RESEARCH STUDY ON STABILIZATION AND CONTROL
MODERN SAMPLED-DATA CONTROL THEORY

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PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA
FINAL REPORT

DIGITAL CONTROLLER DESIGN

subtitle:

ANALYSIS OF THE ANNULAR SUSPENSION POINTING SYSTEM

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1. SIMPLIFIED MODELS OF THE ANNULAR SUSPENSION AND POINTING SYSTEM (ASPS)

1.1 Introduction [1]

The Annular Suspension and Pointing System (ASPS) is a payload auxiliary pointing device of the Space Shuttle. The ASPS is comprised of two major subassemblies, a vernier and a coarse pointing subsystem.

The experiment is attached to a mounting plate/rim combination which is suspended on magnetic bearing/actuators (MBA's) strategically located about the rim. Fine pointing is achieved by "gimballing" the plate/rim within the MBA gaps. Control about the experiment line-of-sight is obtained through the use of a non-contacting rim drive and positioning torquer. All sensors used to close the servo loops on the vernier system are noncontacting elements. Therefore, the experiment is a free-flyer constrained only by the magnetic forces generated by the control loops.

The configuration of the ASPS is shown in Fig. 1-1. The payload/plate/rim combination is mounted on a set of coarse gimbals; an elevation and a lateral coarse gimbal, which provide the slewing and coarse pointing capability of the system. The pointing system concept is unique in that the vernier and coarse pointing subsystem are separate entities. This approach allows for sub-arcsecond pointing of the payload at any coarse gimbal position.

The three functions provided by the ASPS are: (1) pointing the payload, (2) centering the payload in the magnetic actuator assembly, and (3) tracking the payload mounting plate and shuttle motions by the coarse gimbals. Rate and position errors sensed by gyros and celestial sensors located on the payload are processed by a controller which subsequently commands appropriate actuator force to point the payload. Proximeter sensors associated with the actuator clusters detect the payload translation errors which are subsequently processed by the
Figure 1-1. ASPS configuration.

Figure 1-2. Payload and magnetic actuator assembly.
controller and used to ascertain the appropriate centering forces.

Figure 1-2 shows the payload and its mounting surface which is controlled by the magnetic actuator assembly (MAA). The cables shown are for the purpose of connecting electric power from the shuttle to the payload and the MAA on the pallet.

1.2 The Planar Model of the ASPS [2]

In this section the equations of motion of a simplified planar model of the ASPS are derived.

The small-angle, small-displacement model shown in Fig. 1-3 is planar with four degrees of freedom and is composed of a mount, a gimbal assembly (elevation), a pallet with magnetic actuators, and a payload. The pallet has one rotational degree of freedom relative to the mount, and the payload has two translational and one rotational degrees of freedom relative to the pallet.

Let the four degrees of freedom be

\( \phi_1 \) = attitude degree of freedom of the pallet relative to the mount
\( \phi_2 \) = attitude degree of freedom of the payload relative to the pallet
\( x_1 \) = translation degree of freedom of the payload relative to the pallet
\( z_1 \) = translation degree of freedom of the payload relative to the pallet

The following coordinates are defined:

\( (x_0, z_0) \) = inertial axes
\( (x_G, z_G) \) = inertial axes rotated through an angle of \( \phi_M \) relative to the \( (x_0, z_0) \) axes, \( \phi_M \) is defined as the gimbal angle.
\( (x_m, z_m) \) = axes fixed at the pallet center of gravity (CG)
\( (x'_1, z'_1) \) = static axes of the payload
\( (x_j, z_j) \) = axes fixed at the payload center of gravity (CG)
\( (x_i, z_i) \) = axes fixed at the center of the base of the payload.
Figure 1-3. Planar ASPS model.
The following system parameters are defined:

\[ M_i = \text{mass of the payload} = 600 \text{ Kg} \]
\[ M_m = \text{mass of the pallet} = 82 \text{ Kg} \]
\[ J_m = \text{inertia of the pallet about its mass center} = 3.1 \text{ Kg-m}^2 \]
\[ J_i = \text{inertia of the payload about its mass center} = 503 \text{ Kg-m}^2 \]
\[ L = \text{radius of the payload} = 0.406 \text{ m} \]
\[ L_a = \text{distance from the gimbal to the pallet CG} = 0.2064 \text{ m} \]
\[ L_b = \text{distance from the pallet center to the payload CG} = 1.486 \text{ m} \]
\[ r_a = \text{distance from the gimbal assembly to the pallet center} = 0.47 \text{ m} \]
\[ r_b = \text{distance from the gimbal assembly to the payload CG} = 1.956 \text{ m} \]
\[ r_0 = \text{distance from the mount base to the gimbal assembly} = 0.75 \text{ m} \]

Since the payload is suspended with respect to the pallet, there are many ways of fixing its coordinates for the motion of rotation. In other words, the angle \( \phi_2 \) can be defined in a number of ways. Figure 1-4 illustrates the small-angle rotation of the pallet and the payload with \( \phi_2 \) measured as the angle between the coordinate axes of \((x_i,z_i)\) and \((x_j,z_j)\). This configuration is defined as Model I of the ASPS. Figure 1-5 illustrates the model of the ASPS with \( \phi_2 \) measured at the CG of the payload; i.e., between the axes of \((x_i,z_i)\) and \((x_j,z_j)\).

The following coordinate transformations are identified:

Transformation from the static pallet axes to the mount axes:

\[
T_G = \begin{bmatrix}
\cos \phi_M & -\sin \phi_M \\
\sin \phi_M & \cos \phi_M
\end{bmatrix}
\] (1-1)

Transformation from the dynamic pallet axes to the static pallet axis:

\[
T_1 = \begin{bmatrix}
\cos \phi_1 & -\sin \phi_1 \\
\sin \phi_1 & \cos \phi_1
\end{bmatrix} \equiv \begin{bmatrix} 1 & -\phi_1 \\
\phi_1 & 1
\end{bmatrix}
\] (1-2)
Figure 1-4. Planar ASPS Model 1.
Figure 1-5. Planar ASPS Model 2.
Transformation from the dynamic payload axes to the dynamic pallet axes:

\[ T_2 = \begin{bmatrix} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{bmatrix} \begin{bmatrix} 1 \\ \phi_2 \end{bmatrix} \]  \hspace{1cm} (1-3)

The force vectors applied to the payload by the magnetic actuator assembly are defined as:

\[ F_1 = \begin{bmatrix} f_4 \\ f_3 \end{bmatrix} \] = magnetic force applied at the positive \( x_m \) displacement (1-4)

\[ F_2 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \] = magnetic force applied at the negative \( x_m \) displacement (1-5)

The forces \( F_1 \) and \( F_2 \) are illustrated as shown in Fig. 1-5.

The torque applied by the gimbal assembly is designated as \( T_c \), as shown in Fig. 1-5.

The following vector distances are defined for the pallet and the payload.

\[ R_1 = \text{vector distance from the gimbal assembly to the payload point of application of } F_1 \]

\[ R_2 = \text{vector distance from the gimbal assembly to the payload point of application of } F_2 \]

\[ R_3 = \text{vector distance from the gimbal assembly to the pallet point of application of } F_1 \]

\[ R_4 = \text{vector distance from the gimbal assembly to the pallet point of application of } F_2 \]

Equations of Motion of Model 1

Using the degrees of freedom defined in the preceding sections, the kinetic energy of the system in Fig. 1-4 is

\[ T = \text{K.E.} = \frac{1}{2} \dot{R}_m \dot{R}_m + \frac{1}{2} \dot{R}_i \dot{R}_i + \frac{1}{2} J_1 \dot{\phi}_1^2 + \frac{1}{2} J_2 (\dot{\phi}_1 + \dot{\phi}_2)^2 \]  \hspace{1cm} (1-6)

where the primes denote the transpose of a matrix, and
Substitution of Eqs. (1-7) and (1-8) into Eq. (1-6) gives

\[
T = \text{K.E.} = \frac{1}{2} M a \dot{x}_1^2 + \frac{1}{2} M \dot{z}_1^2 + \frac{1}{2} M (\dot{x}_1 - r_b \dot{\phi}_1 - L_b \dot{\phi}_2)^2 + \frac{1}{2} J \dot{\phi}_1^2 + \frac{1}{2} J (\dot{\phi}_1 + \dot{\phi}_2)^2
\]

(1-9)

Let the spring force applied to the system payload due to the cable be designated as

\[
f_s = \begin{bmatrix} f_{sx}(x_1) \\ f_{sz}(z_1) \end{bmatrix}
\]

(1-10)

and the spring torque applied to the payload due to the cable be \( T_s(\phi_2) \). The spring torque applied to the pallet due to the cable is denoted as \( T_p(\phi_1) \).

The relation between the force \( \bar{F} \) and the potential of the system, \( U \), is

\[
\bar{F} = -\nabla U
\]

(1-11)

Thus,

\[
U = U_0 - \int \bar{F} \cdot dx
\]

(1-12)

where \( U_0 = \text{constant} \).

The potential energy of Model I is

\[
U = U_0 - \left[(f_1 - f_4)x_1 + (f_2 + f_3)z_1 + \int_0^{\phi_2} T_s(\phi) d\phi + \int_0^{Z_1} f_{sz}(z) dz + \int_0^{X_1} f_{sx}(x) dx \right. \\
+ (f_3 - f_2) L \phi_2 + \left[\int_0^{\phi_1} T_c(\phi) d\phi + \int_0^{\phi_1} T_p(\phi) d\phi \right]
\]

(1-13)

The Lagrangian is defined as

\[
\mathcal{L} = \text{K.E.} - U
\]

(1-14)

Then from Eqs. (1-9) and (1-13), we get
The Lagrange equation of motion is

\[
\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = 0 \quad i = 1, 2, 3, 4
\] (1-16)

where \( x_1 = x_1, \quad x_2 = z_1, \quad x_3 = \phi_1 \) and \( x_4 = \phi_2 \).

For \( i = 1 \), we have

\[
\frac{\partial \mathcal{L}}{\partial x_1} = f_1 - f_4 + f_{sx}(x_1)
\] (1-17)

\[
\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = M_1 (\ddot{x}_1 - r_b \dot{\phi}_1 - L_b \dot{\phi}_2)
\]

Thus,

\[
\frac{\partial \mathcal{L}}{\partial x_1} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right) = -M_1 \dddot{x}_1 + M_1 r_b \ddot{\phi}_1 + M_1 L_b \ddot{\phi}_2 + (f_1 - f_4) + f_{sx}(x_1) = 0
\] (1-18)

For \( i = 2 \), we have

\[
\frac{\partial \mathcal{L}}{\partial z_1} = f_2 + f_3 + f_{sz}(z_1)
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{z}_1} = M_1 \ddot{z}_1
\]

Then,

\[
\frac{\partial \mathcal{L}}{\partial z_1} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}_1} \right) = -M_1 \dddot{z}_1 + (f_2 + f_3) + f_{sz}(z_1) = 0
\] (1-19)

For \( i = 3 \), we have

\[
\frac{\partial \mathcal{L}}{\partial \phi_1} = T_c(\phi_1) + T_p(\phi_1)
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = M m a z_1 \dot{\phi}_1 + M_1 (-r_b \dot{x}_1 + r_b^2 \dot{\phi}_1 + r_b L b \dot{\phi}_2) + J_1 \dot{\phi}_1 + J_1 \dot{\phi}_1 + J_1 \dot{\phi}_2
\]
\[
\frac{\partial}{\partial \phi_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \frac{\partial \phi_1}{\partial t}}\right) = T_c(\phi_1) + T_p(\phi_1) + M_i r_b \ddot{r}_b \\
- (J_m + J_i + M_m a + M_i r_b^2) \ddot{\phi}_1 - (J_i + M_i r_b^2) \ddot{\phi}_2 = 0 \quad (1-20)
\]

For \( i = 4 \), we have
\[
\frac{\partial L}{\partial \phi_2} = (f_3 - f_2) L + T_s(\phi_2) \\
\frac{\partial L}{\partial \phi_2} = -M_i L b \ddot{x}_1 + r_b L b M_i \ddot{\phi}_1 + M_i \ddot{\phi}_2 + J_i \dot{\phi}_1 + J_i \dot{\phi}_2 \\
\frac{\partial L}{\partial \phi_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \phi_2}\right) = T_s(\phi_2) + (f_3 - f_2) L + M_i L b \ddot{x}_1 \\
- (M_i L_b r_b + J_i \dddot{\phi}_1) = (M_i L_b^2 + J_i) \dddot{\phi}_2 = 0 \quad (1-21)
\]

The Lagrange equations in Eqs. (1-18), (1-19), (1-20) and (1-21) are written in matrix form as follows:
\[
\begin{bmatrix}
M_i & 0 & -M_i r_b & -M_i L_b \\
0 & M_i & 0 & 0 \\
-M_i r_b & 0 & J_i + M_i L^2 a + M_i r_b^2 & J_i + M_i L b r_b \\
-M_i L_b & 0 & J_i + M_i L b r_b & J_i + M_i r_b^2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{z}_1 \\
\ddot{\phi}_1 \\
\ddot{\phi}_2
\end{bmatrix}
= \begin{bmatrix}
f_1 - f_4 + f_{sx}(x_1) \\
f_2 + f_3 + f_{sx}(z_1) \\
T_c(\phi_1) + T_p(\phi_1) \\
(f_3 - f_2) L + T_s(\phi_2)
\end{bmatrix} \quad (1-22)
\]

Equations of Motion of Model 2

For the ASPS system Model 2, the kinetic energy of the system is still given by Eq. (1-6), and \( \dot{R}_m \) is as defined in Eq. (1-7), except that
\[
\dot{R}_1 = \begin{bmatrix}
\dot{x}_1 \\
\dot{z}_1
\end{bmatrix} + \begin{bmatrix}
-\dot{r}_b \\
\dot{\phi}_1
\end{bmatrix} \quad (1-23)
\]

Substitution of Eqs. (1-7) and (1-23) into Eq. (1-6) gives
\[
K.E. = \frac{1}{2} M_m a \ddot{\phi}_1^2 + \frac{1}{2} m \dot{z}_1^2 + \frac{1}{2} M_a (x_1 - r_b \dot{\phi}_1)^2 + \frac{1}{2} J_i \dot{\phi}_1^2 + \frac{1}{2} J_i (\dot{\phi}_1 + \dot{\phi}_2)^2 \quad (1-24)
\]

The potential energy of the Model 2 is
\[ U = U_0 - \{(f_1 - f_4)x_1 + \int_0^1 f_{sx}(x) \, dx + (f_2 + f_3)z_1 + \int_0^1 f_{sz}(z) \, dz + \int_0^\phi_2 T_s(\phi) \, d\phi + \int_0^\phi_1 T_c(\phi) \, d\phi + \int_0^\phi_1 T_p(\phi) \, d\phi \} \]

