

**N79-19044**

Paper No. 36

**AN EFFICIENT NUMERICAL TECHNIQUE FOR CALCULATING  
THERMAL SPREADING RESISTANCE\***

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**ABSTRACT**

An efficient numerical technique for solving the equations resulting from finite difference analyses of fields governed by Poisson's equation is presented. The method is direct (noniterative) and the computer work required varies with the square of the order of the coefficient matrix. The computational work required varies with the cube of this order for standard inversion techniques, e.g., Gaussian elimination, Jordan, Doolittle, etc.

**INTRODUCTION AND BACKGROUND**

**General**

"Thermal spreading resistance" is defined as the conductive thermal resistance between a source region and a sink region in a solid where the geometry is such as to preclude one dimensional heat flow. Knowledge of thermal spreading resistance is needed in two aerospace engineering areas. These are the thermal design of electronic components or equipments and in the prediction and control of thermal contact resistance.

Importance to the Design of Electronic Components and Equipments-The thermal analysis of a power semiconductor or integrated circuit can be reduced to the problem of determining the appropriate spreading and bonding thermal resistances. As an example, the problem of calculating the junction-to-case thermal resistance of a semiconductor bonded to a substrate which is bonded in a metal case will be considered. Figure 1 illustrates this problem.

Heat is generated in a region of known size, the junction region of the semiconductor. The first, and most significant,

\*The work reported in this paper was done under National Aeronautics and Space Administration contract NAS8-28616.

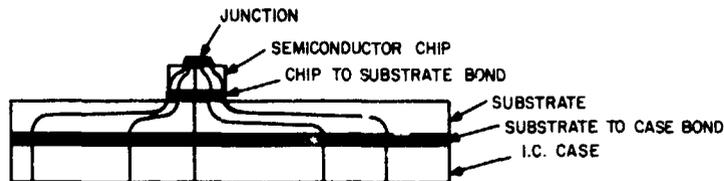


Fig. 1 Semiconductor in an Integrated Circuit

spreading resistance of interest occurs between the junction and the opposite face of the silicon chip. The next thermal resistance of interest is that across the bond between chip and substrate. It is of significance that these thermal resistances are not independent. The thermal conductance of the bond proper can vary several thousandfold depending on the use of a metallic or nonmetallic bonding material. The resistance to heat flow between the semiconductor chip bond region and the rear of the substrate represents a second spreading resistance. In a typical integrated circuit package, the entire bottom region of the substrate would not be available as a sink for a single semiconductor chip due to the presence of other heat dissipating chips. It is usually possible to estimate the size of the effective sink region on the rear of the substrate from considerations of symmetry or because it exceeds dimensions which appreciably affect the thermal spreading resistance. In those few cases where interactions must be considered, the key analytical tool is superposition. Green's function approach may be employed to advantage. For example, see reference 1.

The importance of being able to predict thermal spreading resistances in single and multilayered material in the evaluation of the thermal design of semiconductor or integrated circuits has been shown. Spreading thermal resistances are important in other electrical devices such as phased array antenna elements, Peltier coolers, Seebeck generators, and most devices which utilize conductive heat transfer.

#### Prediction and Control of Thermal Contact Resistance

The resistance to heat flow between two mating (touching, as in a joint) pieces of metal is called thermal contact resistance. When the actual microscopic regions of contact between two mating surfaces are examined, it is found that metal-to-metal contact occurs in small discrete regions where the asperities or microscopic protuberances make contact. References 2 and 3 describe this model of contact in great detail. Figure 2 illustrates this contact model.

