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Analysis of Dynamic System Response to Product Random Processes

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SUMMARY

The response of dynamic systems to the product of two independent Gaussian random processes is developed by use of the Fokker-Planck and associated moment equations. The development is applied to the amplitude modulated process which is used to model atmospheric turbulence in aeronautical applications. The exact solution for the system response is compared with the solution obtained by the quasi-steady approximation which omits the dynamic properties of the random amplitude modulation. The quasi-steady approximation is valid as a limiting case of the exact solution for the dynamic response of linear systems to amplitude modulated processes. In the nonlimiting case the quasi-steady approximation can be invalid for dynamic systems with low damping.

INTRODUCTION

Random processes, which can be represented as the product of two independent random processes, or so-called product processes, include the amplitude modulated process which is used as a model for atmospheric turbulence in aeronautical applications (ref. 1). Atmospheric turbulence is considered to be a local Gaussian random process modulated by a relatively slow random amplitude process. Application of the amplitude modulated process requires methods for the analysis of the response of dynamic systems to the process. The currently used method, termed the quasi-steady approximation in reference 2, considers the dynamic response of the system to the local Gaussian process only and assumes that the amplitude process affects the dynamic system only in a static manner.

In the present study, the Fokker-Planck equation is used to develop a method of analysis that provides exact solution for the response of dynamic systems to the product of two independent Gaussian random processes. The method is used to obtain relations for the statistical moments of the response to the product process for some linear dynamic systems, including a linear time-invariant differential equation of general order. The exact and quasi-steady solutions for the response are compared for both a first-order and a second-order system. The system response to the sum of the product process and an independent Gaussian process, proposed as an atmospheric turbulence model in references 3 to 6, is also examined. Some of the information in this report has been included in reference 7.

SYMBOLS

- a linear filter constant of subscripted process
- b standard deviation of S process
- C () characteristic function

c	standard deviation of M process
$E[]$	ensemble average
F_{jk}	response coefficient of linear state equation for system response
$F\{ \}$	Fourier transformation of indicated quantity
$f()$	response term of state equation
g_j	excitation coefficient of state equation for system response
h_j	$= E[srz_j]$
$h()$	impulsive-response function of linear system
i	unit of imaginaries, $\sqrt{-1}$
k, l, n	integer index (nonnegative)
$M()$	moment of indicated orders
M_4	fourth-order flatness factor or kurtosis (fourth moment/square of second moment)
m, M	mean value random process
N	sum of orders of moments of system response
$p()$	probability density function
$p(r r_0)$	conditional probability density function of r process conditional on value of r_0
q, Q	uncoupled form of general local random process
r, R	local random process
s, S	amplitude random process
t	time
v_j	general state vector random process
x	response variable of second-order differential equation
y, Y	response variable of coupled linear system
z, Z	system response process; response variable of uncoupled linear system
$\alpha_j()$	response term in general state equation

β_{jk}	excitation coefficient in general state equation
γ_j	excitation coefficient in state equation for system response
$\delta(t)$	Dirac delta function
ζ	Fourier transformation variable corresponding to z ; system damping coefficient
η_j	Fourier transformation variable corresponding to y_j
λ, ν	integer index (nonnegative)
μ	linear filter constant for system response (negative of eigenvalue)
$\xi_j(t)$	Gaussian white noise process
ρ	Fourier transformation variable corresponding to r
Σ	covariance matrix of coupled linear system
σ	Fourier transformation variable corresponding to s
τ	time-difference variable
ω	system frequency coefficient

Subscripts:

in	input
j, k, ℓ, m	integer index
o	at an earlier value of time
out	output

The summation convention is used: repeated indices imply a summation unless noted otherwise.

A dot over a symbol indicates a derivative with respect to time.

RESPONSE OF DYNAMIC SYSTEMS TO PRODUCT PROCESSES

Product random processes, particularly the special case of the amplitude modulated process, are described in this section. The quasi-steady approximation for the analysis of the response of dynamic systems to the amplitude modulated random process is examined. The quasi-steady approximation, which ignores the dynamic properties of the amplitude process, is described in qualitative terms. The Fokker-Planck equation is developed for the response of a first-order nonlinear dynamic system to the product process. The response of

the system to the sum of a product process and an independent mean value process is also considered.

Product Processes and the Quasi-Steady Approximation

The product random process is formed by the product of two component processes

$$r(t) s(t) \tag{A}$$

The component processes are specified to be stationary, independent, and Gaussian with zero mean values. The general product process (expression (A)) includes the special case of the amplitude modulated process where the local R component is a rapidly varying process which is modulated by the slowly varying amplitude component S.¹ The product process is developed in general terms without restriction on the relative time variations of the two component processes. Since the probabilistic structures of the two components are completely defined (given their autocovariance functions), the corresponding structure of the product process is determined by the defining relation (expression (A)) and by the appropriate relations of probability theory (ref. 2).

The amplitude modulated process was originally developed by consideration of measured atmospheric turbulence data, segments of which appear to be stationary and Gaussian over small time intervals. Over longer time intervals, the amplitude of the local Gaussian segments appears to vary in a random manner. This interpretation of measured turbulence data led to the development of the amplitude modulated process, which has had extensive application in aeronautical problems associated with atmospheric turbulence (refs. 8 to 10).

The amplitude modulated random process has been applied in the analysis of the response of dynamic systems. The associated analytical technique, termed the quasi-steady approximation, uses the idea of the relatively slow variation of the amplitude component to develop a simple method of analysis. The development of the quasi-steady approximation is shown by consideration of the response or output of a linear system

$$z_{out}(t) = \int_0^{\infty} h(\tau) z_{in}(t-\tau) d\tau \tag{1}$$

The system response is the convolution of the impulsive response function of the linear system and the excitation or input function, which is a product process (expression (A))

¹The terms product process, amplitude modulated process, and quasi-steady process have the following meanings: a product process includes all random processes of the form of expression (A); an amplitude modulated process is the special case with the S component slowly varying with respect to the R component process; a quasi-steady process is the limiting case where the dynamic properties of the S component are omitted.

$$z_{out}(t) = \int_0^{\infty} h(\tau) r_{in}(t-\tau) s_{in}(t-\tau) d\tau \quad (2)$$

Two assumptions are made. First, it is assumed that the impulsive response function is approximately equal to zero after a known time period. Consequently, the integration in equation (2) can be restricted to this period with a negligible change in the computed system response. Second, it is assumed that the amplitude component process is approximately constant during this time period. With these two assumptions the amplitude process can be removed from the convolution integral to give

$$z_{out}(t) \approx s_{in}(t) \int_0^{\infty} h(\tau) r_{in}(t-\tau) d\tau \quad (3)$$

The remaining integral is the response of the linear system to the local R component process. Since the input R process is Gaussian, the output R process is also Gaussian. The approximate system response satisfies the product relation and consequently is an amplitude modulated process

$$z_{out}(t) \approx r_{out}(t) s_{in}(t) \quad (4)$$

The dynamic aspects of the response result from the rapidly varying R component process only. The amplitude component influences the system only in a static manner.

The preceding development shows the conditions that are required for the quasi-steady approximation. The amplitude component must be slowly varying relative to both the local R component and the dynamic properties of the system. The accuracy of the approximation is improved if the system is well damped; the accuracy is questionable if the system is lightly damped. The quasi-steady relation is exact for static linear systems; this follows from a linear transformation of the defining relation (expression (A)). The quasi-steady method of analysis is generally an approximation, however, since development of the exact response requires the consideration of the dynamic response of the system to both component processes. If the exact response is known, then the quasi-steady response is the limiting case as the dynamic properties of the amplitude process become negligible.

An alternate random process has been proposed as a model of atmospheric turbulence in references 3 to 6. The process is formed by the sum of a product process and an independent, random variation of the mean value

$$r(t) s(t) + m(t) \quad (B)$$

The resulting product-plus-mean process is formed from three component processes which are specified to be stationary, independent, and Gaussian with zero mean

values. Since the probabilistic structures of the three component processes are completely defined, the structure of the product-plus-mean process is determined by the defining relation (expression (B)) and the appropriate relations of probability theory.

The response of dynamic systems to the product-plus-mean process can be determined by the quasi-steady approximation. The addition of the mean value component presents no fundamental difficulty for the analysis of linear dynamic systems, since the response to the independent product and mean value processes can be considered separately. The analysis of the response to the Gaussian mean value process requires only the determination of the covariance functions of the response. Using the quasi-steady approximation for the response to the product process, the system response is also a product-plus-mean process (expression (B)). For nonlinear dynamic systems the analysis is inherently more difficult, since the response of the system to the product and the mean value processes cannot be considered separately.

Fokker-Planck Equation for the Response of Dynamic Systems

The general Fokker-Planck equation is applied to the response of dynamic systems forced by the product process. The response of a first-order nonlinear dynamic system is considered in order to illustrate the general procedure. The development is related to that in reference 11.

In the present example the two components of the product process are generated by first-order filtering of independent Gaussian white noise processes. The corresponding differential equations are

$$\dot{s} = -a_s s + \sqrt{2a_s} b \xi_1(t) \quad (5a)$$

$$\dot{r} = -a_r r + \sqrt{2a_r} \xi_2(t) \quad (5b)$$

The two filter constants a_s and a_r determine the dynamic properties of the two component processes. The two independent Gaussian white noise processes have zero mean values and have autocovariance functions which are Dirac delta functions

$$\left. \begin{aligned} E[\xi_k(t)] &= 0 \\ E[\xi_k(t_1) \xi_l(t_2)] &= \begin{cases} \delta(t_1 - t_2) & (k = l) \\ 0 & (k \neq l) \end{cases} \end{aligned} \right\} \quad (6)$$

The corresponding power spectral density functions of the white noise processes have a constant value of $1/2\pi$. The variances of the two component processes are

$$\left. \begin{aligned} E[s^2] &= b^2 \\ E[r^2] &= 1 \end{aligned} \right\} \quad (7)$$

A first-order nonlinear dynamic system with excitation by the product random process (expression (A)) is considered.

$$\dot{z} = -f(z) + rs \quad (8)$$

A composite dynamic system is formed which consists of the differential equations for the dynamic system (eq. (8)) and for the component processes (eqs. (5a) and (5b)). These equations are combined to form the nonlinear state equation for the composite system, which has the general form

$$\dot{v}_j = \alpha_j(\bar{v}) + \beta_{j\ell} \xi_\ell(t) \quad (9)$$

The state vector \bar{v} is the array of the variables of the composite system, which in the present case includes the two components of the excitation process and the system response, that is, the three variables s , r , and z . The excitation term of the composite dynamic system consists of the two Gaussian white noise processes of equations (5a) and (5b). The excitation term of the original dynamic system (eq. (8)) is the product of two response variables of the composite system and, consequently, is included in the response term, that is, the α functions of the state equation (eq. (9)).

The probability density function of the composite system satisfies the associated Fokker-Planck equation (refs. 12 to 15). For a dynamic system satisfying the general state equation (eq. (9)), the Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial v_j} [\alpha_j(\bar{v})p] + \frac{1}{2} \frac{\partial^2}{\partial v_j \partial v_k} [\beta_{j\ell} \beta_{k\ell} p] \quad (10)$$

The Fokker-Planck equation determines the transition probability density function of the state variables,

$$p(\bar{v}, t | \bar{v}_0, t_0)$$

which is the probability density function of the state vector conditional on the given value of the state vector at an earlier value of time. The transition probability density function is generally nonstationary due to the development of the state vector from its initial value.² If the conditional dependence on the initial value is omitted, the resulting first-order probability density function is stationary; the time derivative term in the Fokker-Planck equation is equal to zero.

The general Fokker-Planck equation (eq. (10)) is applied to the analysis of the response of the composite dynamic system to the product process. The associated probability density function is the conditional joint density of the two component processes and the system response

$$p(s, r, z | s_0, r_0, z_0)$$

The Fokker-Planck equation for the composite dynamic system of equations (5a), (5b), and (8) is obtained by application of equations (9) and (10). The resulting equation is

$$\frac{\partial p}{\partial t} = a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] + a_r \left[\frac{\partial}{\partial r} (rp) + \frac{\partial^2 p}{\partial r^2} \right] + \frac{\partial}{\partial z} [f(z)p] - rs \frac{\partial p}{\partial z} \quad (11)$$

The first two terms of the right-hand side correspond to the differential equations of the S and R component processes. The next term corresponds to the nonlinear response term of the original dynamic system (eq. (8)). The final term corresponds to the excitation of the original dynamic system by the product process. This term couples the system response and the two components of the excitation process in the Fokker-Planck equation.

A similar procedure is used for the analysis of the response of dynamic systems to the product-plus-mean process. An additional equation defining the mean value process is introduced. In the present example, the mean value process is generated by a first-order filtering of a Gaussian white noise process in the same manner as the two other component processes

$$\dot{m} = -a_m m + \sqrt{2a_m} c \xi_3(t) \quad (12)$$

²The development of a random process from its initial value introduces the concept of transition properties. These properties are particularly important for the amplitude modulated process due to the significant difference in the dynamic properties of the two component processes. The transition properties explicitly show that short samples of the process appear to be Gaussian with an approximately constant amplitude factor and that long samples have the form of the product process (ref. 2).

