}$$

The characteristic equation of Λ_1 is

$$|sI - \Lambda_1| = \begin{vmatrix} s & -1 \\ a_{21} - (BG)_{21} & s + (BG)_{22} \end{vmatrix}$$

$$= s^2 + (BG)_{22}s - a_{21} + (BG)_{21} = 0 \tag{5-27}$$

Similarly, the decoupled ϕ_1 dynamics with state feedback is described by

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \Lambda_2 \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{5-28}$$

where

$$\Lambda_2 = A_{22} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (BG)_{53} & (BG)_{54} & (BG)_{55} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(BG)_{53} & a_{54} - (BG)_{54} & -(BG)_{55} \end{bmatrix} \tag{5-29}$$

The characteristic equation of Λ_2 is

$$|sI - \Lambda_2| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ (BG)_{53} & -a_{54} + (BG)_{54} & s + (BG)_{55} \end{vmatrix}$$

$$= s^3 + (BG)_{55}s^2 + \left((BG)_{54} - a_{54}\right)s + (BG)_{53} = 0 \qquad (5\text{-}30)$$

For the ϕ_2 component, the decoupled closed-loop state equations are

$$\begin{bmatrix} \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \Lambda_3 \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} \qquad (5\text{-}31)$$

where

$$\Lambda_3 = A_{33} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (BG)_{86} & (BG)_{87} & (BG)_{88} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(BG)_{86} & a_{87} - (BG)_{87} & -(BG)_{88} \end{bmatrix} \qquad (5\text{-}32)$$

The characteristic equation of Λ_3 is

$$|sI - \Lambda_3| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ (BG)_{86} & -a_{87} + (BG)_{87} & s + (BG)_{88} \end{vmatrix}$$

$$= s^3 + (BG)_{88}s^2 + \left((BG)_{87} - a_{87}\right)s + (BG)_{86} = 0 \qquad (5\text{-}33)$$

For pole placement, in order to meet the bandwidth requirements, the corresponding coefficients of the characteristic equations in Eqs. (5-27), (5-30) and (5-33) must match those of Eqs. (5-21), (5-22) and (5-23), respectively.

Thus, for pole-placement,

$$s^2 - 2a_1 s + 2a_1{}^2 = s^2 + (BG)_{22}s + (BG)_{21} - a_{21} \qquad (5\text{-}34)$$

$$s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2b_2)s - 2a_2^2b_2$$
$$= s^3 + (BG)_{55}s^2 + \left((BG)_{54} - a_{54}\right)s + (BG)_{53} \qquad (5\text{-}35)$$

$$s^3 - (2a_3 + b_3)s^2 + (2a_3^2 + 2a_3b_3)s - 2a_3^2b_3$$
$$= s^3 + (BG)_{88}s^2 + ((BG)_{87} - a_{87})s + (BG)_{86} \tag{5-36}$$

Equating the like coefficients in the above three equations, we have

$$\begin{aligned}
(BG)_{22} &= b_{21}g_{12} + b_{22}g_{22} + b_{23}g_{32} = -2a_1 \\
(BG)_{21} &= b_{21}g_{11} + b_{22}g_{21} + b_{23}g_{31} = 2a_1^2 + a_{21} \\
(BG)_{55} &= b_{51}g_{15} + b_{52}g_{25} + g_{53}g_{35} = -(2a_2 + b_2) \\
(BG)_{54} &= b_{51}g_{14} + b_{52}g_{24} + b_{53}g_{34} = 2a_2^2 + 2a_2b_2 + a_{54} \\
(BG)_{53} &= b_{51}g_{13} + b_{52}g_{23} + b_{53}g_{33} = -2a_2^2b_2 \\
(BG)_{88} &= b_{81}g_{18} + b_{82}g_{28} + b_{83}g_{38} = -(2a_3 + b_3) \\
(BG)_{87} &= b_{81}g_{17} + b_{82}g_{27} + b_{83}g_{37} = 2a_3^2 + 2a_3b_3 + a_{87} \\
(BG)_{86} &= b_{81}g_{16} + b_{82}g_{26} + b_{83}g_{36} = -2a_3^2b_3
\end{aligned} \tag{5-37}$$

Notice that the 16 constraint equations on the feedback gains in Eq. (5-10) are conditions on decoupling of the three subsystems, whereas the 8 constraint equations in Eq. (5-37) are the conditions for pole placement. The 24 constraint equations contain 24 unknowns in the elements of the feedback matrix G, g_{ij}, $i=1, 2, 3$, $j=1,2,\ldots,8$.