The Lagrangian \( \mathcal{L} \) is given by

\[ \mathcal{L} = K.E. - U = \frac{1}{2}m(a_1^2 + \frac{1}{2}M_i \dot{x}_1^2 + \frac{1}{2}M_j \dot{\phi}_1^2 + \frac{1}{2}J_1 \dot{\phi}_1^2) + \frac{1}{2}m(a_2^2 + \frac{1}{2}M_i \dot{x}_1^2 + \frac{1}{2}M_j \dot{\phi}_1^2 + \frac{1}{2}J_1 \dot{\phi}_1^2) \]

Following the same procedure as for Model 1, the Lagrange equations of Model 2 are derived by use of Eqs. (1-16) and (1-26), and the result is

\[ \begin{bmatrix} M_i & 0 & -M_i \cdot R_b & 0 \\ 0 & M_i & 0 & 0 \\ -M_i \cdot R_b & 0 & J_m + M_i L^2 a_i b & J_i \\ 0 & 0 & J_i & J_i \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{z}_1 \\ \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} = \begin{bmatrix} f_1 - f_4 + f_{sx}(x_1) \\ f_2 + f_3 + f_{sz}(z_1) \\ T_c(\phi_1) + T_p(\phi_1) \\ (f_3 - f_2) L + (f_1 - f_4) L_b + T_s(\phi_2) \end{bmatrix} \]

In the analysis conducted in the ensuing sections the equations of motions of Model 2 will be used. One reason for this selection is that the mass matrix of Eq. (1-27) is simpler than that of Model 1 in Eq. (1-22). Another reason for using Model 2 is that the model uses the center of gravity of the payload as the reference point of rotation, which is more logical.
1.3 Control of the $z_1$ Dynamics of the Payload

Equation (1-28) indicates that the $z_1$ dynamics of the ASPS are not coupled to the other three degrees of freedom. The $z_1$ dynamics are described by

$$M_{z_1} \ddot{z}_1 = f_2 + f_3 + f_{sz}(z_1) \tag{1-29}$$

The magnetic actuator forces $f_2 + f_3$ are controlled by feeding back the variables $z_1$ and $\dot{z}_1$. The control equation is

$$f_2 + f_3 = -K_p z_1 - K_r \dot{z}_1 \tag{1-30}$$

where $K_p = 37.861$ N/m and $K_r = 211.01$ N/m/sec.

Substitution of Eq. (1-30) into Eq. (1-29), we have

$$M_{z_1} \ddot{z}_1 = -K_p z_1 - K_r \dot{z}_1 + f_{sz}(z_1) \tag{1-31}$$

Figure 1-6 shows the state diagram of the $z_1$ dynamics of the ASPS with the continuous-data position-plus-rate controller. The notation $N_{sz}(z_1)$ in the state diagram represents the functional relation of the wire cable which is attached to the center of the payload mounting surface.

If the wire cable is modelling by a linear spring, $N_{sz}(z_1)$ is simply a constant, $-K_s (N/m)$; that is,

$$f_{sz}(z_1) = -K_s z_1 \tag{1-32}$$

A nonlinear spring characteristic for $N_{sz}(z_1)$ is shown in Fig. 1-7. However, since the mass of the payload is 600 Kg, and the spring constant is
only 0.35 N/m, the effect of the wire cable on the payload dynamics is not

going to be substantial.

The characteristic equation of the continuous-data ASPS \( z_1 \) dynamic system

with the linear wire cable spring characteristic is

\[
M_i s^2 + K_i s + K_p + K_s = 0
\]

or

\[
600s^2 + 211.01s + 38.211 = 0
\]

The damping ratio of the system is

\[
\zeta = 0.6968
\]

and the undamped natural frequency is

\[
\omega_n = 0.2524 \text{ rad/sec}
\]

Analysis of the Digital ASPS \( z_1 \) Dynamics

When the \( z_1 \) dynamics of the ASPS are controlled by a digital position-plus-rate controller, the dynamic equation is

\[
M_i \ddot{z}_1 + K_s z_1 = f_2(t) + f_3(t)
\]

where

\[
f_2(t) + f_3(t) = f_2(kT) + f_3(kT) \quad kT \leq t < (k+1)T
\]

Then the control equation is

\[
f_2(kT) + f_3(kt) = -K_p z_1(kT) - K_r \dot{z}_1(kT)
\]

Figure 1-8 shows the block diagram of the linear digital ASPS payload (\( z_1 \) dynamics).

Since all the system parameters are given, except the sampling period \( T \), we shall study the maximum value of \( T \) for asymptotic stability.

The characteristic equation of the digital system in Fig. 1-8 is

\[
\Delta(z) = 1 + \frac{1}{M_i} (1 - z^{-1})^2 \left( \frac{K_p/s^2}{1 + \frac{K_r}{M_i} s^{-2}} + \frac{K_d/s^3}{1 + \frac{K_s}{M_i} s^{-2}} \right) = 0
\]
The z-transforms in the last equation are evaluated as follows:

\[
\frac{1}{s^2 + \frac{K_s}{M_1} s + \frac{K_s}{M_1}} = \frac{M_1 \zeta \sin \sqrt{\frac{K_s}{M_1} T}}{s \left( s^2 + \frac{K_s}{M_1} s + \frac{K_s}{M_1} \right)} (1-41)
\]

\[
\frac{1}{s(s^2 + \frac{K_s}{M_1})} = \frac{M_1 \zeta \left( \frac{z}{s - 1} - \frac{z(z - \cos \sqrt{\frac{K_s}{M_1} T})}{s - 2 + 2\cos \sqrt{\frac{K_s}{M_1} T} + 1} \right)}{s \left( s^2 + \frac{K_s}{M_1} s + \frac{K_s}{M_1} \right)} (1-42)
\]

Substitution of the last two equations into Eq. (1-40) and simplifying, we have

\[
z^2 + \left( \frac{K_r}{M_1 K_s} \sin \sqrt{\frac{K_s}{M_1} T} - \frac{K_p}{K_s} \cos \sqrt{\frac{K_s}{M_1} T} + \frac{K_p}{K_s} - 2\cos \sqrt{\frac{K_s}{M_1} T} \right) z + 1 + \frac{K_p}{K_s} - \frac{K_p}{K_s} \cos \sqrt{\frac{K_s}{M_1} T} - \frac{K_r}{\sqrt{M_1 K_s} \sin \sqrt{\frac{K_s}{M_1} T}} = 0 (1-43)
\]

Substituting the system parameters into the last equation, yielding,

\[
z^2 + (14.5597\sin 0.02415T - 110.1688\cos 0.02415T + 108.1688) z + 1 - 14.5597\sin 0.02415T - 108.1688\cos 0.02415T + 108.1688 = 0 (1-44)
\]

The roots of the last equation as a function of T are tabulated below and the root locus diagram with T as a variable parameter is shown in Fig. 1-9. The critical value of T for asymptotic stability is approximately 5.7 sec.

<table>
<thead>
<tr>
<th>Sampling period T (sec)</th>
<th>Characteristic Equation</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>( z^2 - 1.96z + 0.965 = 0 )</td>
<td>( 0.98 \pm j0.069 )</td>
</tr>
<tr>
<td>0.5</td>
<td>( z^2 - 1.816z + 0.832 = 0 )</td>
<td>( 0.908 \pm j0.0864 )</td>
</tr>
<tr>
<td>1.0</td>
<td>( z^2 - 1.6163z + 0.680 = 0 )</td>
<td>( 0.808 \pm j0.164 )</td>
</tr>
<tr>
<td>2.0</td>
<td>( z^2 - 1.1686z + 0.4232 = 0 )</td>
<td>( 0.584 \pm j0.286 )</td>
</tr>
<tr>
<td>3.0</td>
<td>( z^2 - 0.657z + 0.2298 = 0 )</td>
<td>( 0.328 \pm j0.349 )</td>
</tr>
</tbody>
</table>
Figure 1-8. Block diagram of the linear digital ASPS payload $z_1$ dynamics.

Figure 1-9. Root loci of $z_1$ dynamics of the digital ASPS payload as the sampling period $T$ varies.
The time responses of the digital system in Fig. 1-8 for various sampling periods are shown in Fig. 1-10. The initial value of $z_1(t)$ is chosen to be 0.002 m, since the static bearing gap of $z_1$ is only 0.0076 m, so that the maximum constraints on the magnitude of $z_1$ are ±0.0038 m. The time responses in Fig. 1-10 substantiates the root locus findings; when $T = 6$ sec, the closed-loop system is unstable. The time responses are quite good for $T$ less than or equal to 3 seconds.

**Effects of Quantization**

The block diagram of the digital ASPS payload $z_1$ dynamics with the quantization effect is shown in Fig. 1-11. The input-output characteristics of the quantizer are illustrated in Fig. 1-12. The quantization level is denoted by $h$ in meter.

The effects of quantization can be classified into three categories: (1) stable system with steady-state error, (2) unstable system with sustained oscillation, and (3) unstable system with unbounded responses. The last case is possible since no saturation is assumed in the system model.

The steady-state error due to quantization can be determined by using the least-upper bound method [3] and the condition of sustained oscillations is found by use of the discrete describing function.

The characteristic equation of the system shown in Fig. 1-11 is written as

$$
\Delta(z) = 1 + Q(z)(1 - z^{-1})\left[\frac{1}{N_i s^2} \frac{K_r + K_p/s}{1 + \frac{K_s}{N_i} s^{-2}}\right] = 0 \quad (1-45)
$$

where $Q(z)$ denotes the discrete describing function of the quantizer.
Figure 1-16. Time responses of the linear digital APS payload $z_1$ dynamics. The initial condition of $z_1$ is 0.002 m.
Figure 1-11. Block diagram of the digital ASPS payload $z_1$ dynamics with quantization.

Figure 1-12. Input-output characteristics of a quantizer.
Figure 1-13. $G_\text{eq}(z)$ plots and critical region bounds of quantizer discrete describing function of ASP payload $z_1$ dynamics; $K_s = 0.35$ N/m.
Figure 1-14. $G_\text{eq}(z)$ plots and critical region bounds of quantizer discrete describing function of ASPS$^q$ payload $z_1$ dynamics; $K_s = 3.5$ N/m.
Figure 1-15. $G_{eq}(z)$ plots for $N = 6$ and $K_s = 0.35$, 3.5 and 35 N/m and critical region bounds of quantizer discrete describing function of ASPS payload $z_1$ dynamics.
The $z$-transform of the last equation is evaluated using the results in Eqs. (1-41) and (1-42). Equation (1-45) becomes

$$\Delta(z) = 1 + Q(z)G_{eq}(z) = 0 \quad (1-46)$$

where

$$G_{eq}(z) = \frac{K_r \sin \sqrt{\frac{K_s}{M_1}} T - K_p \cos \sqrt{\frac{K_s}{M_1}} T + K_p}{K_r \sin \sqrt{\frac{K_s}{M_1}} T - K_p \cos \sqrt{\frac{K_s}{M_1}} T + K_p} \frac{K_s \sin \sqrt{\frac{K_s}{M_1}} T + K_s}{K_r \sin \sqrt{\frac{K_s}{M_1}} T - K_p \cos \sqrt{\frac{K_s}{M_1}} T + K_p}$$

(1-47)

For $K_p = 37.861$, $K_r = 211.01$, $M_1 = 600$, and $K_s = 0.35$, the last equation is simplified to

$$G_{eq}(z) = \frac{14.5597\sin 0.02415T - 108.1688\cos 0.02415T + 108.1688)z - 14.5597\sin 0.02415T}{z^2 - 2z\cos 0.02415T + 1} \quad (1-48)$$

$$-108.1688\cos 0.02415T + 108.1688$$

Figure 1-13 shows the plots of $G_{eq}(z)$ for various periods of sustained oscillations $T_c = NT$, $N = 2, 3, 4, \ldots$. The sampling period $T$ varies along the curves. The square block in the figure which is centered at $-1$ represents the bounds on the critical regions of $-1/Q(z)$ [4]. Theoretically, the intersects between the critical regions of $-1/Q(z)$ and $G_{eq}(z)$ represent conditions of self-sustained oscillations. It is clear from Fig. 1-13 that the system should be free from sustained oscillations for all sampling periods less than 2 seconds.

Figure 1-14 illustrates the $G_{eq}(z)$ plots for $K_s = 3.5 \text{ N/m}$, 10 times the nominal value. As pointed out earlier, since the mass of the payload is so large, the light spring effect of the wire cable does not materially affect the performance of the system. Figure 1-15 further illustrates that even with $K_s = 35 \text{ N/m}$, 100 times the nominal value, the characteristics of the system for sampling periods less than 2 seconds are not significantly affected.

The least-upper bound error analysis of the quantization effect is performed by referring to the system block diagram shown in Fig. 1-16. The quantizer is
Figure 1-16. Block diagram of the digital ASPS payload $z_1$ dynamics for the least-upper bound analysis of quantization effects.

```
DIMENSION PRMT(5),Y(2),DERY(2),AVY(8*2)
EXTERNAL FCT,OUTP
COMMON Z,ZDOT,T,AKIMI,AMINV,Y1,TPRT,TEND,PRTIME,V1S,V1H
COMMON AMI,AMI,KP,KP,TCP,H,AV1H,IV1H,IV1H

ALL TIME INPUTS SHOULD BE INTEGRAL MULTIPLES OF TINT.

H=2.**-4
TCP=1.
TEND=4.00,
TPRT=0.5
TINT=0.5*E-3
Z0=0.002
ZDOT0=0.
ERROR=1.**-5
AMI=6.00
AKS=0.35
AKP=37.861
AKR=811.01
AMI*AMI=AKS/AMI
AMINV=1./AMI
I=0.
```

Figure 1-17. Computer program of the simulation of the ASPS payload $z_1$ dynamics with quantization.
PPTIME=-TPPT
PPMT(1)=T
PPMT(2)=TEND
PRMT(3)=TINT
PRMT(4)=EPROR
DERY(1)=0.5
DERY(2)=0.5
NDIM=2
Y(1)=Z0
Y(2)=ZDOT
CALL RKGS(PRMT,Y,DEPY,NDIM,IHLF,FCT,OUTP,AUX)
END
SUBROUTINE FCT(TIME,Y,DERY)
DIMENSION Y(2),DERY(2)
COMMON Z,ZDOT,T,AKSMI,AMINV,V1,TPRT,TEND,PRTIME,V1Z,V1H
COMMON AMI,AKI,AKP,APR,TSP,H,AV1H,IV1H,V1H0
Z=Y(1)
ZDOT=Y(2)
V1=AKP+2+AKR+ZDOT
IF(TIME-T'40,50,50
T=T+TIP
V1S=V1,
40
V1H=V1H;
IF(V1H.LT.0)GO TO 20
AV1H=(V1H-H')+0.5
GO TO 30
20
AV1H=(V1H-H')-0.5
30
CONTINUE
IV1H=IFIX(AV1H)
V1H0=FLOAT(AV1H,H)
ZDOT=-AKSMI+2-AMINV*V1H0
DERY(1)=ZDOT
DERY(2)=ZDOT
RETURN
END
SUBROUTINE OUTP(TIME,Y,DEPY,IHLF,NDIM,PRMT)
DIMENSION Y(2),DERY(2),PRMT(5)
COMMON Z,ZDOT,T,AKSMI,AMINV,V1,TPRT,TEND,PRTIME,V1Z,V1H
COMMON AMI,AKI,AKP,APR,TSP,H,AV1H,IV1H,V1H0
IF((TIME-PRTIME).LT.TPR)RETURN
PRTIME=PRTIME+TPRT
WRITE(5,100)TIME,V1,V1H,V1H0,(V1(I)),I=1,2),IHLF
100
FORMAT(1X,F5.2,1X,E12.5,1X,E12.5,1X,E12.5,1X,E12.5,1X,2(1X,E12.5),1X,I2)
RETURN
END

Figure 1-17. Computer program of the simulation of the ASPS payload z1 dynamics with quantization.
Figure 1-18. Time responses of the digital ASRS payload, x, dynamics with and without quantization. T = 1 second.
Figure 1-19. Time responses of the digital Asps payload 1 dynamics with and without quantization. *t* = 2 seconds.
replaced by the noise input with an amplitude of \( \pm h/2 \).

