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PERTURBATIONS OF NON-RESONANT SATELLITE ORBITS DUE TO A ROTATING EARTH

ANALYTICAL AND COMPUTATIONAL MATHEMATICS, INC.
PERTURBATIONS OF NON-RESONANT SATELLITE ORBITS DUE TO A ROTATING EARTH

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PERTURBATIONS OF
NON-RESONANT SATELLITE
ORBITS DUE TO A ROTATING EARTH
by
Alan Mueller

1.0 INTRODUCTION

The dominant perturbations of the motion of a satellite near the earth are due to the non-symmetrical gravitational field and the atmospheric drag. The gravitational field may be divided in two classes: terms independent of time (zonal harmonics) and terms which depend explicitly on time. A complete first order solution for satellite motion perturbed by the second zonal harmonic has been developed by Scheifele (reference 1). This solution has been rewritten in the non-singular PSΦ elements by Bond (reference 2). In references 3 and 4, the perturbations due to drag are developed in PSΦ elements and added to this J₂ theory. In reference 5, the long period perturbations due to the additional zonal harmonics are included in the theory. The tesseral harmonics have yet to be treated under this unified theory and are the topic of this report.

The perturbations due to the tesseral harmonics can be placed in four categories:

1) Short period perturbations with a magnitude of

† A first order solution here implies that the solution has a periodic error of second order and a secular error of third order.
2) Intermediate period perturbations with a magnitude of between $J_2$ and $J_2^2$.

3) Along track secular perturbations induced by the periodic perturbations in the mean motion.

4) Resonant perturbations.

For near earth satellites, the atmospheric drag perturbation continually pulls the satellite in and out of the different long period resonant frequencies. The result is that the resonances never become apparent and may be neglected.

Since the $J_2$ theory has been developed only to first order, the second order short period tesseral perturbations may be neglected. For the same reason, the intermediate period perturbations should be included if first order accuracy is to be maintained. The tesseral harmonics have no true secular perturbation but the periodicities in the mean motion induce a secular perturbation in the mean anomaly. This secular perturbation may be determined by simply using the average mean motion instead of the osculating mean motion. Graf (reference 6) finds the average mean motion in a numerical manner. The numerical studies in that reference verify the assumption that use of the average mean motion does account for the apparent secular trend in the mean anomaly. Although the results are good, use of a numerical method would be inconsistent with the idea of having a completely analytical theory.

To complete the solution of the motion of a near earth satellite, the averaged mean motion and the intermediate period perturbations need to be found in a completely analytical manner.

Since the previous PSFS theory has applied Von Zeipel's solution technique, it seems natural to return to this method for a solution of the tesseral perturbation. As in the previous
theory, the solution will be found first in the singular DS$\phi$
elements and then rewritten in the PS$\phi$ elements to remove
singularities. The notation used in the development is de-
scribed in Appendix A.
2.0 FORMULATION OF THE PROBLEM

Only the gravitational field will be considered since its interaction with the drag effects is small. The DSPhamiltonian for the gravitational potential is assumed to have the form

\[
F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 ,
\]  

(1)

where

\[
F_0 = \phi - \frac{\mu}{\sqrt{2L}} , \quad \text{(two-body)} \quad (2)
\]

\[
F_1 = \frac{1}{qr} \left( \left( \frac{z}{r} \right)^2 - \frac{1}{3} \right) , \quad \text{(J}_2 \text{)} \quad (3)
\]

\[
F_2 = F_Z + F_T , \quad \text{(Zonals and Tesserals)} \quad (4)
\]

\[
F_Z = \frac{\mu}{\varepsilon r^2} \sum_{n=3} C_{n,0} \frac{r^n}{r^{n-1}} p^n (\frac{Z}{r}) , \quad (5)
\]

\[
F_T = -\frac{\mu}{\varepsilon r^2} \sum_{n=2} \sum_{m=1}^{n} \frac{r^n}{r^{n-1}} p_m \left( \frac{Z}{r} \right) \left[ C_{n,m} \cos m(\lambda-\theta) + S_{n,m} \sin m(\lambda-\theta) \right] \quad (6)
\]

\[
\varepsilon = -\frac{J_Z}{2} \frac{\mu}{r^2} . \quad (7)
\]
\( p_n^m \) are the associated Legendre polynomials; \( R_e \) is the mean equatorial radius; \( J_2 \), \( C_{n,m} \), \( S_{n,m} \) are the geopotential coefficients; \( \lambda \) is the longitude of the satellite and \( \theta \) is the Greenwich hour angle.