The 24 constraint equations in Eqs. (5-10) and (5-37) can be written as

$$WG = C \tag{5-38}$$

where

$$W = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{51} & b_{52} & b_{53} \\ b_{81} & b_{82} & b_{83} \end{bmatrix}$$

G is the feedback matrix given in Eq. (5-8), and C is a 3x8 matrix whose elements are composed of a_1, a_2, a_3, b_1, b_2, b_3, and a_{ij}, $i=1,2,\ldots,8$, $j=1,2,\ldots,8$.

The feedback gain matrix G can be solved form Eq. (5-38) if W is nonsingular.

Then, $$G = W^{-1}C \tag{5-39}$$

For W to be nonsingular, it is necessary and sufficient for B to have full rank.

For the system parameters given for the ASPS, the feedback matrix is solved from Eq. (5-39), and the results are given as follows:

$g_{11} = 36.449$

$g_{12} = 212.4$

$g_{13} = -9.256981248 \times 10^{8}$

$g_{14} = -2.5471626624 \times 10^{7}$

$g_{15} = -3.3892568 \times 10^{5}$

$g_{16} = 0$

$g_{17} = 0$

$g_{18} = 0$

$g_{21} = -73.35$

$g_{22} = -415.4544$

$g_{23} = 2.2126125552 \times 10^{9}$

$g_{24} = 6.0882526771 \times 10^{7}$

$g_{25} = 8.1012732 \times 10^{5}$

$g_{26} = 3.96750304 \times 10^{5}$

$g_{27} = 1.091703152 \times 10^{5}$

$g_{28} = 1.452664 \times 10^{4}$

$g_{31} = 0$

$g_{32} = 0$

$g_{33} = 3.96750304 \times 10^{8}$

$g_{34} = 1.091703152 \times 10^{7}$

$g_{35} = 1.452664 \times 10^{5}$

$g_{36} = 3.96750304 \times 10^{5}$

$$g_{37} = 1.091703102 \times 10^5$$
$$g_{38} = 1.452664 \times 10^4$$

Substituting G into Eq. (5-5), the elements of the closed-loop A - BG matrix are obtained as follows:

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -6.25\times10^{-2} & -3.5\times10^{-1} \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7.88768\times10^5 & -2.170384\times10^4 & -2.888\times10^2 \end{bmatrix}$$

$$\Lambda_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -788.768 & -217.0384 & -28.88 \end{bmatrix}$$

Thus, the three components of the x, ϕ_1 and ϕ_2 dynamics are completely decoupled, and they have bandwidths of 0.04, 10 and 1 Hz, respectively.

ORIGINAL PAGE IS
OF POOR QUALITY

VI DESIGN OF THE DIGITAL ASPS THROUGH DECOUPLING AND POLE PLACEMENT

In this chapter the digital ASPS is to be designed through decoupling and pole placement. In Chapter IV the digital ASPS was realized by placing sample-and-hold units at the three inputs to the x, ϕ_1 and ϕ_2 components. The feedback gains were the same as those of the analog ASPS.

The dynamics of the system are still described by

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{6-1}$$

where A, B, $\underline{x}(t)$ are as defined in Chapter V. The control vector $\underline{u}(t)$ is given by

$$\underline{u}(t) = -G\underline{x}(kT) \qquad kT \leq t < (k+1)T \tag{6-2}$$

The solution of Eq. (6-1) over the interval $kT \leq t \leq (k+1)T$ is

$$\underline{x}(t) = \phi(t - kT)\underline{x}(kT) + \int_{kT}^{t} \phi(t - \tau)B\underline{u}(\tau)d\tau$$

$$= \left(\phi(t - kT) - \int_{kT}^{t} \phi(t - \tau)d\tau BG\right)\underline{x}(kT) \tag{6-3}$$

where

$$\phi(t) = e^{At} \tag{6-4}$$

Substituting $t = (k+1)T$ into Eq. (6-3), we get

$$x\left((k+1)T\right) = \left(\phi(T) - \int_{0}^{T} \phi(T - \tau)d\tau BG\right)\underline{x}(kT) \tag{6-5}$$