The \( z \)-transform of the displacement \( z_1 \) due to the noise input is

\[
Z_1(z) = \frac{(1 - z^{-1})}{\Delta(z)} \frac{1}{M_i s^3 (1 + K_s s^{-2}/M_i)} \left( \pm \frac{h/2}{z - 1} \right). \tag{1-49}
\]

where \( \Delta(z) \) is as given in Eq. (1-40).

In Eq. (1-49),

\[
(1 - z^{-1})^3 \frac{1}{M_i s^3 (1 + K_s s^{-2}/M_i)} = (1 - z^{-1})^3 \frac{1}{M_i s (s^2 + K_s/M_i)}
= \frac{1}{K_s} \frac{(z + 1)(1 - \cos(K_s/M_i) T)}{z^2 - 2z \cos(K_s/M_i) T + 1} \tag{1-50}
\]

Thus,

\[
Z_1(z) = \frac{1}{K_s} \frac{(z + 1)(1 - \cos(K_s/M_i) T)}{z^2 - 2z \cos(K_s/M_i) T + 1}
\tag{1-51}
\]

The final steady-state value of \( z_1(kT) \) is given by

\[
\lim_{k \to \infty} z_1(kT) = \lim_{z \to 1} (1 - z^{-1})
= \frac{2}{K_s} \frac{(1 - \cos(K_s/M_i) T)(\pm h/2)}{2(1 + \frac{K_p}{K_s})(1 - \cos(K_s/M_i) T)}
= \frac{\pm h/2}{K_s + K_p} \tag{1-52}
\]

This result shows that the least-upper bound on the steady-state value of \( z_1(kT) \) due to quantization is inversely proportional to \( K_s \) and \( K_p \).

For the given values of \( K_s \) and \( K_p \), we have
Thus, for a quantization level of $2^{-4}$, the final error in $z_1$ is $+0.000817827$ m, whereas it is $+0.000051114$ m for a quantization level of $2^{-8}$.

1.4 Computer Simulation of the ASPS Payload $z_1$ Dynamics with Quantization

In this section the $z_1$ dynamics of the ASPS payload are simulated to study the effects of quantization. The computer program is given in Fig. 1-17.

Figure 1-18 illustrates the time responses of $z_1(t)$ of the ASPS payload with and without quantization, for the sampling period of $T = 1$ second. The initial value of $z_1(t)$ was chosen to be $0.002$ m. As predicted by the discrete describing function analysis, the system does not exhibit any sustained oscillations when $T = 1$ sec. However, the nonzero quantization levels did produce steady-state errors in $z_1(t)$. The computer simulated results and the results obtained by the least-upper bound method are tabulated below for comparison. It is expected that the errors predicted by the least-upper bound method will be greater, since it is a worst-case study.

**Sampling Period $T = 1$ sec**

<table>
<thead>
<tr>
<th>Quantization level $h$ (m)</th>
<th>$z_1(\infty)$ least-upper bound (m)</th>
<th>$z_1(\infty)$ simulation (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-4}$</td>
<td>$\pm 0.0008178$</td>
<td>$-0.000444$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>$\pm 0.000051114$</td>
<td>$0.000036$</td>
</tr>
</tbody>
</table>

Figure 1-18 also shows that with the quantization level of $2^{-8}$ (8 bits), the time response of $z_1(t)$ is very close to that of $z_1(t)$ without quantization, so that a larger word length seems unnecessary unless a smaller steady-state error is required.

Figure 1-19 illustrates the time responses of $z_1(t)$ for $T = 2$ sec. For $h = 2^{-4}$, the error is $-0.0003$ at $t = 50$ sec and still increasing. For $h = 2^{-8}$,
the response actually exhibited a sustained oscillation with a peak-to-peak amplitude of 0.000066 m. As shown in Fig. 1-13, when T = 2 sec, the system is marginal in generating sustained oscillations. It should be noted that the digital computer is not the most suitable for simulating digital systems with quantizers, since the computer itself is a digital system with its own quantization levels. However, the results obtained here are conclusive enough to indicate the quantization effects in the ASPS payload, and are useful in the selection of the sampling period and the quantization level.

For sampling periods greater than 2 seconds, the computer simulation results showed that sustained oscillations always existed.

1.5 Nonlinear Spring Effect of Wire Cable on the ASPS Payload $z_1$ Dynamics

In the preceding sections the wire cable attached to the ASPS payload was modelled as a linear spring with a spring constant of $K_s$. In the $z_1$ direction the spring constant $K_s$ is given as 0.35 N/m. It was shown in the last section that since the mass of the payload is so large (600 Kg), the light spring effect of the wire cable will not have a significant effect on the dynamics of the payload. This would be true whether the wire cable is modelled as a linear spring or a nonlinear spring.

The block diagram of the digital ASPS payload $z_1$ dynamics with the nonlinear wire cable spring characteristic is shown in Fig. 1-20. As shown in Fig. 1-20, the nonlinear spring characteristic is defined by the parameters $K_s$ and $h_{wf}$.

The simulation program for the system is given in Fig. 1-21. Figures 1-22 through 1-24 show the time responses of $z_1$ with the initial condition $z_1(0) = 0.002$ m and under the following three conditions:

(a) linear spring, $K_s = 0.35$ N/m
Figure 1-20. Block diagram of the digital ASPS payload $z_1$ dynamics with nonlinear wire cable characteristic.

(b) nonlinear spring, $K_s = 0.35\text{ N/m}$, $h_{wf} = 0.00014\text{ N}$

(c) nonlinear spring, $K_s = 0.35\text{ N/m}$, $h_{wf} = 0.00028\text{ N}$.

Figure 1-22 shows that for the scale chosen for $z_1$ in that figure, the time responses of all the three cases are nearly identical. Figure 1-23 shows the three responses between $t = 10$ sec and $t = 100$ sec with the $z_1$ scale expanded. Figure 1-24 shows the three responses between $t = 25$ sec and $t = 100$ sec with an even more expanded scale for $z_1$. Note that the use of the nonlinear spring model for the wire cable creates a nonzero steady-state.
Figure 1-21. Computer simulation program of the ASPS payload $z_1$ dynamics with nonlinear wire cable characteristic.
Figure 1-22. Time responses of the digital ASPS payload $z_1$ dynamics with nonlinear wire cable characteristic.
Figure 1-23. Time responses of the digital ASPS payload $z_1$ dynamics with nonlinear wire cable characteristic.
Figure 1-2. Time responses of the digital ASPS payload $z_1$ dynamics with nonlinear wire cable characteristic.
error for $z_1$. With $h_{wf} = 0.00014$ N the steady-state error is $0.18 \times 10^{-5}$ m, and
with $h_{wf} = 0.00028$ N the steady-state error is $0.12 \times 10^{-5}$ m. Since these
errors are extremely small, it may be concluded again that the wire cable
effect on the ASPS payload may be neglected whether a linear or a nonlinear
spring model is used.
2.1 Introduction

In the preceding chapter the equations of motion of the ASPS with respect to the planar models of Figs. 1-4 and 1-5 have been derived. The Lagrange equations of the planar ASPS model 2 shown in Fig. 1-5 have been selected for analysis and design of the system. These equations are repeated as follows:

\[
\begin{bmatrix}
    M_i & 0 & -M_i r_b & 0 \\
    0 & M_i & 0 & 0 \\
    -M_i r_b & 0 & J_i + J_i + M_i r_a^2 + M_i r_b^2 & J_i \\
    0 & 0 & J_i & J_i
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{z}_1 \\
    \phi_1 \\
    \phi_2
\end{bmatrix} =
\begin{bmatrix}
    f_1 - f_4 + f_{sx}(x_1) \\
    f_2 + f_3 + f_{sz}(z_1) \\
    T_c(\phi_1) + T_p(\phi_1) \\
    (f_3 - f_2) L + (f_1 - f_4) L_b + T_s(\phi_2)
\end{bmatrix}
\]

The last equation shows that the \( z_1 \) dynamics of the payload are decoupled from the other three degrees of freedom. The \( z_1 \) dynamics have been thoroughly studied in Chapter 1, and both the analog and the digital controllers have been designed. The main objective of this chapter is to study the dynamics related to the \( x_1, \phi_1 \) and \( \phi_2 \) axes which are coupled.

In the ensuing sections an analog controller for the control of the \( x_1, \phi_1 \) and \( \phi_2 \) dynamics is designed by pole-placement techniques.

2.2 Analytical Model of the \( x_1, \phi_1 \) and \( \phi_2 \) Axes

After eliminating the \( z_1 \) dynamics, Eq. (2-1) is reduced to the following form:

\[
\begin{align*}
\dot{x}_1 &= a\phi_1 + b((f_1 - f_4) + f_{sx}(x_1)) \\
\dot{\phi}_1 &= c\ddot{x}_1 - d\phi_2 + e(T_c(\phi_1) + T_p(\phi_1)) \\
\ddot{\phi}_2 &= -\ddot{\phi}_1 + f((f_3 - f_2)L + (f_1 - f_4)L_b + T_s(\phi_2))
\end{align*}
\]

where
In Eq. (2-2) \( f_{sx}(x_1) \) denotes the linear spring force due to the wire cable attached to the payload.

For linear analysis, \( f_{sx}(x_1) \) is assumed to be directly proportional to \( x_1 \); i.e.,

\[
f_{sx}(x_1) = -K_s x_1
\]

where \( K_s = 1.051 \text{ N/m} \).

Similarly, \( T_s(\phi_2) \) denotes the equivalent spring torque due to the wire cable attached to the payload, and \( T_p(\phi_1) \) denotes the spring torque due to the wire cable attached to the pallet. Thus,

\[
T_s(\phi_2) = -K_{s2} \phi_2
\]
\[
T_p(\phi_1) = -K_{s1} \phi_1
\]

where \( K_{s1} = K_{s2} = 0.005 \text{ N/m} \).

Figure 2-1 shows a block diagram of the ASPS dynamics of the \( x_1 \), \( \phi_1 \) and \( \phi_2 \) axes as modelled by Eq. (2-2). Note that the linear system is of the sixth order with three inputs. The input \( e_T \) denotes the torque which is applied to the pallet by the gimbal assembly. The input \( f_1 - f_4 \) represents the net magnetic forces applied at the base of the payload by the magnetic actuator assembly, and the input \( f_L(f_3 - f_2) \) denotes similar magnetic forces due to the
Figure 2-1. Block diagram of the $x_1$, $\phi_1$, $\phi_2$ dynamics of the ASPS.
magnetic actuator assembly.

2.3 Design of An Analog Controller For the ASPS By Pole Placement

In this section we shall design an analog controller for the ASPS to control the dynamics of the $x_1$, $\phi_1$, and $\phi_2$ axes. As shown in Fig. 2-1 there are three inputs that can be used through the magnetic actuator assembly forces and the gimbal torque to control the attitudes of $x_1$, $\phi_1$ and $\phi_2$.

Let us assume that the controls of the $x_1$, $\phi_1$ and $\phi_2$ dynamics of the ASPS can be achieved through state feedback. Specifically, we construct the three inputs as follows:

\[ f_1 - f_4 = -K_p x_1 - K_r \dot{x}_1 \]  
\[ T_c = -K_1 \int \dot{\phi}_1 dt - K_2 \dot{\phi}_2 - K_3 \dot{\phi}_1 \]  
\[ (f_3 - f_2) L + (f_1 - f_4) L_b = -K_4 \int \dot{\phi}_2 dt - K_5 \dot{\phi}_2 - K_6 \dot{\phi}_2 \]

where $K_p$, $K_r$, $K_1$, $K_2$, $K_3$, $K_4$, $K_5$ and $K_6$ are constant gains whose values are to be determined.

The block diagram of the ASPS with state-variable feedbacks as defined in Eqs. (2-6), (2-7) and (2-8) is shown in Fig. 2-2.

In order to evaluate the characteristic equation of the closed-loop ASPS we represent the block diagram of Fig. 2-2 by the simplified signal flow graph in Fig. 2-3.

The branch gains $a$, $c$ and $d$ in Fig. 2-3 have been defined previously.

The loop gains used in Fig. 2-3 are defined as

\[ \Delta_1 = -b(K_r s^{-1} + (K_p + K_s) s^{-2}) \]  
\[ \Delta_2 = -e(K_3 s^{-1} + (K_2 + K_s) s^{-2} + K_1 s^{-3}) \]  
\[ \Delta_3 = -f(K_6 s^{-1} + (K_5 + K_{s2}) s^{-2} + K_4 s^{-3}) \]
Figure 2-2. Block diagram of the ASPS dynamics with state-variable feedback controls.

Figure 2-3. A simplified signal flow graph of the system shown in Figure 2-2.
The design objective is to determine the feedback gain constants $K_p$, $K_r$, $K_1$, $K_2$, $K_3$, $K_4$, $K_5$ and $K_6$ such that the eigenvalues of the closed-loop ASPS are at certain specified locations.