The \( F_2 \) Hamiltonian may be divided in zonal and tesseral harmonics. The functional dependence of \( F_Z \) and \( F_T \) is

\[
F_Z = F_Z(\phi,g,\beta) \quad ,
\]

\[
F_T = F_T(\phi,g,h,\omega_\odot \ell^+,\beta) \quad .
\]

\[
+ \quad \text{The canonical time element } \ell \text{ will always appear, in the problem, premultiplied by the rotation rate of the earth } \omega_\odot .
\]
3.0 VON ZEIPEL'S SOLUTION TECHNIQUE

As was stated earlier, the Von Zeipel solution method is to be used to find the average mean motion and to eliminate the periodicities due to the tesseral harmonics. The generating function $S$ is to be used for the elimination of the short period function of $\phi$ and the intermediate period function of $\omega_\phi$ . The equations governing the transformation in the DS$\phi$ space are

$$\alpha' = \frac{\partial S}{\partial \beta'}, \quad \beta = \frac{\partial S}{\partial \alpha} . \quad (10)$$

As in the hamiltonian, $S$ is assumed to be of the form

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 \quad , \quad (11)$$

$$S_2 = S_Z + S_T \quad , \quad (12)$$

where $S_T$ is a periodic function of $\phi$ and $\omega_\phi$ and $S_1$ and $S_Z$ are periodic functions of $\phi$ only. The development of $S_1$ can be found in reference 1 and therefore the discussion will be restricted to the development of $S_2$ .

The theory of references 1 and 2 neglected $S_2$ to maintain a first order accuracy. But this assumes that $S_2$ is order $O(1)$. This is true for zonal perturbations but the tesseral perturbations result in an $S_2$ which may become larger than $O(1)$. This will be demonstrated in the following analysis. The necessary Von Zeipel equations found from reference 1 are

$$\epsilon^0 : F_0'(\phi',L') = F_0(\phi',L') \quad (13)$$
\[ \epsilon^1 : \frac{\partial F_0}{\partial \phi} \frac{\partial S_1}{\partial \phi} = - F_1 (\beta', \phi, \theta) + F'_1 (\beta', \theta) \quad (14) \]

\[ \epsilon^2 : \frac{\partial F_0}{\partial \phi} \frac{\partial S_2}{\partial \phi} + \frac{\partial F_0}{\partial \lambda} \frac{\partial S_2}{\partial \lambda} = - \frac{1}{2} \sum_{k=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 F_0}{\partial \beta_k \partial \beta_j} \frac{\partial S_1}{\partial \alpha_k} \frac{\partial S_1}{\partial \alpha_j} - \] 
\[ \sum_{k=1}^{4} \frac{\partial F_1}{\partial \beta_k} \frac{\partial S_1}{\partial \alpha_k} + \sum_{k=1}^{4} \frac{\partial F'_1}{\partial \beta_k} \frac{\partial S_1}{\partial \beta_k} - F_Z (\beta', \phi, \theta) - (15) \]

Assuming \( S_2 \) has the form given by (12), then the expressions required to find \( S_Z \) and \( S_T \) are given by

\[ \frac{\partial F_0}{\partial \phi} \frac{\partial S_2}{\partial \phi} = - \frac{1}{2} \sum_{k=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 F_0}{\partial \beta_k \partial \beta_j} \frac{\partial S_1}{\partial \alpha_k} \frac{\partial S_1}{\partial \alpha_j} - \] 
\[ \sum_{k=1}^{4} \frac{\partial F_1}{\partial \beta_k} \frac{\partial S_1}{\partial \alpha_k} + \sum_{k=1}^{4} \frac{\partial F'_1}{\partial \beta_k} \frac{\partial S_1}{\partial \beta_k} - \] 
\[ - F_Z (\beta', \phi, \theta) - F'_2 (\beta', \theta) \quad (16) \]

\[ \frac{\partial F_0}{\partial \phi} \frac{\partial S_T}{\partial \phi} + \frac{\partial F_0}{\partial \lambda} \frac{\partial S_T}{\partial \lambda} = - F_T \quad (17) \]

If \( S_2 \) is to be used in eliminating the intermediate period terms then it is necessary to keep only those terms of order greater than \( O(1) \). If \( S_2 \) is to be used in finding the
mean energy, only those terms which are functions of \( l \) are considered. With these qualifications a much simpler set of equations can be used to find \( S_2 \):

\[
S_2 = 0 ,
\]

\[
\frac{\partial F_0}{\partial \phi} \frac{\partial S_T}{\partial \phi'} + \frac{\partial F_0}{\partial L'} \frac{\partial S_T}{\partial l} = - F_T (\beta', \phi', g, h, \omega, \epsilon) .
\]

\( S_2 \) is set to zero because the right hand sides of equation (16) are of order \( O(1) \) and are independent of \( l \).

Thus the transformation due to the short and intermediate terms is given by

\[
\alpha' = \alpha + \epsilon \frac{\partial S_I}{\partial \beta'} + \epsilon^2 \frac{\partial S_T}{\partial \beta'} ,
\]

\[
\beta = \beta' + \epsilon \frac{\partial S_I}{\partial \alpha} + \epsilon^2 \frac{\partial S_T}{\partial \alpha} .
\]

where

\[ S_T > O(1) . \]

The DS\( \phi \) element \( L \) is the total energy and can be related to the mean motion. The mean total energy \( L' \) can be then related to the instantaneous total energy by

\[
L = L' + \epsilon^2 \frac{\partial S_T}{\partial l} .
\]
where

\[ S_T \geq 0(1) \]

Remarks:

If the Von Zeipel's solution method had been applied to the classical Delaunay variables a much more complicated set of equations would appear. The average mean motion could also be found by differentiating the classical determining function \( S \) with respect to the mean anomaly. But since \( S_1 \) (or \( F_1 \)) is a function of the mean anomaly, \( S_2 \) could not be considered zero since it would also be a function of the mean anomaly. Also the expression for the mean energy would be more complex and difficult to solve since terms of \( O(\varepsilon) \) would appear. All the usual complications of the second order theory would appear just to maintain a first order accuracy. But with the DSO elements the equations for finding the generating function remain concise and relatively easy to solve.

