Reports of the Department of Geodetic Science

Report No. 262

(NASA-CR-157606) THE OBSERVABILITY OF THE CELESTIAL POLE AND ITS NUTATIONS (Ohio State Univ., Columbus.) 103 p HC A06/MF A01 CSCL 03A N78-32946
Unclas
G3/89 30301

THE OBSERVABILITY OF THE CELESTIAL POLE AND ITS NUTATIONS

by

Alfred Leick

Prepared for

National Aeronautics and Space Administration
Goddard Space Flight Center
Greenbelt, Maryland 20770

Grant No. NSG 5265
OSURF Project 711055

OSU

The Ohio State University
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PREFACE

This project is under the supervision of Professor Ivan I. Mueller, Department of Geodetic Science, The Ohio State University, and under the technical direction of Dr. David E. Smith, Code 921, NASA, Goddard Space Flight Center, Greenbelt, Maryland 20771.

This report together with Department of Geodetic Science Report No. 263 was submitted to the Graduate School of The Ohio State University as partial fulfillment of the requirements for the Ph.D. degree.

ABSTRACT

The expected motion characteristics of the real earth are systematically analysed based on available dynamical theories for the rigid model, the elastic earth model and the earth model with liquid core. The various axes which are implicit in the dynamical theories are investigated regarding observability on the basis of astronomical observations and suitability for defining reference directions. The observational insignificance of the "diurnal polar motion" is demonstrated. A special effort is made to clarify customarily used terminology.

ACKNOWLEDGEMENTS

The author wishes to thank Professor Ivan I. Mueller, the supervisor of this work, for the encouragement, patience and confidence he generously offered continually. His readiness to discuss all pertinent problems at any time is especially appreciated.

To Professors Richard Rapp and Urho A. Uotila of the Department of Geodetic Science, and Professor G. Newsom of the Department of Astronomy, sincere appreciation for their time and comments offered, is extended.

Financial support in the form of a One-Year Fellowship given jointly by NASA and ESRO (European Space Research Organization), a Three-Quarter Fellowship from the Federation of German-American Clubs, The Wild Heerbrugg 1974 Geodetic Fellowship, and The Ohio State University Graduate Research Associateship is gratefully acknowledged.

It has been a pleasure and an honor to study at The Ohio State University Department of Geodetic Science. In particular the series of advanced courses greatly deepened the author's understanding of Geodesy.

Finally, a special thanks is given to Maura Taaffe and Irene Tesfai for typing this work.

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LIST OF SYMBOLS

This list of symbols gives definitions for the symbols used throughout the text.

$(X)_E = (X_{1E}, X_{2E}, X_{3E})^T$	Geocentric and inertially oriented coordinate system. The third axis coincides with the north pole of the fixed ecliptic at a standard epoch. The first axis is fixed arbitrarily to the fixed ecliptic.
$(U) = (U_1, U_2, U_3)^T$	Body-fixed geocentric coordinate system. Its orientation within the body is given in a "prescribed" way.
$(U)_F$	The axes of this coordinate system coincide with the principal moments of inertia axes of the earth.
$(X)_F$	Same system as $(U)_F$, but the first axis coincides with the node line of the fixed ecliptic and the equator of figure.
$(X)_H$	The third axis coincides with the angular momentum axis (H). The first axis coincides with the node line of the fixed ecliptic and the plane orthogonal to (H).
$\vec{L}$	Vector of luni-solar torque on the earth.
$\vec{H}$	Angular momentum vector.
$\vec{\omega}$	Instantaneous angular velocity of the earth.
Ω	Mean angular velocity of the earth.
(I)	Instantaneous rotation axis.
(F)	Axis of figure.
(H)	Axis of angular momentum.

(E_0)	Axis of Eulerian pole of rotation.
(C')	Axis of celestial pole for rigid and elastic earth model.
(C'')	Axis of celestial pole for the liquid core model.
(C)	Axis of celestial pole.
$\vec{L}_{X_F} = (L_{1X_F}\ L_{2X_F}\ L_{3X_F})^T$	The components of L in the $(X)_F$ system. The same notation is used regarding other axes and coordinate systems. If no subscript is given, the components refer to the (U)-system.
$L_{X_F} = L_{1X_F} + i L_{2X_F}$	Complex form for equatorial component of torque $\vec{L}$ in $(X)_F$ system. The same notation is used regarding other axes and coordinate systems. If no subscripts are given, the components refer to the (U)-system.
$(u_{1F}, u_{2F}, 1)$	Direction cosines of the axis (F) in the (U)-system whereby second-order terms in u_{iF} are neglected.
$u_F = u_{1F} + i u_{2F}$	Complex notation for the equatorial component of a unit vector along (F) in the (U)-system.
I_U	Inertia tensor in (U)-system.
A_1, A_2, A_3	Least, intermediary, and maximum moments of inertia. These are the elements in the diagonal of the inertia tensor if the coordinate system is selected such that the off-diagonal terms of the inertia tensor are zero.
c_{ij}	Small term in inertia tensor at position (i, j).
U	Tide-generating potential.
GMST	Greenwich mean sidereal time.
k	Tidal effective Love number.
k_S	Secular Love number.

$(\theta_F, \psi_F, \varphi_F)$	Euler angles relating the fixed ecliptic and the $(U)_F$-system. If no subscripts are given, the Euler angles refer to the (U)-system. ψ is counted positive eastward.
$\varkappa_F$	Position of (F) in $(X)_E$ in complex notation. Similar notations are used for other axes.
$\varkappa_{Fh}$	Free motion of (F) in $(X)_E$.
$\varkappa_{Ff}$	Forced motion of (F) in $(X)_E$.
$\varkappa_{FP}$	Precession of (F).
$\varkappa_{FN}$	Nutation of (F).
$u_{Ff} = u_{1Ff} + i u_{1Ff}$	Complex notation for equatorial components of unit vector of forced motion of (F) in (U)-system.
u_{Fh}	Same as above except for free motion of (F).
θ_{Ff}, ψ_{Ff}	Forced motion of (F) in obliquity and longitude.
θ_{FP}, ψ_{FP}	Precession of (F) in obliquity and longitude.
θ_{FN}, ψ_{FN}	Nutation of (F) in obliquity and longitude.
σ_o	Frequency of Euler motion (rigid body).
σ_r	Frequency of Chandler motion (elastic body).

1. INTRODUCTION

The strong increase in measurement accuracy, which is expected from the new laser generation, makes it necessary to look anew at the underlying principle of astronomical frames and their suitability for lunar laser ranging. No attempt is made in this study to introduce a new dynamical theory; rather, extensive use is made of available theories. Special efforts are made to identify the various axes, to discuss their relative body-fixed (with respect to the body) and space-fixed (with respect to inertial space) motions, and to investigate their dependence on defining constants, time varying parameters, suitability for providing a defining direction of reference, etc.

The simplest theory is that of the rigid body earth model. The earliest investigations on this subject are several centuries old. The theory was essentially completed by Oppolzer [1882] when he included small second-order terms in his solution and thus derived the expressions for the diurnal body-fixed motions of the instantaneous rotation axis (I), which are a result of luni-solar attraction. Unfortunately, he re-emphasized an older concept, saying that the astronomical observations refer to that equatorial plane which is orthogonal to the instantaneous rotation axis (I) and neglected to investigate the question of observability anew in view of his expanded and more complete rigid earth theory. See [Oppolzer, 1882, p. 155]. Subsequently, the officially adopted set of nutations has always been given for the instantaneous rotation axis (I). It was only very recently that the question of observability was taken up again by Atkinson [1973, 1975, 1976] and Ooe and Sasao [1974]. Severe difficulties of an habitual as well as a practical nature (since a large amount of astronomical data has been reduced in a certain manner) surfaced at two recent international meetings at which this subject was discussed. In an attempt to correct the situation the General Assembly of the International Astronomical Union at Grenoble in 1976

[IAU Transactions, 1977] passed a resolution to adopt the forced rigid body nutations of the axis of figure (F) instead of those of the instantaneous rotation axis (I). However, in a subsequent meeting [IAU Symposium No. 78, 1977], at which a set for the nutations of the nonrigid earth was adopted, the resolution of Grenoble was ignored. It is thus appropriate to discuss the question of observability in this study and to elaborate on the observational significance of the periodic diurnal body-fixed motions of the instantaneous rotation axis (I). The most accurate rigid earth theory available at this time is that by Woolard [1952] which was officially adopted.

Besides the rigid model the motion characteristics of more elaborate models such as the elastic model and models with liquid core will be investigated. The long history of research in the various earth models has not only produced more realistic models but also an amazing amount of confusion in terminology. In Table 1.1 a summary of those phenomena which will be discussed in great detail in this report is given including a summary of terms which can be found for them in the literature. Note that the terms in each box describe one and the same motion. No guarantee is given that the lists are complete. Table 1.1 is divided into two groups of motions. The body-fixed and space-fixed motions are called polar motion and nutation, respectively. This basic classification is the same as that given in [Munk and MacDonald, 1960]. The characteristic is that polar motion changes latitudes, whereas nutation changes declination. The subdivisions are made according to the earth models and the cause of the motion. The force-free motions are similar in character to constants of integration and can exist independently of the forced motions It is also seen that the term "wobble" is associated with force-free body-fixed motions. The word "free" generally denotes force-free motions. In some cases even the body-fixed motions are called a "nutation." Such terminology is used to illustrate that any periodic motion should be called a nutation. If such conventions were to be followed, a unique identification of the motion components would require information as to whether the motions were body- or space-fixed and as to their periods. The liquid core model introduces new motions in addition to

Table 1 1

Terminology Related to the Motions of the Instantaneous Rotation Axis (I)

Model	Excitation	Polar Motion (Body-Fixed Motion)	Nutation (Space-Fixed Motion)
Rigid or Elastic	force-free	1a. Chandler motion (elastic model) 1b. Euler motion (rigid model) 2. wobble	free nutation
	luni-solar attraction	1. Oppolzer terms 2. forced dirunal motions 3 dynamical variation of latitude (longitude) 4. diurnal polar motion	1. astronomical nutations 2. forced nutations 3. nutations
Liquid Core	force-free	1: nearly diurnal free wobble (NDFW) 2. nearly diurnal free polar motion 3. nearly diurnal free nutation	1. associated free nutation to NDFW 2. free principal core nutation
	core resonance	causes changes in diurnal polar motion and nutation (no name given)	

those of the rigid-body model. These are motions of the shell which result from interactions between the shell and the liquid core. The motions depend on the assumed structure of the shell and the core. If the core is assumed to consist of several layers of equal density then there exists the possibility of several nearly diurnal free wobbles. The free nutation due to the presence of the core is sometimes referred to as "free principle core nutation." The word "core" should be interpreted in such a way that the core is responsible for the particular nutations of the shell. This motion, therefore, should be detectable from instruments located on the surface of the shell. Despite the many names in Table 1.1, the motions in each row are related by very simple kinematical relations. According to the classical theory of Poinsot any rotational motion of a body around a fixed point can be represented by the rolling of a body cone onto a space cone. The line of contact is the instantaneous rotation axis (I). In Section 2.3.5, Poinsot's kinematical representation is discussed in detail.

It is not only the motion of the instantaneous rotation axis (I) which is of concern here. The following axes also have important body-fixed and space-fixed motion characteristics which will be investigated:

(a) angular momentum axis (H)

(b) axis of figure (F)

(c) axis of Eulerian pole of rotation (E_0)

(d) axis of celestial pole (C)

The angular momentum axis (H) has the useful property that its spatial motion is nearly independent of mass redistributions of the magnitude which is expected for realistic earth models [Fedorov, 1958]. The axis of figure (F) coincides at any instant with the direction of the maximum moment of inertia. The term "Eulerian pole of rotation" was introduced by Woolard [1952]. It corresponds to the position which the instantaneous rotation axis (I) would occupy if there were no external forces acting on the body. The term "Celestial Pole C" is used in this study to denote that pole to which astronomical observations and also lunar laser range observations refer. Its defining property is that it has neither periodic diurnal body-fixed nor space-fixed motions. It is now evident that the expression "polar motion" is not unique; rather it is necessary to name the axis to which it refers.

The various motion components are explained most easily in terms of the corresponding mathematical expression. Therefore, a rather complete mathematical derivation is presented for the rigid and elastic models. Both models are treated together. With some simple specifications one can modify the results for the elastic body so as to obtain those of the rigid body. The derivations make heavy use of the works by Fedorov [1958] and McClure [1973] who gave a comprehensive derivation of diurnal polar motion. First, the nutations of the axis of angular momentum (H) are computed. Since the spatial motions of (H) are nearly independent of mass redistributions inside the earth, the simple rigid earth model suffices for the computations. In fact, the derivations were done by Woolard [1952]. However, the most important steps in the derivation will be repeated using Doodson's [1921] tidal development in order to compute the luni-solar torques. This procedure can be found also in Melchior and Georis [1968] It has the advantage that it demonstrates the relationship between the nutations and the earth tides. In a second step the diurnal polar motions of the axes (H), (I) and (F) are derived and their observational significance is analyzed. The spatial motions of the axes (I), (F) and (C) will be obtained by transforming their diurnal polar motions relative to (H) into corrections which are to be added to the space-fixed motions of the angular momentum axis (H).

The liquid core model is treated in a merely descriptive manner. Any rigorous mathematical treatment of such models is beyond the scope of this study. The various new motions are analyzed for their observability and the way in which they influence the choice of a reference axis. Probably the most classical mathematical development for liquid core models in recent times is that of Jeffreys and Vicente [1957a, b].

The analysis regarding observability is based purely on geometric considerations. The observations are assumed to be reduced correctly (tidal corrections, aberrations, etc.). In many cases the question of observability is analyzed conceptually, regardless of whether or not a particular measurement system is capable of reaching the required accuracy. In this report, the classical astronomical observations are analyzed. In [Leick, 1978] the analysis is extended to lunar laser ranging.

2. MOTION SPECTRUM FOR RIGID AND ELASTIC EARTH MODEL

2.1 Fundamental Differential Equations of Motion

2.1.1 Axis of Rotation (I)

In a geocentric inertial frame $(X)_E$ the motion relative to the center of mass is at every instant a rotation around an axis through the center of mass, in which the rate of change of the angular momentum vector $\vec{H}$ about the center of mass is equal to the resultant torque $\vec{L}$.

$$\left(\frac{d\vec{H}}{dt}\right)_{X_E} = \vec{L} \tag{2.1-1}$$

The time derivatives refer to components on the inertial axes since this form of the equation of motion holds only in an inertial system.

The equation of motion involving the time derivatives of components with respect to the moving axes (U) is [Goldstein, 1965]

$$\left(\frac{d\vec{H}}{dt}\right)_{X_E} = \left(\frac{d\vec{H}}{dt}\right)_U + \vec{\omega} \times \vec{H} \tag{2.1-2}$$

$\vec{\omega}$ is the angular velocity vector of the moving frame with respect to the inertial frame. The angular velocity vector lies along the axis of infinitesimal rotation, a direction which is also called the instantaneous axis of rotation (I).

The equations (2.1-2) are a system of three linear first-order differential equations They are usually referred to as Euler's dynamic equations. Substituting equation (2.1-1) in equation (2.1-2) and making use of the usual summation convention, which calls for a summation over repeated indices, equation (2.1-2) is equivalently written as

$$L_i = \frac{dH_i}{dt} + \epsilon_{ijk}\,\omega_j\,H_k \qquad i,j,k = 1,2,3 \tag{2.1-3}$$

All components in the equation above refer to the moving axes. ϵ_{ijk} is the "alternating" tensor, defined by the following properties:

$$\epsilon_{ijk} = \begin{cases} 0 \text{ if any two subscripts are equal} \\ 1 \text{ for even permutation} \\ -1 \text{ for odd permutation} \end{cases}$$

Equations (2.1-3) are quite general. They represent the rotational motion for a rigid or elastic body. The standard expressions for the angular momentum, the tensor of inertia, and the relative angular momentum are, respectively

$$\begin{aligned} H_i &= I_{ij}\,\omega_j + h_i \\ I_{ij} &= \int_M (U_k\,U_k\,\delta_{ij} - U_i\,U_j)\,dm \\ h_i &= \int_M \epsilon_{ijk}\;U_j\,\dot{U}_k\,dm \end{aligned} \tag{2.1-4}$$

The symbol δ_{ij} above denotes the Kronecker delta. The relative angular momentum is caused by internal motions of the mass particles themselves. The dot in $\dot{U}_k$ denotes differentiation of U_k with respect to time. The integration is taken over the whole body of mass M.

Substituting equations (2.1-4) in (2.1-3) leads to yet another form of the equations of motion, usually referred to as the Liouville equation

$$L_i = \frac{d}{dt}(I_{ij}\,\omega_j + h_i) + \epsilon_{ijk}\,\omega_j\,(I_{kl}\,\omega_l + h_k) \tag{2.1-5}$$

The orientation of the rotating coordinate system (U) is arbitrary at this point. An extensive discussion on the possible choices can be found in [Munk and MacDonald, 1960]. Obvious candidate axes are the principal axes for which the products of inertia are zero. The body-fixed system (U) to be used in this study is attached in a "prescribed way" to some observatories, and its origin is at the center of mass. Thus the (U)-system and the axis of figure system $(U)_F$ do not necessarily coincide. The introduction of the (U)-system allows a slightly more general formulation of polar motion inasmuch as it introduces constant components in the off-diagonal positions in the moment of inertia tensor. These terms are responsible for a constant polar motion component which is equal to the angle between the third axis of the (U)-system and the axis of maximum moment of inertia. The periodic polar motion components are independent of the specific choice of the (U)-system This fact justifies the use of the term inertia "tensor." It is thus permissible to select the coordinate system according to its advantages in formulating a specific motion most simply. At this stage of the development it is only required that the third axis of the system (U) be nearly aligned with the rotation axis such that the products and squares of certain small quantities are negligible.

For evaluating the Liouville equation, the inertia tensor has to be known. In order to emphasize that the constant components are included in the off-diagonal terms, the subscript U is used. Thus,

$$I_U = \begin{bmatrix} A_1 + c_{11} & c_{12} & c_{13} \\ c_{12} & A_1 + c_{22} & c_{23} \\ c_{13} & c_{23} & A_3 + c_{33} \end{bmatrix} \qquad (2.1\text{-}6)$$

A_1 and A_3 are the least and the maximum moments of inertia. The c_{ij} are small quantities compared to A_1 and A_3. The angular velocity components in (U) are

$$\omega_U = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \Omega \begin{bmatrix} u_{1I} \\ u_{2I} \\ 1 + u_{3I} \end{bmatrix} \qquad (2.1\text{-}7)$$

Ω is the mean angular velocity of the earth rotation The dimensionless numbers u_{iI} are very small. In fact, u_{iI} as well as c_{ij}/A_3 and $h_i/\Omega A_3$ which appear in the Liouville equation are of the order 10^{-6} or even smaller for realistic earth models. Neglecting second-order terms in these quantities gives a first-order theory accurate to order 10^{-6}. Equation (2.1-5) reduces to

$$\begin{aligned} L_1 &= A_1 \dot{u}_{1I}\Omega + \dot{c}_{13}\Omega + \dot{h}_1 + u_{2I}(A_3 - A_1)\Omega^2 - c_{23}\Omega^2 - h_2\Omega \\ L_2 &= A_1 \dot{u}_{2I}\Omega + \dot{c}_{23}\Omega + \dot{h}_2 - u_{1I}(A_3 - A_1)\Omega^2 + c_{13}\Omega^2 + h_1\Omega \\ L_3 &= \dot{c}_{33}\Omega + A_3 \dot{u}_{3I}\Omega + \dot{h}_3 \end{aligned} \qquad (2.1\text{-}8)$$

u_{1I} and u_{2I} appear only in the first two equations of (2.1-8), whereas u_{3I} is only present in the third equation. This makes it possible to separately solve for the quantities (u_{1I}, u_{2I}) and (u_{3I}), respectively. Using complex notation, the first two equations in (2.1-8) are combined as

$$\dot{u}_I = i\,\sigma_r\,(u_I - \Psi) \tag{2.1-9}$$

The symbols have the following meanings:

$$i = \sqrt{-1}$$

$$u_I = u_{1I} + i\,u_{2I}$$

$$\sigma_r = \frac{A_3 - A_1}{A_1}\,\Omega$$

$$\Psi = \frac{iL}{(A_3 - A_1)\Omega^2} + \frac{c}{A_3 - A_1} - \frac{i\dot{c}}{(A_3 - A_1)\Omega} + \frac{h}{(A_3 - A_1)\Omega} - \frac{i\dot{h}}{(A_3 - A_1)\Omega^2}$$

The complex quantities are:

$$\Psi = \Psi_1 + i\,\Psi_2$$

$$L = L_1 + i\,L_2$$

$$c = c_{13} + i\,c_{23}$$

$$h = h_1 + i\,h_2$$

The dimensionless function Ψ is called the excitation function. It is a function of the torque, the inertia tensor, and the relative angular momentum. The third equation in (2.1-8) becomes

$$\dot{u}_{3I} = \frac{L_3}{A_3\Omega} - \frac{\dot{c}_{33}}{A_3} - \frac{\dot{h}_3}{A_3\Omega} \tag{2.1-10}$$

The complex variable u_I completely describes the body-fixed motion of the instantaneous rotation axis (I) in the frame (U). The direction cosines of (I) are obtained from equation (2.1-7) as

$$\frac{\omega}{|\vec{\omega}|} = (u_{1I},\ u_{2I},\ 1) \tag{2.1-11}$$

where second-order small quantities are omitted.