Let

$$\theta(T) = \int_{0}^{T} \phi(T - \tau)d\tau B \tag{6-6}$$

Then Eq. (6-5) is simplified to

$$\underline{x}\left((k+1)T\right) = \left(\phi(T) - \theta(T)G\right)\underline{x}(kT) \tag{6-7}$$

Equation (6-7) now represents the difference state equations of the closed-loop state-feedback ASPS.

The design objectives are to select the elements of the 3x8 feedback matrix G so that the coefficient matrix of the closed-loop digital system, $\phi(T) - \theta(T)G$, has the following form:

$$\phi(T) - \theta(T)G = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix} \tag{6-8}$$

where Λ_1 is 2x2, Λ_2 and Λ_3 are 3x3 matrices, and the eigenvalues of Λ_1, Λ_2 and Λ_3 are to be placed so that the bandwidth requirements described earlier are satisfied.

Before the design can be carried out, one difficulty remains. It is necessary to first calculate the matrices $\phi(T)$ and $\theta(T)$, knowing A, B and the sampling period T. Since A is 8x8 in the present case, the Laplace transform method becomes a tedious task. The method chosen here is that of approximating $\phi(T)$ by a truncated power series; i.e.,

$$\phi(T) = \sum_{n=0}^{N} A^n T^n / n! \tag{6-9}$$

where N is some positive integer.

The matrix $\theta(T)$ is written

$$\begin{aligned} \theta(T) &= \int_0^T \phi(T - \tau)\, d\tau B \\ &= \int_0^T \sum_{n=0}^{\infty} \frac{A^n (t - \tau)^n}{n!}\, d\tau B \end{aligned} \tag{6-10}$$

If the function $\theta(T)$ is uniformly convergent, the integral and the summation in the last equation can be interchanged. Also, truncating the infinite series at n=N, we have

$$\theta(T) = \left(\sum_{n=0}^{N} A^N T^{n+1} / (n+1)! \right) B \tag{6-11}$$

For T = 0.02 sec, the results of $\phi(T)$ and $\theta(T)$ are given as follows:

$$\phi(0.02)=\begin{bmatrix} 9.998\times10^{-1} & 1.999\times10^{-2} & 0 & -2.968\times10^{-7} & -1.979\times10^{-9} & 0 & 2.968\times10^{-7} & 1.979\times10^{-9} \\ -1.224\times10^{-2} & 9.998\times10^{-1} & 0 & -2.968\times10^{-5} & -2.968\times10^{-7} & 0 & 2.968\times10^{-5} & 2.968\times10^{-7} \\ -4.160\times10^{-7} & -2.080\times10^{-9} & 1 & 1.999\times10^{-2} & 1.999\times10^{-4} & 0 & 1.011\times10^{-9} & 5.059\times10^{-12} \\ -6.240\times10^{-5} & -4.160\times10^{-7} & 0 & 9.999\times10^{-1} & 1.999\times10^{-2} & 0 & 1.517\times10^{-7} & 1.011\times10^{-9} \\ -6.240\times10^{-3} & -6.240\times10^{-5} & 0 & -1.517\times10^{-5} & 9.999\times10^{-1} & 0 & 1.517\times10^{-5} & 1.517\times10^{-7} \\ 4.160\times10^{-7} & 2.080\times10^{-9} & 0 & 1.011\times10^{-9} & 5.059\times10^{-12} & 1 & 1.999\times10^{-2} & 1.999\times10^{-4} \\ 6.240\times10^{-5} & 4.160\times10^{-7} & 0 & 1.517\times10^{-7} & 1.011\times10^{-9} & 0 & 9.999\times10^{-1} & 1.999\times10^{-2} \\ 6.240\times10^{-3} & 6.240\times10^{-5} & 0 & 1.517\times10^{-5} & 1.517\times10^{-7} & 0 & -1.537\times10^{-5} & 9.999\times10^{-1} \end{bmatrix} \tag{6-12}$$