The mean energy is found through a determining function which is defined by a partial differential equation with the anomaly of the satellite \( \phi \) and the changing hour angle \( \omega_0 \). This expression is very similar in form to the partial differential equation found by Graf (reference 6) by applying the "Method of Averages" to the KS differential equations. Maybe it is not so peculiar that the Method of Averages and Von Zeipel's equations result in similar formulas.

If \( F_T \) can be written in a Fourier series of the form

\[
F_T = \sum_k \sum_j \left\{ C_{kj}(\beta',h,g) \cos(\kappa \phi - j \omega_0 \ell) + \right. \\
+ \left. S_{kj}(\beta',h,g) \sin(\kappa \phi - j \omega_0 \ell) \right\}
\]

then the solution for \( S_T \) in equation (19) can be easily shown
to be

\[
S_T = - \sum_k \sum_j \frac{1}{k-\nu_j} \left\{ C_{kj} \sin (k\phi - j\omega_0 t) - S_{kj} \cos (k\phi - j\omega_0 t) \right\} \quad (24)
\]

where \( \nu \) is the ratio of the frequency of the earth's rotation to the revolution frequency of the satellite.

\[
\nu = \frac{\omega_0}{\frac{3F_0}{3L}} = \omega_0 \left. \frac{\nu}{(2L)^{3/2}} \right.
\quad (25)
\]

For near earth satellites this ratio is about one to sixteen,

\[
\nu \approx \frac{1}{16} \quad (26)
\]

Therefore when \( k = 0 \) and for \( j < \frac{1}{\nu} \), the factor \( \frac{1}{k-\nu_j} \) becomes on the order of \( O(\frac{1}{\nu}) \). These terms result in the intermediate period perturbations which are the order of \( O(\frac{\xi^2}{\nu}) \).

In order to complete the solution, the tesseral hamiltonian \( F_T \) must be expressed as a fourier series of the form given by equation (23). As before the solution will then be expressed in the non-singular PS\( \phi \) elements.
4.0 EXPANSION OF THE TIME DEPENDENT GEOPOTENTIAL

The definition of $F_T$ as given by equation (6) is

$$F_T = \frac{\mu}{q^2} \sum_{n=2}^{\infty} \sum_{m=1}^{n} r^2 V_{n,m},$$

(27)

where

$$r^2 V_{n,m} = \frac{r^n}{r^{n-1}} \frac{p_m(\zeta)}{\Gamma_n} \left[ C_{n,m} \cos m(\lambda-\theta) + S_{n,m} \sin m(\lambda-\theta) \right].$$

(28)

We desire an expansion of the above expression in a Fourier series of the canonical DS$\phi$ angular variables $\phi$, $\gamma$, $h$, and $\ell$. The expansion will, in many respects, be similar to Kaula's expansion of the disturbing function in the classical angular variables (reference 7). The important difference is that expansions will now be carried out in the true anomaly instead of the mean anomaly.

As in Kaula, the potential may be expanded using the inclination $F_{\eta mp}$ to a form

$$r^2 V_{n,m} = \frac{r^n}{r^{n-1}} \sum_{p=0}^{n} F_{\eta mp} \left\{ A_{nm} \cos \psi_{\eta mp} + B_{nm} \sin \psi_{\eta mp} \right\}.$$  

(29)

† To maintain the notation of previous authors, this author has decided to keep the notation $p$ as an index in the inclination function. For distinction, the semi-latus rectum will now appear as an italic "p".
where

\[
\psi_{nm} = (n-2p)(\phi + g) + m(h-\theta) , \quad (30)
\]

\[
A_{nm} = \begin{cases} 
C_{nm} & n-m \text{ even} \\
-S_{nm} & n-m \text{ odd}
\end{cases} , \quad (31)
\]

\[
B_{nm} = \begin{cases} 
S_{nm} & n-m \text{ even} \\
C_{nm} & n-m \text{ odd}
\end{cases} . \quad (32)
\]

The recursive relations for the inclination function have been derived by Giacaglia (reference 8). These relations and others appear in Appendix B.

Another transformation is necessary to replace the powers of \( r \) as a Fourier series in \( \phi \):

\[
\frac{1}{r^{n-1}} \begin{bmatrix} \cos \\ \sin \end{bmatrix} \psi_{nm} = \frac{1}{r^{n-1}} \sum_{j=1-n}^{n-1} G_{j}^{n-1} \begin{bmatrix} \cos \\ \sin \end{bmatrix} \psi_{nmj} , \quad (33)
\]

\[
\psi_{nmj} = (n-2p+j) \phi + (n-2p) g + m(h-\theta) . \quad (34)
\]

Note that this is a finite expansion with an eccentricity function \( G_{j}^{n-1}(e) \). The development of \( G_{j}^{n-1} \) is given in Appendix C.