2.1.2 Axis of Angular Momentum (H)

The motion of the angular-momentum vector in the (U)-system is given by equation (2.1-4). Neglecting, once again, the second-order terms, equation (2.1-4) becomes

$$\begin{aligned} H &= A_1\Omega\, u_I + \Omega c + h \\ H_3 &= A_3\Omega(1+u_{3I}) + c_{33}\Omega + h_3 \end{aligned} \qquad (2.1\text{-}12)$$

with

$$H = H_1 + iH_2$$

Thus, substituting the solution of equation (2.1-9) for the instantaneous rotation axis (I) into equation (2.1-12) gives the body-fixed motion of the angular momentum in the (U)-system. The direction cosines, accurate to the first order, are

$$\frac{\vec{H}}{|\vec{H}|} = \left(\frac{H_1}{A_3\Omega}, \frac{H_2}{A_3\Omega}, 1\right) \equiv (u_{1H}, u_{2H}, 1) \qquad (2.1\text{-}13)$$

The position of the angular momentum vector in the terrestrial system is largely a function of the position of the instantaneous rotation axis which is given by u_I. But the relative position of these two axes is, in addition, directly dependent on the disturbances of the tensor of inertia and the relative angular momentum. Equations (2.1-12) and (2.1-13) give

$$u_H - u_I = -\frac{A_3 - A_1}{A_1} u_I + \frac{c}{A_3} + \frac{h}{A_3\Omega}$$

An estimate for the coefficient of u_I on the right-hand side is

$$\frac{A_3 - A_1}{A_1} \cong 0.0033$$

For an elastic earth model or in any other nonrigid model for which c and h are not zero, a more significant separation is expected between the angular momentum axis (H) and the instantaneous rotation axis (I).

2.1.3 Axis of Figure (F)

The motion of the axis of figure is a function of the disturbances of the inertia tensor. In the principal axis system $(U)_F$, in which the third axis coincides with the axis of figure (maximum moment of inertia axis), the off-diagonal elements of the inertia tensor are zero. The third axis of the two systems $(U)_F$ and (U) are related by two small orthogonal rotations as follows:

$$(\vec{U})_F = R_2(u_{1F})\, R_1(-u_{2F})\, (\vec{U}) = R(u_{1F}, u_{2F})\, (\vec{U})$$

u_{1F} and u_{2F} are the direction cosines of the axis of figure (F) in the terrestrial system (U). Next, the inertia tensor I_U of equation (2.1-6) is transformed to the principal axis system. The transformation properties of the components of the inertia tensor are determined by the fact that the matrix I_U transforms under R by a similarity transformation [Goldstein, 1965, p. 147].

$$I_F = R\, I_U\, R^{-1}$$

To the first order, the inertia tensor in the $(U)_F$ system is, therefore,

$$I_F = \begin{bmatrix} A_1 + c_{11} & c_{12} & [u_{1F}(A_1 - A_3) + c_{13}] \\ c_{12} & A_1 + c_{22} & [u_{2F}(A_1 - A_3) + c_{23}] \\ [u_{1F}(A_1 - A_3) + c_{13}] & [u_{2F}(A_1 - A_3) + c_{23}] & A_3 + c_{33} \end{bmatrix} \qquad (2.1\text{-}14)$$

The diagonal elements remain unchanged after the transformation. Since the off-diagonal elements are zero in I_F at all times, the following relations for the direction cosines are taken from (2.1-14):

$$u_{1F} = \frac{c_{13}}{A_3 - A_1}$$
$$u_{2F} = \frac{c_{23}}{A_3 - A_1} \qquad (2.1\text{-}15)$$

The small terms c_{11}, c_{12} and c_{22} which are present in I_F do not enter the excitation function Ψ explicitly. In case $c_{12} \neq 0$, an additional rotation around the third axis will complete the diagonalization of the inertia tensor. Such a rotation, however, is not of concern here when treating only the aspect of polar motion. Equations (2.1-15) clearly show how a constant term in c_{13} and c_{23} causes a constant component in u_{1F}, i.e., a constant offset between the third axis in the (U)-system and the axis of figure. In complex notation the direction cosines are written as

$$u_F = u_{1F} + i\,u_{2F}$$

With the help of (2.1-9), equations (2.1-15) which determine the direction of the axis of figure (F) in the terrestrial system (U) are written as

$$u_F = \frac{c}{A_3 - A_1} \tag{2.1-16}$$

2.1 4 Coplanar Motion

Combining the equations (2.1-11), (2,1-12), (2.1-13) and (2.1-16) gives an interesting relation:

$$u_H - u_F = \frac{A_1}{A_3}(u_I - u_F) + \frac{h}{A_3\Omega} \tag{2.1-17}$$

The presence of the relative angular momentum in equation (2.1-17) is somewhat disturbing in view of the geometrical interpretation. If

$$h = 0 \tag{2.1-18}$$

then the angular momentum vector, the axis of figure, and the instantaneous rotation axis are located in one plane, as is expressed by

$$u_H - u_F = \frac{A_1}{A_3}(u_I - u_F)$$

or (2.1-19)

$$u_I - u_H = \frac{A_3 - A_1}{A_3}(u_I - u_F)$$

Such a motion is certainly true for a rigid body motion where h = c = 0. The separation between the various axes is a function of the moments of inertia. For a spherical body with $A_1 = A_3$, the separation between (I) and (H) vanishes. Since for a realistic earth model the ratio A_1/A_3 is smaller than unity, the angular momentum axis lies always between the axis of figure and the instantaneous axis of rotation. These motion characteristics are valid whether or not external torques act on the body.

2.2 Spatial Motions of the Axis of Angular Momentum

2.2.1 Euler's Kinematic Equations

The relationship between the coordinate system (U) and the inertial system $(X)_E$, i.e., the ecliptic system at a standard epoch T_O, is given by Euler's kinematic equations. The situation is demonstrated in Figure 2.1.

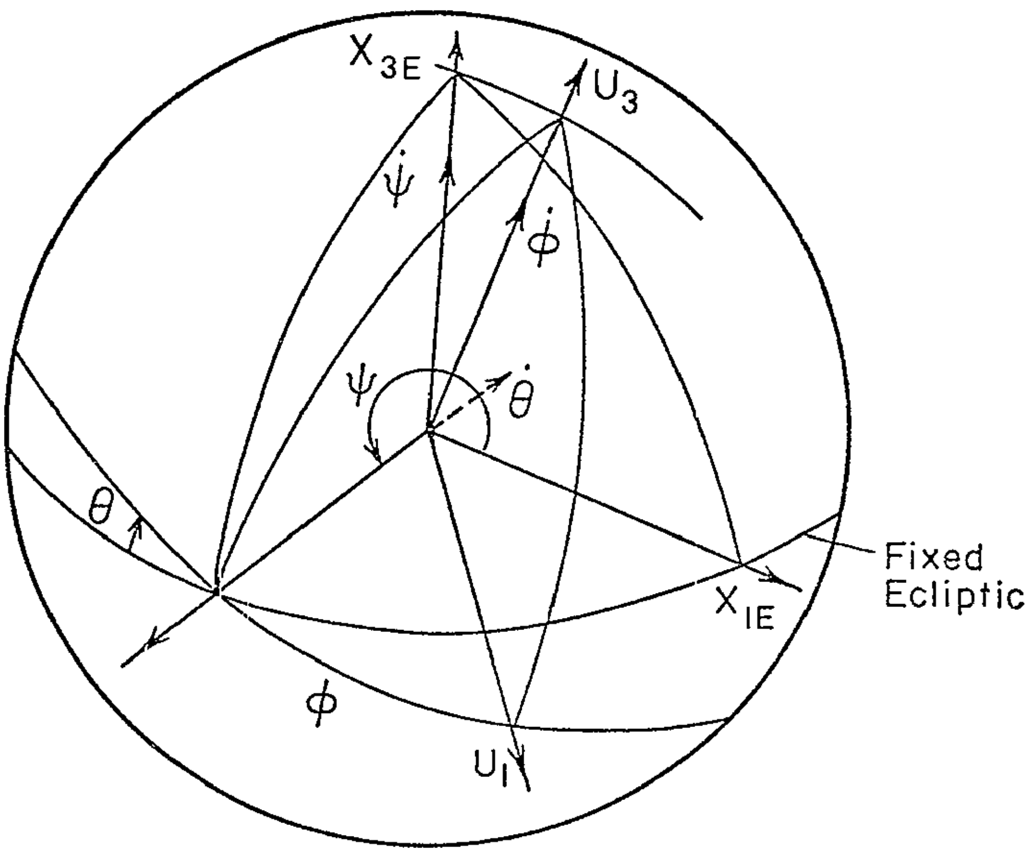


Figure 2.1 Euler's Kinematical Relations

The Euler angles (θ, ψ, φ) and their time derivatives $(\dot{\theta}, \dot{\psi}, \dot{\varphi})$ represent the motion of the (U)-system in space. But since this system is rigidly connected to the body, the angular velocity components ω_i as per equation (2.1-7) describe the same motion. The relation between the two sets of angular velocities is found by resolving each of the angular velocities $\dot{\theta}$, $\dot{\psi}$, $\dot{\varphi}$ along the (U_1, U_2, U_3)-axes and adding the components along each axis as follows:

$$\begin{aligned} \omega_1 &= -\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi \\ \omega_2 &= \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi \\ \omega_3 &= \dot{\psi}\cos\theta + \dot{\varphi} \end{aligned} \qquad (2.2\text{-}1)$$

The inverse relations are

$$\begin{aligned} \dot{\psi}\sin\theta &= -\omega_1\sin\varphi - \omega_2\cos\varphi \\ \dot{\theta} &= -\omega_1\cos\varphi + \omega_2\sin\varphi \\ \dot{\varphi} &= \omega_3 - \dot{\psi}\cos\theta \end{aligned} \qquad (2.2\text{-}2)$$

2.2.2 Poisson's Equation of Motion

The spatial motion of the angular momentum vector is the least sensitive axis regarding internal mass movements. Mass redistributions affect the motion of (H) in space only through their effect on the luni-solar torques. Fedorov [1958] made an extensive analysis of these disturbances. His calculation showed that the luni-solar torques as computed for a rigid and perfectly elastic earth model differ only in the order of 10^{-6}. Within such an accuracy level, therefore, the computation of the nutation of the angular momentum can be based on a rigid earth model.

Here a rigid model with equal least and intermediary moment of inertia is selected. Although the system (U) was used in Figure 2.1 for the purpose of a more general representation, permitting a separation between the U_3-axis and the axis of figure, the following derivation refers to the $(X)_F$-system, whose third axis coincides with the axis of figure (F) and whose first axis is along the node line as defined by the fixed ecliptic and the equator of figure. There should be no confusion about the various coordinate systems used here. The $(X)_F$ or $(U)_F$-systems are especially useful for dynamical purposes since the tensor of inertia is diagonal in this system. $(U)_F$ and (U) are related via equation (2.1-16). A further simplification is achieved by assuming the same magnitude for the least and intermediary moment of inertia, i.e., $A_1 = A_2$. In such a case the selection of the first axis in $(U)_F$ is arbitrary.

The coordinate system $(X)_F$ does not take part in the daily rotation of the earth. Its orientation in space changes only due to precession and nutation, which is being expressed by the small rotational velocity $\vec{\omega}\,'$. The components of $\vec{\omega}\,'$ on the axes of $(X)_F$ are obtained by substituting $\varphi=0$ and $\dot{\varphi}=0$ in equations (2.2-1):

$$\begin{aligned} \omega_1' &= -\dot{\theta}_F \\ \omega_2' &= -\dot{\psi}_F \sin\theta_F \\ \omega_3' &= \dot{\psi}_F \cos\theta_F \end{aligned} \tag{2.2-3}$$

These velocities can be substituted in Euler's dynamic equations (2.1-3). The angular momentum is

$$\begin{aligned} H_i &= A_i\,\omega_i' \qquad i = 1,\ 2 \\ H_3 &= A_3(\omega_3' + \dot{\varphi}) \;=\; A_3\,\omega_3 \end{aligned} \tag{2.2-4}$$

The velocities ω_i' in these equations are due to the motion of the frame $(X)_F$ with respect to $(X)_E$, and ω_3 is the velocity component of the earth rotation about the third axis with respect to the same system. Equations (2.2-4) are valid at any instant. Since the moments of inertia about all the axes perpendicular to the axis of symmetry are the same for the model to be considered here, the moments of inertia A_i do not change with time even though they are not referred to an axis attached to the body.

Euler's dynamic equations (2.1-3) in the $(X)_F$ system are

$$\begin{aligned} A_1\dot{\omega}_1' + \omega_2' A_3(\omega_3' + \dot{\varphi}) - \omega_3' A_1\,\omega_2' &= L_{1x_F} \\ A_1\dot{\omega}_2' - \omega_1' A_3(\omega_3' + \dot{\varphi}) - \omega_3' A_1\,\omega_1' &= L_{2x_F} \\ A_3(\dot{\omega}_3' + \ddot{\varphi}) + \omega_1' A_1\,\omega_2' - \omega_2' A_1\,\omega_1' &= L_{3x_F} \end{aligned} \tag{2.2-5}$$

These equations are rigorous. Their solution, in combination with (2.2-3) would yield the nutation of the axis of figure. No attempt is made to solve this system of equations, rather the motions are investigated which result if certain simplifications are introduced, such as neglecting the product of small quantities ω_i' and their derivatives. The first two equations of (2.2-5) then give

$$\begin{aligned} \omega_2' &= \frac{L_{1x_F}}{A_3\dot{\varphi}} \\ \omega_1' &= -\frac{L_{2x_F}}{A_3\dot{\varphi}} \end{aligned} \tag{2.2-6}$$

The third component L_{3x_F} is identical to zero because of symmetry relative to the equator (compare Section 2.2.3). Therefore, the third equation in (2.2-5) gives

$$A_3\ddot{\varphi} = 0$$

which has the solution

$$\dot{\varphi} = \text{constant}$$

$\dot{\varphi}$ is the mean velocity of the earth rotation. Previously we denoted this mean velocity by Ω so that we have the identity in notation

$$\dot{\varphi} \equiv \Omega \tag{2.2-7}$$

Using equations (2.2-3) and (2.2-7), we obtain from (2.2-6) the well-known Poisson equations of motion

$$\begin{aligned} \dot{\psi}\sin\theta &= -\frac{L_{1x_F}}{A_3\Omega} \\ \dot{\theta} &= \frac{L_{2x_F}}{A_3\Omega} \end{aligned} \tag{2.2-8}$$

These equations are of fundamental importance for describing the orientation of the earth. Indeed, the Poisson equations are the virtually rigorous equations for the nutation of the angular momentum axis (H). To show this, consider the coordinate system $(X)_H$ in which the third axis coincides with (H) and the first axis is along the node line of the fixed eliptic and the plane orthogonal to (H). Introduce the Euler angles ψ_H and θ_H in order to relate the two systems by

$$(\vec{X})_E = R_3(-\psi_H)\, R_1(\theta_H)\, (\vec{X})_H \tag{2.2-9}$$

The components of $\vec{H}$ in the $(X)_E$ system are

$$\vec{H}_{X_E} = A_3\Omega \begin{bmatrix} -\sin\psi_H \sin\theta_H \\ \cos\psi_H \sin\theta_H \\ \cos\theta_H \end{bmatrix}$$

where $|\vec{H}| = A_3\Omega$. Equation (2.1-1) can now be solved directly for the motion of the angular momentum axis in space. The first two equations are

$$\frac{d}{dt}(-A_3\Omega \sin\psi_H \sin\theta_H) = L_{1X_E}$$

$$\frac{d}{dt}(A_3\Omega \cos\psi_H \sin\theta_H) = L_{2X_E}$$

L_{1X_E} and L_{2X_E} are torque components in the ecliptic system $(X)_E$. The two equations above are solved for the rates of the Euler angles.

$$\begin{aligned} \dot{\theta}_H &= (L_{2X_E}\cos\psi_H - L_{1X_E}\sin\psi_H)/(A_3\Omega\cos\theta_H) \\ \dot{\psi}_H &= (-L_{1X_E}\cos\psi_H - L_{2X_E}\sin\psi_H)/(A_3\Omega\sin\theta_H) \end{aligned} \tag{2.2-10}$$

The torque in the $(X)_H$ system is related to the torque in $(X)_E$ according to (2.2-9) as

$$\vec{L}_{X_H} = R_1(-\theta_H)\, R_3(\psi_H)\, \vec{L}_{X_E} \tag{2.2-11}$$

The difference in $\vec{L}_{X_H}$ and $\vec{L}_{X_F}$ is negligible for all practical purposes because the systems $(X)_F$ and $(X)_H$ are closely aligned. In anticipation of later discussions on the free and forced motions, it is noted that in the present context, only the effect due to the difference in forced position enters. With this approximation and knowing that $L_{3X_F} = 0$, equation (2.2-11) gives

$$L_{1X_F} = L_{1X_E}\cos\psi_H + L_{2X_E}\sin\psi_H$$

$$L_{2Y_F} = (-L_{1X_E}\sin\psi_H + L_{2X_E}\cos\psi_H)/\cos\theta_H$$

Solving these equations for L_{1x_E} and L_{2x_E}, and substituting them in equations (2.2-10) results in the Poisson equations (2.2-8).

The most thorough solution of Poisson's equations can be found in [Woolard, 1952]. He made use of a second-order force function for the gravitational attraction. The positions of the moon and the sun were taken from the respective theories of Brown and Newcomb. Woolard's form of the Poisson equation is his equation (30). The verification of identity between his form and equations (2.2-10) can be made almost by inspection. To be more explicit, take the right-hand side of Woolard's equation (4) which gives the torque components in the $(U)_F$-system as follows:

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}_{U_F} = -R_3(\varphi) \begin{bmatrix} \dfrac{\partial \tilde{U}}{\partial \theta} \\ \dfrac{\partial \tilde{U}}{\partial \psi} / \sin\theta \\ 0 \end{bmatrix}$$

where $\tilde{U}$ is his force function. The third torque component is again zero because of symmetry.

The torque components in the $(X)_F$-system are

$$\vec{L}_{X_F} = R_3(-\varphi) \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}_{U_F}$$

or

$$\vec{L}_{X_F} = \begin{bmatrix} -\dfrac{\partial \tilde{U}}{\partial \theta} \\ -\dfrac{\partial \tilde{U}}{\partial \psi} / \sin\theta \\ 0 \end{bmatrix}.$$

Substituting the equation above in (2.2-8) gives exactly Woolard's form of the Poisson equations.

Historically, the Poisson equations have played an important role in deriving a nutation set. The presently adopted set of nutations is that of the angular momentum axis (H) based on Poisson's solution [AENA Supplement, 1961, p. 44]. Officially the adopted set is termed the "nutation of the rotation axis" based on a rigid earth model. But as will be demonstrated in great detail in later sections of this study, the separation between the angular momentum axis (H) and the rotation axis (I) based on a rigid earth model is smaller than 0.″0002, which is the accuracy with which the adopted set of nutations are given. The fact that the angular momentum axis (Poisson solution) is a good approximation to the rotation axis can readily be understood by considering equation (2.1-19). This relationship was first pointed out by Oppolzer [1882]. In fact, in earlier times one dealt only with Poisson's equation when computing the nutation and, therefore, one was not able to discover the nearly diurnal polar motion terms of the instantaneous rotation axis (I). It was Oppolzer [1882] who pointed out the importance of the small terms which were neglected when deriving the Poisson equations from the dynamical equations (2.2-5). He included these small terms in his calculation and demonstrated that the rotation axis and the angular momentum must have a diurnal periodic body-fixed motion. The expressions which describe these periodic motions in a body-fixed frame are since called the "Oppolzer terms" [Woolard, 1952, p. 159]. The corresponding expressions which have to be added to the nutations of the Poisson solution in order to arrive at the nutations of the axis of figure (F) or axis of rotation (I) are unfortunately called "diurnal nutations," although they do not have any diurnal period whatsoever [Woolard, 1952, p. 132]. In Section 2.3 a detailed representation is given for all phenomena mentioned above.

2.2.3 Development of External Torques

The gravitational potential of the earth at a distant point D (r_D, Φ_D, Λ_D) of unit mass is [Heiskanen and Moritz, 1966]:

$$V = \frac{GM}{r_D} \left\{ 1 - \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r_D}\right)^n [J_{nm} R_{nm} (\Phi_D, \Lambda_D) + K_{nm} S_{nm} (\Phi_D, \Lambda_D)] \right\} \qquad (2.2\text{-}12)$$

The symbols have the following meanings:

Φ_D, Λ_D	latitude, longitude of the distant body in the axis of figure system $(U)_F$
r_D	geocentric distance ot the distant body
a	mean earth radius
G	constant of gravitation
M	mass of the earth
J_{nm}, K_{nm}	potential coefficients

The functions R and S are

$$R_{nm}(\Phi_D, \Lambda_D) = P_{nm}(\cos \Phi_D) \cos m \Lambda_D$$
$$S_{nm}(\Phi_D, \Lambda_D) = P_{nm}(\cos \Phi_D) \sin m \Lambda_D$$

where P_{nm} are associated Legendre functions.