The first step of the design problem is concerned with the determination of the characteristic equation of the closed-loop system. This is accomplished by applying the Mason's gain formula to the signal flow graph in Fig. 2-3. The result is

$$\Delta = 1 - \Delta_1 - \Delta_2 - \Delta_3 - c\alpha - d + \Delta_1(\Delta_2 + \Delta_3 + d) + ac\Delta_3 + \Delta_2\Delta_3 - \Delta_1\Delta_2\Delta_3 = 0 \quad (2-12)$$

Substitution of the relations for $\Delta_1$, $\Delta_2$ and $\Delta_3$ into the last equation, and rearranging, we get

$$a_8 s^8 + a_7 s^7 + a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \quad (2-13)$$

where

$$a_8 = 1 - ac - d$$

$$a_7 = b(l - d)K_r + eK_3 + f(l - ac)K_6$$

$$a_6 = b(l - d)K_p + b(l - d)K_s + e(K_2 + K_s) + f(K_5 + K_s) + beKrK_3 + efK_3K_6$$

$$+ bfK_rK_6 - acf(K_2 + K_s)$$

$$a_5 = eK_1 + f(l - ac)K_4 + beK_3(K_p + K_s) + beK_r(K_2 + K_s) + efK_6(K_2 + K_s)$$

$$+ efK_3(K_2 + K_s) + bfK_6(K_p + K_s) + bfK_r(K_5 + K_s) + befK_3K_rK_6$$

$$a_4 = be(K_p + K_s)(K_2 + K_s) + K_1K_r + eK_1K_6 + ef(K_2 + K_s)(K_2 + K_s)$$

$$+ efK_3 + bf(K_p + K_s)(K_5 + K_s) + bfK_rK_4 + bef(K_p + K_s)(K_5 + K_s) + K_rK_3(K_5 + K_s)$$

$$+ K_rK_3(K_5 + K_s)$$
\[ a_3 = b e K_1 (K_p + K_s) + e f K_1 (K_5 + K_{s2}) + e f K_4 (K_2 + K_{s1}) + b f K_4 (K_p + K_s) + k r K_6 (K_2 + K_{s1}) + k r K_3 K_4 (K_p + K_s) K_6 (K_2 + K_{s1}) + (K_p + K_s) K_3 (K_s + K_{s2}) \]

\[ a_2 = e f K_1 K_4 + b e f ((K_p + K_s) K_1 K_6 + (K_p + K_s) (K_2 + K_{s1}) (K_5 + K_{s2}) + (K_p + K_s) K_3 K_4 + K_r K_1 (K_5 + K_{s2}) + K_r K_4 (K_2 + K_{s1})) \]

\[ a_1 = b e f (K_1 K_4 K_r + K_4 (K_p + K_s) (K_2 + K_{s1}) + K_1 (K_p + K_s) (K_5 + K_{s2})) \]

\[ a_0 = b e f K_1 K_4 (K_p + K_s) \]

In these nine coefficients, the values of \( a, b, c, d, e, f, K_{s1}, K_{s2} \) and \( K_s \) are known, and the values of \( K_p, K_r, K_1, K_2, K_3, K_4, K_5 \) and \( K_6 \) are to be determined for a given set of roots of Eq. (2-13).

Let the desired roots of Eq. (2-13) be \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \) and \( \lambda_8 \).

The condition for these roots to be solutions of Eq. (2-13) is

\[ a_8 \prod_{i=1}^{8} (s - \lambda_i) = \sum_{i=0}^{8} a_i s^i \]  

(2-14)

Expanding the left-hand side of Eq. (2-14), we have the equation,

\[ a_8 s^8 + g_7(\lambda) s^7 + g_6(\lambda) s^6 + g_5(\lambda) s^5 + g_4(\lambda) s^4 + g_3(\lambda) s^3 + g_2(\lambda) s^2 + g_1(\lambda) s^1 + g_0(\lambda) = \sum_{i=0}^{8} a_i s^i \]  

(2-15)

where

\[ \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \]

denotes the vector of eigenvalues, and \( g_i(\lambda) \) \( (i = 0, 1, 2, \ldots, 7) \) are known once \( \lambda \) is given. In fact, it is simple to see that

\[ a_i (K_p, K_r, K_1, K_2, K_3, K_4, K_5, K_6) = g_i(\lambda) \]  

(2-16)

for \( i = 0, 1, 2, \ldots, 7. \)
Define
\[ u_1 = f_1 - f_4 \]  
(2-17)
\[ u_2 = T_c \]  
(2-18)
\[ u_3 = (f_3 - f_2)L + (f_1 - f_4)L_6 \]  
(2-19)

Equations (2-17), (2-18) and (2-19) specify the three inputs to the ASPS which is an eighth-order system with the state equation

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{21} & 0 & 0 & a_{24} & 0 & a_{27} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
a_{51} & 0 & 0 & a_{54} & 0 & 0 & a_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{81} & 0 & 0 & a_{84} & 0 & 0 & a_{87} & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
b_{21} \\
b_{22} \\
b_{23} \\
b_{51} \\
b_{52} \\
b_{53} \\
b_{81} \\
b_{82} \\
b_{83} \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{bmatrix} \tag{2-20}
\]

where
\[ x_1 = x \]
\[ x_2 = \dot{x} \]
\[ x_3 = \int \phi_1 dt \]
\[ x_4 = \phi_1 \]
\[ x_5 = \dot{\phi}_1 \]
\[ x_6 = \int \phi_2 dt \]
\[ x_7 = \phi_2 \]
\[ x_8 = \dot{\phi}_2 \]

and the entries of the matrices of the state equation are determined by the open-loop ASPS. The numerical values of these elements are:

\[ a_{21} = -0.6120962971 \]
\begin{align*}
a_{24} &= -0.001484271 \\
a_{27} &= 0.001484271 \\
a_{51} &= -0.312026593 \\
a_{54} &= -0.0007589096 \\
a_{57} &= 0.0007589096 \\
a_{81} &= 0.312026593 \\
a_{84} &= 0.0007589096 \\
a_{87} &= -0.0007688499 \\
b_{21} &= 0.5823745791 \\
b_{22} &= 0.2968854358 \\
b_{23} &= -0.2968854358 \\
b_{51} &= 0.2968854358 \\
b_{52} &= 0.1517819201 \\
b_{53} &= -0.1517819201 \\
b_{81} &= -0.2968854358 \\
b_{82} &= -0.1517819201 \\
b_{83} &= 0.1703268144
\end{align*}

Equation (2-20) describes the system part of the block diagram of Fig. 2-4. In order to use the state feedback to realize a controller, it is necessary that the system is completely controllable. The controllability of the ASPS is determined as follows:

Let Eq. (2-20) be written as
\[ \dot{x}(t) = Ax(t) + Bu(t) \]

The necessary and sufficient conditions for the system to be controllable are that the matrix
\[ U = \begin{bmatrix} B & AB & A^2B & \ldots & A^7B \end{bmatrix} \] (2-21)
Figure 2-4. Block diagram of the ASPS dynamics and the design of the analog controller.
be of rank eight. In the matrix $B$, as shown in Eq. (2-20), there are three non-zero rows. The determinant of these three rows is

$$\text{det} \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{51} & b_{52} & b_{53} \\ b_{81} & b_{82} & b_{83} \end{bmatrix} = 0.0000046913 \neq 0 \quad (2-22)$$

Thus, we have

$$\text{Rank}(B) = 3 \quad (2-23)$$

Also, the matrix

$$AB = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{81} & b_{82} & b_{83} \\ 0 & 0 & 0 \end{bmatrix} \quad (2-24)$$

has three independent columns. These columns are linearly independent on the columns of $B$. Therefore,

$$\text{Rank}(AB) = 3$$

and

$$\text{Rank}(B \ AB) = 6$$

Furthermore, in the matrix $A^2B$, we have
If the rank of $A^2B$ is greater than or equal to two, the matrix $U$ in Eq. (2-21) will have full rank. Consider the two following rows of $AB$:

\[
\begin{bmatrix}
  b_{51} \\ b_{52} \\ b_{53}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  b_{81} \\ b_{82} \\ b_{83}
\end{bmatrix}
\]

It is obvious that these are linearly independent. Therefore,

\[
\text{Rank}(A^2B) \geq 2
\]

Also notice that the third and sixth rows of the matrices $B$ and $AB$ have all zero elements. Therefore, there are at least two linearly independent columns in matrix $A^2B$ which is also linearly independent of the columns of the matrix $(B \ AB)$. Thus, we have the conclusion that

\[
\text{Rank}(U) = 8
\]

and the full rank of matrix $U$ implies the complete controllability of the ASPS.

With the controllability of the system established, it is now possible to select the eigenvalues of the closed-loop ASPS by using state feedback described in Eqs. (2-6), (2-7) and (2-8); i.e.,
\[ u_1 = -K_p \dot{x}_1 - K_r \dot{x}_1 \]
\[ u_2 = -K_1 \int \phi_1 \, dt - K_2 \phi_2 - K_3 \dot{\phi}_1 \]
\[ u_3 = -K_4 \int \phi_2 \, dt - K_5 \phi_2 - K_6 \dot{\phi}_2 \]

The closed-loop control system can be described by

\[ \dot{x}(t) = (A + BG)x(t) = \hat{A}x(t) \]

where the matrices \( A \) and \( B \) are as in Eq. (2-20), and \( G \) is the feedback gain matrix and is given by

\[
G = \begin{bmatrix}
-K_p & -K_r & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -K_1 & -K_2 & -K_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -K_4 & -K_5 & -K_6
\end{bmatrix}
\]

It is the objective of the design problem to choose a set of feedback gains such that the roots of the characteristic equation are as specified. The roots must satisfy the following equation:

\[ \det(sI - \hat{A}) = 0 \]

If a closed-loop system is to be designed such that it has a bandwidth of one Hertz, then the eigenvalues are chosen as

\[ \lambda_1, \lambda_2 = -7.16 \pm j8.75 \]
\[ \lambda_3, \lambda_4 = -42.16 \pm j72.67 \]
\[ \lambda_5 = -11.41 \]
\[ \lambda_6 = -109.15 \]
\[ \lambda_7 = -24.21 \]
\[ \lambda_8 = -6.54 \]

In this case, the eigenvalues must satisfy the following equation:
\[
\det(sI - \hat{A}) = \prod_{i=1}^{8} (s - \lambda_i)
\]  
(2-30)

where \(\lambda_i\) (\(i = 1, 2, \ldots, 8\)) are specified in Eq. (2-29).

After expansion of both sides of Eq. (2-30), the coefficients corresponding to the same powers of \(s\) must be equal. Thus, eight nonlinear algebraic equations are formed which must be solved to give the eight unknown parameters in the feedback matrix \(G\) in Eq. (2-28).

Given the eigenvalues specified in Eq. (2-29), the coefficients of the right-hand side of Eq. (2-20) can be computed. We have

\[
8 \prod_{i=1}^{8} (s - \lambda_i) = s^8 + 2.49936 \times 10^2 s^7 + 2.84334 \times 10^4 s^6 + 1.9438 \times 10^6 s^5 \\
+ 6.6624 \times 10^7 s^4 + 1.2103 \times 10^9 s^3 + 1.2699 \times 10^{10} s^2 + 7.3946 \times 10^{10} s \\
+ 1.78289 \times 10^{11}
\]  
(2-31)

After algebraic manipulation, we have

\[
\det(sI - \hat{A}) = s^8 + \frac{a_7}{a_8} s^7 + \frac{a_6}{a_8} s^6 + \frac{a_5}{a_8} s^5 + \frac{a_4}{a_8} s^4 + \frac{a_3}{a_8} s^3 + \frac{a_2}{a_8} s^2 + \frac{a_1}{a_8} s + \frac{a_0}{a_8}
\]

where \(a_0, a_1, \ldots, a_8\) are defined in Eq. (2-13).

Therefore, a system of equations is obtained,

\[
\begin{align*}
\frac{a_7}{a_8} &= 2.49936 \times 10^2 \times 0.2349 \times 10^{-2} = 0.5871 \\
\frac{a_6}{a_8} &= 2.84334 \times 10^4 \times 0.2349 \times 10^{-2} = 0.6679 \times 10^2 \\
\frac{a_5}{a_8} &= 1.9438 \times 10^6 \times 0.2349 \times 10^{-2} = 0.4566 \times 10^4 \\
\frac{a_4}{a_8} &= 6.6624 \times 10^7 \times 0.2349 \times 10^{-2} = 0.1565 \times 10^6 \\
\frac{a_3}{a_8} &= 1.2103 \times 10^9 \times 0.2349 \times 10^{-2} = 0.2843 \times 10^7 \\
\frac{a_2}{a_8} &= 1.2699 \times 10^{10} \times 0.2349 \times 10^{-2} = 0.2983 \times 10^8 \\
\frac{a_1}{a_8} &= 7.3946 \times 10^{10} \times 0.2349 \times 10^{-2} = 0.1737 \times 10^9 \\
\frac{a_0}{a_8} &= 1.7829 \times 10^{11} \times 0.2349 \times 10^{-2} = 0.4188 \times 10^9
\end{align*}
\]
A computer program was written to solve the set of equations involving \( a_0, a_1, \ldots, a_7 \), using the Brown's method. The feedback gains are solved and are tabulated as follows:

\[
\begin{align*}
K_p &= 91338 \\
K_r &= 10081 \\
K_1 &= 6460200 \\
K_2 &= 569930 \\
K_3 &= -48943 \\
K_4 &= 600900 \\
K_5 &= 129920 \\
K_6 &= 11756
\end{align*}
\]

The computer program is tabulated in Table 2-1.
Table 2-1. Computer Program For the Computation of The Feedback Gains of the Closed-Loop ASPS By Means of the Brown's Method

```fortran
IMPLICIT REAL*(A-H,O-Z)
DIMENSION ROOTS(2,50)
COMMON AKSX, AKSF, AKSS, RB, AMI, A33, A, B, C, D, E, F, J, I, J
TYPE 5
FORMAT(2X, 'THE DEGREE OF P(X) ??? ', I)
ACCEPT 6, N
FORMAT(I2)
TYPE 7

FORMAT(2X, 'SPECIFY THE ROOTS OF P(X) IN THE FOLLOWING: ' / 2X, 50)
** ALL COMPLEX ROOTS MUST BE IN COMPLEX CONJUGATE PAIRS ' ' AND THE ' / 2X, 50 ONE WITH NEGATIVE IMAGINARY PART SHOULD BE ',
** SPECIFIED FIRST. **/'
CONTINUE
CONTINUE

FORMAT(2X, 'REAL (ROOT) ', I2, ' = ', F)
ACCEPT 9, ROOTS(1, I)
TYPE 3, I

FORMAT(2X, 'IMAG (ROOT) ', I2, ' = ', F)
ACCEPT 9, ROOTS(2, I)
CONTINUE

FORMAT(2X, 'LEADING COEFFICIENT = ', F)
ACCEPT 9, COE1
FORMAT(2X, F12.5)
CALL RECON(ROOTS, COE1, W, N)

FORMAT(2X, 'COEFFICIENTS OF P(X) STARTING FROM THE MTH ASK ',
I2, ' ON.')
TYPE 4, (W(I, I=1, N+1))

FORMAT(2X, F12.5)

C-----SYSTEM CONSTANTS
C
AKSX=1.051
AKSF=1.005
AKSS=0.005
RB=1.975
AMI=600.
A33=2805.15
AJ=503.
A=RB
B=1./AMI
C=AMI*RB/A33
D=AJ/A33
E=1./A33
F=1./AJ

CALL SOLVR(N)
STOP
END
```

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Table 2-1. (Continued)