The differential equation for the system response is modified to account for excitation by the combination of the product and the mean value processes

$$\dot{z} = -f(z) + rs + m \quad (13)$$

The Fokker-Planck equation is developed from the general form (eq. (9)) of the state equation for the composite dynamic system. The associated probability density function is the conditional joint density of the three component processes and the system response

$$p(s, r, m, z | s_0, r_0, m_0, z_0)$$

The Fokker-Planck equation for the composite dynamic system of equations (5a), 5(b), (12), and (13) is

$$\begin{aligned} \frac{\partial p}{\partial t} = & a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] + a_r \left[\frac{\partial}{\partial r} (rp) + \frac{\partial^2 p}{\partial r^2} \right] \\ & + a_m \left[\frac{\partial}{\partial m} (mp) + c^2 \frac{\partial^2 p}{\partial m^2} \right] + \frac{\partial}{\partial z} [f(z)p] - rs \frac{\partial p}{\partial z} - m \frac{\partial p}{\partial z} \end{aligned} \quad (14)$$

The resulting Fokker-Planck equation is similar to equation (11) except for the presence of two additional terms: one term corresponds to the differential equation for the mean value process and one term corresponds to the additional excitation of the system by that process.

RESPONSE OF LINEAR DYNAMIC SYSTEMS

Several examples of the response of linear dynamic systems to the product random process are developed in the present section. Exact relations for the statistical moments of the system response are obtained by using the Fokker-Planck equation for the associated composite dynamic system. The quasi-steady approximation is developed as a limiting case of the exact solution for the system response. The analysis shows the conditions required for the approximation in the dynamic systems considered.

The method is first applied to the analysis of the response of a first-order linear dynamic system to the product process. The method is then used to develop the solution for a linear dynamic system of general order, with application to a second-order linear differential equation. One example of the response of a linear dynamic system to the product-plus-mean process is considered. Finally, several possible extensions of the development are outlined.

Response of First-Order Linear System to Product Process

The response of a first-order linear system to the product process, with both components generated by first-order filtering of white noise processes, is considered. The composite dynamic system consists of three first-order differential equations; two are for the component processes and one is for the system response as follows:

$$\dot{s} = -a_s s + \sqrt{2a_s} b \xi_1(t) \quad (15a)$$

$$\dot{r} = -a_r r + \sqrt{2a_r} \xi_2(t) \quad (15b)$$

$$\dot{z} = -\mu z + rs \quad (15c)$$

The three filter constants (a_s , a_r , and μ) determine the dynamic properties of the two components of the product process and those of the system response. The filter constants are positive and are the reciprocals of the time constants of the associated first-order filters. With this system, the dynamic properties of the amplitude process can be varied relative to those of both the R component process and the system response by changing the values of the filter constants.

The frequency response functions of the three state variables of the composite dynamic system are shown in schematic form in figure 1. Since the two excitation components both satisfy first-order linear differential equations with constant coefficients, their frequency response functions have an approximately constant value for frequency values below the corresponding filter constants and decrease as a -1 power of the frequency above that value. The frequency response function of the system response has a similar form.³ The frequency response function of the first derivative of the system response varies as a $+1$ power of the frequency below the value of the filter constant and is constant above that value of the frequency. The frequency response functions in figure 1 are shown for the modulation condition with the amplitude process S slowly varying relative to the local R process, that is, $a_s \ll a_r$.

Using equations (9) and (10), the Fokker-Planck equation for the composite system of equations (15a) to (15c) is

$$\frac{\partial p}{\partial t} = a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] + a_r \left[\frac{\partial}{\partial r} (rp) + \frac{\partial^2 p}{\partial r^2} \right] + \mu \frac{\partial}{\partial z} (zp) - rs \frac{\partial p}{\partial z} \quad (16)$$

³Figure 1 shows the frequency response function for the first-order system of equation (15c), that is, for the system response relative to the excitation product process.

The probability density function is the conditional joint density of the three state variables: s , r , and z . Equation (16) is a special case of equation (11). The first two terms on the right-hand side correspond to the differential equations for the two component processes (eqs. (15a) and (15b)). The last two terms correspond to the differential equation of the system response (eq. (15c)), the last term corresponding to the excitation of the system by the product process.

The Fokker-Planck equation is transformed to give the corresponding equation for the joint characteristic function of the composite dynamic system. The characteristic function is the Fourier transformation of the probability density function over the three state variables

$$\begin{aligned} C(\sigma, \rho, \zeta | s_0, r_0, z_0) &= F\{p(s, r, z | s_0, r_0, z_0)\} \\ &= E[e^{i\sigma s + i\rho r + i\zeta z}] \end{aligned} \quad (17)$$

Taking the Fourier transformation of the Fokker-Planck equation, the resulting characteristic function equation is

$$-\frac{\partial C}{\partial t} = a_s \left[\sigma \frac{\partial C}{\partial \sigma} + b^2 \sigma^2 C \right] + a_r \left[\rho \frac{\partial C}{\partial \rho} + \rho^2 C \right] + \mu \zeta \frac{\partial C}{\partial \zeta} + i \zeta \frac{\partial^2 C}{\partial \sigma \partial \rho} \quad (18)$$

(The development of this equation requires relations which are derived in appendix A.) The terms of the characteristic function equation correspond to the terms of the Fokker-Planck equation. The final term corresponds to the excitation of the dynamic system by the product process.

Relations for the statistical moments of the system response can be developed from either the Fokker-Planck or the characteristic function equation. The following notation is used for the joint moments of the three state variables:

$$E[s^k r^\ell z^n] = M(k, \ell, n) \quad (19)$$

The moment equation is obtained from the characteristic function equation by taking the derivatives of general order with respect to the Fourier transformation variables, which are then set equal to zero to develop the moments from the characteristic function. Using this procedure and some relations developed in appendix A, the equation for the joint moments is

$$\begin{aligned} \frac{\partial}{\partial t} M(k, \ell, n) + (ka_S + \ell a_R + n\mu) M(k, \ell, n) \\ = k(k-1) a_S b^2 M(k-2, \ell, n) + \ell(\ell-1) a_R M(k, \ell-2, n) + nM(k+1, \ell+1, n-1) \end{aligned} \quad (20)$$

The individual terms of the moment equation correspond to the individual terms of the state equation for the composite dynamic system. The first term on the left-hand side corresponds to the transient response of the composite dynamic system. The other terms on the left-hand side correspond to the three linear response terms. The first two terms on the right-hand side correspond to the two white noise excitation terms. These terms pass the moments of the S and R processes from lower to higher order without changing the order of the moment of the system response. The final term corresponds to the product excitation term in the differential equation for the system response. This term introduces coupling between the moments of different orders in the system response.

The moments of the stationary response of the system are obtained by deleting the time-derivative term in the moment equation. The subsequent discussion is restricted to the stationary response. Examination of the moment equation shows that the odd-order stationary moments of the system response are zero. The second moment (which equals the variance in this case) of the system response is

$$E[z^2] = \frac{b^2}{\mu(a_S + a_R + \mu)} \quad (21)$$

The variance of the derivative of the system response is obtained from equations (15c) and (21)

$$E[\dot{z}^2] = \frac{(a_S + a_R)b^2}{(a_S + a_R + \mu)} \quad (22)$$

The variances of the system response and its derivative depend upon the filter constant a_S of the amplitude process. If the value of this filter constant is much smaller than that of the R process, then the variances become those obtained by the quasi-steady approximation. The quasi-steady approximation gives an accurate prediction of the variances of the system response and its first derivative in the limiting case ($a_S/a_R = 0$) irrespective of the value of the filter constant μ of the dynamic system.

The higher moments of the system response are computed from the moment equation (eq. (20)) by a procedure which is described subsequently. In figure 2, the resulting fourth moments of the system response and its first derivative are plotted in normalized form, that is, normalized to their quasi-steady values. The moments are plotted as functions of the filter constant of the amplitude component and for several values of the filter constant of the linear system, both constants given as ratios to the filter constant for the local R component process. The quasi-steady approximation corresponds to the zero

value of the abscissa. The dynamic properties of the amplitude component reduce the value of the fourth moment of the system response. The fourth moment of the derivative can be either decreased or increased by the dynamic properties of the amplitude component. For both the system response and its first derivative the value of the fourth moment is decreased as the value of the filter constant of the dynamic system is decreased.

The moments of the system response are also presented in terms of the associated flatness factors, which indicate the relative occurrence of the extreme values of the random process. The values of the fourth-order flatness factor (or kurtosis) of the system response and its first derivative are plotted in figures 3 and 4. The flatness factors are plotted in the same manner as figure 2, that is, as functions of the filter constant of the amplitude process and for several values of the filter constant of the dynamic system, both constants given as ratios to that of the local R process. In the quasi-steady limit, the values of the two flatness factors both approach the value of 9 for the product of two independent Gaussian processes, which is the quasi-steady approximation (eq. (4)). The values of the flatness factors are reduced by the dynamics of the amplitude process.

The results of figures 2 to 4 indicate whether use of the quasi-steady approximation will either overestimate or underestimate the occurrence of the extreme values of the system response and its first derivative. Since the values of the fourth-order flatness factors are maximum in the quasi-steady limit, use of the quasi-steady approximation in combination with the exact values of the second moments (eqs. (21) and (22)) will tend to overestimate the occurrence of the extreme values. If the quasi-steady values of the second moments are used, then the values of the fourth moments (fig. 2) and not those of the fourth-order flatness factors must be examined. For the system response, use of the quasi-steady approximation in this form will tend to overestimate occurrence of the extreme values. For the derivative of the system response, use of the quasi-steady approximation can either overestimate or underestimate the occurrence of the extreme values, depending upon the value of the ratio of the filter constants of the first-order system and the R process.

The exact solution for the response of the first-order system to the product process is interpreted in the following manner: For the system response (fig. 3), the value of the fourth-order flatness factor is increased and approaches the quasi-steady value of 9 as the value of the system filter constant μ is increased. In the limit of large values of the filter constant, the system response becomes static with respect to both components of the product process, since the frequency response function is approximately constant for frequency values below μ (fig. 1). Consequently, the system response is approximately a product process resulting from a static transformation of the excitation product process. For the derivative of the system response (fig. 4), the trend is reversed. The value of the fourth-order flatness factor increases and approaches the quasi-steady value of 9 as the value of μ is decreased. In this limit, the derivative becomes a static transformation of the product process and is thus a product process. This limiting property of the derivative also follows directly from the differential equation of the original dynamic system, that is, by setting the value of μ to zero in equation (15c).

The stationary forms of the Fokker-Planck and characteristic function equations (eqs. (16) and (18)) can be solved in two special cases. In the first case, the value of the parameter ζ in the characteristic function equation is set to zero. The resulting equation is that for the characteristic function of the two independent Gaussian processes S and R. In the second case, the value of the filter constant a_s of the amplitude process is set to zero in the Fokker-Planck equation to obtain the special form for the quasi-steady approximation, which is

$$a_r \left[\frac{\partial}{\partial r} (rp) + \frac{\partial^2 p}{\partial r^2} \right] + \mu \frac{\partial}{\partial z} (zp) - rs \frac{\partial p}{\partial z} = 0 \quad (23)$$

The probability density function is replaced by the conditional function, which is the joint probability density function of the system response and the R component, conditional on the value of the amplitude parameter s. This conditional joint probability density has the Gaussian functional form with the appropriate covariance matrix for the system response and the R component.

Response of General Linear System to Product Process

The relations for the moments of the response of a linear time-invariant dynamic system of general order are now developed. Relations are developed first for the response of a set of uncoupled first-order systems, the development being a direct extension of that for the single first-order system considered previously. The relations for the uncoupled system can be applied to a general linear system by introducing the eigenvalues and eigenvectors of the coefficient matrix of the corresponding state equation to uncouple the general linear system. The response moments for the general system can thus be computed from those for the uncoupled system. Relations for the covariance matrices of the response of the general linear system are also developed. These relations can be expressed without the use of the eigenvalues and eigenvectors of the state coefficient matrix.