Since the origin ot the coordinate system $(U)_F$ is at the center of mass of the earth, the first-order potential coefficients are zero:

$$J_{10} = J_{11} = K_{10} = 0$$

Using this condition one can rewrite equation (2.2-12) as follows:

$$V = \frac{GM}{r_D} \left\{ 1 - \sum_{n=2}^{\infty} \left(\frac{a}{r_D}\right)^n J_n P_n (\sin \Phi_D) \right.$$
$$\left. - \sum_{n=0}^{\infty} \sum_{m=1}^{n} \left(\frac{a}{r_D}\right)^n P_{nm} (\sin \Phi_D) [J_{nm} R_{nm} + K_{nm} S_{nm}] \right\}$$

The coefficient J_2 is approximately 10^3 times larger than any of the other coefficients. The second-order approximation of the potential is

$$V \cong \frac{GM}{r_D}\left\{1 - \left(\frac{a}{r_D}\right)^2 J_2 P_2 (\sin \Phi_D)\right\}$$

This equation can be modified by using the well-known relations

$$J_2 = \frac{A_3 - \frac{A_1 + A_2}{2}}{M a^2} \tag{2.2-13}$$

and

$$P_2 (\sin \Phi_D) = \frac{1}{2}\left(3 \frac{U_{3F_D}}{r_D} - \frac{1}{2}\right)$$

as

$$V = f(r_D) - \frac{3GM}{2r^5} a^2 J_2 U_{3F_D}^2 \tag{2.2-14}$$

or

$$V = f(r_D) - \frac{3G}{2r^5} (A_3 - A_1) U_{3F_D}^2 \tag{2.2-15}$$

It is assumed, as usual, that the least and intermediary moments of inertia are equal, i.e., $A_1 = A_2$. U_{3F_D} is the third coordinate of the distant body in the $(U)_F$-system. The potential field $f(r_D)$ is a central field; it has no effect on the torques. The expression of the potential (2.2-15) is identical to the one used by Woolard.

The general expression for the torques is now readily obtainable. The gravitational force between the element dM and the mass of the disturbing body M_D, taken as a point mass, according to Figure 2.2, is

$$d\vec{F} = GM_D \, dM \frac{\vec{\ell}}{\ell^3} = - M_D \operatorname{grad} (dV)$$

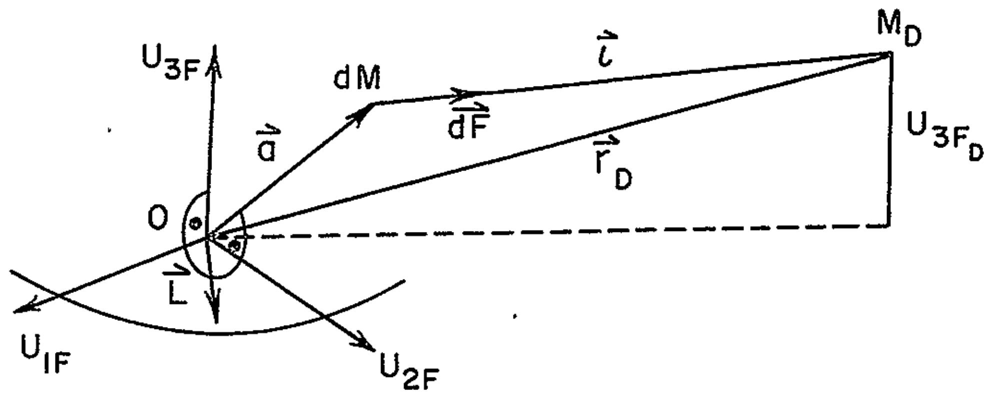


Figure 2.2 Luni-Solar Torque

This differential force causes a torque about the center O

$$d\vec{L} = \vec{a} \times d\vec{F}$$

With

$$\vec{a} = \vec{r}_D - \vec{\ell}$$

the differential torque is $d\vec{L} = r_D \times d\vec{F} - \vec{\ell} \times d\vec{F}$. The last term in the equation above is zero. The total torque is after integration

$$\vec{L}_{U_F} = \int_M \vec{r}_D \times d\vec{F} = \vec{r}_D \times \int_M d\vec{F} = \vec{r}_D \times \vec{F}$$

or

$$\vec{L}_{U_F} = -M_D \vec{r}_D \times \text{grad } V \qquad (2.2\text{-}16)$$

The gradient of V is according to equation (2.2-14)

$$\text{grad } V = -\frac{3GM}{r_D^5} a^2 J_2 \begin{bmatrix} O \\ O \\ U_{3D} \end{bmatrix}$$

Substituting this expression in (2.2-16) one obtains the torque components along the axes of $(U)_F$ as

$$\vec{L}_{U_F} = \frac{3M_D GMa^2 J_2}{r_D^5} (Y_D Z_D\ ,\ -X_D Z_D\ ,\ O)^T \qquad (2.2\text{-}17)$$

The third torque component is zero. The total torque is orthogonal to the earth-moon line, and zero when the moon crosses the equator. Evaluating the torques requires knowledge of the time-dependent positions of the disturbing body in the system $(U)_F$. Such positions are implicit in Doodson's [1921] tidal development. Doodson's development is used here not only because it gives the positions of the disturbing body in the system $(U)_F$, but mainly because it is anticipated to express the torques, and thus the nutations, in terms of harmonic series containing the tidal frequencies. This procedure will make it possible to study the relations between nutation and tidal theory.

In order to make the comparison in the subsequent development easier the Cartesian coordinates in (2.2-17) are replaced by spherical coordinates. Using the well-known relation

$$P_{21}\ (\sin\varphi_D) = 3\sin\varphi_D\ \cos\varphi_D$$

the torque in the $(U)_F$ system is

$$\vec{L}_{U_F} = \frac{G}{r_D^3}\ MM_D a^2 J_2\ [\sin\Lambda_D\ P_{21}(\sin\varphi_D),\ -\cos\Lambda_D\ P_{21}(\sin\varphi_D),\ 0]^T \tag{2.2-18}$$

Doodson's [1921] expansion of the tide generating potential is now given in the notation of McClure [1973]:

$$U = \sum_D \frac{GM_D}{c_D} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{c_D}\right) P_{nm}(\sin\varphi) \sum_J A_{nmJD}\ \cos\left[\alpha_{nmJD} + m\Lambda + (n-m)\frac{\pi}{2}\right] \tag{2.2-19}$$

The tesseral coefficients A_{21jD} are related to Doodson's original coefficients by

$$A_{21jD} = -\tfrac{1}{2}\bar{A}_{21jD} \tag{2.2-20}$$

The index j denotes individual tabular entries. The other symbols in (2.2-19) have the following meaning:

D	disturbing body (sun, moon)
M_D	mass of disturbing body
c_D	mean distance of disturbing body from geocenter
G	constant of gravitation
a	mean earth radius
α_{nmjD}	tidal argument of body D
φ, Λ	latitude and longitude of a station in the axis of figure system $(U)_F$
π	constant 3.14...

The tidal argument can be computed from the relation [McClure, 1973]

$$\alpha_{nmj} + m\Lambda = d_1 \tau + (d_2-5)s + (d_3-5)h$$

$$+ (d_4-5)p + (d_5-5)N' + (d_6-5)p_s \qquad (2.2\text{-}21)$$

where all d_i's are integers. For diurnal tide components d_1 is equal to one. The other symbols on the right hand side of (2.2-21) are Doodson's standard variables, which are sometimes referred to as the mean longitudes. They are related to Brown's fundamental arguments (l, l', F, D, Ω) as

mean longitude of the moon	$s = F + \Omega$	
mean longitude of the sun	$h = F - D + \Omega$	
longitude of lunar perigee	$p = -l + F + \Omega$	
longitude of ascending node of the moon(Ω)	$N' = -\Omega$	(2.2-22)
longitude of the perihelion	$p_s = -l' + F - D + \Omega$	

The mean longitudes are measured from the mean vernal equinox. Expressions for Brown's fundamental arguments are given in AENA Supplement [1961, p. 44] in terms of polynomials of time. τ is the local mean lunar time; it is reckoned from the lower transit of the moon. Finally, the Greenwich mean sidereal time is [Doodson, 1921]

$$GMST = \tau + s - \pi - \Lambda \qquad (2.2\text{-}23)$$

This relation assumes that the zero longitude is at Greenwich. Such an identification is possible because the least and intermediary moments of inertia are equal in our basic dynamical model. For reasons of abbreviation the tidal argument (2.2-21) is sometimes given in the code form

$$d_1\ d_2\ d_3.\ d_4\ d_5\ d_6\,.$$

As can be verified with the help of equations (2.2-21) and (2.2-23), the code form, e.g., 165.555, denotes the tidal argument with sidereal frequency, which sometimes is also symbolized by ${}^{s}K_1$ or ${}^{m}K_1$ depending whether it results from solar or lunar attraction. The positions of the disturbing body in $(U)_F$ are obtained by comparing a formal spherical harmonic expansion of the tide generating potential with equation (2.2-19). Such a formal expansion is [Melchior, 1966, p. 15],

$$U = \sum_{D} \frac{GM_D}{c_D} \sum_{n=2}^{\infty} \frac{a}{c_D}^{\,n}\, P\ (\cos\gamma) \tag{2.2-24}$$

The angle γ is the spherical distance between the station vector and the vector to the disturbing body, so that

$$\cos\gamma = \sin\varphi \sin\varphi_D + \cos\varphi \cos\varphi_D \cos(\Lambda - \Lambda_D)$$

Using the decomposition formula (Heiskanen and Moritz, 1966, p. 33) the tide generating potential (2.2-24) is written in the general form

$$U = \sum \frac{GM_D}{c_D} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r_D}\right)^n W_{nm}\ P_{nm}(\sin\varphi)\, P_{nm}(\sin\varphi_D) \cos m(\Lambda - \Lambda_D), \tag{2.2-25}$$

where $\quad W_{nm} = \dfrac{2(n-m)!}{(n+m)!}$

and $\quad W_{0l} = 1$

Substituting

$$\cos m(\Lambda-\Lambda_D) = \cos m\Lambda \cos m\Lambda_D + \sin m\Lambda \sin m\Lambda_D$$

into equation (2.2-25) and replacing the tidal argument in equation (2.2-19) by

$$\cos\left(\alpha_{nmjD} + m\Lambda + (n-m)\frac{\pi}{2}\right)$$

$$= \cos(m\Lambda)\cos\left[\alpha_{nmjD} + (n-m)\frac{\pi}{2}\right] - \sin(m\Lambda)\sin\left[\alpha_{nmjD} + (n-m)\frac{\pi}{2}\right]$$

a comparison can be made of the coefficients of $\sin m\Lambda$ and $\cos m\Lambda$ in the equations (2.2-25) and (2.2-19). The following relations are found:

$$\left(\frac{1}{r_D}\right)^{n+1} W_{nm}\, P_{nm}\,(\sin\varphi_D)\begin{Bmatrix}\cos m\Lambda_D\\ \sin m\Lambda_D\end{Bmatrix}$$
$$= \left(\frac{1}{c_D}\right)^{n+1}\sum_j A_{nmjD}\begin{Bmatrix}\cos\left[\alpha_{nmjD} + (n-m)\frac{\pi}{2}\right]\\ -\sin\left[\alpha_{nmjD} + (n-m)\frac{\pi}{2}\right]\end{Bmatrix} \qquad (2.2\text{-}26)$$

Only the terms with $n = 2$ and $m = 1$ enter into the torque equations (2.2-18). Combining equations (2.2-18) and (2.2-26) finally gives expressions for the torques

$$L_{1U_F} = GMM_D\frac{a^2}{c_D^3}\cdot 3J_2\sum_j A_{21jD}\cos(\alpha_{21jD})$$

$$L_{2U_F} = GMM_D\frac{a^2}{c_D^3}\, 3J_2\sum_j A_{21jD}\sin(\alpha_{21jD}) \qquad (2.2\text{-}27)$$

$$L_{3U_F} = 0$$

If J_2 is replaced by (2.2-13) and the equation (2.2-20) is used then the equatorial components of the torque in the $(U)_F$-system are in complex form:

$$L_{U_F} = \sum_j A_{jD}\, e^{-i(\alpha_{jD})} \qquad (2.2\text{-}28)$$

where

$$A_{jD} = \frac{3}{2} \frac{GM_D}{c_D^3} (A_3 - A_1) \bar{A}_{21jD} \tag{2.2-29}$$

The subscripts 2 and 1 are omitted in the exponent of (2.2-28). As an intermediary result it is seen that the torques and thus the nutations depend only on the tesseral harmonic coefficients $\bar{A}_{21}$. The torque components in the system $(X)_F$ are

$$\begin{aligned} L_{X_F} &= L_{U_F} e^{i\varphi} \\ &= \sum_j A_{jD} e^{-i(\alpha_{jD} - \varphi)} \end{aligned} \tag{2.2-30}$$

φ is the Euler angle which measures the earth rotation.

The expressions (2.2-30) can be modified to express the torque in terms of frequencies which are symmetric with respect to the earth rotation (sidereal frequency). Combining equations (2.2-21) and (2.2-23) and taking only the second order tesseral, i.e., $n = 2$ and $m = d_1 = 1$, gives the tidal argument

$$\begin{aligned} \alpha_j = {} & GMST + \pi + (d_2 - 6)s + (d_3 - 5)h + (d_4 - 5)p \\ & + (d_5 - 5)N' + (d_6 - 5)p_S \end{aligned} \tag{2.2-31}$$

Combining the latter terms this tidal argument can be rewritten as

$$\alpha_j = GMST + \pi + \Delta\alpha_j \tag{2.2-32}$$

where $\Delta\alpha_j$ is called the nutation argument. The mean sidereal frequency is

$$\Omega = (\dot{GMST}) \tag{2.2-33}$$

Two frequencies are symmetric to Ω if

$$\dot{\alpha}_{j+} = \Omega + \Delta\dot{\alpha}_{j+}$$

$$\dot{\alpha}_{j-} = \Omega + \Delta\dot{\alpha}_{j-} \qquad (2.2\text{-}34)$$

and

$$\Delta\dot{\alpha}_{j-} = -\Delta\dot{\alpha}_{j+}$$

hold. The positive sign in $\Delta\dot{\alpha}_{j+}$ denotes that $\dot{\alpha}_j > \Omega$. In terms of the tidal arguments the symmetry is expressed as

$$\alpha_{j+} = \mathrm{GMST} + \pi + \Delta\alpha_{j+}$$
$$\alpha_{j-} = \mathrm{GMST} + \pi + \Delta\alpha_{j-} \qquad (2.2\text{-}35)$$

Such a splitting up of tidal frequencies is implicit in Doodson's development. The respective symmetric frequencies in the expression (2.2-30) can now be combined with the help of equations (2.2-35) as follows:

$$A_{j+}e^{-i(\alpha_{j+}-\varphi)} + A_{j-}e^{-i(\alpha_{j-}-\varphi)}$$

$$= A_{j+}e^{-i(\mathrm{GMST}-\varphi+\pi+\Delta\alpha_{j+})} + A_{j-}e^{-i(\mathrm{GMST}-\varphi+\pi-\Delta\alpha_{j+})}$$

$$= e^{-i(\mathrm{GMST}-\varphi)}\left[-(A_{j+}+A_{j-})\cos\Delta\alpha_{j+} + i(A_{j+}-A_{j-})\sin\Delta\alpha_{j+}\right] \qquad (2.2\text{-}36)$$

The small difference between the Greenwich mean sidereal time, GMST, and the Euler angle φ is due to luni-solar nutation and planetary precession. It causes the coefficients in the expansion of the torque to have a minor time dependent variation. The most important of these is due to planetary precession because it is secular. It is responsible for small secular terms which can be found in the nutation tables [Woolard, 1952, p. 153]. For the purpose of discussions in this study, the approximation

$$\mathrm{GMST} \cong \varphi \qquad (2.2\text{-}37)$$

is made whenever torques are computed. It may be emphasized that the additional difference between GMST and φ which is due to the free motion is not important in calculating the torques. In fact, all our derivations, so far as they are dealing with the forced motion, correctly assume that polar motion is zero. Polar motion, a result of the homogeneous solution, is independent of the forced solution.

From equations (2.2-30) and (2.2-36) the expression for the torque becomes

$$L_{\chi_F} = \sum_j \left[- (A_{j+} + A_{j-}) \cos \Delta\alpha_{j+} + i\,(A_{j+} - A_{j-}) \sin \alpha_{j+} \right] \qquad (2.2\text{-}38)$$

Thus, two tidal waves which are symmetric to the diurnal frequency form only one constituent in the torque. It is understood that the summation includes both the lunar and solar terms contained in Doodson's tidal expansion.

2.2.4 Integration of Poisson's Equations

The solution of the Poisson equations gives the nutation of the angular momentum axis (H). Denoting the derivatives in the Euler angles by

$$\dot{\varkappa}_H = \dot{\theta}_H + i\,\dot{\psi}_H \sin\theta \qquad (2.2\text{-}39)$$

then from the Poisson's equations (2.2-8) it follows

$$\dot{\varkappa}_H = - \frac{i\,L_{\chi_F}}{\Omega\,A_3}$$

The homogeneous solution (force free, $\vec{L} = 0$) is

$$\varkappa_{Hh} = \tilde{c} \qquad (2.2\text{-}40)$$

where $\tilde{c}$ is a complex constant. This expresses the simple fact that the angular momentum vector forms an invariant direction in space if no external forces act on the body.

A particular solution (forced motion, $\vec{L} \neq 0$) is given by

$$\begin{aligned} \varkappa_{Hf} &= \frac{-i}{A_3\Omega}\int^t L(\tau)\,d\tau \\ &= \frac{-i}{A_3\Omega}\int^t \sum_j \left[-(A_{j+}+A_{j-})\cos\Delta\alpha_{j+} + i(A_{j+}-A_{j-})\sin\Delta\alpha_{j+}\right] d\tau \end{aligned} \tag{2.2-41}$$

where the luni-solar torque of equation (2.2-38) is substituted. $\Delta\alpha_j$ is a polynomial in time. Before integrating, sidereal terms with $\Delta\alpha_j = 0$ are separated. The sidereal terms have per equation (2.2-31) the argument number 165.555. They cause the secular terms in the expressions of the Euler angles and constitute the luni-solar precession. There are two waves with sidereal frequency called ${}^{m}K_1$, and ${}^{s}K_1$, which are due to lunar and solar attraction, respectively. Thus, e.g. (2.2-41) gives for precession and nutation

$$\begin{aligned} \varkappa_{Hf} &= \varkappa_{Hp} + \varkappa_{HN} \\ &= \frac{i}{A_3\Omega}\left(A_{{}^{m}K_1} + A_{{}^{s}K_1}\right)t \\ &+ \frac{1}{A_3\Omega}\sum_j\left\{\frac{-(A_{j+}-A_{j-})}{\Delta\dot{\alpha}_{j+}}\cos\Delta\alpha_{j+} + i\,\frac{(A_{j+}+A_{j-})}{\Delta\dot{\alpha}_{j+}}\sin\Delta\alpha_{j+}\right\} \end{aligned} \tag{2.2-42}$$

Using the torque in the form (2.2-30) and substituting equations (2.2-31) and (2.2-37) an equivalent expression for the nutations can be found

$$\varkappa_{HN} = -\frac{1}{A_3\Omega}\sum_j \frac{A_j}{\Delta\dot{\alpha}_j}\, e^{-i\Delta\alpha_j} \tag{2.2-43}$$

The summation goes over all j, excluding those for which $\Delta\alpha_j = 0$. The luni-solar precession $\varkappa_{HP}$ of the angular momentum axis (H) is according to equations (2.2-39) and (2.2-42)

$$\begin{aligned} \psi_{HP} &= \frac{t}{A_3\,\Omega\,\sin\theta_0}\left(A_{{}^{m}K_1} + A_{{}^{s}K_1}\right) \\ \theta_{HP} &= \theta_0 \end{aligned} \tag{2.2-44}$$

The two tidal amplitudes of equation (2.2-44) are related to the actual values given in Doodson's table by equation (2.2-29). Doodson's coefficients are negative so that one obtains a circular westward motion of the angular momentum axis due to precession. The subscript HP indicates that the elements of precession refer to the angular momentum (H). This is especially important for the obliquity θ_{HP}. It will be demonstrated later that the obliquity of the celestial equator (Section 2.3.4) has a slightly different value. Since θ_{HP} is a constant it appears as if there is no luni-solar precession in obliquity, i.e., the fixed ecliptic and the mean equator always subtend the same angle. But this is only true because the planetary precession was neglected when computing the torques, as is implied by the approximation (2.2-37). The inclusion of the planetary precession results in small non-linear precessional terms in both longitude and obliquity. A more complete expression for luni-solar precession is

$$\begin{aligned} \psi_{HP} &= f_1 t + f_2 t^2 + \cdots \\ \theta_{HP} &= \theta_0 + \theta_2 t^2 + \cdots \end{aligned} \tag{2.2-45}$$

f_1 is called the constant of luni-solar precession and can be identified with the corresponding factor in equation (2.2-44). The coefficients f_2 and θ_2 are computable from theory. The respective expressions are given in [Woolard and Clemence, 1966, p.242]. Note that there is no linear term in obliquity. For the sake of completeness it is mentioned that the use of θ_0 instead of θ in the first equations of (2.2-44) and (2.2-45) implies that the nutations χ_{HN} should be amended by very small periodic terms whose arguments are those of the nutations but whose coefficients change linearly with time. Regarding the nutation, the following observations are made from (2.2-42) and (2.2-43):

1. Each nutation constituent is in general an elliptic motion. It can be separated into two circular motions which have opposite angular velocities. If $A_{j-} = A_{j+}$, the nutation is

circular. There will be no term in the obliquity. An example of such a motion is the annual nutation term due to the solar tides.