SUBROUTINE RECON (ROOTS, 1/N, D(N))
DIMENSION ROOTS(2, 30), D(51)

I = 1, N
U(I) = 0, 0
1 CONTINUE
D(N-I) = 1, 0
NS = N-I
DO 10 J = 1, N
AC = NS - I
IF (ROOTS(2, 1)) 3, 7, 10
3 T = ROOTS(1, I) ** 2 + ROOTS(2, I) ** 2
U = 2, 0 ** ROOTS(1, I)
DO 5 J = NL, N
D(J-I) = T ** 0(J-I) - U ** 0(J) + D(J+1)
5 CONTINUE
D(N) = T ** U(N) - U ** U(N+1)
GO TO 9
7 T = -ROOTS(1, I)
DO 8 J = NL, N
U(J) = T ** 0(J) + D(J+1)
8 CONTINUE
9 D(N+1) = T ** U(N+1)
10 CONTINUE
II = 1, NS
U(II) = 0, 1 ** A
11 CONTINUE
RETURN
END

SUBROUTINE SOLVER(I)
REAL*8 X(50), DUM2(50, 53)
INTEGER*4 DUM1(50, 51), CNAME(50)

C

C-----INITIAL GUESSES OF THE SOLUTION
C
X(1) = 0.105
X(2) = 0.912105
X(3) = -0.48905
X(4) = 0.5716
X(5) = 0.064607
X(6) = 0.117505
X(7) = 0.0306
X(8) = 0.506

C

C-----SPECIFY NAMES FOR EACH COEFFICIENTS FOR PRINTOUT
C
CNAME(1) = 'G'
CNAME(2) = 'H'
CNAME(3) = 'I'
CNAME(4) = 'J'
CNAME(5) = 'K'
CNAME(6) = 'L'
CNAME(7) = 'M'
CNAME(8) = 'N'
Table 2-1. (Continued)

C OTHER PARAMETERS TO BE SPECIFIED

NSI=12 ! THE NUMBER OF SIGNIFICANT ULLIUSUES DESIGN
IP=0 ! PHN IEP NU. ANI OPTIMIZED APPROXIMATION IF IP=1
NIT=500 ! MAXIMUM NUMBER OF ITERATIONS TO BE USED

C

10000 FORMAT(/' **A JACOBIAN RELATED MATRIX WAS SINGULAR---'/
  ' INDICATIVE OF THE PROCESS 'BLOWNG-UP'' **')!
WRITE(5,10000) NIT

10001 FORMAT(' **NUMBER OF ITERATIONS = ',I3,**)
WRITE(5,10001) NIT

10004 FORMAT(' **FEEDBACK COEFFICIENTS; /')
WRITE(5,10004)

10002 FORMAT(10X,A3,=*',D12.5)
RETURN
END

SUBROUTINE VTHF(X,Y,K)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(1),ARSR(I,N,1)
COMMON AXS,AKSP,AKSG,R0,NI,NA33,AJ,N,B,C,Y+E,F,DD(51)

G=X(1)
H=X(2)
P=X(3)
Q=X(4)
R=X(5)
W=X(6)
1=X(7)
0=X(8)
G01=-B0*O
G02=-B*(H+AKSP)
G03=-AK
G04=-E*P
G05=-E*(G+AKSP)
G06=-E*G
G07=O
G08=-F*W
G09=-F*(T+AKSS)
G10=-F*U

C---THE VECTOR ARSR(I,1) CONTAINS THE COEFFICIENTS OF THE N-TH
C---DEGREE POLYNOMIAL ORDERED FROM POWER N TO POWER 0 OF X ---
C---ARSR(1,1) IS THE COEFFICIENT OF X**N, ARSR(2,1) IS THE COEFF-
C---ICIENT OF X**(N-1), ETC.
ARSR(1,1)=G01-G02-G03-G04+G05-G06+G07-G08-G09+G10
GO TO (1,2,3,5,6,7,8,9,10)

1 ARSR(2,1)=-G01-G04-G07+G08+G09
Y=ARSR(2,1)/ARSR(1,1)
RETURN

2 ARSR(3,1)=G02-G05-G08+G01+G02+G03
Y=ARSR(3,1)/ARSR(1,1)
RETURN

3 ARSR(4,1)=G03-G06+G01-G02+G03+G04+G05
Y=ARSR(4,1)/ARSR(1,1)
RETURN

4 ARSR(5,1)=G04-G07+G08-G01+G02+G03+G04+G05
Y=ARSR(5,1)/ARSR(1,1)
RETURN
Table 2-1. (Continued)

Y=ARSR(6,1)/ARSR(1,1)-DD(6)/DD(1)
RETURN

-61*G5*G10
Y=ARSR(7,1)/ARSR(1,1)-DD(7)/DD(1)
RETURN

ARSR(8,1)=62*G6*G9-62*G5*G6+61*G6*G9+G2*G6*G8
Y=ARSR(8,1)/ARSR(1,1)-DD(8)/DD(1)
RETURN

C=G(1)
RETURN

END

SUBROUTINE BROWNZ(N,NIT,NSIG1,ING,X,INTER,COE,IP,L)
REAL X1,COE(L,1),RELH,H,F,FPLUS,DMAX,TEST,CR
DIMENSION INTER(L,1)

C*** THE PARAMETERS NIT,NSIG1,ING,INTER ARE MAXIT,NNSIG1,SINGULAR,POINT-
C*** ER OF THE ALGORITHM 316 RESPECTIVELY AND INTER(1,N+1),COE(I,N+2),
C*** COE(I,N+3) ARE ITSUB(1),I,CHP(I) RESPECTIVELY.
C
NVEOR=1
ING=1
RELH=10.**(-NSIG1)
C1=0.00
C2=1.00
DO 91 M=1,NIT
DO 10 J=1,N
INTER(I,J)=J
DO 50 K=1,N
IF(K.GT.1) CALL BKSUBT(K,N,X,COE,INTER,L)
CALL VHT1F(X,F,K)
FTR=0.0001

10 LLY=0
DO 20 I=K,N
ITEMP=INTER(K,I)
HOLD=X(ITEMP)

H=FTR*HOLD
IF(H.LT.0.00) H=0.0001
X(ITEMP)=HOLD+H
IF(K.GT.1) CALL BKSUBT(K,N,X,COE,INTER,L)
CALL VHT1F(X,FPLUS,K)
COE(ITEMP,N+2)=(FPLUS-F)/H
X(ITEMP)=HOLD
C3=ABS(COE(ITEMP,N+2))
IF(C3.EQ.C1) GO TO 19
C4=ABS(F/C3)
IF(C4.LT.C2) GO TO 20

19 LLY=LLY+1
20 CONTINUE
IF(LLY.LE.(N-K)) GO TO 21
FTR=FTR*10
IF(FTR.GT.0.5) GO TO 100
GO TO 11

21 IF(K.LT.N) GO TO 22
IF(C3.EQ.C1) GO TO 100
COE(K,N+1)=0.00
KMAX=ITEMP
GO TO 41

KMAX=INTER(K,K)
DMAX=ABS(COE(KMAX,N+2))
KPLUS=K+1
Table 2-1. (Continued)

DO 30 I=KPLUS,N
JSUB=INTER(K,I)
TEST=DABS(COE(JSUB,N+2))
IF(TEST.LT.DMAX) GO TO 23
DMIN=TEST
INTER(KPLUS,I)=KMAX
KMAX=JSUB
GO TO 30
30 CONTINUE
IF(DABS(COE(KMAX,N+2)),EQ,C1) GO TO 10
INTER(K,N+1)=KMAX
COE(K,N+1)=0,00
DO 40 J=KPLUS,N
JSUB=INTER(KPLUS,J)
COE(K,JSUB)=-COE(JSUB,N+2)/COE(KMAX,N+2)
40 COE(K,N+1)=COE(K,N+1)+COE(JSUB,N+2)*X(JSUB)
41 COE(K,N+1)=(COE(K,N+1)-F)/COE(KMAX,N+2)+X(KMAX)
50 CONTINUE
X(KMAX)=COE(N,N+1)
IF(N.GT.1) CALL BKSUBT(N,N,X,COE,INTER,L)
IF(M.EQ.1) GO TO 80
DO 40 I=1,N
IF(X(I),EQ,C1) GO TO 58
CR=DABS((COE(I,N+3)-X(I))/X(I))
GO TO 59
50 CR=DABS(COE(I,N+3))
51 CR=CR/RELC
60 CONTINUE
NV=NV+1
IF(NV.GE.3) GO TO 111
GO TO 80
80 DO 90 I=1,N
90 COE(I,N+3)=X(I)
IF(IF.EQ.0) GO TO 91
WRITE(6,200) M,(X(I),I=1,N)
90 FORMAT(110,5D20,10/10X,5D20,10)
91 CONTINUE
RETURN
100 ING=0
RETURN
111 NI=NI-1
RETURN
END

SUBROUTINE BKSUBT(K,N,X,COE,INTER,L)
REAL X(I),COE(L,I)
DIMENSION INTER(L)
KN=K-1
DO 20 I=1,KN
KM=K-I+1
KMAX=INTER(KM-1,N+1)
X(KMAX)=0,00
DO 10 J=KM,N
JSUB=INTER(KM,J)
10 X(KMAX)=X(KMAX)+COE(KM-1,JSUB)*X(JSUB)
20 X(KMAX)=X(KMAX)+COE(KM-1,J+1)
RETURN
END
2.4 Time Responses of the Analog ASPS Designed by Pole Placement

The time responses of the closed-loop ASPS with the analog controller designed by the pole-placement design conducted in the last section are studied by computer simulation.

The listings of the computer simulation program are given in Table 2-2.

The initial values of \( x_1(t), \phi_1(t), \) and \( \phi_2(t) \) are:

\[
\begin{align*}
\phi_1(0) &= 0.013 \\
\phi_2(0) &= 0.013
\end{align*}
\]

It should be noted that no nonlinear characteristics are considered in the system model and the simulations.

Figure 2-5 shows the time response of \( x_1(t) \) with the indicated initial condition. As indicated, the time response of \( x_1(t) \) decays to zero very rapidly, and the vertical axis of Fig. 2-5 is in a logarithmic scale.

Figure 2-6 shows the time response of \( \phi_1(t) \) with the given initial condition. Again, the response decays to zero so rapidly over the 4-second interval that the vertical axis is in logarithmic.

The response of \( \phi_2(t) \) is the fastest of the three, and a time response plot could not be made realistically over a 4-second time interval. Thus, the values of the response of \( \phi_2(t) \) are tabulated in Table 2-3.
Table 2-2. Computer Program For the Simulation of the Analog ASPS with Controller Designed by Pole Placement

```
DIMENSION PRINT(5),Y(6),DERY(4)-H*J(18+2)
EXTERNAL FCY,OUTP
COMMON X,AMI,PHI1,PHI2,PHI3,PHI2IN,PHI3IN,T
& Y(1:12),FCY,PRINT
!
& INY(0:80,FCU),FCF,OUTP
& PLINE
!
NPRT=1
UNIT=6,DEVICE=OSK',FILE='FOR2',INT='AMOS',FRMT='SCQ\OUT')
!
LINE=0
H=0.
TEND=100.
TPRT=0.25
TINT=0.05E-3

C-----SET INITIAL CONDITIONS
!
& X=2.0E-3
& XINT=0.
& PHI1=0.013
& PHI1IN=0.
& PHI2=0.013
& PHI2IN=0.
& PHI3=0.
& PHI3IN=0.
& AKSX=1.051
& AKSP=0.005
& AKSS=0.005
& A=503.
& A33=2805.16
& Ru=1.956
& A0=RB
& B0=1./A0
& CO=AH*RB/A33
& D0=AJ/A33
& E=1./H33
& F=1./AJ
& DELTA=1.-A0*CO-D0
& T=0.
& NPRT=0
& PRMT(3)=TINT
& PRMT(4)=0.05*X
& DO 99 I=1,8:
& DERY(I)=0.125
& NDIM=8
&
& PRMT(1)=T
& PRMT(2)=TEND
& Y(1)=X
& Y(2)=XIOT
& Y(3)=PHI1IN
& Y(4)=PHI1
& Y(5)=PHI1OT
& Y(6)=PHI2IN
& Y(7)=PHI2
& Y(8)=PHI2OT
&
CALL RKGS(PRMT,Y,DERY,NDIM,IFLT,OUTP,AUX)
STOP
END
```
Table 2-2 (continued).

SUBROUTINE FLT(X, Y, DFRY).
DIMENSION Y(8), DERY(8).
COMMON X, XOUT, PHI1, PHI10, PHI20, PHI21, PHI2T, PHI220, T.
V1, V2, V3, A1, AMI, AKS, AKSP, AKSS - 1,3, TFR, B3, DELTY.
A0, A0, CO, DO, E, F, TEND.
NPR = IV, IV, LV, V3, LINE.

V1=0.2, V2=0.3, V3=0.5, Y(1)=0, Y(2)=1, Y(3)=0.
Y(4)=0.566520365*Y(4)+0.489305383*Y(5).
Y(6)=0.466305365*Y(6)+0.129920634*Y(7).
Y(8)=0.117560549*Y(8).

DERY(1)=Y(2).
DERY(2)=(-V1*A0-AKSS*A03*Y(1)+Y(1)-DO)*A0*E*Y(1).

DERY(3)=Y(4).
DERY(4)=Y(5).

DERY(5)=((-V2-AKSP*Y(4))*E-(V1*AKSS*Y(1)+DO*C0)/DELTA).

DERY(6)=Y(7).
DERY(7)=Y(8).

DERY(8)=(V3-AKSS*Y(7))%F*DO+LAC0+V2*AKSP*Y(4)*E.

RETURN.
END.

SUBROUTINE OUTP(XX, Y, DERY, IHLF, NPRT, PRMT).
DIMENSION Y(8), DERY(8), PRMT(5).
COMMON X, XOUT, PHI1, PHI10, PHI20, PHI21, PHI2T, PHI220, T.
V1, V2, V3, A1, AMI, AKS, AKSP, AKSS - A33, TFR, B3, DELTY.
NPR = IV, IV, LV, V3, LINE.

IF(XA, X, TEND) PRMT(5) = -1.
T=XX.