Moment equation.- For a set of uncoupled first-order linear systems, the state equation of the composite dynamic system includes the equations for the S and R component processes and the equation for the uncoupled system.

$$\dot{s} = -a_s s + \sqrt{2a_s} b \xi_1(t) \quad (24a)$$

$$\dot{r} = -a_r r + \sqrt{2a_r} \xi_2(t) \quad (24b)$$

$$\dot{z}_j = -\mu_j z_j + rs \gamma_j \quad (\text{No summation on } j) \quad (24c)$$

This composite system is a direct extension of equations (15a) to (15c). The associated filter constants, which are the negatives of the eigenvalues, can be complex with the restriction that the real parts be positive. The moment equation for the composite system is obtained by developing the associated Fokker-Planck and characteristic function equations. The development is similar to that of equations (16) to (18). The equation for the characteristic function of the composite system of equations (24a) to (24c) is

$$-\frac{\partial C}{\partial t} = a_s \left[\sigma \frac{\partial C}{\partial \sigma} + b^2 \sigma^2 C \right] + a_r \left[\rho \frac{\partial C}{\partial \rho} + \rho^2 C \right] + \mu_j \zeta_j \frac{\partial C}{\partial \zeta_j} + i \gamma_j \zeta_j \frac{\partial^2 C}{\partial \sigma \partial \rho} \quad (25)$$

The ζ_j are the Fourier transformation variables corresponding to the z_j variables. The corresponding moment equation is

$$\left[\frac{\partial}{\partial t} + k a_s + l a_r + \sum_{j=1}^n \nu_j \mu_j \right] M(k; l; \nu_1, \dots, \nu_n) \\ = k(k-1) a_s b^2 M(k-2; l; \nu_1, \dots, \nu_n) + l(l-1) a_r M(k; l-2; \nu_1, \dots, \nu_n) \\ + \sum_{j=1}^n \nu_j \gamma_j M(k+1; l+1; \nu_1, \dots, (\nu_j-1), \dots, \nu_n) \quad (26)$$

where

$$M(k; l; \nu_1, \nu_2, \dots, \nu_n) = E \left[s^k r^l z_1^{\nu_1} z_2^{\nu_2} \dots z_n^{\nu_n} \right]$$

The individual terms of the moment equation correspond to the individual terms of the state equation for the composite system. The terms on the left-hand side, aside from the time-derivative term, correspond to the linear response terms in the set of first-order differential equations. The first two terms on the right-hand side correspond to the two white noise processes. The final set of terms on the right-hand side corresponds to the excitation of the uncoupled dynamic system by the product process.

In the stationary case, the time-derivative term of the moment equation is zero. The stationary moments of the system response can be computed from equation (26) by a procedure which is based upon the sum of the orders of the moments of the original dynamic system

$$\sum_{j=1}^n \nu_j = N \quad (27)$$

The moments are developed by computing all combinations of moments for a given value of the index N , and then increasing the value of that index by one. The procedure starts with the index N having the value zero; the corresponding moments are those of the independent and Gaussian R and S component processes. As the value of the index N is increased, the moments are generated by the first two terms on the right-hand side of equation (26) (which introduce the moments of lower order in the R and S components) and by the final set of terms on the right-hand side (which passes the moments from one value lower in index N (and therefore known) and from one value higher in the indices of the R and S components). This procedure is easily implemented on a digital computer. The only special consideration is the core storage required if the size (index n) of the dynamic system is large.

In the nonstationary case, the moment equation is used to compute the time-dependent transition moments which result from the development of the system response from a given initial condition. Given the moments at one time value, the moment equation is used to compute the first derivatives (with respect to time) of the moments. The derivatives are used to determine the moments at a subsequent time value. The process is then repeated.

The moment equation of the set of uncoupled first-order systems (eq. (24c)) can be used to compute the moments of a coupled linear system

$$\dot{Y}_j = F_{jk}Y_k + rsg_j \quad (28)$$

The transformation between the two systems is accomplished by using the eigenvalues and eigenvectors of the state coefficient matrix F of the coupled system. The moment equation is thus the essential element in the development of the moments of the response of a general time-invariant linear system to excitation by the product of two Gaussian processes.

Covariance matrix equation.- Specific relations can be developed for the covariance matrices of the response of a general linear system. Since the equation has a reasonably simple form, the covariance matrices can be computed directly without the introduction of the eigenvalues and eigenvectors of the system. The development of the covariance matrix equation is a standard method for the analysis of the response of linear dynamic systems to Gaussian white noise processes (refs. 14, 16, and 17).

The state equation of the composite dynamic system is formed from equations (24a), (24b), and (28). The corresponding Fokker-Planck, characteristic function, and moment equations are developed in the same manner as for the first-order systems considered previously. The joint probability density function of the system response and the two component processes is determined from the Fokker-Planck equation for the composite dynamic system

$$\frac{\partial p}{\partial t} = a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] + a_r \left[\frac{\partial}{\partial r} (rp) + \frac{\partial^2 p}{\partial r^2} \right] - \frac{\partial}{\partial y_j} [F_{jk}Y_k p] - rsg_j \frac{\partial p}{\partial y_j} \quad (29)$$

The joint characteristic function is the Fourier transformation over all independent variables of the joint probability density function

$$C(\sigma, \rho, \eta_j | s_0, r_0, y_{j0}) = E\{p(s, r, y_j | s_0, r_0, y_{j0})\} \quad (30)$$

The corresponding characteristic function equation is the Fourier transformation of the Fokker-Planck equation

$$-\frac{\partial C}{\partial t} = a_s \left[\sigma \frac{\partial C}{\partial \sigma} + b^2 \sigma^2 C \right] + a_r \left[\rho \frac{\partial C}{\partial \rho} + \rho^2 C \right] - F_{jk} \eta_j \frac{\partial C}{\partial \eta_k} + i g_j \eta_j \frac{\partial^2 C}{\partial \sigma \partial \rho} \quad (31)$$

The equation for the covariance matrix of the system response variables is obtained by taking the partial derivatives of the characteristic function equation with respect to two system variables and then developing the moments from the characteristic function

$$-\frac{\partial}{\partial t} \Sigma_{j\ell} + F_{jk} \Sigma_{k\ell} + F_{\ell k} \Sigma_{kj} = -g_j h_\ell - g_\ell h_j \quad (32)$$

where

$$\Sigma_{k\ell} = E[Y_k Y_\ell]$$

$$h_j = E[s r y_j]$$

This equation is similar to that for the response of linear dynamic systems to white noise processes and differs only in the form of the term on the right-hand side, which is related to the excitation process. The solution of the equation for the covariance matrix of the response requires the array of triple moments of the two component processes and the array of state variables of the original dynamic system. The equation for these moments is obtained from the characteristic function equation by taking the partial derivatives with respect to σ , ρ , and one of the η variables and then developing the moments from the characteristic function

$$\frac{\partial}{\partial t} h_j + (a_s + a_r) h_j - F_{jk} h_k = b^2 g_j \quad (33)$$

This equation can be solved directly for the array of triple moments, which is then used to determine the covariance matrix of the system response by equation (32).

The covariance matrix of the first derivative of the system response is obtained from the state equation of the system response (eq. (28))

$$E\left[\dot{y}_j \dot{y}_k\right] = F_{j\ell} \Sigma_{\ell m} F_{km} + F_{j\ell} h_{\ell} g_k + F_{k\ell} h_{\ell} g_j + b^2 g_j g_k \quad (34)$$

This covariance matrix can be determined once that of the system response is known.

The equations for the covariance matrix (eqs. (32) and (33)) show the effect of the quasi-steady approximation. The filter constants of the amplitude and local R processes appear only in their sum in equation (33). Thus, if the amplitude process is much slower than the R process, that is, $a_s \ll a_r$, the error in using the quasi-steady approximation ($a_s/a_r = 0$) is small for the covariance matrix. Also, the quasi-steady approximation does not depend on the dynamic properties of the linear dynamic system in the limiting case, although those properties generally affect the accuracy of the quasi-steady approximation in the nonlimiting case.

Stationary Response of Second-Order Linear System

The preceding development is applied to the analysis of the response of a second-order linear system to the product process. The differential equation for the dynamic system is

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2x = rs \quad (35)$$

The state equation for the composite dynamic system is developed by writing equation (35) as a state equation and combining with the first-order equations for the S and R components (eqs. (24a) and (24b)). The covariance matrix equation is then written for the composite dynamic system by using equations (32) and (33). The covariance matrix equation is solved for the variances of the response and its first derivative; this gives

$$E\left[x^2\right] = \frac{b^2}{\beta\omega^2} \left[1 + \frac{(a_s + a_r)}{2\zeta\omega} \right] \quad (36a)$$

$$E\left[\dot{x}^2\right] = \frac{(a_s + a_r)b^2}{2\zeta\omega\beta} \quad (36b)$$

where $\beta = (a_s + a_r)^2 + 2\zeta\omega(a_s + a_r) + \omega^2$. The quasi-steady approximation is obtained from the exact solution by setting the ratio a_s/a_r to zero. The

quasi-steady form of the two variances may be either larger or smaller than the exact values, depending on the values of the system damping and frequency coefficients.

The higher moments of the system response are developed from the general relation (eq. (26)) by using the eigenvalues and eigenvectors of the state equation of the second-order linear system. In figures 5 and 6, the resulting fourth moments of the system response and its first derivative are plotted in normalized form, that is, normalized to their quasi-steady values. The moments are plotted as functions of the filter constant of the amplitude process and for several values of the system frequency coefficient ω , both quantities plotted as ratios to the filter constant of the local R process. The moments are plotted for two values of the system damping coefficient ζ . The values of the fourth moment are usually decreased by the dynamic properties of the amplitude process. The relative decrease is larger for smaller values of the frequency coefficient ω . In the case of large values of the ratio ω/a_r , the dynamic properties of the amplitude process can increase the moments of the response and particularly the first derivative of the system response. The effects of the dynamic properties of the amplitude process are usually more significant for lower values of the system damping coefficient ζ .

The fourth-order flatness factors of the system response and its first derivative are shown in figures 7 to 12. These are plotted in the same manner as the fourth moments in figures 5 and 6. Figures 7 and 8 show the flatness factors of the system response and its first derivative for the system damping coefficient ζ of 0.707. In both cases the fourth-order flatness factor has the value of 9 in the quasi-steady limit ($a_s/a_r = 0$). The values of the flatness factors are reduced by the dynamic properties of the amplitude process. For the system response (fig. 7), the value of the flatness factor is decreased as the value of the system frequency coefficient ω is decreased. As the value of the system frequency is increased the flatness factor approaches the quasi-steady value of 9. The qualitative features of the response of the second-order system are thus similar to those of the response of the first-order system (fig. 3). The flatness factors of the first derivative of the system response (fig. 8) show the same features as the response itself except that the flatness factor is less sensitive to the value of the system frequency coefficient ω . Also, the flatness factor of the first derivative does not approach the quasi-steady value of 9 in the limit of large values of the system frequency coefficient. Figures 9 and 10 show the fourth-order flatness factors of the system response and its first derivative for the system damping coefficient ζ of 0.10. The values of the flatness factor tend to be considerably less than those for the higher value of the damping coefficient in figures 7 and 8. Otherwise the flatness factors show the same general features, including the value of 9 in the quasi-steady limit.

The results of figures 5 to 10 indicate whether use of the quasi-steady approximation will either overestimate or underestimate the occurrence of the extreme values of the system response and its first derivative. Since the values of the fourth-order flatness factors are maximum in the quasi-steady limit, use of the quasi-steady approximation in combination with the exact values of the second moments (eqs. (36a) and (36b)) will tend to overestimate the occurrence of the extreme values of both the system response and its first

derivative. If the quasi-steady values of the second moments are used, then the quasi-steady approximation will tend to overestimate the occurrence of extreme values of the system response (fig. 5). (In the limit of large values of the ratio ω/a_r the quasi-steady relation becomes exact, since the system response becomes a static transformation of the excitation product process.) The same form of the quasi-steady approximation can either overestimate or underestimate the occurrence of extreme values of the first derivative, depending upon the value of the ratio ω/a_r (fig. 6).

The effect of the damping coefficient is shown explicitly in figures 11 and 12 where the fourth-order flatness factors are plotted for several values of the damping coefficient ζ and for a single value of the system frequency coefficient. For both the system response and its first derivative, the flatness factor is strongly dependent upon the value of the damping coefficient. As the value of the damping coefficient is increased, the flatness factors are increased except for values of the damping coefficient near 1. For low values of the damping coefficient, the flatness factors of both the response and its first derivative are close to the Gaussian value of 3. The improvement of the quasi-steady approximation with increased system damping was indicated by the qualitative development of the quasi-steady approximation (eq. (4)). In all cases the appropriate value of 9 is obtained in the quasi-steady limit. For small values of the system damping coefficient, the dynamic properties of the amplitude process invalidate the quasi-steady approximation, that is, for small (but nonzero) values of the ratio of the filter constants a_s/a_r the flatness factors approach the value of 3. Thus for lightly damped second-order linear systems, the dynamic properties of the amplitude component process largely destroy the effect of the amplitude modulation.

A general rule of random process theory is that linear filtering of a non-Gaussian stationary process results in a tendency toward a Gaussian stationary process (refs. 18 and 19). This rule is based upon the expression of the linear system response, using the impulsive response function and the convolution integral, as a weighted sum of the non-Gaussian excitation variables over a range of time values. Since the summation introduces an element of statistical disorder, there is a tendency of the sum toward a Gaussian distribution as indicated by the central limit theorem. Further, this tendency is stronger if the system response has a longer duration, that is, the system frequency is lower or particularly if the system damping is lower. The previous results for the fourth-order flatness factors for the response of the second-order system to the product process are consistent with these rules.