2. Because of the presence of $\Delta \dot{\alpha}_j$ in the denominator the contribution of a particular tidal wave to the nutation depends not only on the magnitude of the tidal wave but also on the location of its frequency relative to the sidereal frequency. Those waves which are close to the sidereal frequency 165.555 have an amplified effect on the nutations.

3. The nutations are officially tabulated in the form

$$\chi_{HN} = \sum_j (a_j \cos \Delta \alpha_{j+} + i b_j \sin \Delta \alpha_{j+}) \qquad (2.2\text{-}46)$$

$$a_j \doteq - \frac{1}{A_3 \Omega} \frac{A_{j+} - A_{j-}}{\Delta \dot{\alpha}_{j+}}$$

with

$$b_j = \frac{1}{A_3 \Omega} \frac{A_{j+} + A_{j-}}{\Delta \dot{\alpha}_{j+}}$$

It is customary in astronomy to count the Euler angles ψ positive westward. This is contrary to the convention adopted here.

4. The method chosen here for deriving the nutation, i.e., using the precise numerical expressions from Doodson's development, concedes an important relationship between the constant of nutation N and the luni-solar precession ψ_{HP}. N is the nutation coefficient in obliquity associated with a 18.6 year period. Using a series expansion for the eccentricity and inclination when computing the

lunar and solar coordinates the following literal expressions can be shown to hold [Kulikov, 1956]:

$$
\begin{aligned}
\psi_{HP} &= \frac{A_3 - A_1}{A_3} \cos\theta_{HP} \left(K \frac{\mu}{1+\mu} + K'\right) \\
&= \frac{A_3 - A_1}{A_3} \left(4871''.140 + 866195''.0 \cdot \frac{\mu}{1+\mu}\right) \\
N &= H \cos\theta_{HP} \frac{\mu}{1+\mu} \cdot \frac{A_3 - A_1}{A_3} \\
&= \frac{A_3 - A_1}{A_3} \cdot 231\,981.8 \cdot \frac{\mu}{1+\mu}
\end{aligned}
\tag{2.2-47}
$$

μ is the mass of the moon in units of the earth mass and the coefficients K, K$'$, and H$'$ are functions of the orbit elements of the moon and the sun. The latter can be computed from theory with sufficient accuracy. The numerical values which are given here refer to the standard epoch 1900.0. Combining equations (2.2-47) gives

$$
\begin{aligned}
N &= \frac{H'}{K'} \psi_{HP} \left(\frac{1}{\mu} + \frac{K}{K'} + 1\right)^{-1} \\
&= \frac{47.6237\, f_1}{\frac{1}{\mu} + 178.822}
\end{aligned}
\tag{2.2-48}
$$

The importance of these formulas is that given the observations for ψ_{HP} and N one can compute the dynamical flattening and the mass ratio μ. However, the mass ratio obtained in this manner is not consistent with the values based on recent data from lunar and planetary space craft. This discrepancy is one of the outstanding problems in the system of astronomical constants. It indicates that the underlying simple nutation theory is not adequate. A partial answer is given when considering earth models with liquid core (See Section 3).

2.3 Relative Motions

2.3.1 The Complete Excitation Function

In order to proceed with the solution of equation (2.1-9) for the motion of the rotation axis (I) the disturbances c in the inertia tensor and the relative angular momentum h, which are to be used in the excitation function (Ψ) need to be computed. For a completely elastic body (Hooke body) the derivations can be found in the standard literature. The following expressions for the disturbances of the inertia tensor are taken from McClure (1973).

The perturbations due to rotational deformation are

$$c_{13\,RD} = \frac{k}{k_s}\,(A_3 - A_1)\,u_{1I}$$

$$c_{23\,RD} = \frac{k}{k_s}\,(A_3 - A_1)\,u_{2I}$$

In complex notation the perturbation is written as

$$c_{RD} = \frac{k}{k_s}\,(A_3 - A_1)\,u_I \qquad (2.3\text{-}1)$$

The secular Love number k_s is

$$k_s = \frac{3G(A_3 - A_1)}{a^5\,\Omega^2} \qquad (2.3\text{-}2)$$

and k is the tidal effective Love number. The perturbations due to tidal deformation are

$$c_{13\,TD} = \frac{kM_D}{c_D^3}\,a^5 \sum_j A_{21\,j}\,\sin\alpha_j$$

$$c_{23\,TD} = \frac{kM_D}{c_D^3}\,a^5 \sum_j A_{21\,j}\,\cos\alpha_j$$

or

$$c_{TD} = \frac{kM_D}{c_D^3} a^5 \sum_j iA_{21j} e^{-i\alpha_j} \tag{2.3-3}$$

The misalignment of the third axis of $(\bar{U})$ with the maximum moment of inertia axis (F) is expressed by the constant corrections

$$c_0 = c_{130} + i\, c_{230} \tag{2.3-4}$$

Only those perturbations are given above which directly effect the equations of motions (2.1-9). These perturbations result from the second degree tesseral harmonic coefficients of the expansion of the earth's gravitational and tidal potential. The zonal coefficients effect only the diagonal terms of the inertia tensor, and according to equation (2.1-10), the velocity of rotation of the earth. The sectorial coefficients contribute only c_{11}, c_{22}, and c_{12}. The excitation due to the relative angular momentum h in Ψ is according to equation (2.1-9)

$$\Psi_h = \frac{h}{(A_3 - A_1)\Omega} - \frac{i\dot{h}}{(A_3 - A_1)\Omega^2}$$

An estimation of Ψ_h in the case of an elastic body is given in [Heitz, 1976, Appendix 3]. He found that Ψ_h is negligibly small.

2.3.2 Body-Fixed Motion

The complete solution for the body-fixed motion of the instantaneous rotation axis is the sum of the homogeneous solution (force free, $\Psi = 0$) and a particular solution (forced solution, $\Psi \neq 0$) of the differential equation (2.1-9)

$$\dot{u}_I = i\,\sigma_r (u_I - \Psi)$$

The complete solution is

$$u_I = u_0 e^{i\sigma_r t} - e^{i\sigma_r t}\, i\sigma_r \int^t \Psi(\tau)\, e^{-i\sigma_r \tau}\, d\tau \tag{2.3-5}$$

u_0 is a complex constant of the homogeneous solution. The excitation function Ψ in (2.1-9) is given by equations (2.2-20), (2.2-28), (2.2-29), and (2.3-1) to (2.3-4). The solution for u_I after integrating (2.3-5) and some lengthy algebraic rearrangement is

$$u_I = u_0 e^{i\sigma_0 t} + \Psi_0 + i \sum_j \frac{s_j^{-1} A_j e^{-i\alpha_j}}{A_3 \Omega^2}, \tag{2.3-6}$$

where

$$\left.\begin{aligned} \sigma_0 &= \frac{\sigma_r\left(1 - \frac{k}{k_s}\right)}{1 + \frac{\sigma_r}{\Omega}\frac{k}{k_s}} \\ \Psi_0 &= \frac{\frac{c_0}{A_3 - A_1}}{1 - \frac{k}{k_s}} \\ s_j &= \frac{\frac{\dot{\alpha}_j}{\Omega}}{1 + \frac{\Delta\dot{\alpha}_j}{\Omega}\frac{k}{k_s}} + \frac{\sigma_r}{\Omega} \end{aligned}\right\} \tag{2.3-7}$$

The motion of the angular momentum axis (H) in the system (U) is according to equation (2.1-12)

$$u_H = \frac{A_1}{A_3} u_I + \frac{c}{A_3} + \frac{h}{A_3 \Omega} \tag{2.3-8}$$

The term due to the relative angular momentum h can be neglected again. Thus, substituting the expressions (2.3-6), (2.3-1), (2.3.3) and (2.3-4) in (2.3-8) gives the body-fixed motion of (H)

$$u_H = \left[\frac{A_1}{A_3} + \left(\frac{A_3 - A_1}{A_3}\right)\frac{k}{k_s}\right] u_0 e^{i\sigma_0 t} + \Psi_0 + i \sum_j s_j^{-1} s_j' \frac{A_j}{A_3 \Omega^2} e^{-i\alpha_j} \tag{2.3-9}$$

where

$$s_j' = \frac{1 - \frac{k}{k_s}}{1 + \frac{\Delta\dot{\alpha}_j}{\Omega}\frac{k}{k_s}} \tag{2.3-10}$$

The motion of the axis of figure is according to equation (2.1-16) and equations (2.3-1) - (2.3-4)

$$u_F = \frac{k}{k_s} u_o \, e^{i\sigma_0 t} + v_b - \frac{ik}{k_s} \sum_j \frac{s_j^{-1} s_j''}{\Omega^2 (A_3 - A_1)} A_j e^{-i\alpha_j} , \quad (2.3\text{-}11)$$

where

$$s_j'' = s_j - \frac{\sigma_r}{\Omega} \quad (2.3\text{-}12)$$

The basic equations (2.3-6), (2.3-9), and (2.3-11) for the respective motions of the instantaneous rotation axis (I), the angular momentum vector (H), and the axis of figure (F) can be re-written in terms of the nutation argument $\Delta\alpha_j$ with the help of equation (2.2-32). The forced components are

$$u_{If} = -i \sum_j \frac{s_j^{-1}}{A_1 \; \Omega^2} A_j \; e^{-i \, (GMST + \Delta\alpha_j)}$$

$$u_{Hf} = -i \sum_j \frac{s_j^{-1} s_j'}{A_3 \; \Omega^2} A_j e^{-i(GMST + \Delta\alpha_j)} \quad (2.3\text{-}13)$$

$$u_{Ff} = i \frac{k}{k_s} \sum_j \frac{s_j^{-1} \; s_j''}{\Omega^2 (A_3 - A_1)} A_j \; e^{-i \, (GMST + \Delta\alpha_j)}$$

The homogeneous, or force-free, motions have a conceptual importance in defining polar motion. They give the hypothetical positions of the axes which would be occupied if there were no external forces acting on the earth. These motions, already contained in the complete solutions, are separately given by

$$
\begin{aligned}
u_{Ih} &= \gamma_0\, e^{i(\sigma_0 t + \Gamma)} + \Psi_0 \\
u_{Hh} &= \gamma_0 \left[\frac{A_1}{A_3} + \left(\frac{A_3 - A_1}{A_3}\right)\frac{k}{k_s}\right] e^{i(\sigma_0 t + \Gamma)} + \Psi_0 \\
u_{Fh} &= \gamma_0\, \frac{k}{k_s}\, e^{i(\sigma_0 t + \Gamma)} + \Psi_0
\end{aligned}
\qquad (2.3\text{-}14)
$$

where the complex constant u_0 is written as

$$u_0 = \gamma_0\, e^{i\Gamma}$$

Γ denotes the phase angle for $t = 0$.

The direction cosines u are transformed to the local components, defined by the direction of the local meridian and the direction orthogonal to it, by

$$u' = u\, e^{-i\Lambda} \qquad (2.3\text{-}15)$$

An equivalent procedure is to replace GMST by mean sidereal time MST in equations (2.3-13). Then, the along-meridian component is $\mathrm{Re}(u') = u'_1$ and the across-meridian component is $\mathrm{Im}\,(u') = u'_2$.

Figure 2.3 displays the body-fixed motions for the elastic model as discussed so far. The figure represents a tangent plane on the unit sphere, the point of tangency is the point where the third axis (U_3) of the coordinate system (U) passes through the sphere. Each of the motions indicated is the equivalent of a mathematical expression developed above. The following observations can be made:

a) The separation $U_3 - \Psi_0$, which is given by the second equation of (2.3-7), is a linear function of the constant component c_0 in the inertia tensor. (Ψ_0) is not identical with the axis of figure (F). For the rigid ($k = 0$) model the axis of figure (F) will be at T at any time.

b) The radius of the circle with center at Ψ_0 and which passes through (S) is given by the first term of u_{Fh} in equation (2.3-14). Thus, the position of (S) is given by u_{Fh} in (2.3-14). Similarly the points (C$'$) and (E_0) are found by plotting u_{Hh} and u_{Ih}, respectively. E_0 has been termed in Woolard [1952, p. 160] the "Eulerian Pole of Rotation". It corresponds to the hypothetical surface point which the instantaneous rotation axis (I) would occupy if no external forces were present. The prograde (in the sense of the earth rotation) body-fixed motion of (E_0) and (C$'$) is sometimes called "wobble". The amplitude and phase cannot be predicted from theory. The period is either the Chandler (elastic) or the Euler (rigid) period, as is well known.

c) The nearly diurnal motion of the axes (I) and (H) around (E_0) and (C) are retrograde motions. They are a result of luni-solar attraction as expressed by equations (2.3-13). In case of the elastic body even the axis of figure (F) has a non-zero diurnal motion u_{Ff}. The motions in Figure 2.3 are approximated by the exact circular diurnal constituent corresponding to the tide K_1 which causes the precession. In actuality each tidal frequency has a corresponding diurnal motion component. Numerical values for these motions are given in McClure [1973] and Woolard [1952]. The equations (2.2-29) and (2.3-13) show that these forced motions are proportional to the difference in the principal moments of inertia.

d) The complete solution as well as the homogeneous and forced solution separately fulfill the relation (2.1-19)

$$u_H - u_F = \frac{A_1}{A_3} (u_I - u_F)$$

as can be verified from the respective equations. Thus, all three axes are in a plane at all times.

The axes through the points (S), (C$'$), and (E_0), which result from the homogeneous solution, exhibit no periodic diurnal body-fixed motions.

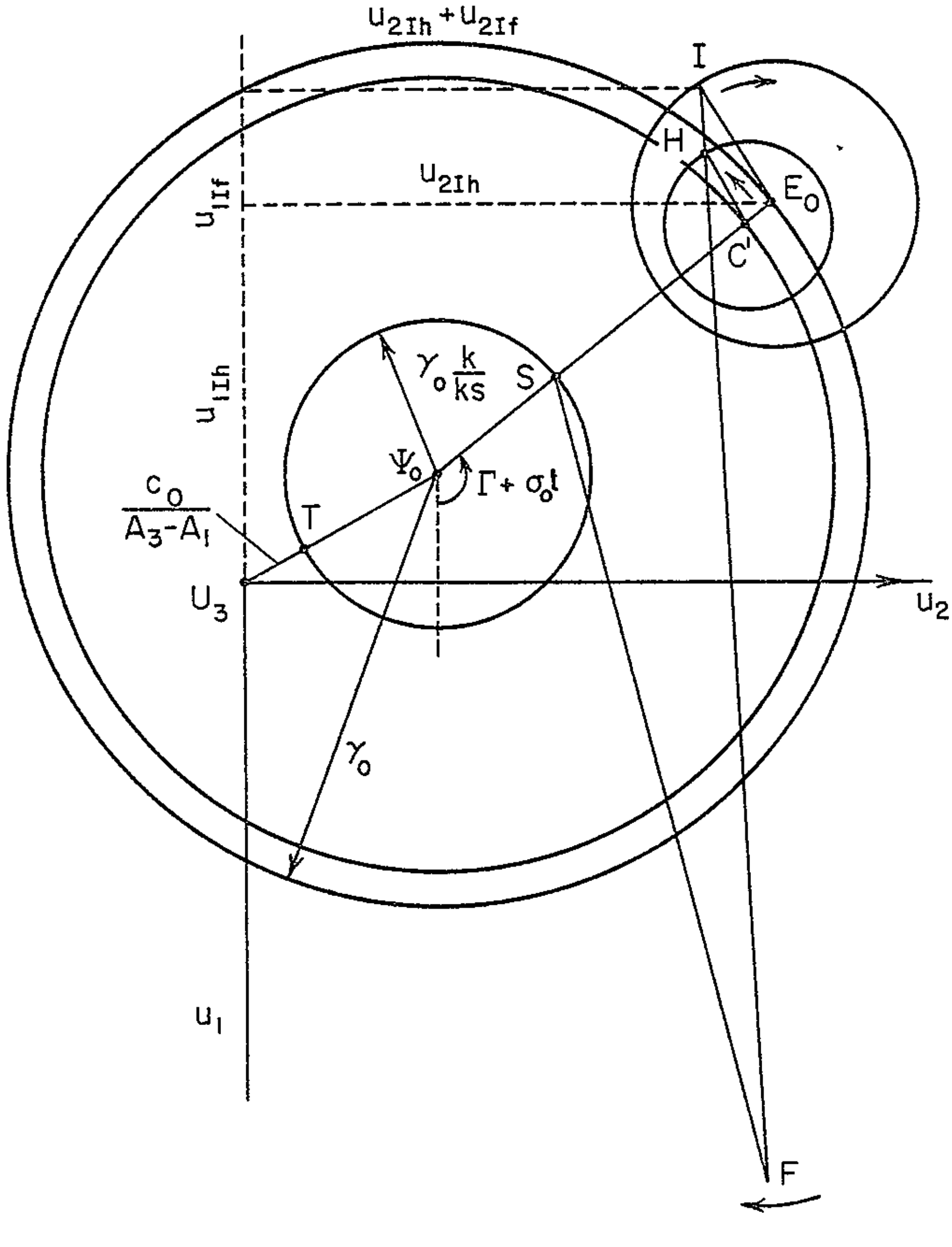


Figure 2.3 Body-Fixed Motions for the Elastic Body

e) The magnitudes of the various motions in Figure 2.3 are given in Table 2.1.

Table 2.1

Magnitudes in Polar Motion

Forced Motion			
Model	I - E_o	I - H	S - F
elastic rigid	$\leq$ 62 cm $\leq$ 61 cm	$\leq$ 21 cm 1.5 cm	$\leq$ 60 m F = T
Free Motion			
Model	Radius	Frequency	C′ - E_o
elastic	γ_o	σ_o	$\gamma_o \frac{A_3 - A_1}{A_3}\left(1 - \frac{k}{k_s}\right)$
rigid	γ_o	σ_r	$\gamma_o \frac{A_3 - A_1}{A_3}$

f) The separation E_o - C′ is due to the difference in the homogeneous solutions of the rotation axis (I) and the angular momentum axis (H). Its magnitude

$$\gamma_o \left(\frac{A_3 - A_1}{A_1}\right) \left(1 - \frac{k}{k_s}\right) \cong 1.5 \text{ cm}$$

can only be indirectly obtained through the observed value of γ_o. In the next section this distance will be verified as the amplitude of the free nutation.

As already mentioned the motions of the rigid earth model are obtained by putting k = 0. Equations (2.3-6), (2.3-9), and (2.3-11) are then

$$u_I = \gamma_0 e^{i(\sigma_r t + \Gamma)} + \Psi_r - i \sum_j \frac{A_j}{A_1 \Omega (\dot{\alpha}_j + \sigma_r)} e^{-i\alpha_j}$$

$$u_H = \gamma_0 \frac{A_1}{A_3} e^{i(\sigma_r t + \Gamma)} + \Psi_r - i \sum_j \frac{A_j}{A_3 \Omega (\dot{\alpha}_j + \sigma_r)} e^{-i\alpha_j} \qquad (2.3\text{-}16)$$

$$u_F = \Psi_r = \frac{c_0}{A_3 - A_1}$$

For a rigid model the axis of figure (F) is, of course, body-fixed. Figure 2.3 can easily be modified for the rigid body case. The separation between the instantaneous rotation axis (I) and the angular momentum axis (H) is much smaller than for the elastic body because the coefficient s'_j in u_{Hf} of equation (2.3-13) is equal to 1. The following ratio holds:

$$\frac{u_{Hf}\text{, elastic}}{u_{Hf}\text{, rigid}} = 1 - \frac{k}{k_s}$$

The "forced" terms of u_{If} are usually called the Oppolzer terms. The most important ones are for the rigid model [Woolard, 1952]:

$$\begin{aligned} u_{1If} = \; & 0.0087 \sin(\mathrm{GMST}) \\ & -0.0062 \sin(\mathrm{GMST}-2s) \\ & -0.0029 \sin(\mathrm{GMST}-2h) \\ & +0.0012 \sin(\mathrm{GMST}-\Omega) \\ u_{2If} = \; & 0.0087 \cos(\mathrm{GMST}) \\ & -0.0062 \cos(\mathrm{GMST}-2s) \\ & -0.0029 \cos(\mathrm{GMST}-2h) \\ & +0.0012 \cos(\mathrm{GMST}-\Omega) \end{aligned} \qquad (2.3\text{-}17)$$

The arguments are explained in (2.2-22).