$$\theta(0.02) = \begin{bmatrix} 1.164\times10^{-4} & 5.937\times10^{-5} & -5.937\times10^{-5} \\ 1.164\times10^{-2} & 5.937\times10^{-3} & -5.937\times10^{-3} \\ 3.958\times10^{-7} & 2.023\times10^{-7} & -2.023\times10^{-7} \\ 5.937\times10^{-5} & 3.035\times10^{-5} & -3.035\times10^{-5} \\ 5.937\times10^{-3} & 3.035\times10^{-3} & -3.035\times10^{-3} \\ -3.958\times10^{-7} & -2.023\times10^{-7} & 2.050\times10^{-7} \\ -5.937\times10^{-5} & -3.035\times10^{-5} & 3.075\times10^{-5} \\ -5.937\times10^{-3} & -3.035\times10^{-3} & 3.075\times10^{-3} \end{bmatrix} \tag{6-13}$$

For the same bandwidth requirements described in Chapter 5, the following eigenvalues are selected in the s-plane:

x component: (0.04 Hz bandwidth)

$$p_1, p_2 = -0.178 \pm j0.178$$

ϕ_1 component: (10 Hz bandwidth)

$$p_3 = -200$$

$$p_4,\ p_5 = -44.436 \pm j44.436$$

ϕ_2 component: (1 Hz bandwidth)

$$p_6 = -20$$

$$p_7,\ p_8 = -4.4436 \pm j4.4436$$

Using the transformation $z = e^{Ts}$, these eigenvalues are transformed into the z-plane. The corresponding z-plane eigenvalues are:

x: $z_1,\ z_2 = 0.9964457 \pm j0.0035417$

ϕ_1: $z_3 = 0.01831564$

$z_4,\ z_5 = 0.2592945 \pm j0.3191948$

ϕ_2: $z_6 = 0.67032$

$z_7,\ z_8 = 0.911366 \pm j0.096987$

The condition of decoupling of the closed-loop digital system given in Eq. (6-8) results in 42 constraint equations. This is due to the fact that Eq. (6-8) represents 64 scalar equations, but there are 42 zero elements in the matrix. The pole placements of the eighth-order system would produce 8 more constraint equations, so there are only 24 unknowns in the feedback matrix G. Therefore, in general, there would not be a set of solutions with more equations than unknowns. However, a closer look at the elements of $\phi(T)$ and $\theta(T)$ reveals that some of the elements are extremely small so that a good approximation can be obtained by assuming that these small elements are zeros. In other words, the state transition matrix $\phi(0.02)$ given in Eq. (6-12) can be approximated by the following matrix.

$$\phi(0.02) \simeq \left[\begin{array}{cc|ccc|ccc}
9.998\times10^{-1} & 1.999\times10^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.224\times10^{-2} & 9.998\times10^{-1} & 0 & -2.968\times10^{-5} & -2.968\times10^{-7} & 0 & 2.968\times10^{-5} & 2.968\times10^{-7} \\
\hline
0 & 0 & 1 & 1.999\times10^{-2} & 1.999\times10^{-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 9.999\times10^{-1} & 1.999\times10^{-2} & 0 & 0 & 0 \\
-6.240\times10^{-5} & -4.160\times10^{-7} & 0 & -1.517\times10^{-5} & 9.999\times10^{-1} & 0 & 1.517\times10^{-5} & 1.517\times10^{-7} \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1.999\times10^{-2} & 1.999\times10^{-4} \\
0 & 0 & 0 & 0 & 0 & 0 & 9.999\times10^{-1} & 1.999\times10^{-2} \\
6.240\times10^{-3} & 6.240\times10^{-5} & 0 & 1.517\times10^{-5} & 1.517\times10^{-7} & 0 & -1.537\times10^{-5} & 9.999\times10^{-1}
\end{array}\right] \tag{6-14}$$