Response of First-Order System to Product-Plus-Mean Process

An example of the analysis of the response of a linear dynamic system to the product-plus-mean process (expression (B)) is briefly considered. Since the dynamic system is linear, the response to the independent product and mean value processes can be considered separately. Since the mean value process is Gaussian, the corresponding linear system response is also Gaussian. This response is added to that for the product random process.

The linear system of equation (15c), modified by the addition of the mean value component into the excitation process, is considered for an example. The mean value process is formed by a first-order filtering of an independent white noise process in the same manner as the amplitude and the local R components

$$\dot{m} = -a_m m + \sqrt{2a_m} c \xi_3(t) \quad (37)$$

The dynamic system is a first-order linear system excited by the product-plus-mean process

$$\dot{z} = -\mu z + rs + m \quad (38)$$

The analysis of the system response can be developed by using the Fokker-Planck, characteristic function, and moment equations for the composite dynamic system. However, it is easier to consider the response to the independent product and mean value processes separately

$$z = z_1 + z_2 \quad (39)$$

The first term on the right-hand side is the response to the product process; the moments are given by equations (21) and (22). The second term is the response to the mean value process. The second moment (which equals the variance in this case) of the total response is

$$E[z^2] = \frac{b^2}{\mu(a_s + a_r + \mu)} + \frac{c^2}{\mu(a_m + \mu)} \quad (40)$$

The covariance of the derivative of the response is found by using the differential equation of the original system (eq. (38)) and equation (22)

$$E[\dot{z}^2] = \frac{(a_s + a_r)b^2}{(a_s + a_r + \mu)} + \frac{a_m c^2}{(a_m + \mu)} \quad (41)$$

These two relations show the relative influence of the product and the mean value processes, which are identified by their standard deviations b and c , respectively. All other moments of the system response can be developed in a similar manner. The main problem is the analysis of the system response to the product process; the analysis of the additional response to the mean value process introduces no fundamental difficulty for linear systems.

Possible Extensions of the Analysis

The present development of the exact analysis of the response of dynamic systems to product random processes can be extended in several ways. One extension is the development of the analysis for more general forms of the two component processes. The primary requirement is that the components be generated by linear time-invariant filtering of Gaussian white noise processes. The differential equations for the components become more complicated, but the formulation procedure for the associated Fokker-Planck, characteristic function, and moment equations is not changed. The relations for the response of a linear system to a product process having one component formed by a general-order filtering of Gaussian white noise are developed in appendix B.

The present development can be extended to the analysis of dynamic system response to more general forms of the product process. One possible form, for example, is the sum of several independent Gaussian and product processes

$$\sum_{j=1}^{N_1} m_j(t) + \sum_{j=1}^{N_2} r_j(t) s_j(t)$$

All component processes are specified to be stationary, Gaussian, and independent. The general form of the preceding expression allows considerable flexibility in modeling both the dynamic and probabilistic structures of a random process. The response of a linear dynamic system to the general random process is the sum of the terms for the response to the individual terms of the excitation process, where the individual terms of both the excitation and response processes are independent. The moments of the system response to the individual terms and thus to the total excitation process can be determined by the present analytical method.

The present development can also be extended to the analysis of the non-stationary response of dynamic systems to product processes. The moments are functions of time due to the nonstationarity of the system response. One example of the nonstationary case is the transient response of linear time-invariant systems due to initial conditions (refs. 14 and 16). A second example is the analysis of the response of linear dynamic systems with time dependent coefficients (ref. 17).

CONCLUDING REMARKS

Exact and approximate solutions for the response of dynamic systems to the product of two independent and stationary Gaussian processes are examined. The Fokker-Planck equation is used to develop the exact solution for the system response. For linear time-invariant systems, the associated moment equation can be formulated and solved for all statistical moments of the system response. The development thus presents a method for the analysis of the response of dynamic systems to a class of strongly non-Gaussian random processes.

The validity of the quasi-steady approximation for the analysis of the response of dynamic systems to amplitude modulated processes is examined. The quasi-steady solution is developed from the exact solution by omitting the dynamic properties of the modulating amplitude random process. The quasi-steady approximation requires that the amplitude process be slowly varying with respect to the local random process. The properties of the dynamic systems can strongly influence the accuracy of the approximation, although the approximation is accurate in the limiting case of slow variation of the amplitude process relative to the local process. For linear systems with low damping, the quasi-steady approximation is not valid in the nonlimiting case, since the values of the fourth-order flatness factor of the response are much closer to those of a Gaussian process than those of an amplitude modulated process.

The response of dynamic systems to the sum of a product process and an independent Gaussian mean value process is examined. For linear systems the response can be developed by considering the response to the independent product and mean value processes separately. For nonlinear systems the coupled response can be developed by using the Fokker-Planck equation.

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APPENDIX A

RELATIONS FOR CHARACTERISTIC FUNCTIONS

Several relations which are used in the development of the characteristic function equations and the corresponding moment equations are derived in this appendix. The characteristic function is the Fourier transformation of the probability density function

$$C(\zeta) = E[e^{i\zeta z}] = \int_{-\infty}^{\infty} e^{i\zeta z} p(z) dz = F\{p(z)\} \quad (A1)$$

The moments of a random process are obtained from the characteristic function

$$\left. \frac{d^l}{d\zeta^l} C(\zeta) \right|_{\zeta=0} = i^l E[z^l] \quad (A2)$$

The development of the characteristic function equation from the Fokker-Planck equation requires several relations associated with the Fourier transformation. The first relation is the Fourier transformation of the derivatives of the probability density function

$$F\left\{\frac{d^l}{dz^l} p(z)\right\} = i^{-l} \zeta^l C(\zeta) \quad (A3)$$

The second relation is the Fourier transformation of the product of the independent variable and its probability density function

$$F\{z^l p(z)\} = i^{-l} \frac{d^l}{d\zeta^l} C(\zeta) \quad (A4)$$

Both equations follow from the definition of the characteristic function (eq. (A1)). Another relation is obtained by combining equations (A3) and (A4)

$$F\left\{\frac{d}{dz}[z p(z)]\right\} = -\zeta \frac{dC(\zeta)}{d\zeta} \quad (A5)$$

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The development of the moment equation requires the following relations for the derivatives of the products of the independent variable and the characteristic function:

$$\left. \frac{d^l}{d\zeta^l} [\zeta C(\zeta)] \right|_{\zeta=0} = l \left. \frac{d^{l-1}}{d\zeta^{l-1}} C(\zeta) \right|_{\zeta=0} \quad (\text{A6a})$$

$$\left. \frac{d^l}{d\zeta^l} [\zeta^2 C(\zeta)] \right|_{\zeta=0} = l(l-1) \left. \frac{d^{l-2}}{d\zeta^{l-2}} C(\zeta) \right|_{\zeta=0} \quad (\text{A6b})$$

Equations (A6a) and (A6b) can also be expressed in terms of the moments of the random process by using equation (A2)

$$\left. \frac{d^l}{d\zeta^l} [\zeta C(\zeta)] \right|_{\zeta=0} = l i^{l-1} E[z^{l-1}] \quad (\text{A7a})$$

$$\left. \frac{d^l}{d\zeta^l} [\zeta^2 C(\zeta)] \right|_{\zeta=0} = l(l-1) i^{l-2} E[z^{l-2}] \quad (\text{A7b})$$

APPENDIX B

RESPONSE OF LINEAR DYNAMIC SYSTEMS - GENERAL R PROCESS

The development in the body of the paper, which considers first-order filtering of white noise for the two components of the product process, can be extended to allow more general forms of the component processes. The relations for the response of linear systems to a product process whose R component process is generated by a general linear filtering of Gaussian white noise are developed in this appendix. This general form includes both the Dryden and, by use of rational approximations, the von Kármán forms which are used to model atmospheric turbulence in aeronautical applications (ref. 10). The analytical development considers the response of a general linear system, both in coupled form for the development of the covariance matrix equation and in uncoupled form for the development of the general moment equation.

Differential Equations

The local R component of the product process is generated by a general-order linear filtering of a Gaussian white noise process. The corresponding state equation of the vector R process is

$$\dot{r}_\alpha = A_{\alpha\beta} r_\beta + d_\alpha \xi_2(t) \quad (B1)$$

where A is the response coefficient matrix and d is the excitation coefficient matrix of the state equation. The amplitude component is generated by a first-order linear filtering of an independent Gaussian white noise process

$$\dot{s} = -a_s s + \sqrt{2a_s} b \xi_1(t) \quad (B2)$$

The excitation function of the dynamic system is the product of the amplitude component process and a linear combination of the r_α variables, thus allowing a general form for the product process. The differential equation for the dynamic system is written as a linear state equation either in coupled or in uncoupled form

$$\dot{y}_j = F_{jk} y_k + g_j s \phi_{\alpha} r_\alpha \quad (B3a)$$

$$\dot{z}_j = -\mu_j z_j + \gamma_j s \phi_{\alpha} r_\alpha \quad (\text{No summation on } j) \quad (B3b)$$

where ϕ is a coefficient matrix which allows a general form for the contribution of the vector R process to the product process. The two sets of

coordinates are related by a linear transformation using the eigenvectors of the state coefficient matrix of the coupled system.

Covariance Matrix Equation

The differential equations for the components of the product process and for the coupled linear system are combined to form the state equation for the composite dynamic system. The conditional joint probability density function of this system satisfies the associated Fokker-Planck equation

$$\begin{aligned} \frac{\partial p}{\partial t} = a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] - A_{\alpha\beta} \frac{\partial}{\partial r_\alpha} (r_\beta p) + \frac{1}{2} d_{\alpha d\beta} \frac{\partial^2 p}{\partial r_\alpha \partial r_\beta} \\ - F_{jk} \frac{\partial}{\partial y_j} (y_k p) - s\phi_\alpha r_\alpha g_j \frac{\partial p}{\partial y_j} \end{aligned} \quad (B4)$$

The corresponding characteristic function is the joint Fourier transform of the probability density function

$$C(\sigma, \rho_\alpha, \eta_j | s_0, r_{\alpha 0}, y_{j0}) = F\{p(s, r_\alpha, y_j | s_0, r_{\alpha 0}, y_{j0})\} \quad (B5)$$

The equation for the characteristic function is the joint Fourier transform of the Fokker-Planck equation

$$\begin{aligned} - \frac{\partial C}{\partial t} = a_s \left[\sigma \frac{\partial C}{\partial \sigma} + \sigma^2 b^2 C \right] - A_{\alpha\beta} \rho_\alpha \frac{\partial C}{\partial \rho_\beta} + \frac{1}{2} d_{\alpha d\beta} \rho_\alpha \rho_\beta C \\ - F_{jk} \eta_j \frac{\partial C}{\partial \eta_k} + ig_j \eta_j \phi_\alpha \frac{\partial^2 C}{\partial \sigma \partial \rho_\alpha} \end{aligned} \quad (B6)$$

The equation for the covariance matrix of the system response is obtained by taking the derivative of the characteristic function equation with respect to two of the transformation variables corresponding to the system response variables. The resulting covariance equation is

$$- \frac{\partial}{\partial t} \Sigma_{jl} + F_{jk} \Sigma_{kl} + F_{lk} \Sigma_{kj} = -g_j \phi_\alpha H_{\alpha l} - g_l \phi_\alpha H_{\alpha j} \quad (B7)$$

where

$$\begin{aligned} \Sigma_{kl} &= E[y_k y_l] \\ H_{\alpha l} &= E[s r_\alpha y_l] \end{aligned}$$

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The response terms on the left-hand side are identical to those in the corresponding equation for the response to Gaussian white noise excitation. The terms on the right-hand side are different, involving the H matrix of triple moments between the S process and one element of each of the R and Y arrays. The equation for these triple moments is obtained from the characteristic function equation by taking the derivatives with respect to the three transformation variables corresponding to the variables of the triple moments. The resulting equation is

$$\frac{\partial}{\partial t} H_{\alpha\lambda} + a_S H_{\alpha\lambda} - A_{\alpha\beta} H_{\beta\lambda} - F_{lk} H_{\alpha k} = b^2 g_l \phi_{\beta} R_{\beta\alpha} \quad (B8)$$

where

$$R_{\beta\alpha} = E[r_{\beta} r_{\alpha}]$$

The covariance matrix for the vector R process is determined from the covariance matrix equation for the linear system of equation (B1)

$$-\frac{\partial}{\partial t} R_{\alpha\beta} + A_{\alpha\gamma} R_{\gamma\beta} + A_{\beta\gamma} R_{\gamma\alpha} = -d_{\alpha} d_{\beta} \quad (B9)$$

For the special case of a scalar R process, the preceding relations reduce to those developed in the body of the paper. The covariance matrix of the system response is obtained by solving the preceding relations in reverse order: equation (B9) gives the covariance matrix of the vector R process; equation (B8) gives the H matrix of the triple moments; and equation (B7) gives the covariance matrix of the system response.