A final transformation is required to place the disturbing function in the form of equation (23). In the expression of the angle \( \psi_{nmj} \), the hour angle may be written as

\[
\theta = \theta_o + \omega_c t , \quad (35)
\]
where $\theta_0$ is the hour angle at time $t = 0$. Time is now a dependent variable in the DS system given by:

$$t = t_0 + \frac{\mu}{(2L)^{1/2}} \left[ M(\phi) - \phi \right],$$

(36)

where

$$M(\phi) = E - e \sin E.$$

(37)

Replacing equations (35) and (36) into equation (34) one finds

$$\psi_{nmpjk} = (n-2p+j) \phi + mv \phi + (n-2p) g +$$

$$+ m (h-\omega \ell - \theta_0).$$

(38)

Thus we define a new function $N^m_k(v,e)$ which gives

$$\begin{bmatrix} \cos \\ \sin \end{bmatrix} \psi_{nmpjk} = \sum_k N^m_k \begin{bmatrix} \cos \\ \sin \end{bmatrix} \psi_{nmpjk},$$

(39)

where

$$\psi_{nmpjk} = (n-2p+j+k) \phi + (n-2p) g + m (h-\omega \ell - \theta_0).$$

(40)

The development of $N^m_k$ can be found in Appendix D.

Accumulating the results we arrive at the final expression for the disturbing function $F_T$:

-19-
\[ F_T = \frac{\mu}{\epsilon^2 q} \sum_{n=2}^n \sum_{m=1}^n \sum_{p=0}^n \sum_{j=1-n}^{n-1} \sum_{k=-\infty}^{\infty} V_{nmpjk}, \quad (41) \]

where one element is given by

\[ V_{nmpjk} = J_{nmpjk} \left\{ A_{nm} \cos \psi_{nmpjk} + B_{nm} \sin \psi_{nmpjk} \right\} \quad (42) \]

and

\[ J_{nmpjk} = \frac{R_n}{\rho^{n-1}} F_{nmpjk} G_{n-1}^{n-1} N_{jk}^m. \quad (43) \]

Thus one element of the determining function \( S_T \) defined by equation (19) becomes

\[ S_{Tnmpjk} = \frac{-J_{nmpjk}}{(n-2p+j+k-vm)} \left\{ A_{nm} \sin \psi_{nmpjk} - B_{nm} \cos \psi_{nmpjk} \right\}. \quad (44) \]

The mean energy can then be found from equations (44) and (22):

\[ L' = L + \frac{\mu}{q} \sum_{n} \sum_{m} \sum_{p} \sum_{j} \sum_{k} \Delta L_{nmpjk}. \quad (45) \]

where \( \Delta L_{nmpjk} \) is expressed as
\[ \Delta L_{nmpjk} = \omega_0 \frac{J_{nmpjk}}{(n-2p+j+k-vm)} \left\{ A_{nm} \cos \psi_{nmpjk} + B_{nm} \sin \psi_{nmpjk} \right\} \]

The elimination of the intermediate period terms may also be found by determining the partial derivatives of each element of \( S_{nmpjk} \) in which

\[ k = 2p - n - j \]
5.0 EXPRESSING THE GEOPOTENTIAL IN \( PS_\phi \) ELEMENTS

In the expansion of the geopotential, the angle \( \psi_{nmpjk} \) becomes undefined for small eccentricities and inclinations. The Fourier expansion of equations (41), (42) and (43) should be transformed into a Fourier series in the true longitude \( \sigma_1 \) and the angle \( \omega_\phi \ell + \theta_\phi \), and polynomials of the well defined functions

\[
\eta = e \sin (g+h) \quad \delta = s \cdot \sin h \\
\zeta = e \cos (g+h) \quad \beta = s \cdot \cos h .
\]

With the definition of the true longitude \( \sigma_1 = \phi + g + h \), the angle \( \psi_{nmpjk} \) becomes

\[
\psi_{nmpjk} = \theta_{nmpjk} - (j+k)(g+h) + (m-n+2p) h ,
\]

\[
\theta_{nmpjk} = (n-2p+j+k) \sigma_1 - m(\omega_\phi \ell + \theta_\phi) .
\]

Keeping the notation of Giacaglia (reference 9) we make new definitions

\[
q^+ = j + k \quad , \quad \alpha = m - n + 2p \quad .
\]

so that \( \psi_{nmpjk} \) becomes

\[
\psi_{nmpjk} = \theta_{nmpjk} - q(g+h) + \alpha h .
\]