Similarly, we can approximate $\theta(0.02)$ by the following matrix:

$$\theta(0.02) \simeq \left[\begin{array}{ccc}
0 & 0 & 0 \\
1.164\times10^{-2} & 5.937\times10^{-3} & -5.937\times10^{-3} \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
5.937\times10^{-3} & 3.035\times10^{-3} & -3.035\times10^{-3} \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
-5.937\times10^{-3} & -3.035\times10^{-3} & 3.075\times10^{-3}
\end{array}\right] \tag{6-15}$$

ORIGINAL PAGE IS
OF POOR QUALITY

Then, the matrix $\theta(0.02)G$ becomes

$$\theta(0.02)G \simeq \left[\begin{array}{cc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\theta G)_{21} & (\theta G)_{22} & (\theta G)_{23} & (\theta G)_{24} & (\theta G)_{25} & (\theta G)_{26} & (\theta G)_{27} & (\theta G)_{28} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\theta G)_{51} & (\theta G)_{52} & (\theta G)_{53} & (\theta G)_{54} & (\theta G)_{55} & (\theta G)_{56} & (\theta G)_{57} & (\theta G)_{58} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\theta G)_{81} & (\theta G)_{82} & (\theta G)_{83} & (\theta G)_{84} & (\theta G)_{85} & (\theta G)_{86} & (\theta G)_{87} & (\theta G)_{88} \end{array}\right] \quad (6\text{-}16)$$

Based on Eqs. (6-14), (6-15) and (6-16), the following 16 equations are obtained for decoupling the ϕ - θG matrix:

ORIGINAL PAGE IS
OF POOR QUALITY

$$(\theta G)_{23} = \theta_{21}g_{13} + \theta_{22}g_{23} + \theta_{23}g_{33} = \phi_{23} = 0$$

$$(\theta G)_{24} = \theta_{21}g_{14} + \theta_{22}g_{24} + \theta_{23}g_{34} = \phi_{24} = -2.968\times10^{-5}$$

$$(\theta G)_{25} = \theta_{21}g_{15} + \theta_{22}g_{25} + \theta_{23}g_{35} = \phi_{25} = -2.968\times10^{-7}$$

$$(\theta G)_{26} = \theta_{21}g_{16} + \theta_{22}g_{26} + \theta_{23}g_{36} = \phi_{26} = 0$$

$$(\theta G)_{27} = \theta_{21}g_{17} + \theta_{22}g_{27} + \theta_{23}g_{37} = \phi_{27} = 2.968\times10^{-5}$$

$$(\theta G)_{28} = \theta_{21}g_{18} + \theta_{22}g_{28} + \theta_{23}g_{38} = \phi_{28} = 2.968\times10^{-7}$$

$$(\theta G)_{51} = \theta_{51}g_{11} + \theta_{52}g_{21} + \theta_{53}g_{31} = \phi_{51} = -6.240\times10^{-5}$$

$$(\theta G)_{52} = \theta_{51}g_{12} + \theta_{52}g_{22} + \theta_{53}g_{32} = \phi_{52} = -4.160\times10^{-7}$$

$$(\theta G)_{56} = \theta_{51}g_{16} + \theta_{52}g_{26} + \theta_{53}g_{36} = \phi_{56} = 0$$

$$(\theta G)_{57} = \theta_{51}g_{17} + \theta_{52}g_{27} + \theta_{53}g_{37} = \phi_{57} = 1.517\times10^{-5}$$

$$(\theta G)_{58} = \theta_{51}g_{18} + \theta_{52}g_{28} + \theta_{53}g_{38} = \phi_{58} = 1.517\times10^{-7}$$

$$(\theta G)_{81} = \theta_{81}g_{11} + \theta_{82}g_{21} + \theta_{83}g_{32} = \phi_{81} = 6.240\times10^{-3}$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$(\theta G)_{82} = \theta_{81}g_{11} + \theta_{82}g_{22} + \theta_{83}g_{32} = \phi_{82} = 6.240\times10^{-5}$$

$$(\theta G)_{83} = \theta_{81}g_{13} + \theta_{82}g_{23} + \theta_{83}g_{33} = \phi_{83} = 0$$

$$(\theta G)_{84} = \theta_{81}g_{14} + \theta_{82}g_{24} + \theta_{83}g_{34} = \phi_{84} = 1.517\times10^{-5}$$

$$(\theta G)_{85} = \theta_{81}g_{15} + \theta_{82}g_{25} + \theta_{83}g_{35} = \phi_{85} = 1.517\times10^{-7}$$