Moment Equation

The general moment equation for the composite dynamic system is developed by writing the equations for both the dynamic system and the vector R process in uncoupled form. The resulting composite dynamic system consists of the first-order differential equation for the amplitude component process (eq. (B2)), the set of equations for the uncoupled variables (q) of the vector R process, and the set of equations for the uncoupled linear system

$$\dot{q}_{\alpha} = -\tau_{\alpha} q_{\alpha} + \delta_{\alpha} \xi_2(t) \quad (\text{No summation on } \alpha) \quad (B10a)^4$$

$$\dot{z}_j = -\mu_j z_j + \gamma_j s \psi_{\beta} q_{\beta} \quad (\text{No summation on } j) \quad (B10b)$$

⁴The transverse velocity component of the Dryden spectral function requires special consideration since it introduces multiple eigenvalues.

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where the $-\tau$, δ , and ψ arrays are obtained by transformation of the A , d , and ϕ arrays of equations (B1) and (B3b).

The joint probability density function of this composite dynamic system satisfies the associated Fokker-Planck equation

$$\begin{aligned} \frac{\partial p}{\partial t} = a_s \left[\frac{\partial}{\partial s} (sp) + b^2 \frac{\partial^2 p}{\partial s^2} \right] + \tau_\alpha \frac{\partial}{\partial q_\alpha} (q_\alpha p) + \frac{1}{2} \delta_\alpha \delta_\beta \frac{\partial^2 p}{\partial q_\alpha \partial q_\beta} \\ + \mu_j \frac{\partial}{\partial z_j} (z_j p) - s \psi_\alpha q_\alpha \gamma_j \frac{\partial p}{\partial z_j} \end{aligned} \quad (B11)$$

The moment equation of the composite dynamic system is developed either from the Fokker-Planck equation or from the corresponding equation for the characteristic function. The following notation is used for the joint statistical moments:

$$E \left[s^k q_1^{\lambda_1} \dots q_\ell^{\lambda_\ell} z_1^{\nu_1} \dots z_n^{\nu_n} \right] = M(k; \lambda_1, \dots, \lambda_\ell; \nu_1, \dots, \nu_n) \quad (B12)$$

The resulting moment equation for the composite dynamic system is

$$\begin{aligned} \left[\frac{\partial}{\partial t} + k a_s + \sum_{\alpha=1}^{\ell} \lambda_\alpha \tau_\alpha + \sum_{j=1}^n \nu_j \mu_j \right] M(k; \lambda_1, \dots, \lambda_\ell; \nu_1, \dots, \nu_n) \\ = k(k-1) a_s b^2 M(k-2; \lambda_1, \dots, \lambda_\ell; \nu_1, \dots, \nu_n) \\ + \frac{1}{2} \sum_{\alpha=1}^{\ell} \lambda_\alpha (\lambda_\alpha - 1) \delta_\alpha^2 M(k; \lambda_1, \dots, \lambda_{\alpha-2}, \dots, \lambda_\ell; \nu_1, \dots, \nu_n) \\ + \frac{1}{2} \sum_{\alpha=1}^{\ell} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\ell} \lambda_\alpha \lambda_\beta \delta_\alpha \delta_\beta M(k; \lambda_1, \dots, \lambda_{\alpha-1}, \dots, \lambda_{\beta-1}, \dots, \lambda_\ell; \nu_1, \dots, \nu_n) \\ + \sum_{\alpha=1}^{\ell} \sum_{j=1}^n \psi_\alpha \nu_j \gamma_j M(k+1; \lambda_1, \dots, \lambda_{\alpha+1}, \dots, \lambda_\ell; \nu_1, \dots, \nu_{j-1}, \dots, \nu_n) \end{aligned} \quad (B13)$$

APPENDIX B

The first term on the left-hand side of the moment equation corresponds to the transient response of the composite dynamic system. The remaining three sets of terms on the left-hand side correspond to the linear response terms of the composite dynamic system. The first term on the right-hand side corresponds to the Gaussian white noise excitation term of the amplitude component process. The next two sets of terms on the right-hand side correspond to the white noise excitation terms of the vector Q process. These terms on the right-hand side pass the moments from lower to higher in the components of the excitation processes without changing the order in the system response variables. The final set of terms corresponds to the excitation term of the linear dynamic system. This set of terms passes the moments of given order in the response variables (and higher order in the amplitude and one component of the vector Q process) to the moments of higher order in the response variables. The moment equation can be solved numerically by the procedure described in the body of the paper for the case of a scalar R process. The resulting moments in the uncoupled response variables are transformed to those of the original dynamic system by using the linear transformation of the eigenvectors of the coefficient matrix of the original dynamic system.

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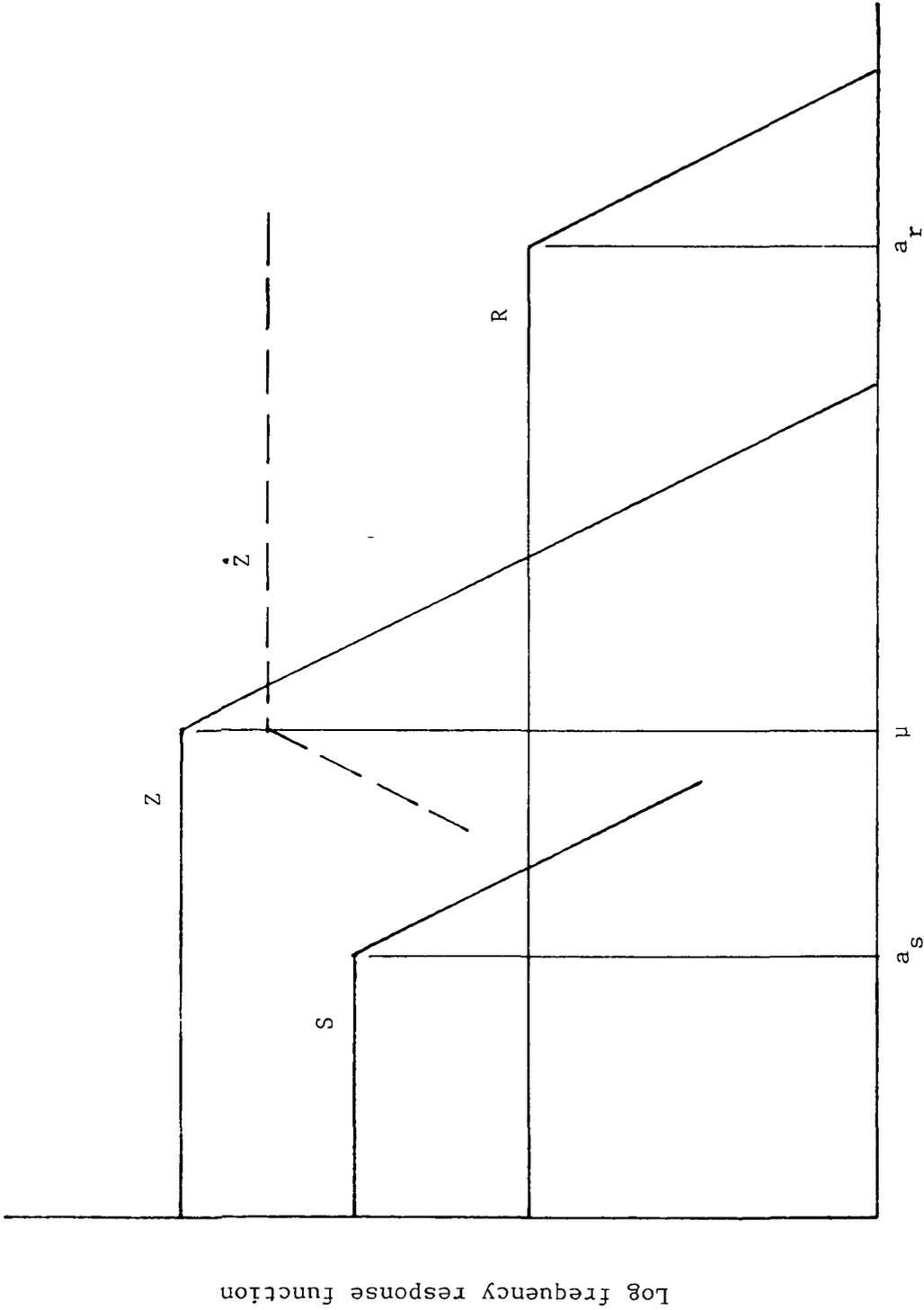
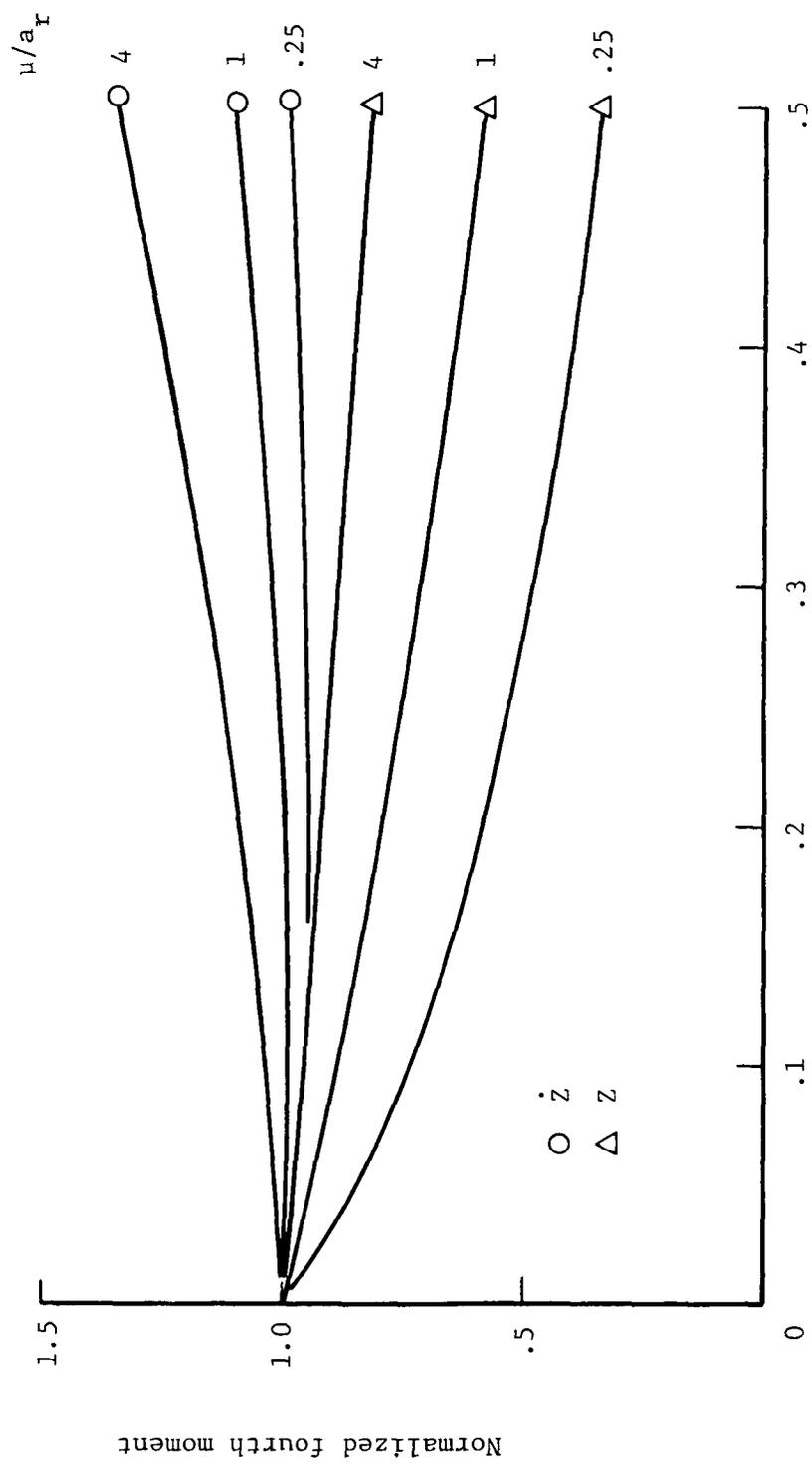
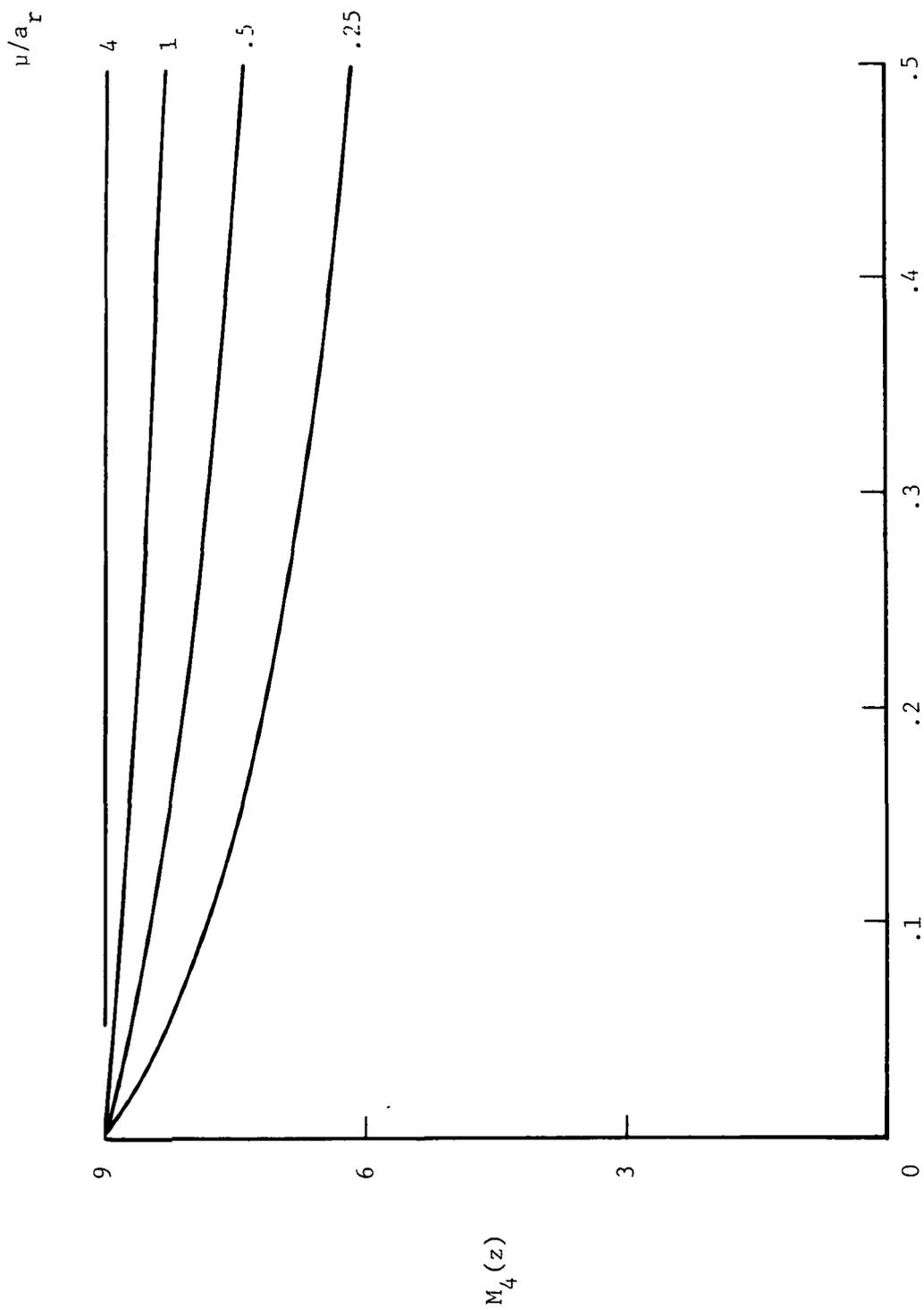