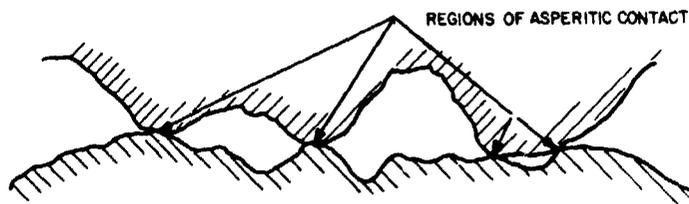


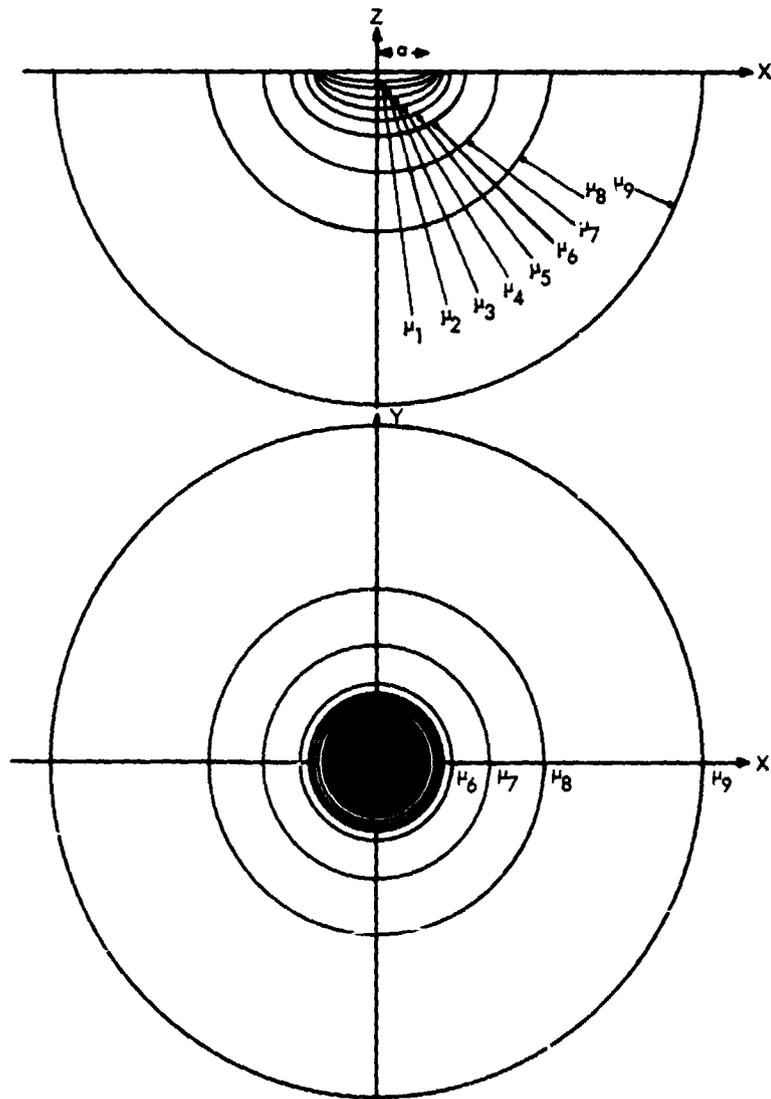
Fig. 2. Microscopic View of the Joint of Contacting Pieces of Metal

The heat flow to and from a region of asperitic contact is seen to be of the "spreading" type. In fact, the effective thermal contact resistance of any contact may be considered as the sum of the parallel microscopic spreading resistances in the contacts themselves. References 2 and 3 deal largely with isotropic contacts in which the thermal conductivity within the bodies of both contacts is uniform.

Analysis has shown that the bulk of the spreading resistance occurs close to the region of actual asperitic contact and that the spreading resistance in any region varies inversely with the thermal conductivity of the material. Figures 3 and 4 illustrate the first of these points. Figure 3, drawn to scale, shows the equipotential lines about a circular contact region each drawn to show one-tenth of the total spreading resistance between the circular contact region and a body of semi-infinite extent. It is seen that half of this total resistance occurs within one contact radius from the circular contact or source region and 80 percent occurs within three contact radii. Figure 4 illustrates these relationships. Figures 3 and 4 are taken from reference 4.

The thermal conductivity of the contact material close to the contact region is of such importance that even a thin 45 Angstrom thick layer of oxide on an aluminum contact can contribute measurably to the thermal contact resistance. This has been shown by actual measurement; see reference 4.