\( ^+ \) To maintain the notation of previous authors, this author has decided to keep the notation of \( q \) as an index in the expansion. Please observe the definition of \( q \) given in Appendix A.
By defining

\[ \eta_q = e|q| \ sin q (g+h) \]
\[ \delta_a = s|a| \ sin a h \]
\[ \zeta_q = e|q| \ cos q (g+h) \]
\[ \beta_a = s|a| \ cos a h \] \hspace{1cm} (54)

\[ R_{aq} = \zeta_q \beta_a + \eta_q \delta_a \] \hspace{1cm} (55)

\[ I_{aq} = \zeta_q \delta_a - \eta_q \beta_a \] \hspace{1cm} (56)

the expressions for the functions of \( \psi_{nmpjk} \) become

\[ e|q|s|a| \ cos \ \psi_{nmpjk} = R_{aq} \ cos \ \theta_{nmpjk} + \]
\[ + I_{aq} \ sin \ \theta_{nmpjk} \] \hspace{1cm} (57)

\[ e|q|s|a| \ sin \ \psi_{nmpjk} = R_{aq} \ sin \ \theta_{nmpjk} + \]
\[ - I_{aq} \ cos \ \theta_{nmpjk} \] \hspace{1cm} (58)

Recursive relations exist for the definitions of equation (54). They are

\[ \eta_q = \eta_{q-1} \zeta_1 + \zeta_{q-1} \eta_1 \] \hspace{1cm} (59)

\[ \zeta_q = \zeta_{q-1} \zeta_1 - \eta_{q-1} \eta_1 \] \hspace{1cm} (60)

\[ \eta_{-q} = - \eta_q \] \hspace{1cm} (61)

\[ \zeta_{-q} = \zeta_q \] \hspace{1cm} (62)

\[ \zeta_1 = Q \nu_2 \] \hspace{1cm} \[ \eta_1 = - Q \alpha_2 \]
\[ \delta_{\alpha} = \delta_{\alpha-1} \beta_{1} + \beta_{\alpha-1} \delta_{1} \quad . \] (63)

\[ \beta_{\alpha} = \beta_{\alpha-1} \beta_{1} - \delta_{\alpha-1} \delta_{1} \quad . \] (64)

\[ \delta_{-\alpha} = -\delta_{\alpha} \quad , \quad \beta_{-\alpha} = \beta_{\alpha} \quad , \] (65)

\[ \beta_{1} = P_{\rho_{3}} \quad , \quad \delta_{1} = -P_{\sigma_{3}} \quad . \] (66)

With the expressions (57) and (58), one is now able to write each element of the disturbing function given by equation (42) entirely in non-singular coordinates:

\[ V_{nmpjk} = \mathcal{J}_{nmpjk} \left\{ \left( A_{nm} \alpha_{aq} - B_{nm} \beta_{aq} \right) \cos \theta_{nmpjk} + \right. \]
\[ \left. + \left( A_{nm} \beta_{aq} + B_{nm} \alpha_{aq} \right) \sin \theta_{nmpjk} \right\} \quad . \] (67)

\[ \mathcal{J}_{nmpjk} = \frac{R_{n}}{p^{n-1} \tilde{F}_{nmp} \tilde{G}_{j}^{n-1} N_{k}^{m} e^{(|j|+|k|-|j\cdot k|)}} \quad . \] (68)

\[ \tilde{F}_{nmp} = \frac{F_{nmp}}{s \sqrt{q_{1}}} \quad . \] (69)

\[ \tilde{G}_{j}^{n-1} = \frac{G_{i}^{n-1}}{e^{1\sqrt{j}} \sqrt{1j}} \quad . \] (70)

\[ \tilde{N}_{k}^{m} = \frac{N_{k}^{m}}{e^{1\sqrt{k}} \sqrt{1k}} \quad . \] (71)

The expressions of the barred values are found from the relations of the unbarred values. These barred expressions
are free of singularities. The relations for $\bar{F}_{nmp}$, $\bar{C}_j^{m-1}$ and $\bar{N}_k^m$ may be found in Appendices B, C and D, respectively.
6.0 CONCLUSIONS

Orbit perturbations due to the time dependent harmonics of the gravitational field of the earth have been studied for the case of near-earth satellites. Since for these orbits, the atmospheric drag perturbations never permits the satellite to remain long in a resonant frequency, the resonant perturbations have been neglected. Because the short period perturbations due to the tesseral harmonics are of second order, these terms have also been neglected.

The solution for the satellite motion which includes the intermediate period perturbations of between first and second order have been found by using the Von Zeipel solution method with a general, recursive expansion of the geopotential in the non-singular PSω elements. Also, the along track secular perturbations induced by the periodic perturbations in the mean motion are eliminated by the computation of the average energy using the same theory.