Figure 1.- Schematic form of frequency response function of three state variables of equations (15a) to (15c).



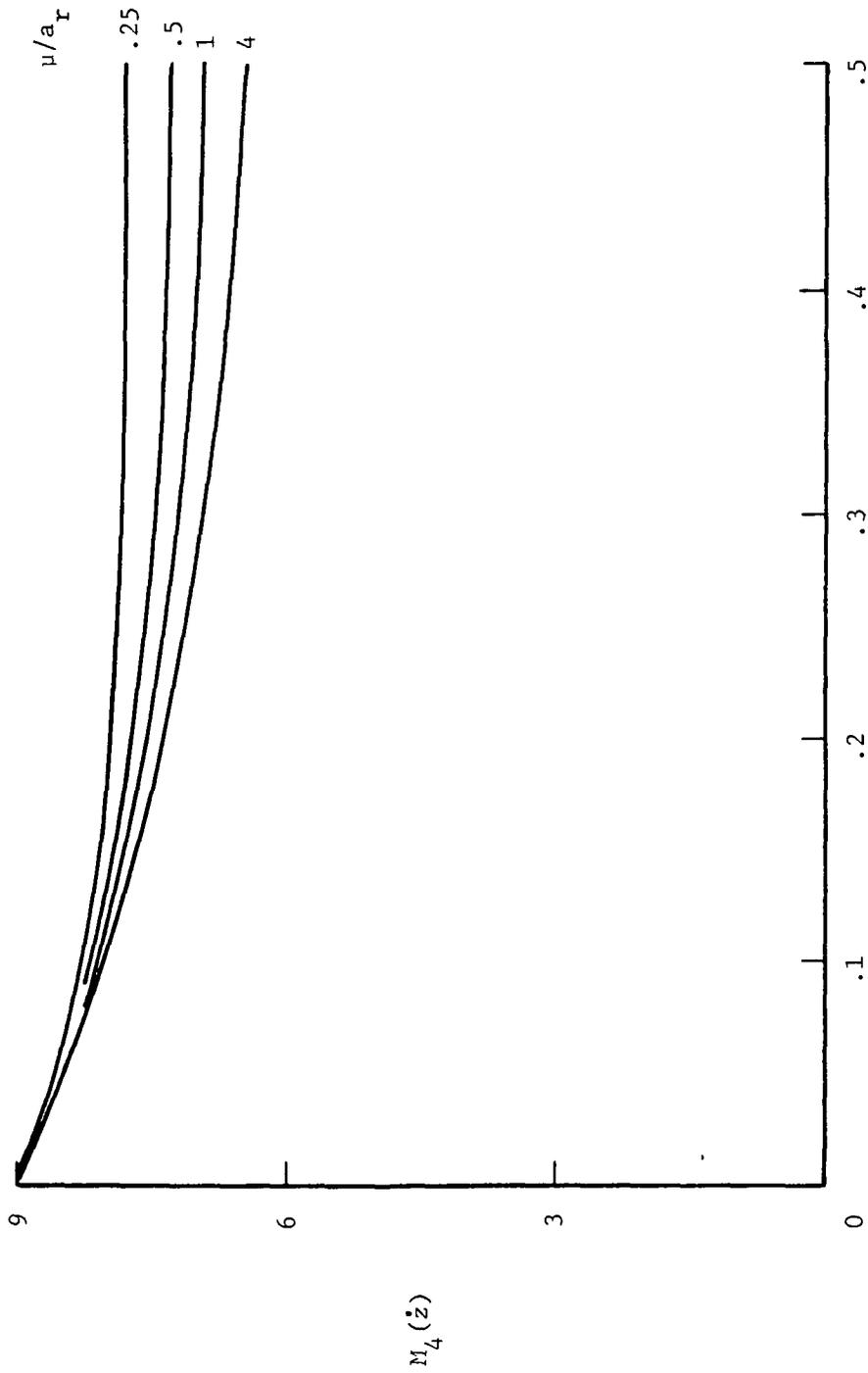
Ratio of filter constants of S and R processes, a_s/a_r

Figure 2.- Normalized fourth moments of response and first derivative of response of first-order system to product process.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 3.- Fourth-order flatness factor of response of first-order system to product process.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 4.- Fourth-order flatness factor of first derivative of response of first-order system to product process.

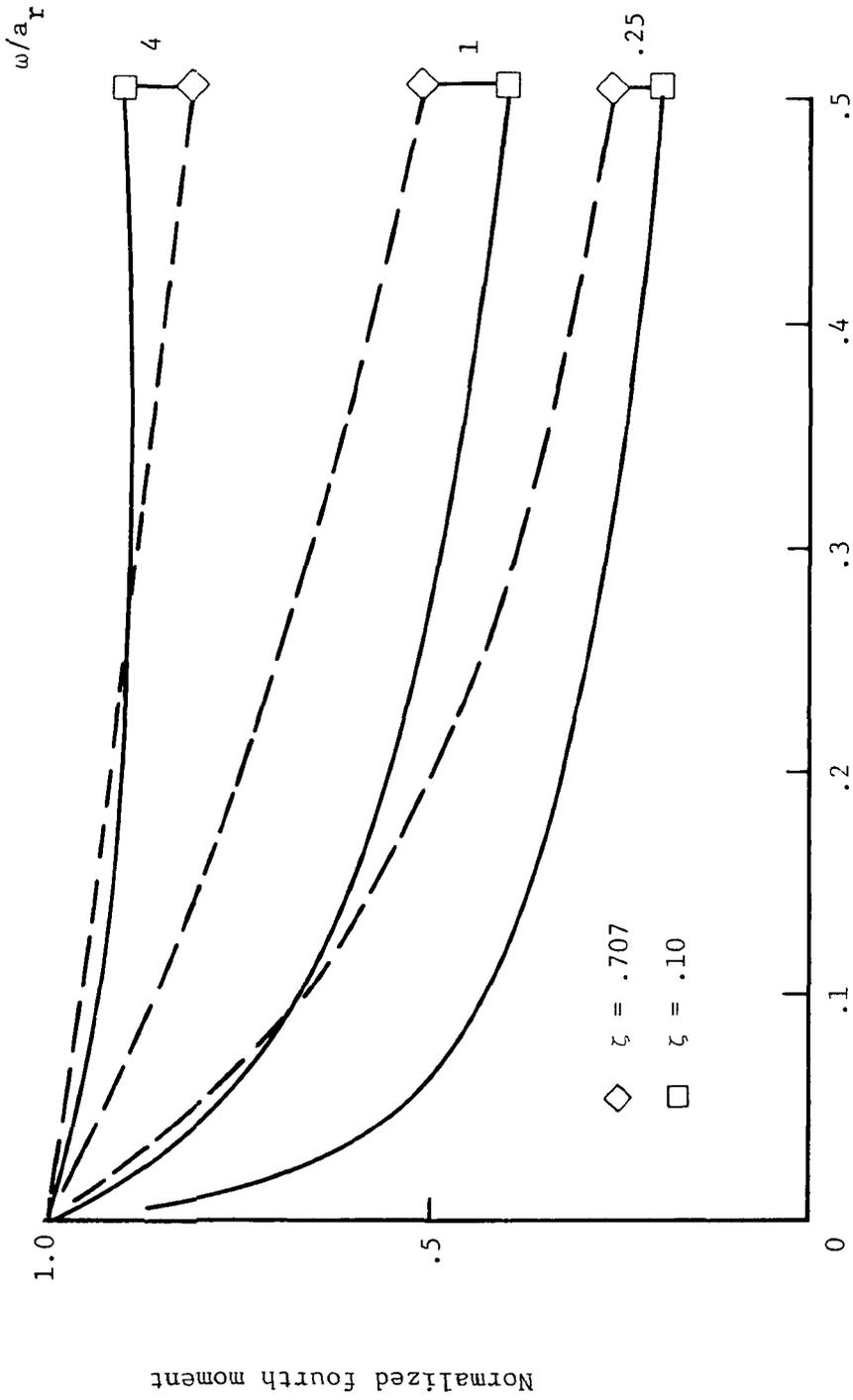
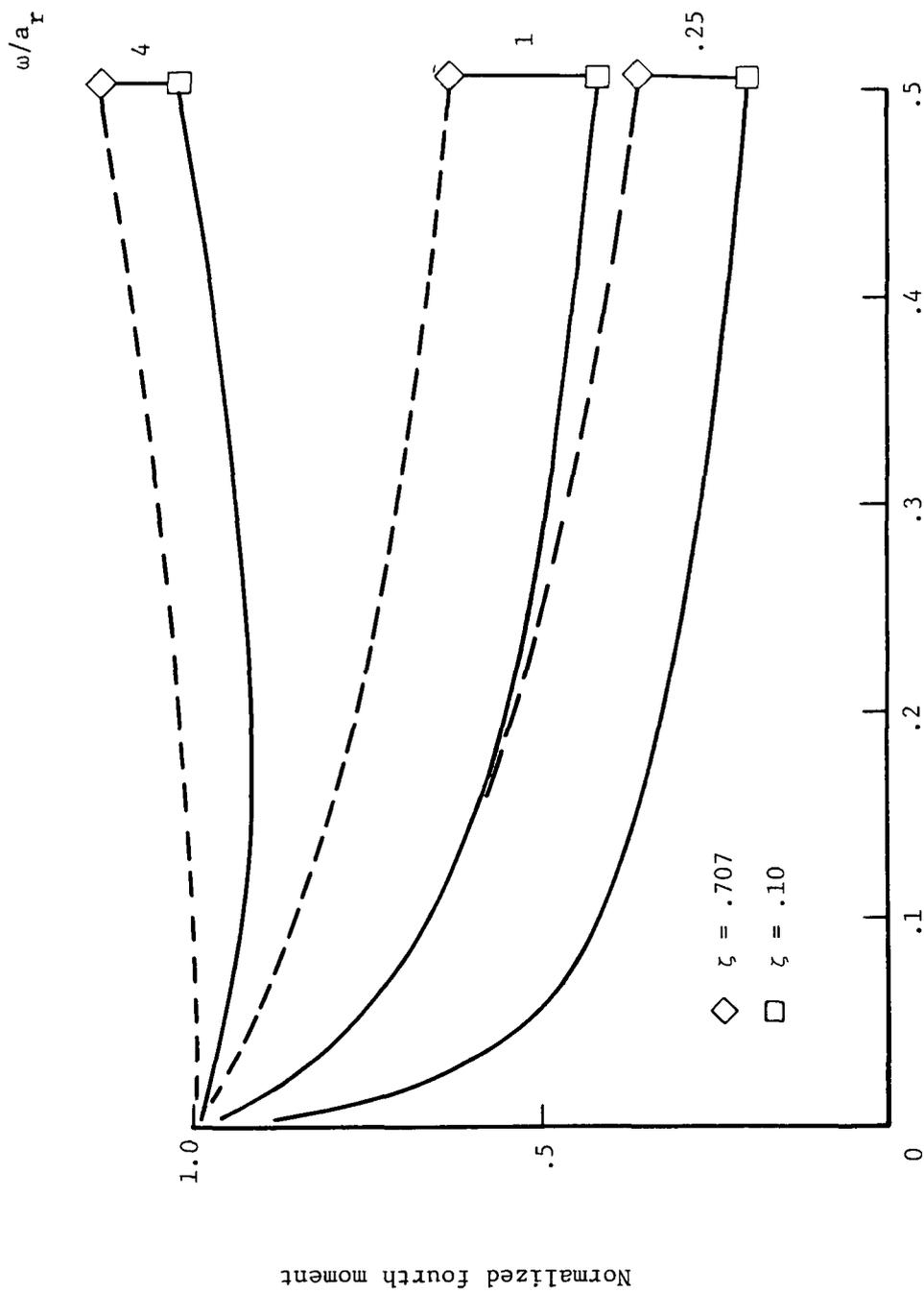
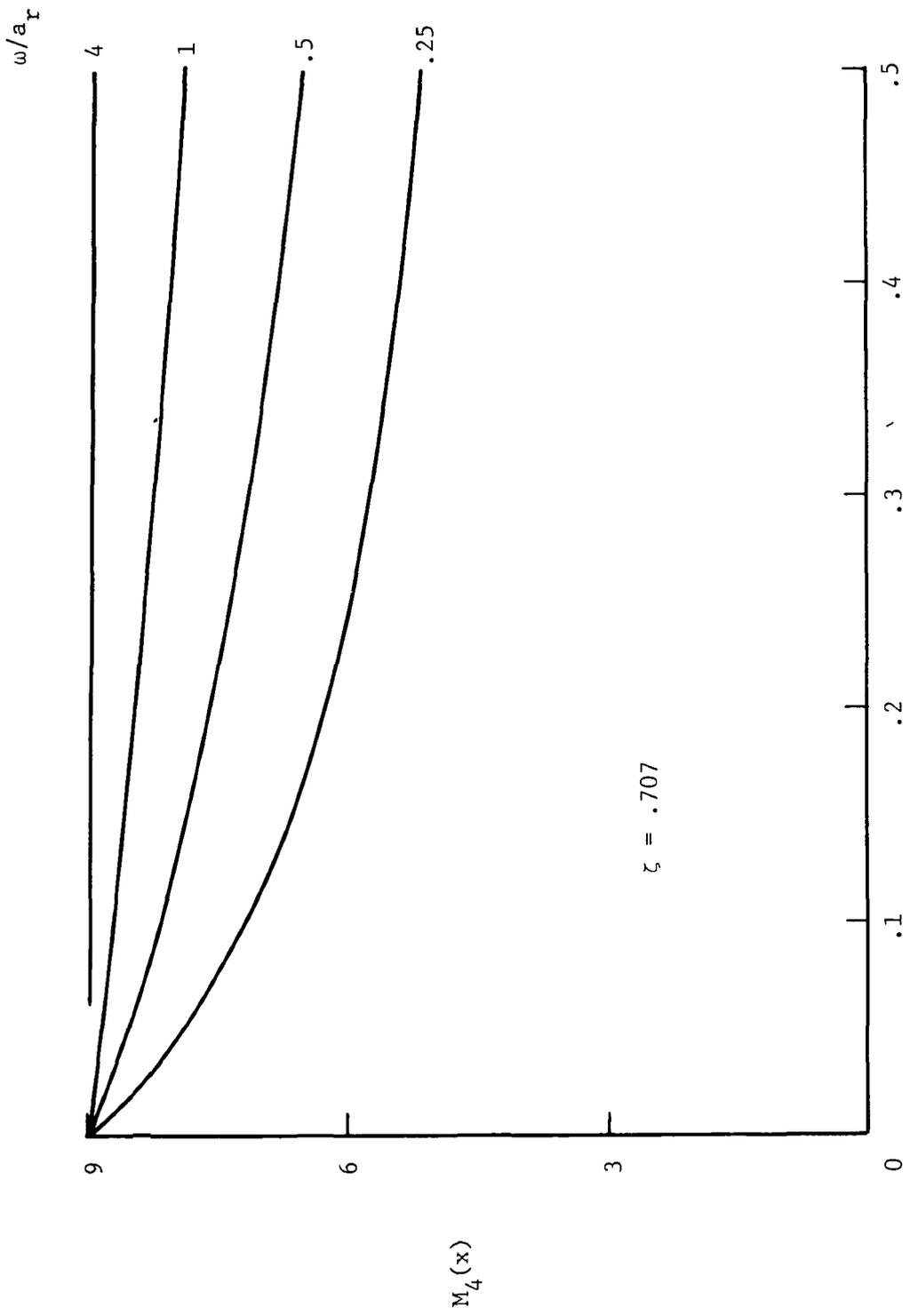