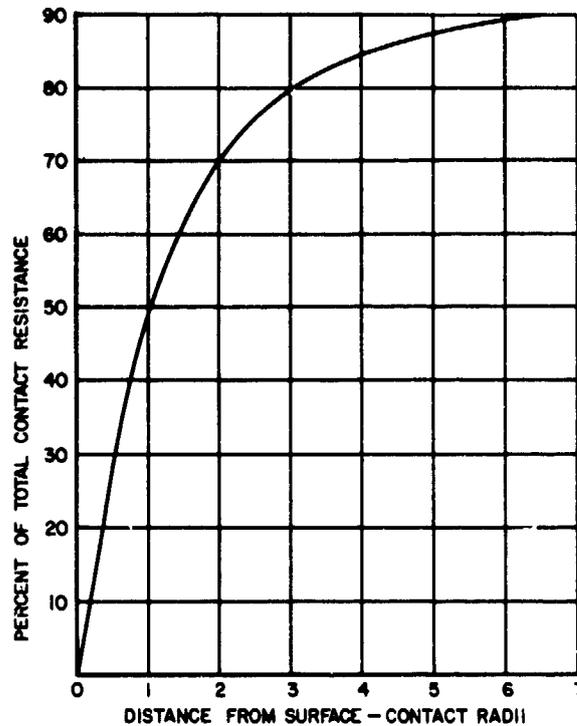
Mikic and Carnasciali, reference 5, have utilized the above facts to enhance thermal contact conductance by plating materials of higher conductivity on the contacting faces of a metallic joint. They present an approximate analysis of spreading resistance from a circular contact into a contact region composed of two layers of materials with different conductivities. An exact boundary value solution of this basic problem has proven impossible as no mathematical function has been found which will satisfy all the boundary conditions between the plating and the body materials.



**Fig. 3. Temperature Profiles Described by Holm's Equation for Isothermal Circular Source on a Semi-Infinite Slab**

Professor C.J. Moore, Jr.\* in his discussion at the end of reference 5, felt this two layered spreading resistance problem

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**Fig. 4. Percent of Total Constriction Resistance for a Single Isothermal Circular Source on a Semi-Infinite Slab as a Function of Distance into Body of Contact**

could best be handled by a "well-conditioned finite difference computer code." Mikic and Carnasciali then question the economic feasibility of such calculations.

Attempts by the author of this study to solve the two layer thermal spreading resistance problem using a finite difference approach utilizing Gauss-Seidel iteration have shown the cost of digital computer calculation to be excessive for large nodal systems.

## **METHOD**

### **General**

The governing differential equation for the thermal spreading resistance problem is Poisson's equation. For those spread-

ing resistance problems that are two-dimensional\* or may be reduced to two-dimensional problems, the equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = Q_t''' \quad (1)$$

Consider a rectangular field subdivided into rectangular subregions as illustrated in Figure 5. The heat balance equation describing the heat flow among element  $m, n$  and its four principal neighbors is:

$$\begin{aligned} & (T_{m,n} - T_{m,n+1})H_{m,n} + (T_{m,n} - T_{m-1,n})V_{m,n} \\ & + (T_{m,n} - T_{m,n-1})H_{m,n-1} + (T_{m,n} - T_{m+1,n})V_{m+1,n} \\ & = Q_{m,n} \end{aligned} \quad (2)$$

where

$T$  is temperature

$Q_t'''$  is heat generated per unit volume

$x, y, z$  are spatial coordinates

$H, V$  are horizontal and vertical conductances, respectively

$m$  is row index

$n$  is column index

$Q_{m,n}$  is heat generated in node  $m, n$

The convention for the horizontal and vertical conductances used is shown in Figure 6.

Each of the following observations will be helpful in understanding the discussion which follows:

- (1) When any temperature  $T_{m,n}$  is known (e.g., as a boundary condition), it will affect equation  $m, n$  by yielding a virtual heat generation term  $Q'_{m,n}$  which is subtracted from the right-hand side of equation (2) where  $Q'_{m,n}$  is:

$$Q'_{m,n} = T_{m,n}(H_{m,n} + V_{m,n} + H_{m,n-1} + V_{m+1,n}) \quad (3)$$

\*The method developed is applicable to three-dimensional problems as discussed in this last part of this section.