This theory has been implemented in an operational computer program ASOP (reference 12) and numerical experiments verify the expected accuracy.
REFERENCES


APPENDIX A - NOTATION
The definitions of the DSφ elements are as follows:

**The angular variables**

- $\alpha_1 = \phi$  true anomaly
- $\alpha_2 = g$  argument of pericenter
- $\alpha_3 = h$  ascending node
- $\alpha_4 = l$  time element

**The action variables**

- $\beta_1 = \phi$  related to two-body energy
- $\beta_2 = G$  angular momentum magnitude
- $\beta_3 = H$  Z component of angular momentum
- $\beta_4 = L$  the total energy

These may be transformed to the canonical PSφ elements:

\[
\begin{align*}
\sigma_1 &= \phi + g + h \\
\sigma_2 &= -\sqrt{2(\phi-G)} \sin (g+h) \\
\sigma_3 &= -\sqrt{2(G-H)} \sin h \\
\sigma_4 &= l \\
\rho_1 &= \phi \\
\rho_2 &= \sqrt{2(\phi-G)} \cos (g+h) \\
\rho_3 &= \sqrt{2(G-H)} \cos h \\
\rho_4 &= L
\end{align*}
\] (A3)

Abbreviations used in the text are

\[
\rho = \frac{1}{\mu} \left( G - \phi + \frac{\mu}{\sqrt{2L}} \right)^2 \quad \text{(semi-latus rectum)}
\]

\[
q = G - \frac{1}{2} \left( \phi - \frac{\mu}{\sqrt{2L}} \right)
\]
\[ e = \sqrt{1 - \frac{2L}{\mu} p} \quad \text{(eccentricity)} \]

\[ s = \sin I = \sqrt{1 - \frac{H^2}{G^2}} \quad \text{(I is the inclination)} \]

\[ c = \cos I = \frac{H}{G} \]

\[ Q = \frac{\sqrt{p_4}}{\mu} \left[ \frac{2\mu}{\sqrt{2}\sqrt{p_4}} + G - \phi \right]^{\frac{3}{2}} \]

\[ P = \frac{\sqrt{2(G + H)}}{2G} \]
Recursive relations for the Inclination Function $F_{nmp}$ have been derived by Giacaglia (reference 8). The first recurrence relation given in that reference is simple and valid for all $n$, $m$, $p$ except when $n = m$:

$$F_{nmp} = (-1)^{n-m+1} \frac{(2n-1)s}{2(n-m)} \left[ F_{n-1,m,p} - F_{n-1,m,p-1} \right]$$

$$- \left( \frac{n-1+m}{n-m} \right) F_{n-2,m,p} \quad (B1)$$

The other relations derived in that reference were not suitable because they had singularities for zero inclinations. A new recurrence relation derived through induction by this author completes the definition of the inclination function:

$$F_{nmp} = \frac{2n-1}{2} \left\{ s^2 (1+c)^{-1} F_{n-1,n-1,p-1} + \right.$$  

$$+ (1+c) F_{n-1,n-1,p} \} \quad (B2)$$

In all the recurrence relations

$$F_{nmp} = 0 \quad \text{for} \quad p < 0 \quad \text{and} \quad p > n \quad \text{and} \quad m < 0 \quad \text{and} \quad m > n \quad . \quad (B3)$$
The definition of the barred inclination function is

\[ \overline{F}_{nmmp} = \frac{F_{nmmp}}{s|\alpha|} \quad \alpha = m - n + 2p . \quad (B4) \]

From equation (B1) one finds the relation for \( \overline{F}_{nmmp} \) is

\[ \overline{F}_{nmmp} = (-1)^{n-m+1} \frac{2n-1}{2(n-m)} \left[ a \overline{F}_{n-1,m,p} - b \overline{F}_{n-1,m,p-1} \right] - \]

\[ - \frac{(n-1+m)}{(n-m)} \overline{F}_{n-2,m,p-1} . \quad (B5) \]

The values of \( a \) and \( b \) are found from the conditions

\[
\begin{array}{c|c|c|c}
\text{n > m+2p} & \text{n < m+2p} & \text{n = m+2p} \\
\hline
a = & 1 & s^2 \\
b = & s^2 & 1 & s^2 \\
\end{array}
\]

From equation (B2) the definition of \( \overline{F}_{nmmp} \) becomes

\[
\overline{F}_{nmmp} = \frac{2n-1}{2} (1+c)^{-1} \left\{ \overline{F}_{n-1,n-1,p-1} + \\
+ (1+c)^2 \overline{F}_{n-1,n-1,p} \right\} . \quad (B6)
\]
As before the following applies

\[ \bar{F}_{np} = 0 \quad \text{for } p < 0 \text{ and } p > n \quad \text{(B7)} \]
\[ m > 0 \text{ and } m > n \]

The starting value of the recurrence relation is

\[ \bar{F}_{000} = 1 \quad \text{.} \quad \text{(B8)} \]
APPENDIX C - ECCENTRICITY FUNCTION
APPENDIX C

ECCENTRICITY FUNCTION

The eccentricity function $G^n_j$ is defined by the relation

$$\left(\frac{p}{r}\right)^n \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \phi = \sum_{j=-n}^{n} G^n_j \begin{bmatrix} \sin \phi \\ \cos \phi \end{bmatrix} \phi$$

(C1)

where $\frac{p}{r} = 1 + e \cos \phi$.