2.3.3 Space-Fixed Motion

The spatial positions of the various axes are determined with respect to the angular momentum axis (H), whose position (θ_H, ψ_H) is known from the solution of Poisson's equations. The direction cosines u_i which were derived in the previous section are transformed into changes of the corresponding Euler angles.

Consider the previously used coordinate system $(X)_H$ and (U). The position of the first system relative to the system $(X)_E$ is given by the pair of Euler angles (θ_H, ψ_H) and the relation (2.2-9)

$$(\vec{X})_E = R_3(-\psi_H)\, R_1(\theta_H)\, (\vec{X})_H$$

The body-fixed system (U) is related to the inertial system by

$$(\vec{X})_E = R_3(-\psi)\, R_1(\theta)\, R_3(-\varphi)\, (\vec{U})$$

These two equations readily give the direction cosines of the axis (H) in the (U) system. For small differences in the Euler angles the direction cosines of the angular momentum vector $(\vec{H})$ are to the first order of small quantities

$$\begin{bmatrix} u_{1H} \\ u_{2H} \\ u_{3H} \end{bmatrix} = R_3(\varphi) \begin{bmatrix} \delta\psi \sin\theta_H \\ -\delta\theta \\ 1 \end{bmatrix} \qquad (2.3\text{-}18)$$

where

$$\delta\psi = \psi - \psi_H$$

$$\delta\theta = \theta - \theta_H$$

The first two equations of (2.3-18) can be written in complex notation as follows:

$$u_{1H} + i u_{2H} = e^{-i\varphi} (\delta\psi \sin\theta_H - i\delta\theta),$$

which is rewritten as

$$\delta\varkappa \equiv \delta\theta + i\,\delta\psi \sin\theta_H = i\, e^{i\varphi}(u_{1H} + i u_{2H}) \qquad (2.3\text{-}19)$$

Note that the correction is given in the sense of

$$\delta\varkappa_{U_3} = \varkappa_{U_3} - \varkappa_H$$

where the subscript U_3 is used to emphasize that $\varkappa_{U_3}$ determines the space-fixed position of the U_3 - axis relative to H.

The space-fixed position of the various axes can now be given. Using (2.3-19) the instantaneous rotation axis (I) is obtained as follows:

$$\begin{aligned}\varkappa_I &= \varkappa_H + \delta\varkappa_I \\ &= \varkappa_H + \delta\varkappa_{Ih} + \delta\varkappa_{If} \\ &= \varkappa_H + \delta\theta_I + i\,\delta\psi_I \sin\theta_H \\ &= \varkappa_H + i\,e^{i\varphi}(u_H - u_I)\end{aligned}$$

Substituting equations (2.3-6) and (2.3-9) the Euler angles become

$$\varkappa_I = \varkappa_H - i\left(\frac{A_3-A_1}{A_1}\right)\left(1-\frac{k}{k_s}\right)\gamma_0\, e^{i(\sigma_0 t+\Gamma+\varphi)} - \sum_j \frac{A_3-A_1}{A_1} B_j e^{-i\Delta\alpha_j} \qquad (2.3\text{-}20)$$

where

$$B_j\left(\frac{A_3-A_1}{A_1}\right) \equiv s_j^{-1}(A_3 - A_1 s_j')\,\frac{A_j}{A_1 A_3\,\Omega^2}$$

The space-fixed motion of the axis of figure (F) is with equations (2.3-9) and (2.3-11)

$$\begin{aligned}\varkappa_F &= \varkappa_H + \delta\varkappa_F \\ &= \varkappa_H + i\,\frac{A_1}{A_3}\left(1-\frac{k}{k_s}\right)\gamma_0\, e^{i(\sigma_0 t+\Gamma+\varphi)} + \sum_j B_j e^{-i\Delta\alpha_j} \qquad (2.3\text{-}21)\end{aligned}$$

Finally, the spatial motion of U_3 - axis is

$$\begin{aligned}\varkappa_{U_3} &= \varkappa_H + \delta\varkappa_{U_3} \\ &= \varkappa_H + i\,e^{i\varphi}\,u_H \\ &= \varkappa_H + i\left[\frac{A_1}{A_3} + \left(\frac{A_3-A_1}{A_3}\right)\frac{k}{k_s}\right]\gamma_0 e^{i(\sigma_0 t+\Gamma+\varphi)} + \sum_j \frac{s_j^{-1}s_j'}{A_3\,\Omega^2} A_j e^{-i\Delta\alpha_j} \qquad (2.3\text{-}22) \\ &\quad + i\,\Psi_0\, e^{i\varphi}\end{aligned}$$

Note that during the derivation of these equations we have used the approximation (2.2-37), i.e., the Euler angle φ has been equated to Greenwich sidereal time.

The following observations can be made regarding the space-fixed motions:

a) The homogeneous term

$$\delta \varkappa_{Ir} = -i\left(\frac{A_3 - A_1}{A_3}\right)\left(1 - \frac{k}{k_s}\right)\gamma_0 e^{i(\sigma_0 t + \dot{\Gamma} + \varphi)}$$

of equation (2.3-20) is called <u>free nutation</u>. It gives the hypothetical path of the instantaneous rotation axis (I) in space which would occur if no external forces were present. Its frequency is nearly diurnal. The amplitude, based on the observed values of γ_0, is approximately 1.5 cm. The free nutation is <u>not</u> included in the officially adopted set of nutations of the rotation axis. Special considerations will be given in Section 2.4.2 as to the observational significance of the free nutation. In any case, a frame connected to the instantaneous rotation axis (I) will perform a nearly diurnal "rocking" in inertial space.

b) The forced periodic terms in equation (2.3-20) which are to be added to those of equation (2.2-43) of the angular momentum axis (H) in order to obtain the forced motion of the instantaneous axis of rotation (I), are of significantly large magnitude for the elastic model. They account for the 21 cm separation between (I) and (H) already indicated in Figure 2.3. The ratio with respect to the motion of the angular momentum axis

$$\frac{|\delta \varkappa_{If}|}{|\varkappa_H|}_j = \frac{\left[A_3\left(1 + \frac{\Delta\dot{\alpha}_j}{\Omega}\right)\frac{k}{k_s} - A_1\left(1 - \frac{k}{k_s}\right)\right]\Delta\dot{\alpha}_j}{A_1\left[\dot{\alpha}_j + \sigma_r\left(1 + \frac{\Delta\dot{\alpha}_j}{\Omega}\frac{k}{k_s}\right)\right]}$$

demonstrates the dependencies on the elastic properties.

c) The axis of figure (F) has, of course, a diurnal nutation component too, which results from the homogeneous solution. The forced motion is very large; it describes the 60 m motion indicated in Figure 2.3. The axis of figure is the most sensitive axis with respect to mass redistribution among all the axes considered so far. It is, therefore, unlikely to be chosen as a defining direction for a reference frame.

d) The equations give immediately the ratios:

$$\left|\frac{\delta\varkappa_{Ih}}{\delta\varkappa_{Fh}}\right| = \left|\frac{\delta\varkappa_{If}}{\delta\varkappa_{Ff}}\right| = \frac{A_3 - A_1}{A_1}$$

The motions for a rigid model are obtained by putting k = 0. They are:

$$\left.\begin{aligned}
\varkappa_I &= \varkappa_H + \delta\varkappa_{Ih} + \delta\varkappa_{If} \\
&= \varkappa_H - i\left(\frac{A_3 - A_1}{A_3}\right)\gamma_o\, e^{i(\sigma_r t+\Gamma+\varphi)} - \sum_j \frac{A_3 - A_1}{A_1}\,\overline{B}_j\, e^{-i\Delta\alpha_j} \\
&\text{where} \\
\overline{B}_j &= \frac{1}{A_3\Omega}\,\frac{A_j}{(\dot{\alpha}_j + \sigma_r)} \\
\varkappa_F &= \varkappa_H + \delta\varkappa_{Fh} + \delta\varkappa_{Ff} \\
&= \varkappa_H + i\frac{A_1}{A_3}\gamma_o\, e^{i(\sigma_r t+\Gamma+\varphi)} + \sum_j \overline{B}_j\, e^{-i\Delta\alpha_j} \\
\varkappa_{U_3} &= \varkappa_H + \delta\varkappa_{U_3h} + \delta\varkappa_{U_3f} \\
&= \varkappa_H + i\frac{A_1}{A_3}\gamma_o\, e^{i(\sigma_r t+\Gamma+\varphi)} + i\Psi_o\, e^{i\varphi} + \sum_j \overline{B}_j\, e^{-i\Delta\alpha_j}
\end{aligned}\right\} \quad (2.3\text{-}23)$$

In this case the ratio

$$\left|\frac{\delta\varkappa_{If}}{\varkappa_H}\right|_j = \frac{A_3 - A_1}{A_3}\cdot\frac{\Delta\dot{\alpha}_j}{\dot{\alpha}_j + \sigma_r}$$

is extremely small. The largest periodic term in $\delta\varkappa_{If}$ has a coefficient of 0.″00002 [Woolard, 1952, p. 133]. The terms are, consequently, not included in the officially adopted set of nutations for the instantaneous rotation axis (I) since terms smaller than 0ʺ.0002 are omitted [AENA Supplement, 1961, p. 44]. The coefficients $\bar{B}_j$ for the rigid model are given in Woolard [1952, p. 132]. The largest terms are

$$
\begin{aligned}
\delta\psi_{Ff} &= 0.01615 \ \sin(2s) \\
& +0.00753 \ \sin(2h) \\
& +0.00338 \ \sin(\Omega) \\
& + \ldots \\
\delta\theta_{Ff} &= -0.00868 \\
& +0.00590 \ \cos(2s) \\
& +0.00275 \ \cos(2h) \\
& -0.00100 \ \cos(\Omega)
\end{aligned}
\qquad (2.3\text{-}24)
$$

The arguments are explained in equations (2.2-22). The constant term in $\delta\theta_{Ff}$ is caused by the sidereal term in $\delta\varkappa_{Ff}$. It is actually a combination of two sidereal terms, one resulting from the moon, the other from the sun. The motions $\delta\varkappa_{If}$ and $\delta\varkappa_{Ff}$ are referred to as "luni-solar diurnal nutations" in the astronomical literature [Woolard, 1952]. This terminology seems an unfortunate choice since these motions have no diurnal period whatsoever.

2.3.4 The Celestial Pole (C′)

The pole to which the nutations refer is denoted by (C′). This pole fulfills the following criteria:

1) The pole (C′) should exhibit no nearly diurnal periodic body-fixed motions.

2) The position of the (C′) in space should be computable at any time from the motion theory of the underlying models, i.e., no free solution component is permissible.

For ideal bodies, such as the rigid or perfectly elastic earth model, both criteria are easy to fulfill. Considering Figure 2.3, there appear to be several choices for selecting the (C') . All points along the line Ψ_0-E_0 in Figure 2.3 satisfy criterion 1, e.g. equations (2.3-14) give

$$u_{C'} = u_{Hh} = \left[\frac{A_1}{A_3} + \left(\frac{A_3 - A_1}{A_3}\right) \frac{k}{k_s}\right] \gamma_0 \, e^{i(\sigma_0 t + \Gamma + \varphi)} + \Psi_0$$

and

$$u_{E_0} = u_{Ih} = \gamma_0 e^{i(\sigma_0 t + \Gamma + \varphi)} + \Psi_0$$

The spatial motions of the axes passing through these points are

$$\begin{aligned} \varkappa_{C'} &= \varkappa_H + i e^{i\varphi}(u_H - u_{Hh}) = \varkappa_H + i e^{i\varphi}(u_{Hf}) \qquad (2.3\text{-}25) \\ &= \varkappa_H + \delta \varkappa_{U_3 f} \end{aligned}$$

and

$$\begin{aligned} \varkappa_{E_0} &= \varkappa_H + i e^{i\varphi}(u_H - u_{Ih}) = \varkappa_H + i e^{i\varphi}(u_{Hf} + u_{Hh} - u_{Ih}) \\ &= \varkappa_{C'} + \delta \varkappa_{Ih} \end{aligned}$$

The expressions are arrived at by identifying the body-fixed components in Figure 2.3 and using their transformed values in terms of corrections to the Euler angles.

Only the celestial pole (C') qualifies under criterion 2. Its position in space can be computed from equation (2.2-43) for $\varkappa_H$ and from equation (2.3-22) for $\delta\varkappa_{U_3 f}$. The north celestial pole thus defined has no periodic diurnal body-fixed motion due to external forces. Its body-fixed motion results solely from force-free motions. Its space-fixed motion is entirely due to external forces; it has no space-fixed motion resulting from a homogeneous solution component. The point, E_0 , which is called by Woolard the Eulerian pole of rotation, exhibits a diurnal motion in space which is exactly equal to the free nutation given in equation (2.3-20). The amplitude and phase of $\delta\varkappa_{Ih}$ cannot be determined from theory.

The term $\delta\varkappa_{U_3 f}$ is the transformation of u_{Hf} in equation (2.3-13). As equations (2.2-29) and (2.3-22) show, the amplitude of $\delta\varkappa_{U_3 f}$ is proportional to the mass of the disturbing body M_D and the form factor $(A_3 - A_1)/A_3$. Another interesting feature of the pole (C') is seen if we, for reasons of simplicity, assume a zero free motion. This is the case of the steady state motion. The pole (C′) has no body-fixed motion in this case. But, the angular momentum axis (H) and the instantaneous rotation axis (I) still have the nearly diurnal motions u_{hf} and u_{If} , whose magnitude is proportional to M_D and $(A_3-A_1)/A_3$. If one constructs various models with different form factors, i.e., adding mass points symmetrically around the equator, the pole (C′) will remain at the same body-fixed position whereas (H) and (I) change the radius of their circular motions. It is thus clear that the celestial pole (C′) and the instantaneous rotation axis (I) do have different properties and should conceptually always be separated. However, from a merely descriptive point of view, the pole (C′) may be considered as a "rotation axis" as well.

It is worth noting that in the case of the rigid earth model the spatial motion of the north celestial pole is

$$\varkappa_{c'\,rigid} = \varkappa_H + \delta\varkappa_{Ff\,rigid} \tag{2.3-26}$$

since according to equations (2.3-23) the equality

$$\delta\varkappa_{u_3 f\,rigid} = \delta\varkappa_{Ff\,rigid}$$

holds for the rigid body. The spatial separation between the pole (C′) of the elastic and the rigid model is

$$\Delta(\varkappa_{c'}) = \delta\varkappa_{u_3 f\,elastic} - \delta\varkappa_{u_3 f\,rigid}$$

The maximum separation is 21 cm (see Figure 2.3).

It is probably because of equation (2.3-26) that in the literature one finds frequently the statement that the nutation should be computed for the axis of figure [Atkinson, 1973, 1975, 1976].

There is no doubt that in all cases the axis as expressed in equation (2.3-26) was intended. But, once again, for clarity's sake, one ought to distinguish between the axis of figure (F), which is defined as the direction of the maximum moment of inertia at any epoch, and the north celestial pole (C$'$) as has been defined above. Even for the rigid earth model the axis (C$'$) generally does not coincide with the axis of figure (F). Only when the homogeneous solution is zero the identity

$$\varkappa_F \equiv \varkappa_{C'}$$

holds. In case of rigid motion the intermediary role of the instantaneous axis of rotation becomes quite obvious. When setting up the Euler dynamic equations (2.1-3) or (2.2-5)the concept of instantaneous rotation axis is needed since ω_i are the velocity components of the instantaneous rotation axis (I) in the frame $(U)_F$. Symbolically Euler's dynamical equations can be written as

$$\text{EDEq} \equiv f_i(\omega_i, I_F) = L_{iF}$$

Euler's geometric equations relate the velocity componets ω_i to the derivatives of the Euler angles $\dot{\theta}_F$, $\dot{\psi}_F$, $\dot{\varphi}_F$. The Euler angles relate the frame $(U)_F$ to the inertial system. Substituting

$$\omega_i = g_i(\dot{\theta}_F, \dot{\psi}_F, \dot{\varphi}_F)$$

in the dynamical equations gives

$$\text{EDEq} \equiv f_i(g(\dot{\theta}_F, \dot{\psi}_F, \dot{\varphi}_F), I_F) = L_{iF} \qquad (2.3\text{-}27)$$

These equations can be solved for the Euler angles. The angles θ_F and ψ_F give the "forced position" of the axis of figure in space. For a rigid body the axis of figure (F) has no body-fixed motion, hence

$$u_{Fi\,\text{rigid}} = 0$$

The space-fixed motion is strictly computable from equation (2.3-27). Therefore, the so-defined axis fulfills both requirements for the pole (C$'$) which were set up above. In fact, the solution of equation (2.3-27) is identical to equation (2.3-26). One can now add the homogeneous solution, which results in the deviation of the (C$'$) from the axis of figure, but this aspect is not of concern here. Rather, it is noted that after substituting Euler's kinematic equation into equation (2.3-27) one does not have to worry about the instantaneous rotation axis (I) anymore. The instantaneous rotation axis (I) is conceptually needed only at the initial stage when formulating the dynamical equations.

At present the rotation axis (I) is adopted as the reference pole instead of (C$'$). The practical complications arising from that convention are dealt with in Section 2.4.4 In Section 2.4.2 the observational significance of the homogeneous component which separates (C$'$) and (E_0) in Figure 2.3 is discussed. The additional complications arising from the liquid core are the subject of Section 3.

2.3.5 Poinsot's Kinematical Representation

Poinsot showed in 1857 that a continuous rotational motion of a rigid body about a fixed point is always geometrically equivalent to the rolling without slippage of a body cone (polhode cone) on a space-fixed cone (herpolhode cone). The line of contact between these two cones is the instantaneous rotation axis (I) of the body. Mathematically, Poinsot's representation is related to the motion of the inertia ellipsoid. An extensive treatment on this subject is given in the standard literature. Here only a simple intuitive explanation is given.

Any continuous motion of a body about a fixed point can be represented by dividing the time into infinitesimal elements and considering the motion of the body during each element of time as a rotation about the corresponding instantaneous axis of rotation. During the motion, the position of the instantaneous axis in the body and in space is gradually changing. To show this, consider the case of a cone rolling on a plane and rotating about its fixed vertex 0 (Figure 2.4):

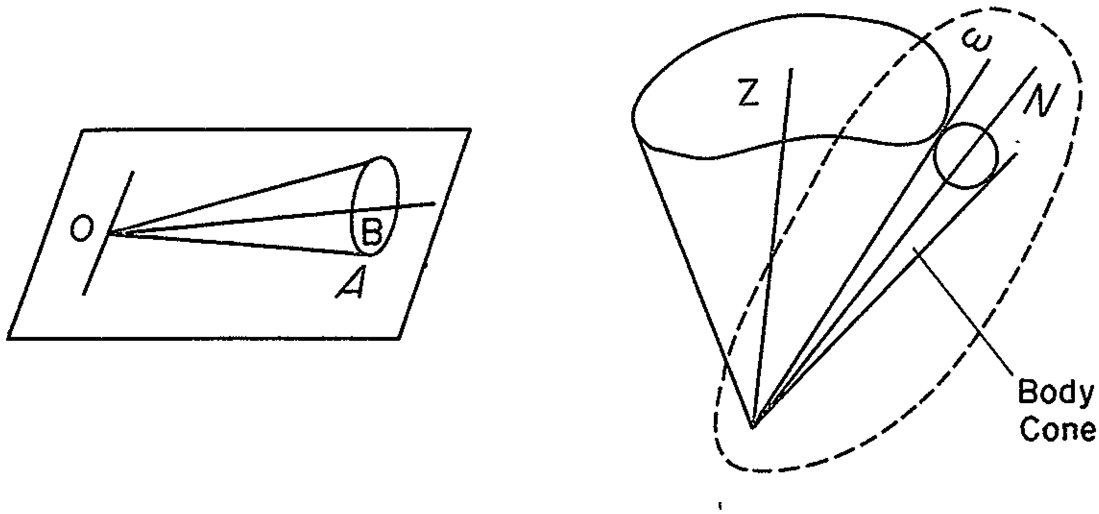


Figure 2.4 Poinsot's Kinematical Representation

At any instant, the motion of the cone consists of a rotation about the line of tangency OA, which is the instantaneous axis in this case. As the cone rolls, various lines of its surface come into contact with the plane, and the instantaneous axis has different positions in the moving cone. At the same time, consecutive positions of the instantaneous axis in space form the plane on which the cone is rolling. This example can be generalized. Instead of a circular cone rolling on a plane, a cone of arbitrary shape is rolled on a surface of another cone. One can assume also that the rolling cone is not a physical body but a geometrical surface formed by consecutive positions of the instantaneous axis in the moving body of any shape as indicated by dotted lines. Likewise, the space cone is formed by successive positions of the instantaneous axis in space, and the motion of the body is visualized by rolling the cone connected with the body <u>on</u> the cone fixed in space. By varying the shapes of both cones, all possible motions of a rigid body about a fixed point can be obtained.