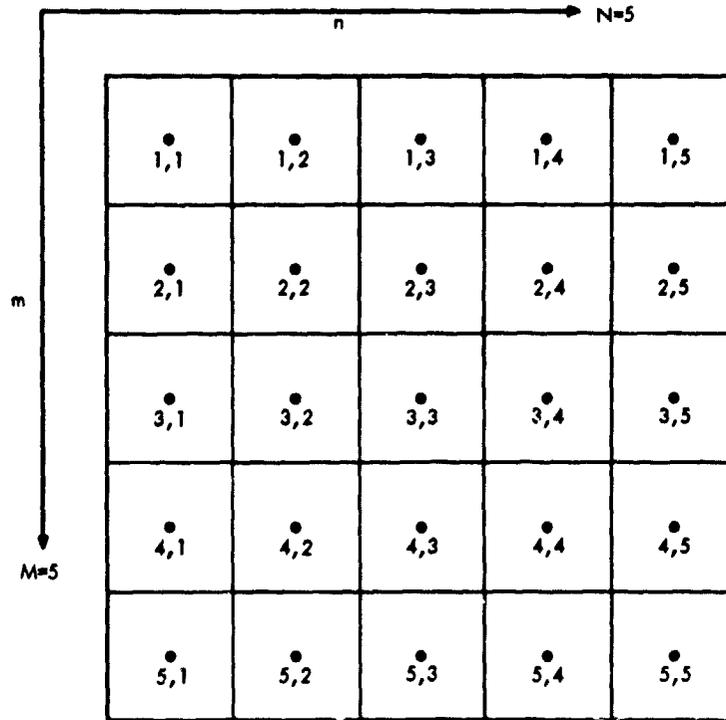


Figure 5. Rectangular Field Divided into 25 Finite Elements

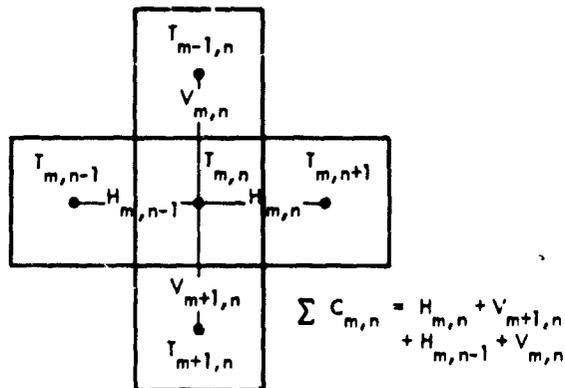


Figure 6. Nomenclature for Nodal Interconductances

- (2) If the original physical field is divided into nodes of  $M$  rows and  $N$  columns, then:
- (a) There will be  $N \cdot M$  linear equations.
  - (b) There will be not more than  $N \cdot M$  unknowns (fewer if some temperatures are initially prescribed).

- (c) There can be as many different and distinct nodal conductances as there are interconnections between adjacent nodes.

Now, if the system of linear finite difference equations is written in matrix form (using the nodes of Figure 5) from left to right, top row to bottom row, as in reading English, a coefficient matrix results that has a pattern characteristic for field problems described by Poisson's or LaPlace's equations. This pattern is illustrated in Figure 7.

It was noted by Karlqvist (reference 6) that the matrix in Figure 7 may be partitioned as shown. It can be seen that each of the submatrices is  $\sqrt{NM} \times \sqrt{NM}$  and the size of the coefficient matrix is  $NM \times NM$  where the original finite element matrix was  $N \times M$  in size.

#### Derivation of an Efficient Technique for Exact Solution of this System of Equations

The submatrices shown in Figure 7 are defined in Figure 8. Expanding the partitioned matrices (Figure 8) into a system of equations, having normalized each equation with respect to the diagonal element:

$$\begin{aligned} T_1 - A_1 T_2 &= C_1 \\ -B_2 T_1 + T_2 - A_2 T_3 &= C_2 \\ -B_3 T_2 + T_3 - A_3 T_4 &= C_3 \\ \vdots & \\ -B_n T_{n-1} + T_n &= C_n \end{aligned}$$

the general equation has the form:

$$-B_i T_{i-1} + T_i - A_i T_{i+1} = C_i \quad (4)$$

The first equation can be solved for  $T_1$ :

$$T_1 = C_1 + A_1 T_2 \quad (5)$$

and the i-th for  $T_i$ :

$$T_i = C_i + A_i T_{i+1} + B_i T_{i-1} \quad (6)$$

The goal is to find a recursion relationship built upon successive substitutions, which provides a solution for the i-th



$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix}$$

Fig. 8. System of Submatrices in Matrix Notation

unknown in terms of the  $(i+1)$ -th. That is:

$$T_i = A'_i T_{i+1} + B'_i \quad (7)$$

Examining equation (5) for  $T_1$  above, it can be seen that:

$$A'_1 = A_1 \quad \text{and} \quad B'_1 = C_1$$

The equation for  $T_2$  is:

$$T_2 = C_2 + A_2 T_3 + B_2 T_1 \quad (8)$$

which, when written in terms of the equation for  $T_1$ , becomes:

$$T_2 = [I - B_2 A'_1]^{-1} A_2 T_3 + [I - B_2 A'_1]^{-1} [C_2 + B_2 B'_1] \quad (9)$$

where  $I$  is the unit matrix.