Figure 5.- Normalized fourth moment of response of second-order system to product process.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 6.- Normalized fourth moment of first derivative of response of second-order system to product process.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 7.- Fourth-order flatness factor of response of second-order system to product process for $\zeta = 0.707$.

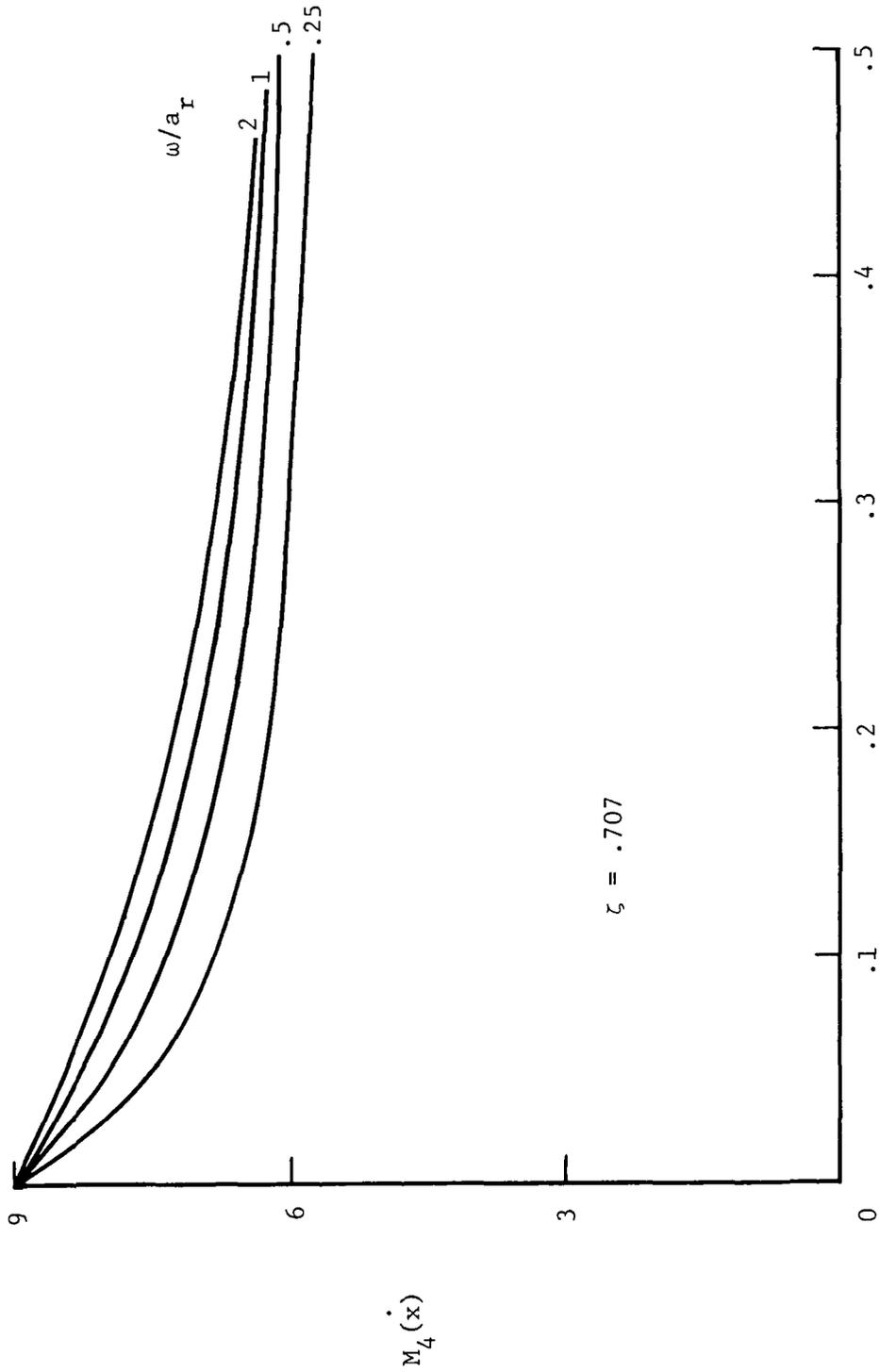
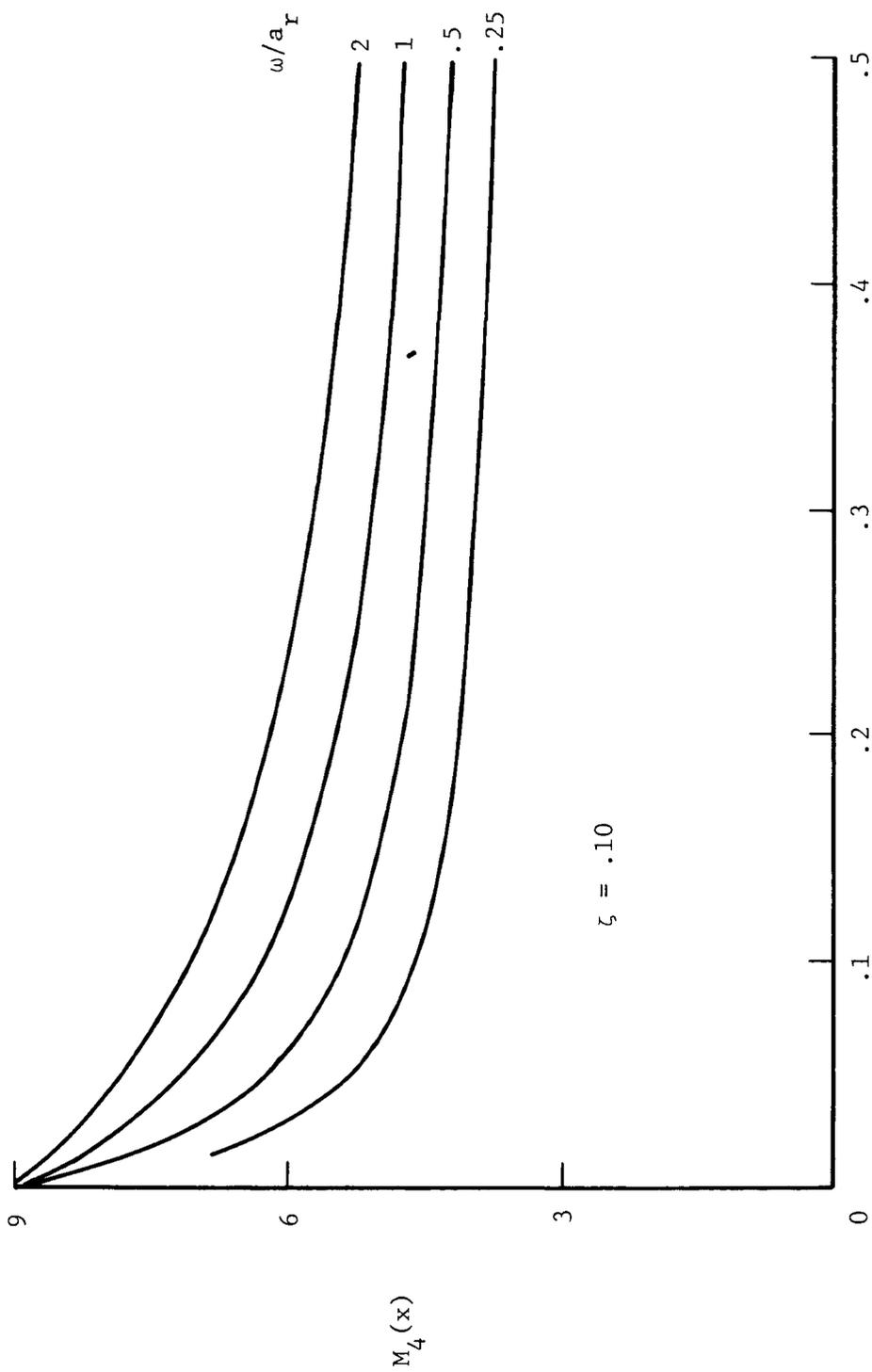


Figure 8.- Fourth-order flatness factor of first derivative of response of second-order system to product process for $\zeta = 0.707$.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 9.- Fourth-order flatness factor of response of second-order system to product process for $\zeta = 0.10$.