The binomial expansion of $\left(\frac{p}{r}\right)^n$ becomes

$$\left(\frac{p}{r}\right)^n = \sum_{s=0}^{n} \binom{n}{s} \left(\frac{e}{2}\right)^s \sum_{t=0}^{s} \binom{s}{t} \cos (s-2t)$$

(C2)

Using equation (C2) and the definition of equation (C1)
the expression for $G^n_j$ becomes

$$G^n_j = e^{|j|} \sum_{s=|j|}^{n} \binom{n}{s} e^{s-j} \left(\frac{s}{2}\right) \left(\frac{1}{2}\right)^s \quad \text{for } j > 0$$

(C3)

$$G^n_{-j} = G^n_j$$
Recurrence relations exist for the binomial coefficient

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

(C4)

The expression for \( \bar{G}^n_j \) becomes

\[
\bar{G}^n_j = \frac{G^n_j}{e^{\frac{j^2}{4}}} = \sum_{k=|j|}^{n} \binom{n}{k} e^{k-j} \left( \frac{k}{k-1} \right) \left( \frac{1}{2} \right)^k
\]

(C5)

\[
\bar{G}^n_{-j} = \bar{G}^n_j
\]
It is desired to make the following expansion

\[
\begin{bmatrix}
\cos \\
\sin
\end{bmatrix} \sin \left( \phi - M \right) = \sum_k N_k^m \begin{bmatrix}
\cos \\
\sin
\end{bmatrix} k \phi \quad .
\]

To develop this expansion, a complex notation is adopted which has the same meaning

\[
\exp \left[ i \psi (\phi - M) \right] = \sum_k N_k^m \exp (i k \phi) \quad ,
\]

where \( N_k^m \) is defined by

\[
N_k^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ i \psi (\phi - M) - i k \phi \right] d\phi \quad .
\]

As in the development of Hansen coefficients (reference 10) some abbreviations are useful

\[
\frac{\sqrt{1+e}}{\sqrt{1-e}} = \frac{1+\beta}{1-\beta} \quad \text{or} \quad \beta = \frac{e}{1+\sqrt{1-e^2}} \quad .
\]

\[
z = \exp i E \quad .
\]
The relation between the true and eccentric anomaly

\[
\tan \phi = \frac{1}{2} \tan \frac{E}{2} = \frac{1}{\sqrt{1+e}}
\]

(D6)

can be written in another form by equations (D4) and (D5) as

\[
\exp i\phi = \frac{z-\beta}{i-\beta z} = z \left(1 - \frac{\beta}{z}\right) \left(1 - \beta z\right)^{-1}
\]

(D7)

Similarly the relation between mean and eccentric anomalies

\[
M = E - e \sin E
\]

(D8)

may be written as

\[
\exp iM = z \cdot \exp \left[\frac{e}{2} \left(z - \frac{1}{z}\right)\right]
\]

(D9)

By differentiating equation (D7), the expression for \(d\phi\) becomes

\[
d\phi = \frac{(1-\beta^2)}{i} (1-\beta z)^{-1} z^{-1} (1 - \frac{\beta}{z})^{-1} dz
\]

(D10)
Thus the integrand in equation (D3) becomes

$$\exp i \left[ \frac{(\nu-k)\phi - \nu k}{1} \right] \frac{1}{(1-\beta z)^{k-m\nu-1}} \left(1 - \frac{\beta}{z}\right)^{-(k+1-m\nu)} z^{-1-k}.$$  

(D11)

$$\cdot \exp \left[ \frac{m \nu e}{2} \left( \frac{1}{z} - \frac{1}{z} \right) \right] dz .$$

The powers of $(1-\beta z)$ and $1 - \frac{\beta}{z}$ can be expanded with small parameter $\beta$

$$f_s = (1-\beta z)^{k-m\nu-1}$$

$$\frac{df}{dz} = -\beta (1-\beta z)^{k-m\nu-2} (k-m\nu-1)$$

$$\frac{d^2 f}{dz^2} = (-\beta)^2 (1-\beta z)^{k-m\nu-3} (k-m\nu-1) (k+m\nu-2)$$  

(D12)

$$\frac{d^5 f}{dz^5} = (-\beta)^5 (1-\beta z)^{k-(m\nu+s+1)} \prod_{j=0}^{s-1} [k-(m\nu+j+1)]$$

$$\frac{d^s f}{dz^s} (z=0) = -(-\beta)^s \prod_{j=0}^{s-1} [k-(m\nu+j+1)]$$
By Taylor's expansion

\[ f_s = 1 + \sum_{s=1}^{\infty} \frac{d^s f(z=0)}{dz^s} \frac{1}{s!} z^s \]

or

\[ f_s = \sum_{s=0}^{\infty} a_s \beta^s z^s \]  \hspace{1cm} (D13)

where

\[ a_0 = 1 \quad a_s = \frac{(-1)^s}{s!} \prod_{j=0}^{s-1} [k-(mv+j+1)] \]

or

\[ a_s = -\left(\frac{k-mv-s}{s}\right) a_{s-1} \]