The general coefficients found in this manner become:

$$A'_i = [I - B_i A'_{i-1}]^{-1} A_i \quad (\text{a square matrix})$$

and

$$B'_i = [I - B_i A'_{i-1}]^{-1} [B_i B'_{i-1} + C_i] \quad (\text{a column matrix})$$

Therefore:

$$T_i = A'_i T_{i+1} + B'_i \quad (10)$$

The temperature matrices (columns) are found starting at the  $\sqrt{NM}$  row (or 5th row of Figure 8).

$$T_{\sqrt{MN}} = B'_{\sqrt{MN}} \quad \text{as} \quad A'_{\sqrt{MN}} = 0 \quad (11)$$

Defining  $O$  as the order of the coefficient matrix (e.g., square matrix of Figure 7), the system of equations has been solved by operating on  $3\sqrt{O} - 2$  submatrices, each of which is the square root of the size of the original  $NM \times NM$  coefficient

matrix. Three  $\sqrt{O}$  inversions of these submatrices are required. The total number of multiplications - a good measure of the computer effort required during solution - is:

$$\text{No. of Multiplications} = 3O^2 + O^{3/2} - O + O^{1/2} \quad (12)$$

This may be compared against other direct methods (see ref. 7):

| <u>Method</u>        | <u>Number of Multiplications<br/>Required during Solution</u> |
|----------------------|---|
| Gaussian Elimination | $\frac{1}{3}O^3 + O^2 - \frac{1}{3}O$                         |
| Jordan               | $\frac{1}{2}O^3 + O^2 - \frac{1}{2}O$                         |
| Doolittle            | $\frac{1}{3}O^3 + O^2 - \frac{1}{3}O$                         |
| Cholesky             | $\frac{1}{6}O^3 + \frac{3}{2}O^2 + \frac{1}{3}O$              |

Cornock's method (ref. 8), a triangulation type, also makes use of the characteristic pattern of submatrices which results during a finite difference solution for fields described by Poisson's equation. When the field properties are homogeneous and isotropic, Cornock's method is very powerful since only one of the above submatrices of order  $\sqrt{O}$  need be inverted. However, for the general solution of the nonhomogeneous field, the number of multiplications required is:

$$\frac{13}{2}O^2 - \frac{5}{2}O^{3/2} - 2O - 5 \quad (13)$$

Cornock's method does not lend itself to ready general programming for matrices of variable size as does the method described in this paper.

That equation (12) is indicative of the computer effort required for solution has been substantiated in practice. Figure 9 shows the variation in cost realized in the solution of very large matrices using the method developed in this paper. The measured slope of the line in Figure 9 is very close to 2.

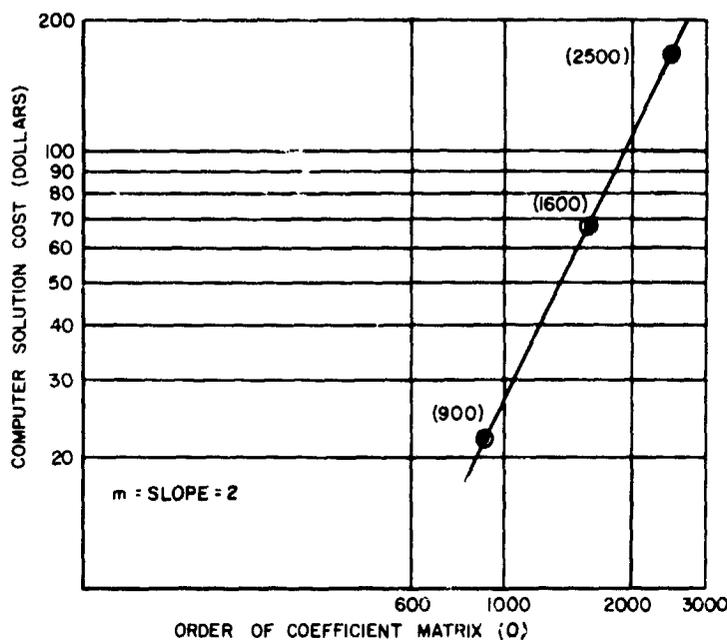
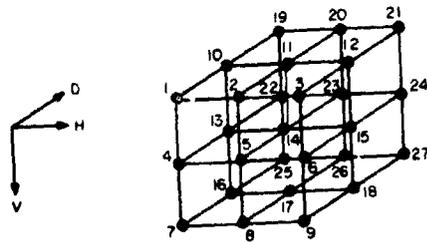