RETURN.
END.
Table 2-3. Time Response of $\phi_2$ of The ASPS With The Analog Controller Designed by Pole Placement

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>$\phi_2(t)$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.3 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.25</td>
<td>$-4.342 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.50</td>
<td>$-3.43 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.75</td>
<td>$1.626 \times 10^{-4}$</td>
</tr>
<tr>
<td>1.00</td>
<td>$-1.30 \times 10^{-5}$</td>
</tr>
<tr>
<td>1.25</td>
<td>$5.361 \times 10^{-7}$</td>
</tr>
<tr>
<td>1.50</td>
<td>$7.984 \times 10^{-7}$</td>
</tr>
<tr>
<td>1.75</td>
<td>$-6.162 \times 10^{-8}$</td>
</tr>
<tr>
<td>2.00</td>
<td>$3.742 \times 10^{-9}$</td>
</tr>
<tr>
<td>2.25</td>
<td>$-2.212 \times 10^{-10}$</td>
</tr>
<tr>
<td>2.5</td>
<td>$9.579 \times 10^{-11}$</td>
</tr>
<tr>
<td>2.75</td>
<td>$1.834 \times 10^{-11}$</td>
</tr>
<tr>
<td>3.00</td>
<td>$-2.312 \times 10^{-13}$</td>
</tr>
<tr>
<td>3.25</td>
<td>$7.007 \times 10^{-13}$</td>
</tr>
<tr>
<td>3.75</td>
<td>$9.804 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
III. SAMPLED-DATA CONTROL OF THE ANNULAR SUSPENSION AND POINTING SYSTEM (ASPS)

3.1 The Sampled-Data Controller

The sampled-data controller for the ASPS is obtained by incorporating sample-and-hold devices in the control inputs for the dynamics of \( x_1 \) and \( x_2 \), using the same feedback gains as designed in the analog controller. The design objective is to select an appropriate sampling period \( T \) for the sampled-data ASPS.

The sampled-data ASPS is represented by the block diagram of Fig. 3-1. The \( Q \) blocks represent the effect of quantization which will not be considered at present.

3.2 Stability Analysis of the Sampled-Data ASPS by the z-Transform

The signal flow graph representation of Fig. 3-1 is shown in Fig. 3-2. The following equations are written directly from the signal flow graph.

\[
\begin{align*}
\dot{u}_1 &= \left[ -bG_h \left( \frac{K_p}{s^2} + \frac{K_r}{s} \right) \left( 1 - G_3 - G_4 - G_5 + G_3G_5 \right) u_1^* - eG_h \left( \frac{K_p}{s^2} + \frac{K_r}{s} \right) \left( 1 - G_5 \right) u_2^* \\
&+ aG_h \left( \frac{K_p}{s^2} + \frac{K_r}{s} \right) u_3^* \right] / \Delta \\
\dot{u}_2 &= \left[ -bG_h \left( \frac{K_3}{s} + \frac{K_2}{s} + \frac{K_1}{s^2} \right) \left( 1 - G_5 \right) u_1^* - eG_h \left( \frac{K_2}{s^2} + \frac{K_2}{s} + \frac{K_1}{s^2} \right) \left( 1 - G_1 - G_5 + G_1G_5 \right) u_2^* \\
&+ aG_h \left( \frac{K_3}{s} + \frac{K_2}{s} + \frac{K_1}{s^2} \right) \left( 1 - G_1 \right) u_3^* \right] / \Delta \\
\dot{u}_3 &= \left[ bG_h \left( \frac{K_6}{s} + \frac{K_5}{s^2} + \frac{K_4}{s^3} \right) u_1^* + eG_h \left( \frac{K_6}{s} + \frac{K_5}{s^2} + \frac{K_4}{s^3} \right) \left( 1 - G_1 \right) u_2^* \\
&- fG_h \left( \frac{K_6}{s} + \frac{K_5}{s^2} + \frac{K_4}{s^3} \right) \left( 1 - G_1 - G_2 + G_1G_3 \right) u_3^* \right] / \Delta 
\end{align*}
\]
Figure 3-1. Block diagram of the ASPS with the sampled-data controller.
Figure 3-2. Digital signal flow graph of the ASPS.
where
\[ G_1 = -bK_s/s^2 \]
\[ G_2 = ac \]
\[ G_3 = -eK_s/s^2 \]
\[ G_4 = d \]
\[ G_5 = -fK_s/s^2 \]
\[ G_h = (1 - e^{-Ts})/s \]

\[ \Delta = 1 - G_1 - G_2 - G_3 - G_4 - G_5 + G_1G_3 + G_1G_4 + G_1G_5 + G_3G_5 + G_2G_5 - G_1G_3G_5 \]

Taking the z-transform on both sides of Eqs. (3-1) through (3-3), we have:

\[ U_1(z) = \mathcal{Z}\left(\frac{G_A}{\Delta}\right) U_1(z) + \mathcal{Z}\left(\frac{G_B}{\Delta}\right) U_2(z) + \mathcal{Z}\left(\frac{G_C}{\Delta}\right) U_3(z) \quad (3-4) \]
\[ U_2(z) = \mathcal{Z}\left(\frac{G_D}{\Delta}\right) U_1(z) + \mathcal{Z}\left(\frac{G_E}{\Delta}\right) U_2(z) + \mathcal{Z}\left(\frac{G_F}{\Delta}\right) U_3(z) \quad (3-5) \]
\[ U_3(z) = \mathcal{Z}\left(\frac{G_G}{\Delta}\right) U_1(z) + \mathcal{Z}\left(\frac{G_J}{\Delta}\right) U_2(z) + \mathcal{Z}\left(\frac{G_K}{\Delta}\right) U_3(z) \quad (3-6) \]

where
\[ G_A = -bG_h\left(\frac{K_p}{s^2} + \frac{K_r}{s}\right) \left(1 - G_3 - G_4 - G_5 + G_3G_5\right) \]
\[ G_B = -eG_h\left(\frac{K_p}{s^2} + \frac{K_r}{s}\right) \left(1 - G_5\right) \]
\[ G_C = aG_h\left(\frac{K_p}{s^2} + \frac{K_r}{s}\right) \]
\[ G_D = -bcG_h\left(\frac{K_3}{s} + \frac{K_2}{s^2} + \frac{K_1}{s^3}\right) \left(1 - G_5\right) \]
\[ G_E = -eG_h\left(\frac{K_3}{s} + \frac{K_2}{s^2} + \frac{K_1}{s^3}\right) \left(1 - G_1 - G_5 - G_1G_5\right) \]
\[ G_F = dG_h\left(\frac{K_3}{s} + \frac{K_2}{s^2} + \frac{K_1}{s^3}\right) \left(1 - G_1\right) \]
\[ G_G = bcG_h\left(\frac{K_6}{s} + \frac{K_5}{s^2} + \frac{K_4}{s^3}\right) \]
\[ G_J = eG_h\left(\frac{K_6}{s} + \frac{K_5}{s^2} + \frac{K_4}{s^3}\right) \left(1 - G_1\right) \]
\[ G_K = -fG_n \left( K_5 \frac{1}{s^2} + K_4 \frac{1}{s^3} \right) \left( I - G_1 - G_2 - G_3 + G_1 G_3 \right) \]

Substitution of \( G_1, G_2, G_3, G_5 \) and \( G_6 \) into the last nine equations, we can show by partial fraction expansion that the functions \( G_A/\Delta, G_B/\Delta, G_C/\Delta, G_D/\Delta, G_E/\Delta, G_F/\Delta, G_I/\Delta, G_J/\Delta, \) and \( G_K/\Delta \) are of the following form:

\[ \frac{G_i(s)}{\Delta} = \frac{2(1 - e^{-Ts})}{s} \left( \frac{a_{i1}s + b_{i1} \omega_1}{s^2 + \omega_1^2} + \frac{a_{i2}s + b_{i2} \omega_2}{s^2 + \omega_2^2} + \frac{a_{i3}s + b_{i3} \omega_3}{s^2 + \omega_3^2} + \frac{a_{i4}}{s} \right) \quad (3-7) \]

where \( i = A, B, C, D, E, F, I, J \) and \( K. \)

\[ \omega_1 = 1.31014 \times 10^{-3} \]
\[ \omega_2 = 3.55991 \times 10^{-3} \]
\[ \omega_3 = 3.77726 \times 10^{-2} \]

\[ a_{A1} = 7.6593401 \times 10^{-6} \quad b_{A1} = 5.2968403 \times 10^{-2} \]
\[ a_{A2} = -5.2124686 \times 10^{-4} \quad b_{A2} = -1.3266373 \]
\[ a_{A3} = 6.894976 \quad b_{A3} = 1.653877 \times 10^3 \]
\[ a_{B1} = -3.177573 \times 10^{-3} \quad b_{B1} = -2.1974837 \times 10^1 \]
\[ a_{B2} = -7.8558545 \times 10^{-3} \quad b_{B2} = -1.999412 \times 10^1 \]
\[ a_{B3} = 3.5257182 \quad b_{B3} = 8.4570434 \times 10^2 \]
\[ a_{C1} = -6.6321553 \times 10^{-4} \quad b_{C1} = -4.5865363 \]
\[ a_{C2} = 3.6432745 \times 10^{-2} \quad b_{C2} = 9.272583 \times 10^1 \]
\[ a_{C3} = -3.5504543 \quad b_{C3} = -8.5163773 \times 10^2 \]
\[ a_{D1} = 1.1863223 \times 10^6 \quad b_{D1} = -1.3711851 \times 10^2 \]
\[ a_{D2} = 3.9724468 \times 10^5 \quad b_{D2} = -1.2475924 \times 10^2 \]
\[ a_{D3} = -1.583584 \times 10^6 \quad b_{D3} = 5.2770228 \times 10^3 \]
\[ a_{E1} = -6.1833492 \times 10^8 \quad b_{E1} = 7.1468909 \times 10^4 \]
\[ a_{E2} = -2.7868194 \times 10^7 \quad b_{E2} = 8.7523256 \times 10^3 \]
\[ a_{E3} = 1.843583 \times 10^5 \quad b_{E3} = -6.143425 \times 10^2 \]
\[ a_{E4} = 1.2920375 \times 10^9 \]
\[ a_{F1} = -1.2905753 \times 10^8 \quad b_{F1} = 1.4916845 \times 10^4 \]
\[ a_{F2} = 1.2924317 \times 10^8 \quad b_{F2} = -4.0590328 \times 10^4 \]
\[ a_{F3} = -1.8564023 \times 10^5 \quad b_{F3} = 6.1865076 \times 10^2 \]
\[ a_{i1} = 2.3031281 \times 10^4 \quad b_{i1} = -6.5239325 \]
\[ a_{i2} = -1.7136123 \times 10^5 \quad b_{i2} = 1.318941 \times 10^2 \]
\[ a_{i3} = 1.4832586 \times 10^5 \quad b_{i3} = -1.2113775 \times 10^3 \]
\[ a_{J1} = -1.2004364 \times 10^7 \quad b_{J1} = 3.4004041 \times 10^3 \]
\[ a_{J2} = 1.202163 \times 10^7 \quad b_{J2} = -9.2528628 \times 10^3 \]
\[ a_{J3} = -1.7267858 \times 10^4 \quad b_{J3} = 1.4102663 \times 10^2 \]
\[ a_{K1} = -5.80131 \times 10^7 \quad b_{K1} = 4.4651788 \times 10^4 \]
\[ a_{K2} = -2.5188685 \times 10^6 \quad b_{K2} = 7.1350445 \times 10^2 \]
\[ a_{K3} = 4.4225387 \times 10^5 \quad b_{K3} = -3.6118882 \times 10^3 \]
\[ a_{K4} = 1.2017944 \times 10^8 \]
\[ a_{A4} = a_{B4} = a_{C4} = a_{D4} = a_{F4} = a_{J4} = a_{J4} = 0 \]

Equations (3-4), (3-5) and (3-6) are represented by the digital flow graph of Fig. 3-3. The characteristic equation of the sampled-data ASPS is determined by evaluating the \( \Delta \) of the signal flow graph of Fig. 3-3. We have

\[
\Delta(z) = 1 - \gamma \left( \frac{G_A}{\Delta} \right) - \gamma \left( \frac{G_E}{\Delta} \right) - \gamma \left( \frac{G_K}{\Delta} \right) - \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) - \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) - \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_J}{\Delta} \right) - \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_E}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_J}{\Delta} \right) + \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) + \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) + \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) + \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_C}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_J}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) + \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_J}{\Delta} \right) + \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_K}{\Delta} \right) \gamma \left( \frac{G_I}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_D}{\Delta} \right) \gamma \left( \frac{G_B}{\Delta} \right) + \gamma \left( \frac{G_A}{\Delta} \right) \gamma \left( \frac{G_F}{\Delta} \right) \gamma \left( \frac{G_J}{\Delta} \right)
\]

\[(3-8)\]
Figure 3-3. Digital signal flow graph of Eqs. (3-3), (3-4) and (3-5).
Since \( \omega_1, \omega_2 \) and \( \omega_3 \) are small, and \( b_{ij} \omega_j \ll a_{ij} \) for \( i = A, B, C, D, E, F, I, J, K, \) and \( j = 1, 2, 3 \), the z-transform of \( G_i/\Delta \) can be approximated and simplified. This is justified as follows:

\[
\mathcal{Z}\left[ \frac{G_A}{\Delta} \right] = 2(1 - z^{-1}) \left( \frac{a_{A1}s + b_{A1}\omega_1}{s(s^2 + \omega_1^2)} + \frac{a_{A2}s + b_{A2}\omega_2}{s(s^2 + \omega_2^2)} + \frac{a_{A3}s + b_{A3}\omega_3}{s(s^2 + \omega_3^2)} \right)
\]  

(3-9)

Then

\[
\mathcal{Z}\left[ \frac{a_{A1}s + b_{A1}\omega_1}{s(s^2 + \omega_1^2)} \right] = \frac{a_{A1}zs\sin\omega_1T}{\omega_1(z^2 - 2z\cos\omega_1T + 1)} + \frac{b_{A1}}{\omega_1} \left( \frac{z - z(z - \cos\omega_1T)}{z^2 - 2z\cos\omega_1T + 1} \right)
\]  

(3-10)

For small \( \omega_1T \),

\[
\sin\omega_1T \approx \omega_1T
\]

\[
\cos\omega_1T \approx 1
\]

Equation (3-10) becomes

\[
\mathcal{Z}\left[ \frac{a_{A1}s + b_{A1}\omega_1}{s(s^2 + \omega_1^2)} \right] = \frac{a_A z^T}{(z - 1)^2}
\]  

(3-11)

Thus, Eq. (3-9) becomes

\[
\mathcal{Z}\left[ \frac{G_A}{\Delta} \right] = \frac{2T}{z - 1} \left( a_{A1} + a_{A2} + a_{A3} \right) = \frac{13.7889T}{z - 1}
\]  

(3-12)

Similarly,

\[
\mathcal{Z}\left[ \frac{G_B}{\Delta} \right] = \frac{2T}{z - 1} \left( a_{B1} + a_{B2} + a_{B3} \right) = \frac{7.029369T}{z - 1}
\]  

(3-13)

\[
\mathcal{Z}\left[ \frac{G_C}{\Delta} \right] = \frac{2T}{z - 1} \left( a_{C1} + a_{C2} + a_{C3} \right) = \frac{-7.029369T}{z - 1}
\]  

(3-14)

\[
\mathcal{Z}\left[ \frac{G_D}{\Delta} \right] = \frac{2T}{z - 1} \left( a_{D1} + a_{D2} + a_{D3} \right) = \frac{-34.04T}{z - 1}
\]  

(3-15)

\[
\mathcal{Z}\left[ \frac{G_E}{\Delta} \right] = \frac{T}{z - 1} \left( 2a_{E1} + 2a_{E2} + 2a_{E3} + a_{E4} \right) = \frac{-11T}{z - 1}
\]  

(3-16)
\[
\begin{align*}
\left[ \frac{G_F}{\Delta} \right] &= \frac{2T}{z - 1} (a_{F1} + a_{F2} + a_{F3}) = \frac{-0.46T}{z - 1} \\
\left[ \frac{G_I}{\Delta} \right] &= \frac{2T}{z - 1} (a_{I1} + a_{I2} + a_{I3}) = \frac{-8.18T}{z - 1} \\
\left[ \frac{G_J}{\Delta} \right] &= \frac{2T}{z - 1} (a_{J1} + a_{J2} + a_{J3}) = \frac{-3.71T}{z - 1} \\
\left[ \frac{G_K}{\Delta} \right] &= \frac{T}{z - 1} (2a_{K1} + 2a_{K2} + 2a_{K3} + a_{K4}) = \frac{10.7T}{z - 1}
\end{align*}
\] (3-17) (3-18) (3-19) (3-20)

Substituting these terms into Eq. (3-8) and simplifying, we have the characteristic equation as

\[
z^3 - (3 + 13.49T)z^2 + (3 + 26.978T + 361.59T^2)z - (1 + 13.49T + 361.59T^2 + 687T^3) = 0
\] (3-21)

It should be noted that the digital ASPS is originally a twelfth-order system. Because \(\omega_1, \omega_2,\) and \(\omega_3\) are very small, the analysis conducted here results in a third-order approximation with the characteristic equation given in Eq. (3-21). When we neglected \(\omega_1, \omega_2\) and \(\omega_3\), in essence we have made a variation on the unit circle in the \(z\)-plane. Thus, the stability criterion should be modified to \(|z| < 1\pm\epsilon\) where \(\epsilon\) depends on the value of \(\omega_1, \omega_2\) and \(\omega_3\). As a matter of fact, for small sampling periods, the characteristic roots of the system are all very close to the \(z = 1\) point in the \(z\)-plane.