In case of the forced motion of the earth, the Z-axis coincides with the pole of the ecliptic and the space fixed cone has a mean radius equal to the obliquity. The cone is not circular but has "ripples" superimposed on it due to the nutation. The time varying size of the body-fixed cone is determined at any instant through the diurnal polar motion terms of the rotation axis, e.g., u_{I_r} of equation (2.3-13). It is understood that the term "body-fixed" cone refers only to an infinitesimal time span. With this in mind, the two cones describe the forced position of the instantaneous rotation axis completely at any instant. The center of the body cone can be associated with the pole (C′). But it is known that the body cone or its representative, the axis (C′), moves within the earth due to the Chandler motion (force-free motion). This motion itself can be represented by a pair of rolling cones. This leads to an alternative representation of the motions, i.e., instead of using one pair of general cones in infinitesimal time intervals several pairs of circular cones are used in a finite time interval. This allows the separate interpretation of the various motion components. Each of these pairs of cones defines a particular component $\vec{\omega}_j$ of the instantaneous rotation axis in magnitude and in direction. The real physical instantaneous rotation axis in space is the sum of all components:

$$\vec{\omega} = \sum_j \vec{\omega}_j .$$

Besides separating the free and forced motions the latter is once more split up in precession and circular nutations. Thus

$$\vec{\omega} = \vec{\omega}_h + \vec{\omega}_p + \sum_j \vec{\omega}_{j\ N} .$$

It is more convenient to deal in this context with the circular nutations rather than with their elliptical combination since the former are directly accessible to the cone representation. As a type of classification a motion is called prograde if it occurs in the direction of the earth rotation, otherwise it is called retrograde. The three types of motions which occur in the earth motion spectrum are shown in Figure 2.5. The angles s and r are the vertex semi-angles of the space-fixed and body-fixed cones, and ν and μ are the angular velocities of the instantaneous rotation axis component on the respective cone. The vertex of all cones is at the geocenter. The magnitude of the resulting rotation axis component is

$$\omega_1 = \mu^2 + \nu^2 - 2\mu\nu \cos \delta$$

The two equations, as depicted in Figure 2.5,

$$\frac{\sin r}{\nu} = \frac{\sin \delta}{\omega}$$

$$\frac{\sin s}{\mu} = \frac{\sin \delta}{\omega}$$

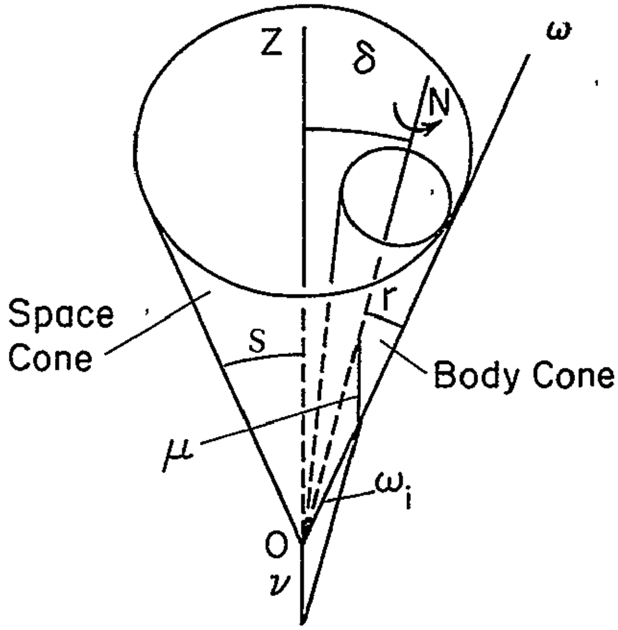
give the relation between the vertex semi-angles and the angular velocities

$$\frac{\sin r}{\sin s} = \frac{\nu}{\mu} \tag{2.3-28}$$

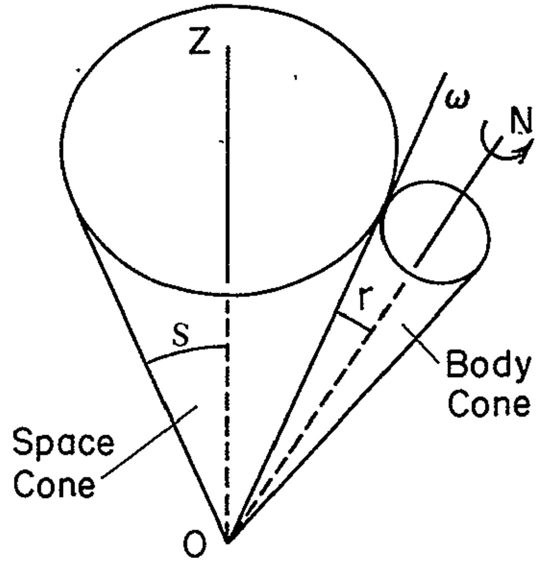
This equation implies that the cones move on each other without slipping. For small vertex angles the relation simplifies to

$$\frac{r}{s} = \frac{\nu}{\mu} \tag{2.3-29}$$

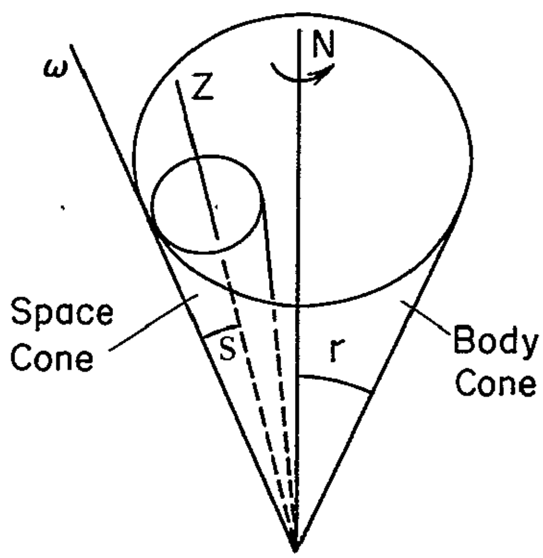
The various motions can now be classified.



Case 1:
Retrograde internal rolling



Case 2:
Prograde external rolling



Case 3: Prograde rolling on the outside of the space cone

Figure 2.5 Cone Representation of the Earth's Motion Spectrum

A) Force-free Motion: Case 3

$Z =$ angular momentum axis (H)

$N =$ axis of figure (F)

From the first equation of (2.3-14) and equation (2.3-20) the ratio (2.3-29) is

$$\frac{\nu_0}{\gamma_0\left(\frac{A_3-A_1}{A_3}\right)\left(1-\frac{k}{k_s}\right)} = \frac{\sigma_0+\Omega}{\sigma_0} \quad . \tag{2.3-30}$$

Here we have used $\Omega = \dot{\varphi}$ which results from the approximation GMST $= \varphi$. The validity of (2.3-30) can be verified by rearrangements on the left hand side.

B) Precession: Case 1 ; $\Delta\alpha_{K_1} = 0$

$Z =$ fixed ecliptic pole

$N = C'$ (precession only)

From u_{II} of equation (2.3-13) the vertex semi-angle of the body cone for $\Delta\alpha_{K_1} = 0$ is

$$r = \frac{A_{K_1}}{\Omega^2 A_3}$$

The vertex semi-angle of the space cone is θ_0 as is seen from simple geometric considerations. The spatial frequency ν is taken from equation (2.2-44) as

$$\nu = \dot{\psi}_{HP} = \frac{A_{K_1}}{A_3\Omega\sin\theta_0}$$

The body-fixed frequency is, of course, equal to the sidereal frequency Ω. Thus, equation (2.3-28) gives

$$\frac{\dfrac{A_{K_1}}{\Omega^2 A_3}}{\sin\theta_0} = \frac{\dfrac{A_{K_1}}{A_3\Omega\sin\theta_0}}{\Omega} \; .$$

C) Nutation: Case 1 if $\Delta\dot{\alpha}_j > 0$

Case 2 if $\Delta\dot{\alpha}_j < 0$

Z = is located on the space cone of the precession

N = C′ (nutation only)

$\Delta\dot{\alpha}_j$ is the nutation frequency. It was first defined in equation (2.2-32). The nutation frequency is either positive or negative, depending whether the corresponding tidal frequency is larger or smaller than the sidereal frequency Ω. The vertex semi-angle and frequency for the body-fixed motion is given by equation (2.3-6), and those for the space-fixed motion are taken from equations (2.2-43) and (2.3-20). Inserting these quantities in (2.3-29) gives the ratio:

$$\frac{\dfrac{A_j\, s_j^{-1}}{A_1\, \Omega^2}}{\dfrac{A_j}{A_3\, \Omega\, \Delta\dot{\alpha}_j} + \dfrac{A_j\, s_j^{-1}\,(A_3 - A_1\, s_j')}{\Omega^2 A_1\, A_3}} = \frac{\Delta\dot{\alpha}_j}{\dot{\alpha}_j} \qquad (2.3\text{-}31)$$

The validity of equation (2.3-31) can be verified by re-arrangements on the left-hand side. The equation holds for the rigid and elastic models.

In general, then, all circular nutations and precession will result in retrograde body motions because the exponent in the expressions for u_{Ir} in equation (2.3-6) is always negative. Similarly, those circular nutations for which $\Delta\dot{\alpha}_j < 0$ will cause prograde spatial motions of the rotation axis, whereas the others give retrograde spatial motions.

It has been demonstrated above that all known motions for the rigid and elastic model can be represented by pairs of cones rolling on each other without slipping. Each type of motion is completely identifiable by its period and its occurrence in either space or body.

2.4 Coordinate System Definition and Observability

2.4.1 Relationship Between Coordinate Systems

The Poisson equation for the spatial motion of the angular momentum axis was solved in an inertial frame which was taken to be the ecliptic at a fixed (standard) epoch. The physical realization of the ecliptic can be thought of as the plane defined by the solar center, the earth-moon barycenter and the velocity of the barycenter. The orientation of this plane changes secularly and periodically due to the planetary perturbations. The periodic variations are computable from theory. That fictitious plane which has only a secular motion in inertial space is the exact definition of the ecliptic.

The relationship between the ecliptic and the celestial system whose pole is (C′) is given by the dynamical theory of the rotation of the earth in combination with some constants of definition.

The mean position of the celestial equator (C′) relative to the fixed ecliptic is implicit in equation (2.3-25). Using equation (2.2-45) and the constant term of $\delta\varkappa_{U_3 f}$ in (2.3-22), which is identical to the constant term in (2.3-24) in case of the rigid body, the precession of the celestial pole (C′) is

$$\begin{aligned} \psi_{C'P} &= \psi_{HP} = f_1 t + f_2 t^2 + \ldots \\ \theta_{C'P} &= \theta_{HP} + \delta\theta^c = \theta_0 + \delta\theta^c + \theta_2 t^2 + \ldots \end{aligned} \qquad (2.4\text{-}1)$$

f_1 is the constant of luni-solar precession. It can be derived from observations or from theory. The latter possibility is indicated through equation (2.2-47). The mean obliquity at the standard epoch is found by observations. This constant includes the small constant $\delta\theta^c$ of equation (2.3-22) or (2.3-24) which is equal to the mean separation between the angular momentum (H) and the celestial pole (C′). For abbreviation, the equations (2.4-1) can be rewritten as follows:

$$\varkappa_{C'P} = \varkappa_{HP} + \delta\varkappa^c_{U_3 f} \qquad (2.4\text{-}2)$$

The superscript c denotes the constant term in $\delta\varkappa_{U_3 f}$.

The nodal line Υ, in Figure 2.6 between the fixed ecliptic and the mean equator of date does not coincide with the direction of the mean vernal equinox Υ_M. The latter is equal to the nodal line of the ecliptic of date and the mean equator of date. The motion of the ecliptic on the moveable mean equator is called the planetary precession in right ascension a, with

$$a = g_1 t + g_2 t^2 + \ldots$$

The g_1's follow from the planetary theory. The transformation due to precession is

$$S = R_3(a)\ R_1(-\theta_{c'p})\ R_3\ (-\psi_{c'p})\ R_1(\theta_0 + \delta\theta^c) \qquad (2.4\text{-}3)$$

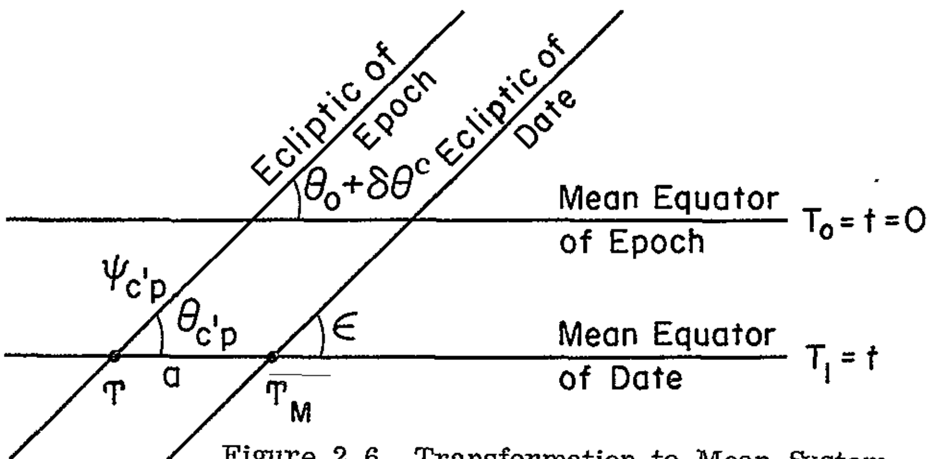


Figure 2.6 Transformation to Mean System

and the remaining transformation from the mean celestial system to the true celestial system $(X)_{c'}$ is

$$N = R_3(a_1)\ R_3(-\theta_{c'p} - \theta_{c'N}) R_3(\psi_{c'N}) R_1(\theta_{c'p}) R_3(-a) \qquad (2.4\text{-}4)$$

where

$$a_1 = a + \Delta a$$

and Δa is a small computable term [Woolard and Clemence, 1966, p.240]. $\theta_{c'N}$ and $\psi_{c'N}$ are the nutation angles of the celestial pole (C′). With equation (2.3-25), the nutation angles can be written as

$$\chi_{c'N} = \chi_{HN} + \delta\chi^p_{U_3 f}$$

where the superscript denotes the periodic terms in $\delta \varkappa_{U_3 t}$.
With a few minor changes, Figure 2.6 can also serve for setting up equation (2.4-4).

In practical astronomical work one tends to select a different set of rotation angles in the transformation (2.4-3) [Mueller, 1969, p. 65]. One also prefers to use the so-called "reduced" nutation in longitude and obliquity, $\psi'_{C'/N}$ and $\theta'_{C'/N}$, which relate the true equator to the mean equator of date. Then the transformation N becomes

$$N = R_1(-\epsilon - \theta'_{C'/N})R_3(-\psi'_{C'/N})R_1(\epsilon)$$

The reduction of the nutations is carried out in Woolard (1952, p. 157). ϵ is the mean obliquity of date,

$$\epsilon = \theta_0 + \delta\theta^c + a_1 t + a_2 t^2 + \dots$$

The coefficients a_i follow again from theory.

Finally, the motion of the vernal equinox due to the combined motion of the ecliptic and the equator is called general precession in longitude ψ_1,

$$\psi_1 = h_1 t + h_2 t^2 + \dots$$

where

$$h_1 = f_1 - g_1 \cos(\theta_0 + \delta\theta^c)$$

Since g_1 and h_1 follow from theory there is no additional new constant which is to be determined by observation.

2.4.2 Fundamental Declinations and Latitudes

The fundamental declinations and latitudes are the true observables in observatory astronomy. They give the position of the observatory and the star in the true celestial system $(X)_{C'}$. The procedure consists of observing zenith distances of stars at upper and lower culmination. At no stage of the procedure will the star position or an adopted series of nutations be needed.

For the following description it is assumed that the Chandler motion is zero for the period of 12 hours. This assumption, of course, will not be fulfilled since one has no control over the actual free motion. This simply demonstrates that all efforts to determine the orientation of the earth in space are ultimately limited by the amount of Chandler motion which occurs during the interval which is needed to carry out the basic observations. The progressive Chandler motion is approximately 5 cm for 12 hours.

The left and the right picture in Figure 2.7 show the situation at upper and lower culmination of the same star. The pole (C') remains body-fixed during the 12 hour interval. In particular, it has no nearly diurnal free nutation as was shown in Section 2.3.4. The body-fixed motion of the angular momentum axis (H) is represented as a circle in the lower two pictures. There is no need for (H) or the axis (U_3) to be in the meridian during the observations. The body-fixed position of (H) for upper transit at epoch T is shown in the lower left picture. The lower right picture shows the body-fixed position of (H) at lower and upper transit. For the sake of completion, a possible body-fixed position of the Eulerian pole of rotation (E_0) is also shown. The angle γ is the along-meridian polar motion component. γ_1 and γ_2 are the periodic diurnal body-fixed motion components of (H) along the meridian. Φ_U denotes the adopted station latitude in the (U)-system. δ is the declination in the true celestial system $(X)_{C'}$ whose third axis is identical with (C'). The observed zenith distances at culmination are z_1 and z_2. From the upper two pictures in Figure 2.7, the following two basic relations are readily seen:

$$
\begin{aligned}
90 - \delta &= (z_2 - z_1)/2 \\
90 - (\Phi_{U_3} + \gamma) &= (z_2 + z_1)/2
\end{aligned}
\qquad (2.4\text{-}5)
$$

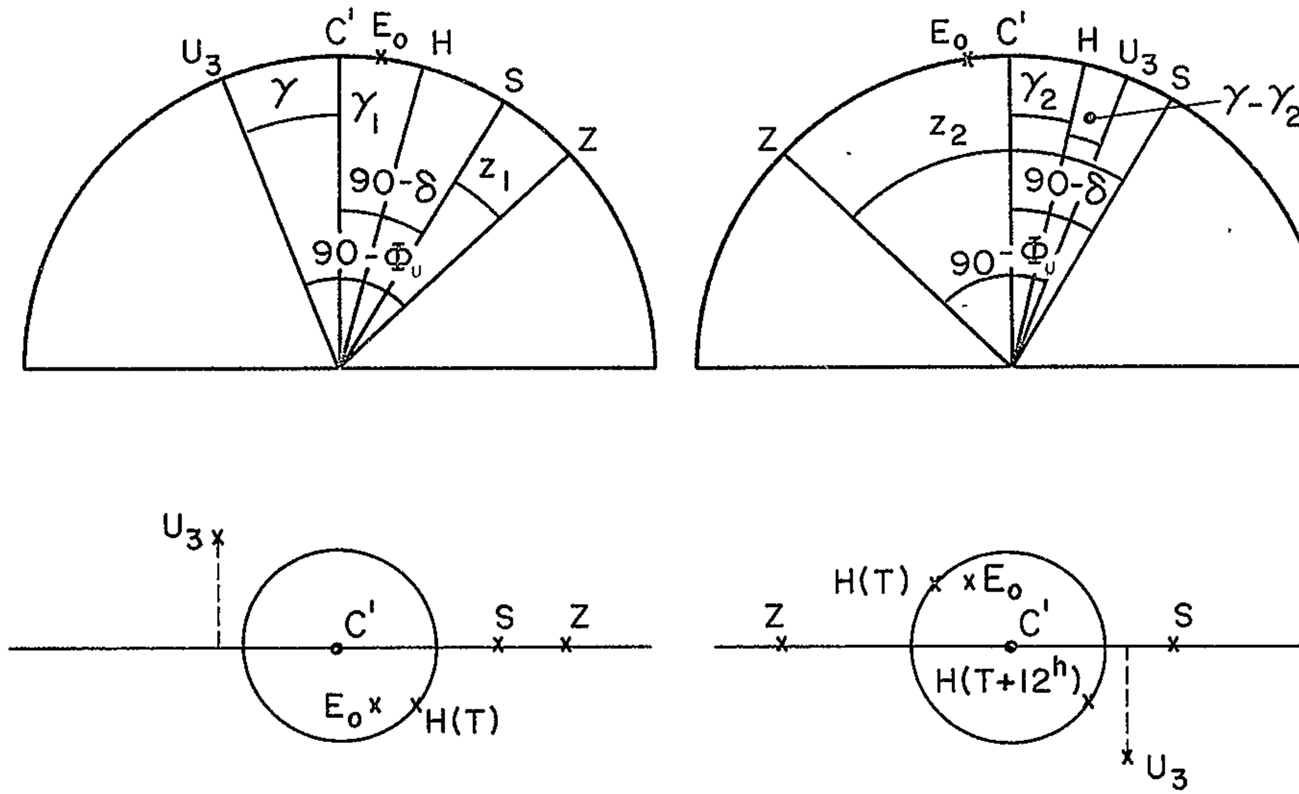


Figure 2.7 Fundamental Declinations and Latitudes

It is now clear that the declination and the latitude in the true celestial system $(X)_{C'}$ are the observables. The position of (H) at the time of observation has no importance in this context. More importantly, the instantaneous rotation axis (I) does not enter the observation geometry at all. It is therefore concluded that, on the basis of the observational procedure discussed above, the instantaneous rotation axis (I) and the angular momentum axis (H) are not observable.