The eight additional constraint equations for the solution of the 24 elements of the feedback gain matrix G come from the pole placement requirements on the submatrices, Λ_1, Λ_2 and Λ_3.

After the matrix $\phi - \theta G$ has been decoupled, it can be written as

$$\phi(0.02) - \theta(0.02)G = \left[\begin{array}{c|c|c} \Lambda_1 & 0 & 0 \\ \hline 0 & \Lambda_2 & 0 \\ \hline 0 & 0 & \Lambda_3 \end{array}\right] \tag{6-18}$$

where

$$\Lambda_1 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ (\theta G)_{21} & (\theta G)_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} - (\theta G)_{21} & \phi_{22} - (\theta G)_{22} \end{bmatrix} \tag{6-19}$$

$$\Lambda_2 = \begin{bmatrix} \phi_{33} & \phi_{34} & \phi_{35} \\ \phi_{43} & \phi_{44} & \phi_{45} \\ \phi_{53} & \phi_{54} & \phi_{55} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\theta G)_{53} & (\theta G)_{54} & (\theta G)_{55} \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{33} & \phi_{34} & \phi_{35} \\ \phi_{43} & \phi_{44} & \phi_{45} \\ \phi_{53} - (\theta G)_{53} & \phi_{54} - (\theta G)_{54} & \phi_{55} - (\theta G)_{55} \end{bmatrix} \tag{6-20}$$

$$\Lambda_3 = \begin{bmatrix} \phi_{66} & \phi_{67} & \phi_{68} \\ \phi_{76} & \phi_{77} & \phi_{78} \\ \phi_{86} & \phi_{87} & \phi_{88} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\theta G)_{86} & (\theta G)_{87} & (\theta G)_{88} \end{bmatrix}$$

ORIGINAL PAGE IS OF POOR QUALITY

$$= \begin{bmatrix} \phi_{66} & \phi_{67} & \phi_{68} \\ \phi_{76} & \phi_{77} & \phi_{78} \\ \phi_{86} - (\theta G)_{86} & \phi_{87} - (\theta G)_{87} & \phi_{88} - (\theta G)_{88} \end{bmatrix} \tag{6-21}$$

The characteristic equation of the decoupled x dynamics is

$$|zI - \Lambda_1| = \begin{vmatrix} z - \phi_{11} & -\phi_{12} \\ -\phi_{21} + (\theta G)_{21} & z - \phi_{22} + (\theta G)_{22} \end{vmatrix}$$

$$= z^2 + \left((\theta G)_{22} - \phi_{11} - \phi_{22}\right)z + \phi_{11}\phi_{22} - \phi_{11}(\theta G)_{22} + \phi_{12}\left((\theta G)_{21} - \phi_{21}\right)$$

$$= (z + a_1 + jb_1)(z + a_1 - jb_1) \tag{6-22}$$

where $a_1 = -0.9964457$ and $b_1 = -0.0035417$.

Now solving for the unknowns in Eq. (6-22), we have

$$(\theta G)_{21} = \theta_{21}g_{11} + \theta_{22}g_{21} + \theta_{23}g_{31}$$

$$= \frac{1}{\phi_{12}}\left(a_1^2 + b_1^2 + \phi_{11}^2 + 2a_1\phi_{11} + \phi_{12}\phi_{21}\right) \tag{6-23}$$

$$(\theta G)_{22} = \theta_{21}g_{12} + \theta_{22}g_{22} + \theta_{23}g_{32} = \phi_{11} + \phi_{12} + 2a_1 \tag{6-24}$$