The roots of Eq. (3-21) for \(T = 0.001, 0.01\) and \(0.1\) sec are tabulated as follows:

- \(T = 0.001\) sec \(~z = 0.9372, 1.038 \pm j0.0604\)
- \(T = 0.01\) sec \(~z = 1.0192, 1.0578 \pm j0.175\)
- \(T = 0.1\) sec \(~z = 1.203, 1.573 \pm j1.7478\)

Although in a strict sense these roots are all outside the unit circle, and the corresponding system are all unstable, as discussed above, the roots for \(T = 0.001\) sec are so close to the unit circle that if enough accuracy
were carried out in the analysis without approximation, these roots could actually be inside the unit circle. In any case, for $T = 0.1$ sec or greater the digital system is definitely unstable. As will be verified by computer simulation results in the next chapter, the digital ASPS is stable for $T = 0.001$ sec and unstable for $T = 0.1$ sec, with the stability boundary being somewhere around $T = 0.0075$ sec.

Therefore, the approximation conducted on the z-transform analysis still gave useful results on the stability condition of the digital ASPS with respect to the sampling period $T$. Because the characteristic roots of the system are all very close to the $z = 1$ point for stable values of $T$, an accurate analysis using the exact model is almost impossible even with the aide of a digital computer.
IV. COMPUTER SIMULATION OF THE DIGITAL ASPS

4.1 The Digital ASPS With The Sampled-Data Controller

It was discussed in the preceding chapter that the digital ASPS was obtained by placing sample-and-hold units in the control inputs of the $x_1$, $\phi_1$ and $\phi_2$ components, using the same feedback gains as designed for the continuous-data system. The z-transform analysis conducted in the last chapter showed that the digital ASPS is unstable for sampling periods greater than 0.01 seconds approximately.

The computer listing of the simulation program for the digital ASPS without quantization effects is given in Table 4-1.

The simulation results verified that the digital ASPS neglecting quantization is unstable for sampling periods greater than 0.01 second. When the sampling period is less than or equal to 0.075 seconds, the simulated responses remained stable.

Figure 4-1 illustrates the time response of the $x_1$ dynamics when $T = 0.005$ seconds. The initial conditions are: $x_1(0) = 2 \times 10^{-3}$ and $\dot{x}_1(0) = 0$. Figure 4-2 shows the same response of $x_1$ for $t$ greater than 6 seconds. The vertical scale is changed to show the oscillatory characteristics as the response decays to zero.

Figures 4-3 and 4-4 illustrate the time responses of $\phi_1$ and $\phi_2$, respectively, when $T = 0.005$ seconds.

Although the overshoots of the time responses of the digital ASPS are quite large for $T = 0.005$ sec., it can be shown that these overshoots are substantially reduced by reducing the sampling period $T$.

4.2 The Digital ASPS With Quantization

In this section the effect of quantization in the sample-and-hold channels of $x_1$, $\phi_1$ and $\phi_2$ is investigated by means of computer simulation.
Table 4-1. Computer Simulation Program of the Digital ASPS.

```
DIMENSION PRMT(5),Y(1:8),DERY(8),AUX(8,8)
EXTERNAL STD,OUTP
COMMON X,XDOT,PHII,PHI10,PHII1TN,PHII2,PHI20T,PHI2Y,PHI2N
1 V1,V2,V3,AJ,AM=AKSX,AKSP,AKSS,A33,TPRT,R8,DELTA
2 NPRINT,V1,V2,V3,LINE
OPEN UNIT=6,DEVICE='DSK',FILE='FOR20,DAT',ACCESS='UNKNOWN'

LINE=V
TSP=0.005
M=0,
TEND=100.
TPRT=0.05
TINT=0.5E-3
C--SET INITIAL CONDITIONS
X=2.0E-3
XDOT=0.
PHII=0.013
PHI10=0.
PHII1N=0.
PHI2=0.013
PHI20T=0.
PHI2IN=0.
AM=600.
AKSX=1.051
AKSP=0.005
AKSS=0.005
AJ=503.
A33=2805.1
R8=1.756
A0=R8
B0=1./AM
C0=AM*R8/A33
D0=AJ/A33
E=1./A33
F=-1./AJ
DELTA=1.-A0*C0-D0
T=0.
NPRINT=0
PRMT(1)=TINT
PRMT(2)=0.01*X
DO 99 1=1,8
99 DERY(I)=0.125
NPRINT=8
10 IF(T.GB.TEND) CALL EXIT
V1=0.9133885*X+0.1001835*XDOT
V2=0.4646227*PHII1N+0.5649360*PHI1-0.4894035*PHI10
V3=0.6009006*PHI21N+0.1299216*PHI2+0.1175605*PHI20T

PRMT(1)=T
PRMT(2)=T+TSP
Y(1)=X
Y(2)=XDOT
Y(3)=PHII1N
Y(4)=PHI1
Y(5)=PHI10
Y(6)=PHI21N
Y(7)=PHI2
Y(8)=PHI20T
```
CALL RKGS(PRNT,Y,DERY,ITAM=IHLF,FTC,OUTF,NPRT)
GO TO 10
END

"SUBROUTINE FTC(X,Y,DERY)"
DIMENSION Y(8),DERY(8)
COMMON X,XDOT,PHI1,PHI1D,T,PHI2,PHI2D,T,PHI3,PHI3D,T,PHI4,PHI4D,
& V1,V2,V3,V4,AM1,AKSX,AKSF,AKSS,A33,TPRT,TPRT,ITAM
& A0,B0,C,CO,E,F,TEND,
& NPRT,IV1,IV2,IV3,LINE
DERY(1)=Y(2)
DERY(2)=(-V1*AM1*AKSX*AKSF*AKSS-A33*TPRT,B0*C0)/DELTA
& 2+(-V1*AKSX*AKSF*AKSS-A33*TPRT,B0*C0)/DELTA
DERY(3)=Y(4)
DERY(4)=Y(5)
DERY(5)=(-V2-AKSS*Y(7))*E+(-V1-AKSS*Y(1))*B0*C0
& 2+(-V1-AKSS*Y(7))*B0*C0/DELTA
DERY(6)=Y(7)
DERY(7)=Y(8)
DERY(8)=(-V2-AKSS*Y(7))*E+(-V1-AKSS*Y(1))*B0*C0
& 2+(-V1-AKSS*Y(7))*B0*C0/DELTA
RETURN
END

SUBROUTINE OUTF(X,Y,DERY,IHL,F,NDIM,PRMT)
DIMENSION Y(8),DERY(8),PRMT(5)
COMMON X,XDOT,PHI1,PHI1D,T,PHI2,PHI2D,T,PHI3,PHI3D,T,PHI4,PHI4D,
& V1,V2,V3,V4,AM1,AKSX,AKSF,AKSS,A33,TPRT,TPRT,ITAM
& A0,B0,C,CO,E,F,TEND,
& NPRT,IV1,IV2,IV3,LINE
I=XX,GT,TEND) PRMT(5)=1,
Y=XX,
TF(X,Y,1,TFRT,FLOAT(NPRT)) RETURN
LINE=LINE+1
WRITE(6,100) IHL,F,XX,(Y(I),I=1,8)
FORMAT(1X,I2,3X,1PE10.6,1PE11.3)
X=Y(1)
XDOT=Y(2)
PHI1IN=Y(3)
PHI1=Y(4)
PHI1D=YT(5)
PHI2IN=Y(6)
PHI2=Y(7)
PHI2D=YT(8)
NPRT=NPRT+1
IF(LINE,LT,10) GO TO 200
WRITE(5,100) IHL,F,XX,(Y(I),I=1,8)
LINE=0
200 RETURN
END
Figure 1-2: Time response of $x_2$ of the digital filter with $T = 0.005$ sec.
The computer program listing is given in Table 4-2.

When 8-bit quantizers are placed in any or all of the input channels of the digital ASPS the system is still stable, and no sustained oscillations were observed. Figure 4-5 shows the phase plane trajectories of $x_1$ versus $\dot{x}_1$ of the digital ASPS with and without an 8-bit quantizer. The quantizer is placed only in the $x_1$ channel.

Figures 4-6 and 4-7 illustrate the time responses of $\phi_1$ and $\phi_2$, respectively, with the 8-bit quantizer for $x_1$, and $T = 0.005$ seconds.

Figure 4-8 gives the phase plane trajectory of $x_1$ versus $\dot{x}_1$ when a 16-bit quantizer is placed in the $x_1$ channel only. The trajectory of $x_1$ with the 8-bit quantizer is also shown in the same figure for comparison. Note that for the same time instants the response of $x_1$ with the 8-bit quantizer is greater than that of the response with the 16-bit quantizer. The phase plane representation of $x_1$ is chosen because of the high oscillations in the responses which make the time response difficult to plot.

Figures 4-9 and 4-10 illustrate the time responses of $\phi_1$ and $\phi_2$, respectively, for $T = 0.005$ sec and a 16-bit quantizer in the $x_1$ channel. The overshoots in these responses are smaller than the corresponding ones in Figs. 4-6 and 4-7 which are for the 8-bit quantizer.

Figures 4-11, 4-12 and 4-13 show that results on the responses of $x_1$, $\phi_1$ and $\phi_2$, respectively, when an 8-bit quantizer is placed in the $\phi_1$ channel only. The sampling period is 0.005 seconds.

Figures 4-14, 4-15 and 4-16 show the time responses of $x_1$, $\phi_1$ and $\phi_2$, respectively, when an 8-bit quantizer is placed in the $\phi_2$ channel only. The sampling period is 0.005 seconds.

Figures 4-17, 4-18 and 4-19 give the time responses of $x_1$, $\phi_1$ and $\phi_2$, respectively.
respectively, when an 8-bit quantizer is placed in each of the three control input channels, and the sampling period is 0.002 seconds. As mentioned earlier, the oscillations and overshoots are substantially reduced when the sampling period is smaller.
Table 4-2. Computer Simulation Program of the Digital ASPS with Quantization.

```
DIMENSION PRMT(5),Y(8),DERY(7),XNL(5*2)
EXTERNAL FCT+OUTP
COMMON X+XDOT,PH1,PHI1OT,PHI1IN,PHI2,PHI2OT,PHI2IN+T
=ALCP\$V+V5+A1+AIL+AKS+ASK+ASK$+N3+TPI+R5+DELTAS
=ALCP\$W+CO+00+0+TENL+T
&NPRT,J+IV2,IV3+LINE
OPEN(UNIT=0,DEVICE='BSK',FILE='FOR2G.csv',ACCIRDY='SCQOUT')

LINE=0
TSP=0.02
H=2.00-8
TEND=100
IPRT=0.05
TINT=0.5E-4

C----SET INITIAL CONDITIONS
X=1.0E-3
XDOT=0
PHI1=0.002
PHI1DT=0,
PHI1IN=0,
PHI2=0.002
PHI2UT=0,
PHI2IN=0,
AHI=600.
AKSX=1.051
AKSP=0.005
AKSS=0.005
AJ=503.
AJ=2905.15
RB=1,958
AO=RB
BO=1.+AMI
CO=AH1+RB/AJ3
=AO/AJ3
E=1.+AJ3
R=1.+AJ
DELTAS=1.0A+CO-BO
T=0
NPRT=1
PRMT(1)=TINT
PRMT(4)=0.05*X
T=97
I=1.8
99
DERV(1)=0.125
NDOM=8
10
IF(T,UE,TEND) CALL L00,
V1=0.91J8D5*X+0.1061J5*XDOT
V2=0.9480217%PHI1IN+0.567936%PHI1-0.1084305%PHI1OT
V3=0.9969986%PHI2IN+0.1279286%PHI2+0.1175685%PHI2OT

CALL QUANTZ(V1)
CALL QUANTZ(V2)
CALL QUANTZ(V3)

PRMT(1)=T
PRMT(2)=T/TSP
```
Y(1)=X
Y(2)=XI
Y(3)=PHI1
Y(4)=PHI1
Y(5)=PHI1
Y(6)=PHI2
Y(7)=PHI2
Y(8)=PHI2

CALL RKGS(PRMT, Y, VERY, NOIN, IHLF, FCT, OUTP, AUX)
GO TO 10
END

SUBROUTINE QUANTZ(XX)
COMMON X, XOUT, PHI1, PHI1DT, PHI1TH, PHI2, PHI2DT, PHI2TN, T,
      V1, V2, V3, A1, A1T, AK, AKP, AKS, N3, TPRT, K, DELTA,
   N, A0, B0, C0, D0, E, F, TEND, H,
NPRM, IV1, IV2, IV3, LTNE

IF(XX, LT, 0.) GO TO 20
IV=(XX/H)+0.5
GO TO 30
IV=(XX/H)-0.5
XX=FLOAT(IV)*H
RETURN
END
Figure 4.3: The response of $F$ in the digital test with $4$-bit quantizer for $10.0000$ bits. The quantizer is only in the $k$th channel.
Figure 14.8: Time response in the phase-plane of $x_1$ of the discrete APS with an 8-bit quantizer in the $x_1$ channel. 
$t = 0.001$ sec.
Figure 4-10. Time response of $B_2$ of the digitizer with an external signalizer in the X channel: $T = 0.005$ sec.
Figure 7: The phase plane trajectory of $x$ versus $y$ of the digital APS with a 8-bit quantizer in the $y$ channel. $T = 0.005$ sec.
Figure 24: The response of \( x \) of the digital APSF-4A at 8 sec.