The observational insignificance of the Eulerian pole of rotation E_0 is readily demonstrated with the help of Figure 2.7. As has been pointed out previously, the only difference between the north celestial pole (C') and the Eulerian pole of rotation (E_0) is a homogeneous solution component, so that the latter still exhibits a nearly diurnal space-fixed motion due to free nutation. Consider the body-fixed point E_0 in Figure 2.7. In the upper figures, E_0 performs a similar motion relative to the (C') as does, for example, the axis (U_3). But the presence of such a fixed crust point, and there are infinitely many possible choices all laying on the line $\Psi_0 - C'$ of Figure 2.3, does not effect the principles expressed by equations (2.4-5) at all. Thus, the Eulerian pole of rotation is not observable either.

2.4.3 Determination of the Constants of Definition

The relationship between the celestial system $(X)_{C'}$ and the fixed ecliptic system $(X)_E$ is expressed by the standard expressions

$$\sin\delta = \cos\beta \sin\lambda \sin\theta_{C'} + \sin\beta \cos\theta_{C'}$$

and

$$\cos\delta \sin\alpha = \cos\beta \sin\lambda \cos\theta_{C'} - \sin\beta \sin\theta_{C'} \tag{2.4-6}$$

α and δ are the right ascension and declination in $(X)_{C'}$. λ and β are the ecliptic longitude and latitude in the fixed ecliptic system. $\theta_{C'}$ is the true obliquity of date, which is split up according to equations (2.4-1) and (2.4-2) as

$$\theta_{C'} = \theta_{C'P} + \theta_{C'N} \tag{2.4-7}$$

whereby the small constant $\delta\theta^c$ is included in $\theta_{c'p}$. Eliminating the longitude λ in (2.4-6) gives

$$\sin\alpha \tan \theta_{c'} = \tan\delta - \frac{\sin\beta}{\cos\delta \cos\theta_{c'}} \qquad (2.4\text{-}8)$$

The ecliptic latitude β of objects close to the ecliptic such as the sun, planets, etc. are known from theory accurately enough in order to be treated as a known quantity. Since at the meridian transit the right ascension α is equal to the sidereal time, the right ascension is expressed in terms of the reading, t', of a clock which keeps uniform time, as

$$\alpha = k_1 + k_2 t' + t' \qquad (2.4\text{-}9)$$

where k_1 and k_2 are the constant and the rate correction, respectively. Equations (2.4-7) to (2.4-9) can serve as a mathematical model to compute the right ascension and obliquity $\theta_{c'p}$ from repeated observations of declinations according to the procedure given in Section 2.4.2. Consequently, the right ascension will refer to the true celestial equator defined by the pole (C'), and the true vernal equinox, i.e., it will be independent of the instantaneous rotation axis (I). In view of equation (2.4-7), the solution for the constant $\theta_{c'p}$ will include the small constant $\delta\theta^c$.

The constant of luni-solar precession is principally derivable from the secular variation in right ascension. Detailed procedures for determining the constants of precession are found in the literature.

The constant of nutation, N, which is the coefficient of the 18.6 year nutation in obliquity is another constant needed in order to be able to completely give the true position of (C') in space at any time. It is a function of the astronomical constants $(A_3 - A_1)/A_3$, the lunar mass, and the obliquity. The constant of nutation determines the common multiplier (scalar) for all other nutation terms. Various determinations of the constant of nutation are discussed in the literature. One of the most complete determinations is that by Fedorov [1958]. One should note that the observational determination of N includes the small term with the same period in $\delta\theta_{F_F}$ (equation 2.3-24). This fact has to be taken into account when computing the common multiplier of the nutations or when one attempts to verify a relation of the type of equation (2.2-47).

2.4.4 Practical Aspects for Adapting a Set of Nutations

In the usual astronomical work one does not observe the same star at upper and lower culmination as discussed earlier; rather, one observes stars at either their upper or lower culmination. The latitudes resulting from such a scheme are

$$\Phi = \delta \pm z \tag{2.4-10}$$

in case the upper culmination occurs south or north of the observatory zenith. Latitudes computed by this formula always refer to the equator of date to which the declinations are counted, i.e., the colatitude is the instantaneous angular distance between the observers' zenith and the pole as defined by the set of adopted nutations. According to the definition of the celestial pole (C′) any latitude computed by

$$\Phi_{C'_P} = \delta_{C'} \pm z$$

varies only due to the progressive Chandler motion. The use of any other set of nutations will result in a computable variation of the latitude.

Consider the change in declination due to nutation [Mueller, 1969, p. 74],

$$\Delta\delta_{C'} = -\psi_{C'_N} \sin\theta_{C'_P} \cos\alpha + \theta_{C'_N} \sin\alpha \tag{2.4-11}$$

This formula is in agreement with our sign convention for ψ, i.e., positive from west to east. α is the right ascension of the star.

Equation (2.4-11) can be written as

$$\Delta\delta_{c'} = \mathrm{Re}(i\,\varkappa_{f'}\,e^{-i\alpha}) \qquad (2.4\text{-}12)$$

The difference in latitude based on, for example, the nutations of the pole (C') and the instantaneous rotation axis (I) is

$$\begin{aligned} \Phi_{c'} - \Phi_I &= \delta_{c'} - \delta_I \\ &= \mathrm{Re}\,[i\,(\varkappa_{c'N} - \varkappa_{IN})\,e^{-i\alpha}] \\ &= \mathrm{Re}\,[i\,(\varkappa_{HN} + \delta\varkappa^{p}_{U_3f} - \varkappa_{HN} - \delta\varkappa^{p}_{If})\,e^{-i\alpha}] \\ &= \mathrm{Re}\,[i\,(\delta\varkappa^{p}_{U_3f} - \delta\varkappa^{p}_{If})\,e^{-i\alpha}] \end{aligned} \qquad (2.4\text{-}13)$$

With equation (2.3-20) and (2.3-22) this difference becomes

$$\Phi_{c'} - \Phi_I = \mathrm{Re}\; i \sum_j \frac{A_j\, s_j^{-1}}{A_1\,\Omega^2}\; e^{-i(\Delta\alpha_j + \alpha)} \qquad (2.4\text{-}14)$$

The summation is to be taken over all tidal waves except those having sidereal frequency. As was mentioned earlier, the waves of sidereal frequency are responsible for the small constant from $\overline{\delta\theta^c}$. In case of meridian observations the right ascension α in (2.4-14) can be replaced by the mean sidereal time MST. Thus, from equation (2.4-14), the first equation in (2.3-13) and equation (2.3-15) the latitude difference is the negative of the meridional diurnal motion component of the rotation axis (I)

$$\Phi_{c'} - \Phi_I = -\mathrm{Re}\,(u'_{If}) = -u'_{1If} \qquad (2.4\text{-}15)$$

The difference is usually referred to as "<u>dynamical variation of latitude</u>." Again, the term with sidereal frequency is excluded in (2.4-15). Equation (2.4-14) clearly shows that latitude determinations as derived from zenith distance measurements to stars at transit exhibit a nearby diurnal variation. If one observes the same star at consecutive transits, the latitude variation shows periods equal to those of the nutations and with respective amplitudes as given in equation (2.4-14).

The situation is very similar for longitude observations. The usual formula for correcting the right ascensions due to nutation is

$$\Delta\alpha_{C'} = -\psi_{C'/N} \cos\theta_{C'/P} + (-\psi_{C'/N} \sin\theta_{C'/P} \sin\alpha - \theta_{C'/N} \cos\alpha)\tan\delta \quad (2.4\text{-}16)$$

where ψ is again counted positively eastward. The first term, which is a correction to the vernal equinox, is called the equation of the equinox

$$\mathrm{Eq.\ E} = -\psi_{C'/N} \cos\theta_{C'/P}$$

It is equal to the difference between the mean and true right ascension of a body on the equator. The other terms, which are a function of the position of the star, can be rewritten as

$$\Delta\tilde{\alpha}_{C'} = \mathrm{Im}(-i\,\varkappa_{C'/N}\, e^{-i\alpha})\tan\delta \quad (2.4\text{-}17)$$

The basic longitude equation is

$$\begin{aligned}\Lambda &= \mathrm{MST} - \mathrm{GMST}\\ &= \mathrm{AST} - \mathrm{GAST}\end{aligned}$$

For reasons of compatability both the local and the Greenwich sidereal times are corrected for polar motion. Polar motion, of course, is understood to consist only of the motion of the pole (C'). Meridian observations give

$$\begin{aligned}\Lambda_{C'} &= \alpha - \mathrm{GAST}\\ &= \alpha_M + \mathrm{Eq.\ E} + \mathrm{Im}(-i\,\varkappa_{C'/N}\, e^{-i\alpha})\tan\delta - \mathrm{GMST} - \mathrm{Eq.E}\\ &= \alpha_M + \mathrm{Im}(-i\,\varkappa_{C'/N}\, e^{-i\alpha})\tan\delta - \mathrm{GMST}\end{aligned}$$

α_M denotes the mean right ascension of date. The longitude computed in this manner will vary only due to the progressive Chandler motion.

Computing the longitude based on the nutations of the instantaneous rotation axis (I) gives the difference

$$\Lambda_{c'} - \Lambda_I = \mathrm{Im}\,[-i\,(\varkappa_{c'N} - \varkappa_{IN})]\, e^{-i\alpha} \tan\delta$$

$$= \mathrm{Im}\left(-i\sum_j \frac{A_j\, s_j^{-1}}{A\,\Omega^2}\, e^{-i\,(\Delta\alpha_j + \alpha)}\right)\tan\delta \qquad (2.4\text{-}18)$$

For transit observations, this difference, according to the first equation in (2.3-13) and equation (2.3-15) becomes

$$\Lambda_{c'} - \Lambda_I = \mathrm{Im}\,(u'_{If})\tan\delta = u'_{2If}\tan\delta \qquad (2.4\text{-}19)$$

where $\dot{u}'_{2If}$ is the component of the diurnal motion of the rotation axis (I) orthogonal to the local meridian. When verifying the longitude difference from observations, one has to account again for the fact that the obliquity, as derived from observations, contains the small constant term $\delta\theta^c$. Thus the sidereal terms are excluded in (2.4-18) and (2.4-19).

3. MODELS WITH LIQUID CORE

3.1 Brief Historical Review

The desire to explain the discrepancies between the observed and predicted motion of the earth resulted in a long history of a search for better earth models. The first efforts to investigate earth models with liquid core date back to Hopkins [1839]. After Chandler's explanation regarding the basic period in polar motion, Hough [1895] and Sludskii [1896] discovered independently the possibility of a nearly diurnal free wobble (NDFW) for a rotating container with liquid core. Poincaré [1910] published his investigation on the precession of the deformable earth, giving an elegant accounting for possible movements of the core. His theory is reproduced in [Melchior, 1966]. Takeuchi [1950] carried out numerical integration of the equations for a heterogeneous and compressible globe by utilizing different models of the earth's interior, these models being constructed according to seismological results. Jeffreys and Vicente [1957 a, b] continued the studies of Sludskii and Poincaré and proved, based on the work by Takeuchi, that an effect of resonance due to the movements in the liquid core (core resonance) appears on the waves whose period is close to that of the NDFW. Molodenskii [1961] published an analogous study with results similar to those of Jeffreys and Vicente. This theory is also reproduced in [Melchior, 1966]. Recently, Shen and Mansinha [1976] presented an extension of Molodenskii's theory. McClure [1976] studies the effect of transverse meridional core-flow relative to the mantle on the separation between the total angular momentum vector and the rotation vector of the shell based on a generalization of Poincaré's model. Smith [1977] investigated, theoretically and numerically, the free modes based on geophysically plausible rotating, slightly elliptical earth models.

The above review is necessarily incomplete in view of the large amount of work which has been devoted to the subject. The degree of difficulties in the mathematical treatment depends on the specific assumptions regarding the structure of the core and shell, the core boundary, etc. In this section only a discussion of the additional characteristic motions is given. For any derivations the reader is referred to the literature.

3.2 Shell and Core Interactions

The interactions between the core and the shell effect the orientation of the shell in space. Determination of the orientation of the shell, therefore, can give valuable information about the core motion spectrum.

3.2.1 Free Mode

The nearly diurnal free wobble (NDFW) is a possible free mode which is given by the homogeneous solution of the equations of the combined motion of shell and core. It can be categorized as a body (shell)-fixed motion of the instantaneous rotation axis (I) having a nearly diurnal period. In the astronomical literature it is sometimes referred to simply as nearly diurnal free "nutation." Its observational significance has been explained recently. It was Toomre's [1974] contribution to strengthen the point that even this nearly diurnal free "wobble" must be accompanied by a free nutation in space whose amplitude and period is approximately 460 times larger than that of the NDFW. These motions can formally be visualized again by Poinsot's kinematical representation. The motions occur regardless of whether the shell is taken rigid or elastic. The mathematical derivations of this mode can be found in, for example, [Rochester, et al., 1974] or [McClure, 1976]. The frequency of the nearly diurnal free wobble, α'', is primarily a function of the ellipticity of the core as is indicated by the relation

$$\alpha'' = -\Omega\left(1 + \frac{A_1}{A^o} \cdot \frac{A_3' - A_1'}{A_1'}\right) ,$$

which holds for a rigid shell and ideal fluid core. The symbols A_1' and A_3' denote the moments of inertia of the core, A^o is the equatorial moment of inertia of the shell alone, and the other symbols have unchanged meaning. The frequency of the accompanying free nutation is equal to α'' plus the earth's sidereal motion. This is a property of Poinsot's kinematical representation. Thus, the ratio (2.3-29) is

$$\frac{r}{s} = \frac{\alpha'' + \Omega}{\alpha''} \cong \frac{1}{460} \qquad (3.2\text{-}1)$$

Both the nearly diurnal free wobble and the accompanying nutation are retrograde motions as shown in Case 1 of Figure 2.5. These notions are super-imposed on those discussed earlier. In particular, the shell-fixed motion of the instantaneous rotation axis of the shell is

$$u_{Is} = u_I + r e^{i\alpha'' t} \qquad (3.2\text{-}2)$$

The space-fixed motion of the instantaneous rotation axis of the shell is

$$\varkappa_{Is} = \varkappa_I + 460 \ r e^{i\frac{\alpha'' t}{460}} \qquad (3.2\text{-}3)$$

The spatial change, as expressed in equation (3.2-3) is the same for the north celestial pole (C′); thus,

$$\varkappa_{C''} = \varkappa_{C'} + 460 \ r e^{i\frac{\alpha'' t}{460}} \qquad (3.2\text{-}4)$$

C″ stands for "north celestial pole <u>of the shell</u>." Probably the most meaningful and unique labeling of the two harmonic motions discussed here is "retrograde free principal core nutation." The word "core" indicates the origin of the motion and the adjective "principal" implies that more motions of this type are possible depending on the assumed core structure. Jeffreys and Vicente [1957b], for example, found another free but prograde mode because of the density structure in their core model. This clearly demonstrates the importance of the assumptions about the form and structure of the core in this type of calculation.

3.2.2 Core Resonance

Another geophysical phenomenon responsive to the presence of the free mode is the tides of the solid earth. Jeffreys and Vicente pointed out that nearly diurnal tides whose frequency is close to α'' should experience amplifications. This phenomenon is called "core resonance." It is presented mathematically in terms of a factor which depends on the tidal frequency and by which the tidal amplitudes A_j are multiplied. The factor increases as the tidal frequency approaches α''. Jeffreys and Vicente, as well as Molodenskii, gave estimates for the change in the tidal amplitudes A_j, which can, via equation (2.2-43), easily be converted to changes in nutation. An extensive analysis of these effects based on various models can be found in Melchior [1971]. Some of the larger effects are given in Table 3.1. The values are to be added to the nutation of the rigid earth. The signs are such that the amplitudes of the nutations increase in magnitude. The first column in Table 3.1 denotes the tidal waves in terms of their code number, which is explained in Section 2.2. In addition, the commonly used symbol is given for each tide. The significance of these symbols in terms of systematizing the tidal waves is found in [Melchior, 1966]. Thus the waves OO_1 and O_1 form the semi-monthly elliptical nutation.

Table 3.1 Corrections for Rigid Earth Nutations in Arcsec Due to Core Resonance

Period (days)	$d\psi \sin\theta$	$d\theta$
semi-monthly (13.7) OO_1 ; 185.555 O_1; 145.555	0.0020	0.0026
semi-annual (183) φ_1; 167.555 P_1; 163.555	0.016	0.020
annual (365) ψ_1; 166.554 S_1; 164 556	-0.0077	0.0056

As was mentioned above, it is the relative position of the tidal frequency with respect to the frequency of the nearly diurnal wobble, α'', which determines the magnification of the tidal amplitude. α'' is not identical to the diurnal frequency of the K_1 tide but it is closer to that of the ψ_1 tide. Since the elliptical nutations consists of two tidal waves whose frequencies are symmetric with respect to K_1, the two waves experience dissymmetric resonance effects. An example is the annual nutation which is formed by the tidal waves ψ_1 and S_1. Both tidal amplitudes are of equal magnitude, which results in a zero annual nutation in obliquity according to equation (2.2-42). But the liquid core model requires a nutation in obliquity solely due to the dissymmetric correction.

The corrections of Table 3.1 have to be added to both the nutations of (I) and (C$'$). This is so because tidal analysis gives corrections to the tidal amplitudes which enter in both sets of nutations according to equations (2.3-23).

3.3 The North Celestial Pole of the Shell (C'') and Its Generalization

The discussion of the section above demonstrates that for the real earth somewhat different terrestrial motions of the instantaneous rotation axis and a slightly different orientation in space have to be expected compared to ideal models, such as the rigid and elastic model or even the model with liquid core. Since all geometric observations (directions, ranges) take place on the shell one may define the orientation of the earth in terms of the orientation of the shell.

Similar to the definitions of Section 2.3.4, the orientation of the shell in space is based on the direction of the pole (C''), i.e., the direction which has neither periodic diurnal motions relative to the shell nor to space. A generalization of this definition is obvious. Instead of investigating specific models for an axis which has the desired properties of the celestial pole one can simply define the celestial pole without reference to any model. Thus,

> the orientation of the earth is based on the direction of the celestial pole (C), i.e., the axis which has neither periodic diurnal body-fixed nor space-fixed motions. The body-fixed motions of (C) are called polar motion.

This definition can be related to the actual measurement procedures. One can now say that the observations take place on the surface of the earth and that the term "body-fixed" is to be understood as a motion relative to a "rigid" surface on which the stations are located. It is noted that the effect of the motion of the crust due to rigid body tides has been assumed to be removable by computations. Another type of relative station motion, which is very slow and occurs over long periods of time usually referred to simply as "crustal motion" or "plate motion," is left out of consideration here. Their implications regarding coordinate system definition are studied extensively in [Leick, 1977]. The realization of the direction (C) in the body-fixed frame and in space is principally possible through observation of fundamental latitudes and declinations (Section 2.4.2). Each term in the set of nutations for the pole (C) is empirical and has to be determined from observations; whereas, the relative magnitude of the nutation coefficients for a specific model result from the motion theory. Fortunately, the rigid or the elastic model approximates the actual motion of the earth very well. It is, therefore, not necessary to attempt to observe the whole spectrum of nutations; rather, one can limit the investigations to specific frequencies whose amplitudes are most likely to deviate from the model values. The frequencies of Table 3.1 are certainly among such "dangerous" frequencies.

Formally, one can write the space-fixed position of the celestial pole (C) as

$$\varkappa_C = \varkappa_{C'} + \Delta\varkappa_{FCN} + \underline{\Delta\varkappa_{CR}} + \Delta\varkappa_U \tag{3.3-1}$$

where $\varkappa_{C'}$ (equation 2.3-25) can be considered the first order approximation of the pole (C); whereas, the remaining empirical terms are of second order. The term $\Delta\varkappa_{FCN}$ denotes the contribution of the free core nutation. This term is entirely empirical. Neither its magnitude nor its phase is known since it is a result of the homogeneous solution. The term $\Delta\varkappa_{CR}$ denotes the resonance effect on the nutation due to core motions. In this case, the frequencies and estimates of their amplitudes are obtained from theory (Table 3.1), but depend strongly on the assumptions of the core structure. Therefore, these terms are essentially also of empirical nature. The last term, $\Delta\varkappa_U$, stands for all effects which are not yet accounted for.

In the usual latitude observation work where a star is observed only at one culmination, as represented by equation (2.4-10),

$$\Phi = \delta \pm z$$

the computed colatitude refers to the axis defined by the adopted set of nutations. This axis has no physical meaning. It only serves to transform the actual, observable into a latitude-like quantity. Thus, the additional shell-fixed motion of the instantaneous rotation axis (I) which is due to the nearly diurnal free wobble (equation 3.2-2) is irrelevant for this observational procedure. Therefore, regarding the determination of the nearly diurnal free wobble, the only thing one can do is to determine the associated free nutation from the analysis of the observed zenith distance and conclude then via equation (3.2-1) the actual size of the nearly diurnal free wobble. Using equations (2.4-10), (2.4-12), and (3.3-1), the computed latitude variation will be

$$\begin{aligned}
\Delta\Phi &= \Phi_C - \Phi_{C'} \\
&= \mathrm{Re}\; i(\Delta\varkappa_{FCN} + \Delta\varkappa_{CR} + \Delta\varkappa_U)\; e^{-i(MST)} \\
&= \Delta\delta_{FCN} + \Delta\delta_{CR} + \Delta\delta_U \\
&= -u'_{1C'\,\mathrm{diurnal}}
\end{aligned} \tag{3.3-2}$$

In principle, any adopted set of nutations can serve in this type of analysis. Instead of taking equation (2.3-25) for the pole (C′), one could just as well select the nutations of the rotation axis (I) of (2.3-20) or the angular momentum axis (H) of (2.2-43). In these cases, additional computable terms would appear on the right-hand side of equation (3.3-2).