For the ϕ_1 dynamics, the characteristic equation of Λ_2 is

$$|zI - \Lambda_2| = \begin{vmatrix} z - \phi_{33} & -\phi_{34} & -\phi_{35} \\ -\phi_{43} & z - \phi_{44} & -\phi_{45} \\ -\phi_{53} + (\theta G)_{53} & -\phi_{54} + (\theta G)_{54} & z - \phi_{55} + (\theta G)_{55} \end{vmatrix}$$

$$= z^3 + \left((\theta G)_{55} - \phi_{55} - \phi_{44} - \phi_{33}\right) z^2 + \left(-(\phi_{44} + \phi_{33})(\theta G)_{55} + \phi_{45}(\theta G)_{54}\right.$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$+ \phi_{35}(\theta G)_{53} - \phi_{54}\phi_{45} + \phi_{33}\phi_{55} + \phi_{33}\phi_{44} - \phi_{43}\phi_{34} - \phi_{35}\phi_{53} + \phi_{44}\phi_{45}$$

$$+ \left((\phi_{33}\phi_{44} - \phi_{43}\phi_{34})(\theta G)_{55} + (\phi_{35}\phi_{43} - \phi_{33}\phi_{45})(\theta G)_{54}\right.$$

$$+ (\phi_{34}\phi_{45} - \phi_{35}\phi_{44})(\theta G)_{53} - \phi_{33}\phi_{44}\phi_{55} + \phi_{33}\phi_{54}\phi_{45} + \phi_{43}\phi_{34}\phi_{55}$$

$$\left. - \phi_{43}\phi_{35}\phi_{54} - \phi_{34}\phi_{45}\phi_{53} + \phi_{35}\phi_{44}\phi_{53}\right) \tag{6-25}$$

The desired characteristic equation is written as

$$(z + a_2 + jb_2)(z + a_2 - jb_2)(z + c_2) = 0 \tag{6-26}$$

where

$$c_2 = -0.01831564$$

$$a_2 = -0.2592945$$

$$b_2 = -0.3191948$$

Equating Eqs. (6-25) and (6-26), we get

$$(\theta G)_{55} = \theta_{51}g_{15} + \theta_{52}g_{25} + \theta_{53}g_{35} = \phi_{55} + \phi_{44} + 1 + 2a_2 + c_2 \tag{6-27}$$

$$(\theta G)_{54} = \theta_{51}g_{14} + \theta_{52}g_{24} + \theta_{53}g_{34}$$

$$= \frac{-\phi_{35}(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{45}(\phi_{35} + \phi_{34}\phi_{45} - \phi_{35}\phi_{44})} + \frac{1}{\phi_{45}}\left((\phi_{44} + 1)(\phi_{55} + \phi_{44}\right.$$

$$\left. + 1 + 2a_2 + c_2) - \phi_{44}\phi_{45} + \phi_{45}\phi_{54} - \phi_{55} - \phi_{44} + 2a_2c_2 + a_2^2 + b_2^2\right) \tag{6-28}$$

$$(\theta G)_{53} = \theta_{51}g_{13} + \theta_{52}g_{23} + \theta_{53}g_{33}$$

$$= \frac{(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{35} + \phi_{34}\phi_{45} - \phi_{35}\phi_{44}} \tag{6-29}$$

For the ϕ_2 dynamics, the characteristic equation of Λ_3 is

$$|zI - \Lambda_3| = \begin{vmatrix} z - \phi_{66} & -\phi_{67} & -\phi_{68} \\ -\phi_{76} & -\phi_{77} & -\phi_{78} \\ -\phi_{86} + (\theta G)_{86} & -\phi_{87} + (\theta G)_{87} & z - \phi_{88} + (\theta G)_{88} \end{vmatrix}$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$= z^3 + ((\theta G)_{88} - \phi_{88} - \phi_{77} - \phi_{66})z^2 + (-(\phi_{77} + \phi_{66})(\theta G)_{88} + \phi_{78}(\theta G)_{87}$$

$$+ \phi_{68}(\theta G)_{86} - \phi_{87}\phi_{78} + \phi_{66}\phi_{88} + \phi_{66}\phi_{77} - \phi_{76}\phi_{67} - \phi_{68}\phi_{86} + \phi_{77}\phi_{78})z$$