Similarly

\[ f_t = (1-\beta w)^{-(k+1-mv)} \] \hspace{1cm} \text{where} \quad w = \frac{1}{z}

\[ \frac{df_t}{dw} = \beta (1-\beta w)^{-(k-mv+2)} (k-mv+1) \]

\[ \frac{d^2 f_t}{dw^2} = \beta^2 (1-\beta w)^{-(k-mv+3)} (k-mv+1)(k-mv+2) \]  \hspace{1cm} (D14)
\[
\frac{d^t f}{dw^t} = \beta^t (1-\beta w)^{-(k-mv+1+t)} \prod_{j=0}^{t-1} (k-mv+1+j)
\]

\[
\frac{d^t f (w=0)}{dw^t} = \beta^t \prod_{j=0}^{t-1} (k-mv+1+j)
\]

Therefore:

\[
f_t = \sum_{t=0}^{\infty} b_t \beta^t w^t \quad \text{or} \quad \sum_{t=0}^{\infty} b_t \beta^t z^{-t}
\]

(D15)

where

\[
b_0 = 1 \quad b_t = \frac{1}{t} \prod_{j=0}^{t-1} (k-mv+1+j)
\]

or

\[
b_t = \left(\frac{k-mv+t}{t}\right) b_{t-1}
\]

With equations (D13) and (D15) the integral becomes

\[
N_k^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ (mv-k) \phi - mvM \right] d\phi
\]

(D16)

\[
N_k^m = (1-\beta^2) \sum_{s=0} a_s b_t \beta^{s+t} \int \frac{1}{2\pi i} z^{-1-(k-s+t)} \exp \left[ \frac{mv}{2} \left( \frac{1}{z} - \frac{1}{z} \right) \right] dz.
\]
But the Bessel function is defined as

\[ J_n(a) = \frac{1}{2\pi i} \int_{C} z^{-1-n} \exp \left[ \frac{a}{2} \left( \frac{1}{z} - \frac{1}{2} \right) \right] \, dz \quad \text{, (D17)} \]

so that the final form becomes

\[ N^m_k = (1-\beta^2) \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} a_s b_t \beta^{s+t} \, J_{k-s+t}(mve) \quad \text{, (D18)} \]

where

\[ a_0 = 1 \quad a_s = -\frac{(k-mv-s)}{s} a_{s-1} \quad \text{ (D19)} \]

\[ b_0 = 1 \quad b_t = \frac{(k-mv+t)}{t} b_{t-1} \quad \text{ (D20)} \]

The barred function \( \bar{N}_k^m \) can be found from its definition and equation (D18)

\[ \bar{N}_k^m = \frac{N_k^m}{|k|} \quad \text{ (D21)} \]

If one notes the definition of \( \beta \) in equation (D4), one finds

\[ \bar{N}_k^m = (1-\beta^2) \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{a_s b_t}{(1+1/e^2)^{s+t}} e^{2\sigma(mv)(k-s+t)} J_{k-s+t}(mve) \quad \text{ (D22)} \]
The value of \( \hat{J}_n(\alpha) \) is determined by the following conditions

<table>
<thead>
<tr>
<th>( k \geq 0 )</th>
<th>( k &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k - s + t \geq 0 )</td>
<td>( \sigma = t )</td>
</tr>
<tr>
<td>( k - s + t &lt; 0 )</td>
<td>( \sigma = s - k )</td>
</tr>
</tbody>
</table>

(D23)

Note that \( \sigma \) can never be negative.

The function \( \hat{J}_n(\alpha) \) is defined from the Bessel function as

\[
\hat{J}_n(\alpha) = \frac{J_n(\alpha)}{\alpha^n}. \quad (D24)
\]

It can be found from following recursive relation

\[
\hat{J}_{n-1} = 2n \hat{J}_n - \alpha^2 \hat{J}_{n+1}. \quad (D25)
\]

with starting values

\[
\hat{J}_{n+1} = \left( \frac{1}{2} \right)^{n+1} \frac{1}{(n+1)!}.
\]

(D26)

\[
\hat{J}_n = \left( \frac{1}{2} \right)^n \left\{ \frac{1}{n!} - \frac{\alpha^2}{(n+1)!} \right\}.
\]
The expressions for $N_k^m$ were determined in another manner by Bond (reference 11) up to order $\epsilon^3$. These expressions are given here in the barred form.

\[
\begin{align*}
\bar{N}_0^m &= 1 - (mv\epsilon)^2 \\
\bar{N}_{-1}^m &= -m\nu \left[ 1 - m\nu \left( \frac{3}{8} + \frac{1}{2m\nu} \right) \right] \\
\bar{N}_1^m &= m\nu \left[ 1 + m\nu \left( \frac{3}{8} - \frac{1}{2m\nu} \right) \right] \\
\bar{N}_{-2}^m &= \frac{m\nu}{2} \left( \frac{3}{4} + m\nu \right) \\
\bar{N}_2^m &= -\frac{m\nu}{2} \left( \frac{3}{4} - m\nu \right) \\
\bar{N}_{-3}^m &= \frac{-m\nu}{2} \left( \frac{1}{3} + \frac{3}{4m\nu} + \frac{1}{3} (m\nu)^2 \right) \\
\bar{N}_3^m &= \frac{m\nu}{2} \left( \frac{1}{3} - \frac{3}{4m\nu} + \frac{1}{3} (m\nu)^2 \right)
\end{align*}
\]