Figure 9. Cost of Computation versus Size of Coefficient Matrix

#### Applicability of Technique to Three-Dimensional Problems

The technique discussed above is suited to the solution of field problems having three or more dimensions. Figure 10 illustrates the characteristic pattern of the coefficient matrix for a three-dimensional finite element array. It is seen that the submatrices are  $\sqrt[3]{O} \times \sqrt[3]{O}$  in size for an  $O \times O$  coefficient matrix. The same block tridiagonal pattern of submatrices is seen to occur as in the two-dimensional case so the derivation above for the technique of solution for two-dimensional matrices is still applicable. Since the submatrices are  $\sqrt[3]{O} \times \sqrt[3]{O}$  rather than  $\sqrt{O} \times \sqrt{O}$  in size, the technique is even more powerful for three-dimensional problems. The number of multiplications required for solution of the three-dimensional problem is a function of  $O^{4/3}$  as opposed to  $O^2$  for the two-dimensional array where  $O$  is the order of the coefficient matrix.

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|    | N |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |   |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|---|
|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |   |   |
| 1  | Σ | H | 0 | V | 0 | 0 | 0 | 0 | 0 | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | C |   |
| 2  | H | Σ | H | 0 | V | 0 | 0 | 0 | 0 | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 3  | 0 | H | Σ | 0 | V | 0 | 0 | 0 | 0 | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 4  | V | 0 | 0 | Σ | H | 0 | V | 0 | 0 | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 5  | 0 | V | 0 | H | Σ | H | 0 | V | 0 | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 6  | 0 | 0 | V | 0 | H | Σ | 0 | V | 0 | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 7  | 0 | 0 | 0 | V | 0 | 0 | Σ | H | 0 | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 8  | 0 | 0 | 0 | 0 | V | 0 | H | Σ | H | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | C |   |
| 9  | 0 | 0 | 0 | 0 | 0 | V | 0 | H | Σ | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 10 | D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 11 | 0 | D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | H  | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 12 | 0 | 0 | D | 0 | 0 | 0 | 0 | 0 | 0 | 0  | H  | Σ  | 0  | V  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 13 | 0 | 0 | 0 | D | 0 | 0 | 0 | 0 | 0 | V  | 0  | 0  | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0 |   |
| 14 | 0 | 0 | 0 | 0 | D | 0 | 0 | 0 | 0 | 0  | V  | 0  | H  | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0 |   |
| 15 | 0 | 0 | 0 | 0 | 0 | D | 0 | 0 | 0 | 0  | 0  | V  | 0  | H  | Σ  | 0  | V  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0 |   |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | D | 0 | 0 | 0  | 0  | V  | 0  | 0  | Σ  | H  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0 |   |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D | 0 | 0  | 0  | 0  | V  | 0  | 0  | H  | Σ  | H  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0 |   |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D | 0  | 0  | 0  | 0  | V  | 0  | H  | Σ  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0  | 0 |   |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | H  | Σ  | H  | 0  | V  | 0  | 0  | 0  | 0  | 0 |   |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | H  | Σ  | 0  | V  | 0  | 0  | 0  | 0  | 0 |   |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | V  | 0  | 0  | Σ  | H  | 0  | V  | 0  | 0  | 0 |   |
| 23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | V  | 0  | H  | Σ  | H  | 0  | V  | 0  | 0 |   |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | 0  | 0  | V  | 0  | 0  | Σ  | H  | 0 |   |
| 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | 0  | V  | 0  | H  | Σ  | H  | 0 |   |
| 26 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | V  | 0  | H  | Σ  | 0 |   |
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | D  | 0  | 0  | 0  | V  | 0  | H  | Σ | 0 |

WHERE  $\Sigma_M = \Sigma + V + H + D$

Fig. 10 Matrix Representation of System of Equations Describing Three-Dimensional Field Showing Pattern Established by Submatrices of Coefficient Matrix

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