$$+ ((\phi_{66}\phi_{77} - \phi_{76}\phi_{67})(\theta G)_{88} + (\phi_{68}\phi_{76} - \phi_{66}\phi_{78})(\theta G)_{87}$$

$$+ (\phi_{67}\phi_{78} - \phi_{68}\phi_{77})(\theta G)_{86} - \phi_{66}\phi_{77}\phi_{88} + \phi_{66}\phi_{87}\phi_{78} + \phi_{76}\phi_{67}\phi_{88}$$

$$- \phi_{76}\phi_{68}\phi_{87} - \phi_{67}\phi_{78}\phi_{86} + \phi_{68}\phi_{77}\phi_{86}) \qquad (6\text{-}30)$$

The desired characteristic equation is written as

$$(z + a_3 + jb_3)(z + a_3 - jb_3)(z + c_3) = 0 \qquad (6\text{-}31)$$

where

$$c_3 = -0.67032$$

$$a_3 = -0.911366$$

$$b_3 = -0.096987$$

Equating Eqs. (6-30) and (6-31), we get

$$(\theta G)_{86} = \theta_{81}g_{16} + \theta_{82}g_{26} + \theta_{83}g_{36}$$

$$= \frac{(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{68} + \phi_{67}\phi_{78} - \phi_{68}\phi_{77}} \qquad (6\text{-}32)$$

$$(\theta G)_{87} = \theta_{81}g_{17} + \theta_{82}g_{27} + \theta_{83}g_{37}$$

$$= \frac{-\phi_{68}(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{78}(\phi_{68} + \phi_{67}\phi_{78} - \phi_{68}\phi_{77})} + \frac{1}{\phi_{78}}((\phi_{77} + 1)(\phi_{88} + \phi_{77} + 1 + 2a_3 + c_3)$$

$$- \phi_{77}\phi_{78} + \phi_{78}\phi_{87} - \phi_{88} - \phi_{77} + 2a_3c_3 + a_3^2 + b_3^2) \qquad (6\text{-}33)$$

$$(\theta G)_{88} = \theta_{81}g_{18} + \theta_{82}g_{28} + \theta_{83}g_{38}$$

$$= \phi_{88} + \phi_{77} + 1 + 2a_3 + c_3 \qquad (6\text{-}34)$$

Equations (6-23), (6-24), (6-27), (6-28), (6-29), (6-32), (6-33) and (6-34) form the additional eight constraint equations, together with the 16 equations in Eq. (6-17) are adequate for the solution of the 24 unknown feedback gains in G.

The solution of the 24 constraint equations gives the following results:

$$
\begin{aligned}
g_{11} &= 35.432513595904985 \\
g_{12} &= 209.57461847387285 \\
g_{13} &= -9.368484250209332\times10^{7} \\
g_{14} &= -8.1139770162127 1032\times10^{6} \\
g_{15} &= -1.4453443216549823\times10^{5} \\
g_{16} &= g_{17} = g_{18} = 0 \\
g_{21} &= -71.361744259885901 \\
g_{22} &= -409.94846384061196 \\
g_{23} &= 2.2392641102247482\times10^{8} \\
g_{24} &= 1.9394105849068884\times10^{7} \\
g_{25} &= 3.454675889387745\times10^{5} \\
g_{26} &= 3.578272520216161\times10^{5} \\
g_{27} &= 1.3239693135355619\times10^{6} \\
g_{28} &= 1.2749737170875715\times10^{4} \\
g_{31} &= g_{32} = 0 \\
g_{33} &= 4.015292755500438\times10^{7} \\
g_{34} &= 3.4776162570406019\times10^{6} \\
g_{35} &= 6.194684677849835\times10^{4} \\
g_{36} &= 3.578272527392904\times10^{5} \\
g_{37} &= 1.323969309412944\times10^{6} \\
g_{38} &= 1.2749737129324844\times10^{4}
\end{aligned}
\qquad (6\text{-}35)
$$

ORIGINAL PAGE IS
OF POOR QUALITY