The corrections in declination in equation (3.3-2) are identical to the negative of the component of the diurnal motions of the adopted pole (C′) around the pole (C) along the local meridian. It is emphasized that this diurnal motion strictly refers to the pole (C′) as defined by the adopted set of nutations. It has nothing to do with the diurnal body-fixed motion of the actual instantaneous rotation axis (I) or angular momentum axis (H). These latter motions are not observable!

It is realized that the latitude variation of equation (3.3-2) is very slow if zenith distances are observed at culmination of the same star. It is, therefore, permissible to approximate the diurnal motion by a circle of constant radius over an interval of, say, one day, five days, etc. $\Delta\Phi$ has then the same magnitude for each station's observation. This difference is part of the Kimura [1902] term which was introduced in order to represent the "non-polar variation in latitude." The Kimura term also absorbs other constant effects which are specific to astronomical observation, such as errors in proper motion, aberration, etc. In any case, it is important to note that the introduction of Kimura's term makes it, at least conceptually, possible to determine the celestial pole (C) even from observations at one culmination only. One only needs the zenith distance observations from several stations of the same star at culmination in order to determine polar motion and the constant non-polar term in the least squares sense. Let Φ_U be the adopted latitude, then the equations are in the usual notation:

$$\begin{aligned} \Phi_U &= \Phi_C - u_{1C} \cos\Lambda - u_{2C} \sin\Lambda \\ &= \Phi_{C'} - u_{1C} \cos\Lambda - u_{2C} \sin\Lambda + \Delta\delta_{FCN} + \Delta\delta_{CR} + \Delta\delta_U \end{aligned}$$

This expression can be rewritten as

$$\Phi_U - \Phi_{C'} = -u_{1C} \cos\Lambda - u_{2C} \sin\Lambda + Z$$

where

$$Z = -u'_{1C'}{}_{\text{diurnal}}$$

is the Kimura term. The subscript C' denotes the adopted set of nutations.

Since in all computations, the rigid earth nutations of (I) have been used, the analysis of Kimura's term should give information on core resonance, etc. In fact, the International Latitude Service introduced in 1955 a special observation program, which is called the three-group observations, in order to make an effective analysis of the Z-term possible.

3.4 Observational Evidence - IAU (1977) Proposed Nutation Series

Reports claiming to have observed the nearly diurnal free wobble, NDFW, based on above or below pole observations, were given at several occasions. A summary is given in Yatskiv [1972]. The earliest observation, related to this second free mode, seems to have been reported by Popov [1963]. He found an amplitude of 0".016 from a long series of latitude observations of two stars culminating approximately 10 hours apart. Until the clarification made by Toomre [1974] the astronomers, apparently, being unaware of the associated large free nutation, interpreted their observations indeed as those of the NDFW. Examining the method of analysis of Popov along the lines discussed in previous sections, it becomes immediately clear that the reported variations actually represent the change in nutation. The latest report on the NDFW is [Yatskiv et al., 1975]. They not only confirm the presence of the retrograde mode, but also find that the existence of the prograde mode is quite possible on the basis of the available data.

Some of the effects of core resonance have also been confirmed by observations. A correction of 0".02 for the semi-annual nutation was obtained from analysis of Kimura's term [Wako, 1970]. The presently adopted constant of nutation for the epoch 1900.0 is N = 9".21. Using the current best estimates of the mass of the moon and the constant of precession, the relation (2.2-48) yields a somewhat larger value of N = 9".22. But from observations one arrives constantly at a smaller value N = 9".20 [Fedorov, 1958]. Applying the correction due to core-resonance to the computed value reduces the latter to the observed value. This is certainly one of the nicest features of the liquid core models. It explains one of the longest known incoherencies in the system of fundamental constraints.

An astronomical determination of the fortnightly term was reported by Guinot [1970] for longitude observations and by Morgan [1952] for latitude observation. Morgan's analysis was based on P.Z.T. observations at Washington. While using the adopted set of nutations of the rotation axis (I) for the rigid earth, he obtained a fortnightly correction equal to that of the Oppolzer term, just as rigid model theory predicts. More complete analyses were carried out by Fedorov [1958] in terms of the fortnightly diurnal motion term, and recently by McCarthy [1976], who computed the corrections to the fortnightly nutation directly. Both obtained values which are in agreement with those predicted by the co-resonance model (Table 3.1). They used the nutations of the angular momentum (H) as computed from Poisson's equation, as their reference pole. One may add that McCarthy used (H) as an approximation to the rotation axis of the rigid model and corrected the observations for the Oppolzer terms (equation 2.4-15); whereas, Fedorov specifically intended to use (H) because its position in space is relatively independent of mass redistribution. But from the merely analysis point of view, this distinction is unimportant since the adopted set of nutations (Poisson solution) serves only as an intermediary reference standard. McCarthy's corrections should be interpreted as corrections to the nutations of (C′) in order to get the nutation of (C). Fedorov's corrections need to be added to the nutation of (H) in order to get the nutation of (C). This is so because astronomical observations give the motion of the pole, as defined by the adopted set of nutations, with respect to (C). The fortnightly correction term of Table 3.1 can be readily converted to a corresponding diurnal body-fixed motion term. The derivation of its radius follows from equation (2.2-46)

$$\varkappa_{2s} = a_{2s} \cos (2s) + i\, b_{2s} \sin (2s) \qquad (3.4\text{-}1)$$

where $2s$ denotes the fortnightly term, and from equation (2.3-19),

$$u_{2s} = -i\,\Delta\,\varkappa_{2s}\; e^{-i\varphi} \tag{3.4-2}$$

Neglecting the subscripts and combining equations (3.4-1) and (3.4-2), one gets

$$\begin{aligned} u_{2s} = -\left[\frac{a+b}{2}\sin(\varphi - 2s) + \frac{a-b}{2}\sin(\varphi + 2s)\right] \\ - i\left[\frac{a+b}{2}\cos(\varphi - 2s) + i\,\frac{a-b}{2}\cos(\varphi + 2s)\right] \end{aligned} \tag{3.4-3}$$

Substituting the coefficients from Table 3.1 and neglecting their differences gives

$$\delta u_{2s} = -0.0023\ \sin(\varphi - 2s) - i\ 0.0023\ \cos(\varphi - 2s) \tag{3.4-4}$$

Fedorov found a radius for the fortnightly term of 0.″009. As far as the model is concerned, this radius formally corresponds to the rigid earth nutation plus the core resonance effects, i.e., as predicted by Jeffreys and Vicente [1957, a, b]. Equations (2.3-17) and (3.4-4) give

$$|u_{2s}| = 0.0052 + 0.0023 = 0.''0085$$

The perfectly elastic model, however, predicts a fortnightly diurnal term of approximately $\frac{2}{3}$ 0.″0062, as is seen from the second equation in (2.3-13). We may therefore conclude that, at least for the fortnightly nutation, the perfectly elastic model does not seem to conform with observation. Since the BIH [1975] has currently adopted the nutation of the pole (C′) of the elastic model (equation (2.3-25)), their procedure may require revision at some future time.

It appears, based on the results of Table 3.1 and the method implied, that the actual diurnal radius of the motion of (I) around the pole (C) does not change significantly because of core resonance effects. This statement is possible because it can be shown that the same corrections to the tidal amplitudes cause nearly identical corrections for both (I) and (C′) (see equations 2.3-23). However, this fact cannot be proven from astronomical observations since (I) is not observable.

The IAU-Symposium No. 78 (1977) recommended that the following set of coefficients be substituted for the corresponding coefficients in Woolard's series for the nutations in order to provide a more accurate representation of the forced nutation of the axis of rotation of the earth due to the luni-solar perturbing forces:

Table 3.2 IAU (1977) Proposed Nutation Coefficients

Period (days)	ψ sin θ	θ
6798	-6.843	9.206
3399	0.083	-0.091
365	0.058	0.006
183	-0.520	0.569
122	-0.020	0.022
27.6	0.028	0.000
13.7	-0.083	0.091

The sign convention in Table 3.2 is the same as in the AENA Supplement [1961, p. 44]. The coefficients are obtained by adding the tidal corrections to the nutation of the instantaneous rotation axis (I) of the rigid body. The thus modified nutation series does not describe the position of the celestial pole (C). The theoretical corrections for the dynamical variations are still needed!

4. SUMMARY AND RECOMMENDATIONS

The motion characteristics of the rigid, the elastic earth model and the model with liquid core have been reviewed. An attempt has been made to strictly distinguish between motions due to external forces and the free motions. The dynamical theories of the rigid and elastic earth model involve various axes, such as the axis of rotation (I), the angular momentum axis (H), the axis of figure (F) and an axis called the celestial pole (C'). Special efforts have been made to investigate the significance of these axes regarding observability. It has been found that even for these ideal models, where the stations do not change due to crustal motions, only the celestial pole (C') is observable through fundamental astronomical observations. "Observable" is to be understood in the sense that the direction between the observatory and the celestial pole (C') can be measured without using any hypotheses or models. The pole (C') moves with respect to the body only because of the progressive Chandler motion. It is understood that fundamental astronomical observations, where the same star is observed at both culminations, give only the mean positions for the twelve hour time span.

The diurnal motions of the instantaneous rotation axis (I), and the angular momentum (H) have been investigated. The periodic diurnal polar motions of (I) are strictly related to the spatial nutations of (I). The correspondence between those motions can be demonstrated most easily in terms of two cones rolling on each other, having characteristic vertex angles and speeds. It has been shown that the instantaneous rotation axis (I) is needed only at the initial step when formulating the motions. From the descriptive point of view, one can associate with a rotating body which is subject to external forces two instantaneous rotation axes. One of the two axes is (I), which must have periodic diurnal body-fixed motions. The other axis is (C'); it is effected only by the progressive motion (free motion). Since the Chandler (Euler) motion, at least conceptually, represents the initial conditions, it can assume any magnitude. (Please remember that we are talking about ideal bodies.) If its magnitude is zero, then the celestial pole (C') does not move with respect to the model surface; whereas, (I) still has the diurnal motions around (C'). This type of explanation is strictly valid since the forced and free motions are independent, as they represent two independent solution components to the differential equation of motion. As for adopting a set of nutations, clearly, preference has to be given to the celestial pole C', not only because it is observable (in case of rigid and elastic bodies) but also because the positional elements which refer to it have no diurnal periodic variations and do not require a correction for the so-called dynamical variations.

The differences and commonalities between the rigid and perfectly elastic model have been discussed at length, the main characteristic being that the angular momentum axis remains virtually unchanged in space whereas the direction of the (C') changes somewhat. But it was pointed out that the observational evidence does not entirely confirm the predicted changes.

The liquid core model is the most general model considered in this study. It introduces a new spectrum of motions, such as the nearly diurnal free wobble (NDFW) and its associated change in nutation, both resulting from the free solution. In addition, certain frequencies change their amplitudes due to core resonance. The observations fit such a model quite well. Similarly, as in the case of the rigid and elastic bodies, the celestial pole of the shell (C'') was defined as that pole having neither shell-fixed nor space-fixed periodic diurnal motions. Thus, in the case of the liquid core model, the celestial pole of the shell (C'') is observable.

Above it has been said that the celestial pole (C') or (C'') is observable depending on the mathematical model under consideration. The real earth, of course, does not behave exactly as these models indicate although the liquid core model fits the observations better than the other two models. Yet, the concept of the celestial pole, or even better that of the celestial pole of the shell can be extended so as to denote that pole which is observable in actuality. This pole can then simply be called the "Celestial Pole (C)." It has the property of having no body-fixed and no space-fixed diurnal periodic motions. The naming of this observable pole correctly does not give any hint as to the best fitting theoretical model since, anyhow, each model is only an approximation. The point of view is taken that the nutations and, of course, the polar motions of the celestial pole (C) can only be determined from observations. The possible adoption of the IAU - 1977 set of nutations is a step in this direction. Unfortunately, this set of nutations does not include the theoretical Oppolzer terms; therefore, the dynamical corrections are still needed.

It is recommended that in the future the terms "polar motion" and "nutation" be only associated with the body-fixed and space-fixed motions of the celestial pole (C), respectively. If the motion of any other axis is meant, the name of this axis should be given explicitly.

The celestial pole (C) gives the natural reference direction not only for astronomical observations but also for laser ranging to lunar reflectors. More details on this subject are given in [Leick, 1978].

Finally, it is sometimes suggested that the nutations be given for the angular momentum (H). But it is clear that such a procedure violates the concept of observability. It may be true that the direction of (H) in space is the same for the rigid, elastic, or any other reasonable earth model. But this property is not of much interest to the astronomer or geodesist who tries to determine the orientation of the earth. It is conceptually simpler to refer to an axis which is observable. Both representations are actually equivalent, i.e., determining the nutations of the celestial pole (C) or using the adopted (rigid model) nutations of (H) but then determining the periodic diurnal body-fixed motions of the angular momentum (H).

C-2

REFERENCES

AENA Supplement, 1961. Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac. H.M. Stationary Office, London (Reprint, 1974).

Atkinson, R. d'E. 1973. "On the Dynamical Variations of Latitude and Time," Astr. J., 78, 1, pp. 147-151.

Atkinson, R. d'E. 1975. "On the Earth's Axes of Rotation and Figure," Mon. Not. R. Astr. Soc, 71, pp. 381-386.

Atkinson, R. d'E. 1976. "On the Earth's Axes of Rotation and Figure," paper presented at the XVIth General Assembly of the IAU, Grenoble.

BIH. 1975. Bureau International de l'Heure - Annual Report 1975, Paris.

Bursa, M. 1975. "Effect of the Third Product of Inertia and the Equatorial Flattening of the Earth on the Free Nutation of the Rotation Axis," Bull. Astron. Inst. Czech., 26, 4.

DeSitter, von W. 1904. "Ueber die von der Anziehung von Sonne und Mond Herruehrenden Breiten Variationen." Astr. Nachr., 166, 331.

Doodson, A.T. 1921. "The Harmonic Development of the Tide-generating Potential," Proc. Roy. Soc. Land., Ser. A. 100, pp. 305-329.

Fedorov, YE. P. 1958. Nutation and Forced Motion of the Earth's Pole, The MacMillan Company, New York (Translation from the Russian by Jeffreys, B.S., 1963).

Goldstein, H. 1965. Classical Mechanics. Addison-Wesley Publishing Co., Reading, Massachusetts.

Guinot, B. 1970. "Short-period Terms in Universal Time." Astron. & Astrophys. 8., pp. 26-28.

Heiskanen, W.A., Moritz, H. 1966. Physical Geodesy. Freeman, San Francisco.

Heitz, S. 1976. "Mathematische Modelle der geodaetischen Astronomie," Deutsche Geodaetische Kommission, Reihe A. Heft Nr. 85.

Hopkins, W. 1839. "Researches in Physical Geology," Phil. Trans. R. Soc. Lond., 129, pp. 381-423.

Hough, S.S. 1895. "The Oscillations of a Rotating Ellipsoidal Shell Containing Fluid," Phil. Trans. R. Soc. Lond., A, 186, pp. 469-506.

IAU. 1977. Symposium No. 78, Nutation and the Earth's Rotation, Kiev, Russia. Reidel, Dordrecht, Holland, (in press).

IAU. 1977. Transactions of the International Astronomical Union, Vol. XVI B. "Proceedings of the Sixteenth General Assembly of the IAU," Grenoble, France. Edited by E.A. Müller and A. Jappel. Reidel, Dordrecht, Holland.

Jeffreys, H., Vicente, R.O. 1957a. "The Theory of Nutation and the Variation of Latitude," Mon. Not. R. Astr. Soc., 117, pp. 142-161.

Jeffreys, H., Vicente, R.O. 1957b. "The Theory of Nutation and the Variation of Latitude. The Roche Model Core," Mon. Not. R. Astr. Soc., 117, pp. 162-173.

Kimura, H. 1902. "New Annual Term in the Variation of Latitude, Independent of the Components of the Pole's Motion," Astr. J., 27, 107.

Kulikov, K.A. 1956. Fundamental Constants of Astronomy. Translated from the Russian by the Israel Program for Scientific Translations, Jerusalem, 1964. U.S. Department of Commerce, Washington, D.C.

Leick, A. 1978. "Earth Orientation from Lunar Laser Range-Differencing." The Ohio State University, Department of Geodetic Science Report No. 263.

McCarthy, D.D. 1976. "Observation of the Fortnightly Nutation Term and the 'Dynamical Variation of Latitude' with Photographic Zenith Tubes," Astr. J. 81, 6, p. 482.

McClure, P. 1973. "Diurnal Polar Motion," Goddard Space Flight Center Report No. X-592-73-259.

McClure, P. 1976. "Core Resonance Effects on the Earth's Angular Momentum Vector and Rotation Axis - A Generalized Model," Report of the Department of Aerospace Engineering and Engineering Mechanics, The University of Texas, Austin.

Melchior, P. 1966. The Earth Tides, Pergamon Press.

Melchior, P., Georis, B. 1968. "Earth Tides, Precession-Nutation and the Secular Retardation of the Earth's Rotation," Phys. Earth Planet. Interiors 1, pp. 267-287.

Melchior, P. 1971. "Precession-Nutation and Tidal Potential." Celestial Mechanics 4, pp. 190-212.

Melchior, P. 1975. "General Report on Earth Tides (1971-1975)," XVI General Assembly of the IUGG, Grenoble, France.

Molodenskii, M.S. 1961. "The Theory of Nutation and Diurnal Earth Tides," Comm. Obs. R. Belgique, 188 S. Geoph., No. 58, pp. 25-56.

Morgan, H.R. 1952. "The Short Period Terms in the Nutation as Given by Observations," Astron. J. 57, p. 232.

Mueller, Ivan I. 1969. Spherical and Practical Astronomy as Applied to Geodesy. Frederick Ungar Publishing Co., New York.

Munk, W.H., MacDonald, G.J.F. 1960. The Rotation of the Earth, The University Press, Cambridge.

Ooe, M., Sasao, T. 1974. "Polar Wobble, Sway and Astronomical Latitude-Longitude Observations," Publ. Int. Lat. Obs. Mizusawa, 9, 223.

Oppolzer, T.R. 1882. Lehrbuch zur Bahnbestimmung der Kometen und Planeten, Erster Band, Verlag von Wilhelm Englemann, Leipzig.

Poincaré, H. 1910. "Sur la precession des corps deformables," Bull. Astr., 27, pp. 321-356.

Popov, N. 1963. "Nutational Motion of the Earth's Axis," Nature, 198, p. 1153.

Rochester, M.G., Jensen, O.G., Smylie, D.E. 1974. "A Search for the Earth's Nearly Diurnal Free Wobble," Geophys. J.R., 38, pp. 349-363.

Shen, P.Y., Mansinha, L. 1976. "Oscillation, Nutation and Wobble of an Elliptical Rotating Earth with Liquid Core," Geophys. J.R. Astr. Soc. 46, pp. 467-496.

Sludskii, F. 1896. "De La Rotation de la Terre Supposée Fluide à son Intérieur," Bul. Soc. Natur. Moscou. 9. pp. 285-318.

Smith, M. 1977. "Wobble and Nutation of the Earth," Geophys. J.R. Astr. Soc. 50, 103-140, Royal Astronomical Society, London.

Takeuchi, H. 1950. "On the Earth Tide of the Compressible Earth of Variable Density and Elasticity," Transactions American Geophysical Union, Vol. 31, No. 5, Part 1.

Toomre, A. 1974. "On Nearly Diurnal Wobble of the Earth," Geophys. J. R. Astr. Soc. 38. pp. 335-348.

Wako, Y. 1970. "Interpretation of Kimura's Annual Z-Term," Publ. Astr. Soc. Japan 22, pp. 525-544.

Woolard, E.W. 1952. "Theory of the Rotation of the Earth Around Its Center of Mass," Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac, Vol. XV, Part 1, U.S. Government Printing Office, Washington, D.C.

Woolard, E.W., Clemence, G.M. 1966. Spherical Astronomy. Academic Press, New York.

Yatskiv, Ya. S. 1972. "On the Comparison of Diurnal Nutation Derived from Separate Series of Latitude and Time Observations," Rotation of the Earth, IAU Symposium No.48, edited by P. Melchior and S. Yumi. Reidel, Dordrecht Holland.

Yatskiv, Ya. S., Wako Y., Kaneko Y. 1975. "Study of the Nearly Diurnal Free Nutation Based on Latitude Observations of the ILS Stations (I)." Publication of the International Latitude Observatory of Mizusawa, X, 1, Japan.

Related Literature

IAU. 1974. Colloquium No. 26, On Reference Coordinate Systems for Earth Dynamics, Torun, Poland. Edited by B. Kolaczek and G. Weiffenbach. Institute of Higher Geodesy and Geodetic Astronomy, Warsaw Technical University, Warsaw, and the Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.