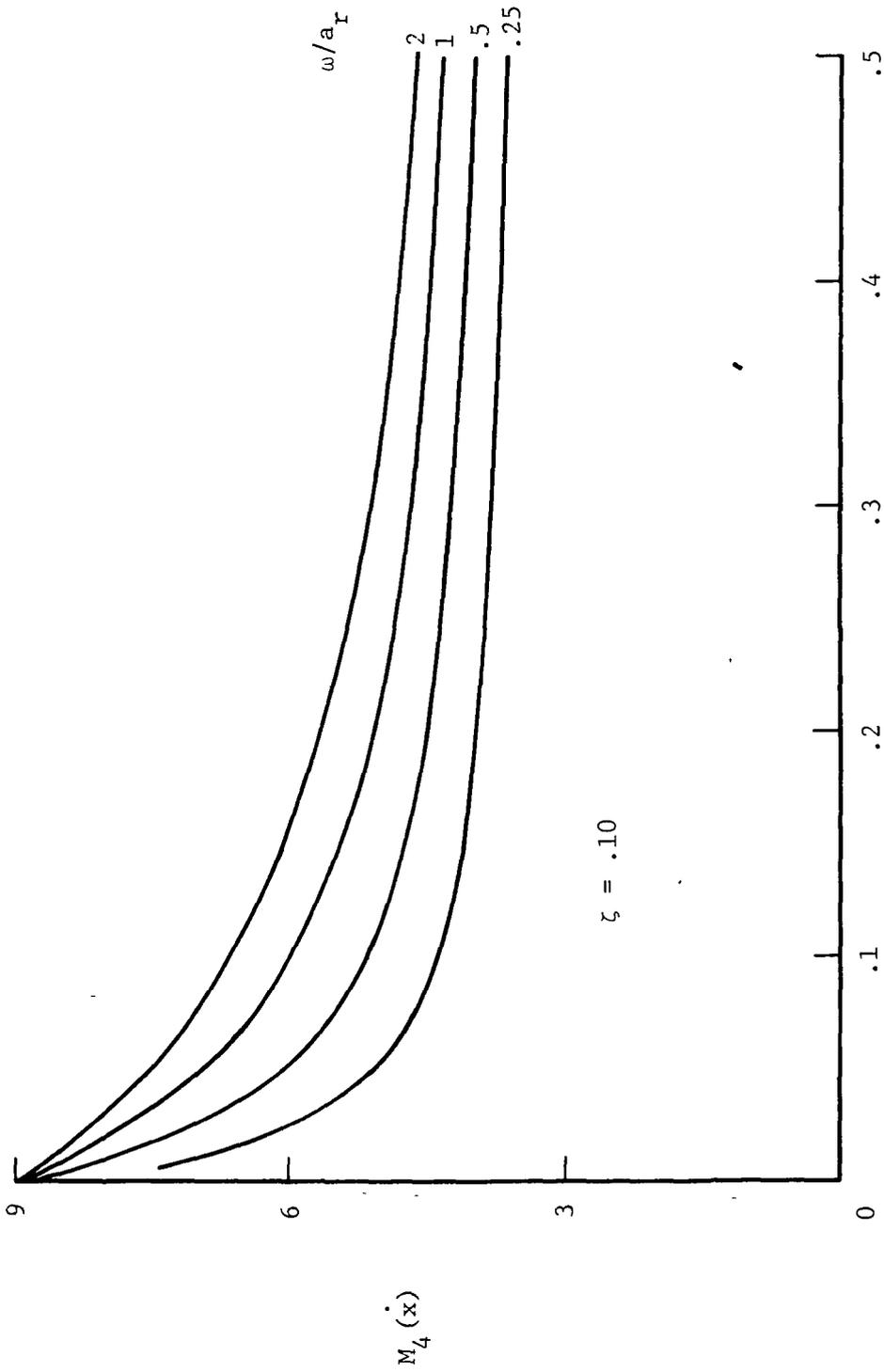
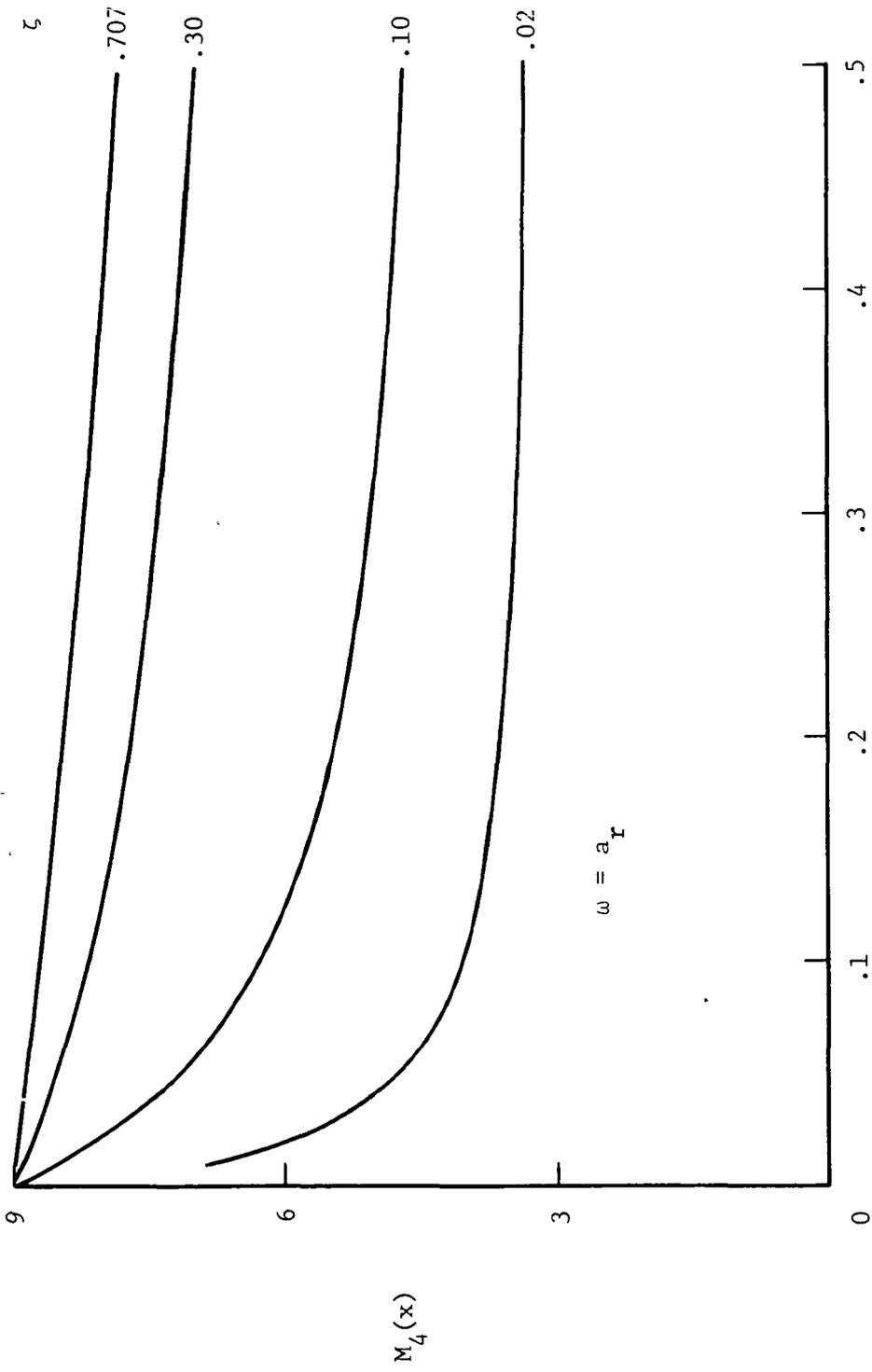
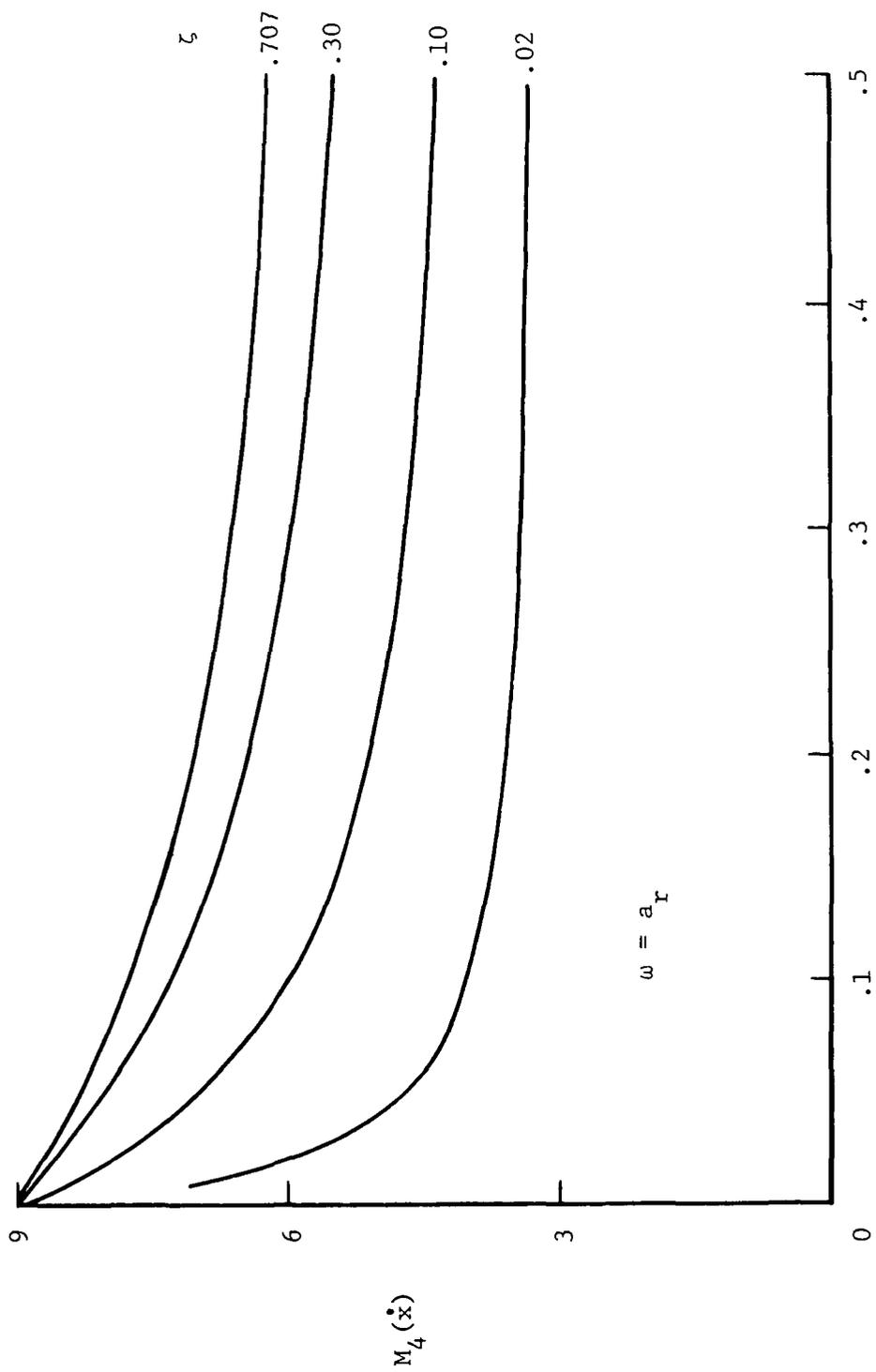


Figure 10.- Ratio of filter constants of S and R processes, a_s/a_r
 Fourth-order flatness factor of first derivative of response of
 second-order system to product process for $\zeta = 0.10$.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 11.- Fourth-order flatness factor of response of second-order system to product process.



Ratio of filter constants of S and R processes, a_s/a_r

Figure 12.- Fourth-order flatness factor of first derivative of response of second-order system to product process.

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16 Abstract The response of dynamic systems to the product of two independent Gaussian random processes is developed by use of the Fokker-Planck and associated moment equations. The development is applied to the amplitude modulated process which is used to model atmospheric turbulence in aeronautical applications. The exact solution for the system response is compared with the solution obtained by the quasi-steady approximation which omits the dynamic properties of the random amplitude modulation. The quasi-steady approximation is valid as a limiting case of the exact solution for the dynamic response of linear systems to amplitude modulated processes. In the nonlimiting case the quasi-steady approximation can be invalid for dynamic systems with low damping.			
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