Labelled: in the P3 channel.
Figure 4.17: Time response of $x_1$ of the digital ASFS with a quantizer in each of the three channels. $\Delta = 0.8$, $T = 0.007$ sec.
Figure 4.10: Time response of \( \tau \) of the digital ASPS with an FF A/L switch, quarter in each of the three input channels. \( \tau = 0.002 \) sec.
V. DESIGN OF THE ANALOG ASPS THROUGH DECOUPLING AND POLE PLACEMENT

5.1 Introduction

In Chapter II the analog controllers of the ASPS are designed for the control of the \( x, \phi_1 \) and \( \phi_2 \) dynamics. The eight eigenvalues of the system are assigned so that the overall system has an equivalent bandwidth of 2 Hz. We shall see later that this is not an accurate description of the bandwidth requirements of the system.

The analog controllers consist of eight state-feedback gains with no coupling between \( x, \phi_1 \) and \( \phi_2 \). In other words, the input \( u_1 \) of the \( x \) component is affected only by feedbacks from \( x \) and \( x \), the input \( u_2 \) of the \( \phi_1 \) component is affected only by feedbacks from \( J\phi_1, \phi_1 \) and \( \dot{\phi}_1 \). Similarly, the input \( u_3 \) of the \( \phi_2 \) component is realized by feedback from the states \( J\phi_2, \phi_2 \) and \( \dot{\phi}_2 \). Therefore, the three independent controllers are essentially PID controllers.

The values of the eight feedback gains are determined by using the Brown's method for pole-placement. The sampled-data version of the \( x, \phi_1 \) and \( \phi_2 \) dynamics of the ASPS was obtained by inserting sample-and-hold devices in the three input channels. It was shown in Chapter III that using the feedback gains designed for the analog system, the sampled-data system is stable for sampling periods less than or equal to 0.0075 seconds. However, this sampling period is still considered to be too small to be economical and practical for the ASPS.

5.2 Design of the Analog ASPS Through Decoupling And Pole Placement

As it turns out the desired bandwidths requirement of the ASPS is as follows:

\[
\begin{align*}
x_1 \text{ dynamics} & : 0.04 \text{ Hz} \\
\phi_1 \text{ dynamics} & : 10 \text{ Hz}
\end{align*}
\]
Since the $\phi_1$ dynamics is 250 times faster than the $x$ dynamics, it is virtually impossible to find an equivalent bandwidth of the overall system and establish a general eigenvalue requirement for the system. Therefore, it is necessary to decouple the $x$, $\phi_1$ and $\phi_2$ dynamics through state feedback and simultaneously place the poles to realize the desired bandwidths for all three system components. The problem is stated as:

Given the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(5-1)

where

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \end{bmatrix}$$

(5-2)

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

(5-3)

$A$ is the $8 \times 8$ coefficient matrix and $B$ is the $8 \times 3$ input matrix. The state feedback is defined by

$$u(t) = -Gx(t)$$

(5-4)

where $G$ is the $3 \times 8$ feedback matrix. The elements of $G$ are to be selected so that the coefficient matrix of the closed-loop system, $A - BG$, has the following form:
A - BG = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix} \quad (5-5)

where

- $\Lambda_1$ is a 2x2 matrix for $x$
- $\Lambda_2$ is a 3x3 matrix for $\phi_1$
- $\Lambda_3$ is a 3x3 matrix for $\phi_2$.

The eigenvalues of $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are so selected that the bandwidths requirements are satisfied.

Referring to Fig. 2-I the A matrix of the ASPS has the following form:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{24} & 0 & 0 & a_{27} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
a_{51} & 0 & 0 & a_{54} & 0 & 0 & a_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a_{81} & 0 & 0 & a_{84} & 0 & 0 & a_{87} & 0
\end{bmatrix} \quad (5-6)

The B matrix is written as

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
b_{21} & b_{22} & b_{23} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
b_{51} & b_{52} & b_{53} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
b_{81} & b_{82} & b_{83}
\end{bmatrix} \quad (5-7)
where the \( a_{ij} \)'s and \( b_{ij} \)'s in the A and B matrices are given as

\[
\begin{align*}
a_{21} &= -0.61207568 \\
a_{24} &= -1.4844272 \times 10^{-3} \\
a_{27} &= 1.4844272 \times 10^{-3} = -a_{24} \\
a_{51} &= -0.31202659 \\
a_{54} &= -7.589096 \times 10^{-4} \\
a_{57} &= 7.589096 \times 10^{-4} = -a_{54} \\
a_{81} &= 0.31202659 \\
a_{84} &= 7.589096 \times 10^{-4} \\
a_{87} &= -7.6884996 \times 10^{-4} \\
b_{21} &= 0.58237458 \\
b_{22} &= 0.29688544 \\
b_{23} &= -0.29688544 \\
b_{51} &= 0.29688544 \\
b_{52} &= 0.15178192 \\
b_{53} &= -0.15178192 = -b_{52} \\
b_{81} &= -0.29688544 \\
b_{82} &= -0.15178192 \\
b_{82} &= 0.15376999
\end{align*}
\]

It is simple to show that B has full rank (rank = 3).

Let the feedback matrix be represented by

\[
G = \begin{bmatrix}
g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\
g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\
g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38}
\end{bmatrix}
\]  \hspace{1cm} (5-8)

Then BG becomes
where

\[(BG)_{ij} = \sum_{k=1}^{3} b_{ik} g_{kj} \quad i=1,2,\ldots,8, \quad j=1,2,\ldots,8\]

In order for \(A - BG\) to be a diagonal matrix, the following relations must hold:

\[(BG)_{23} = a_{23} = 0\]
\[(BG)_{24} = a_{24}\]
\[(BG)_{25} = a_{25} = 0\]
\[(BG)_{26} = a_{26} = 0\]
\[(BG)_{27} = a_{27}\]
\[(BG)_{28} = a_{28} = 0\]
\[(BG)_{51} = a_{51}\]
\[(BG)_{52} = a_{52} = 0\]
\[(BG)_{56} = a_{56} = 0\]
\[(BG)_{57} = a_{57}\]
\[(BG)_{58} = a_{58} = 0\]
\[(BG)_{81} = a_{81}\]
\[(BG)_{82} = a_{82} = 0\]
\[(BG)_{83} = a_{83} = 0\]
\[(BG)_{84} = a_{84}\]
\[(BG)_{85} = a_{85} = 0\]

These equations represent 16 equations with 24 variables in the feedback gains \(g_{ij}\) \((i=1,2,3; j=1,2,...,8)\).

For the present design, the \(x\) dynamics is specified to have a bandwidth of 0.04 Hz. Since the \(x\) dynamics are represented by a second-order system, the bandwidth of the system is given by

\[
BW = \omega_n \left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{\frac{1}{2}}
\]  

(5-11)

If we choose the damping ratio \(\zeta\) to be 0.707, then

\[
BW = \omega_n
\]  

(5-12)

where \(\omega_n\) is the natural undamped frequency.

Therefore, for the \(x\) dynamics, the eigenvalues are selected to be at

\[
p_1, p_2 = a_1 \pm ja_1
\]  

(5-13)

where

\[
a_1 = -\frac{0.04 \times 2\pi}{\sqrt{2}} = -0.178
\]  

(5-14)

The \(\phi_1\) and \(\phi_2\) dynamics are of the third order. However, we can use second-order approximations by placing the real root far to the left on the real axis in the \(s\)-plane. For this case we let the real root of the \(\phi_1\) dynamics be placed at

\[
p_3 = b_2 = -200
\]  

(5-15)

Then, the complex roots are:

\[
p_4, p_5 = a_2 \pm ja_2
\]  

(5-16)

where

\[
a_2 = -\frac{2\pi \times 10}{\sqrt{2}} = -44.436
\]  

(5-17)

Similarly, for the \(\phi_2\) dynamics, we let

\[
p_6 = b_3 = -20
\]  

(5-18)
Then,
\[ p_7, p_8 = a_3 + ja_3 \]  \hspace{1cm} (5-19)

where
\[ a_3 = -\frac{2\pi \times 1}{\sqrt{2}} = -4.4436 \]  \hspace{1cm} (5-20)

The characteristic equations of the three decoupled subsystems are as follows:

\textbf{x component:}
\[
(s - a_1 - ja_1)(s + a_1 + ja_1) = s^2 - 2a_1 s + 2a_1^2 = 0 \]  \hspace{1cm} (5-21)

\textbf{\( \phi_1 \) component:}
\[
(s - b_2)(s - a_2 - ja_2)(s - a_2 + ja_2) = (s - b_2)(s^2 - 2a_2 s + 2a_2^2) = s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2 b_2)s - 2a_2^2 b_2 = 0 \]  \hspace{1cm} (5-22)

\textbf{\( \phi_2 \) component:}
\[
(s - b_3)(s - a_3 - ja_3)(s - a_3 + ja_3) = (s - b_3)(s^2 - 2a_3 s + 2a_3^2) = s^3 - (2a_3 + b_3)s^2 + (2a_3^2 + 2a_3 b_3)s - 2a_3^2 b_3 = 0 \]  \hspace{1cm} (5-23)

If there exists a set of feedback gains such that the decoupling of the subsystems \( x, \phi_1 \) and \( \phi_2 \) can be accomplished, then the decoupled closed-loop system will have the independent characteristic equations of Eqs. (5-21), (5-22) and (5-23).

Let the matrix \( A \) of Eq. (5-6) be written as
\[
A = \begin{bmatrix}
A_{11} & X & X \\
-1 & - & -1 \\
X & - & A_{22} \\
-1 & X & -1 \\
X & - & A_{33}
\end{bmatrix} \]  \hspace{1cm} (5-24)

where \( A_{11} \) denotes a 2x2 matrix, \( A_{22} \) and \( A_{33} \) are 3x3 matrices. The submatrices denoted by \( X \) contain elements which are unimportant for the immediate development.

In view of Eqs. (5-5) and (5-9), the decoupled \( x \) dynamics with state feedback can be expressed as
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \Lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]  

(5-25)

where

\[ \Lambda_1 = A_{11} - \begin{bmatrix} 0 & 0 \\ (BG)_{21} & (BG)_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21} - (BG)_{21} & -(BG)_{22} \end{bmatrix} \]  

(5-26)

The characteristic equation of \( \Lambda_1 \) is

\[ |sI - \Lambda_1| = \begin{vmatrix} s & -1 \\ a_{21} - (BG)_{21} & s + (BG)_{22} \end{vmatrix} = s^2 + (BG)_{22} s - a_{21} + (BG)_{21} = 0 \]  

(5-27)

Similarly, the decoupled \( \phi_1 \) dynamics with state feedback is described by

\[ \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \Lambda_2 \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \]  

(5-28)

where

\[ \Lambda_2 = A_{22} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (BG)_{53} & (BG)_{54} & (BG)_{55} \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(BG)_{53} & a_{54} - (BG)_{54} & -(BG)_{55} \end{bmatrix} \]  

(5-29)

The characteristic equation of \( \Lambda_2 \) is

\[ |sI - \Lambda_2| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ (BG)_{53} & -a_{54} + (BG)_{54} & s + (BG)_{55} \end{vmatrix} \]
For the \( \phi_2 \) component, the decoupled closed-loop state equations are

\[
\begin{bmatrix}
    x_6 \\
    x_7 \\
    x_8 \\
\end{bmatrix}
= \begin{bmatrix}
    x_6 \\
    x_7 \\
    x_8 \\
\end{bmatrix}
= A_3 \begin{bmatrix}
    x_6 \\
    x_7 \\
    x_8 \\
\end{bmatrix}
\]  

(5-31)

where

\[
A_3 = A_{33} - \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    (BG)_{86} & (BG)_{87} & (BG)_{88} \\
\end{bmatrix}
\]

(5-32)

The characteristic equation of \( A_3 \) is

\[
\begin{vmatrix}
    s & -1 & 0 \\
    0 & s & -1 \\
    (BG)_{86} & -(BG)_{87} & s + (BG)_{88} \\
\end{vmatrix}
= s^3 + (BG)_{88}s^2 + ((BG)_{87} - a_{87})s + (BG)_{86} = 0
\]  

(5-33)

For pole placement, in order to meet the bandwidth requirements, the corresponding coefficients of the characteristic equations in Eqs. (5-27), (5-30) and (5-33) must match those of Eqs. (5-21), (5-22) and (5-23), respectively.

Thus, for pole-placement,

\[
s^2 - 2a_1s + 2a_1^2 = s^2 + (BG)_{22}s + (BG)_{21} - a_{21}
\]  

(5-34)

\[
s^3 - (2a_2 + b_2)s^2 + (2a_2^2 + 2a_2b_2)s - 2a_2^2b_2
= s^3 + (BG)_{55}s^2 + ((BG)_{54} - a_{54})s + (BG)_{53}
\]  

(5-35)
\[ s^3 - (2a_3 + b_3)s^2 + (2a_3^2 + 2a_3b_3)s - 2a_3^2b_3 = s^3 + (BG)_{88}s^2 + ((BG)_{87} - a_{87})s + (BG)_{86} \]  
(5-36)

Equating the like coefficients in the above three equations, we have

\[
\begin{align*}
(BG)_{22} &= b_{21}g_{12} + b_{22}g_{22} + b_{23}g_{32} = -2a_1 \\
(BG)_{21} &= b_{21}g_{11} + b_{22}g_{21} + b_{23}g_{31} = 2a_1^2 + a_{21} \\
(BG)_{55} &= b_{51}g_{15} + b_{52}g_{25} + b_{53}g_{35} = -(2a_2 + b_2) \\
(BG)_{54} &= b_{51}g_{14} + b_{52}g_{24} + b_{53}g_{34} = 2a_2^2 + 2a_2b_2 + a_{54} \\
(BG)_{53} &= b_{51}g_{13} + b_{52}g_{23} + b_{53}g_{33} = -2a_2^2b_2 \\
(BG)_{88} &= b_{81}g_{18} + b_{82}g_{28} + b_{83}g_{38} = -(2a_3 + b_3) \\
(BG)_{87} &= b_{81}g_{17} + b_{82}g_{27} + b_{83}g_{37} = 2a_3^2 + 2a_3b_3 + a_{87} \\
(BG)_{86} &= b_{81}g_{16} + b_{82}g_{26} + b_{83}g_{36} = -2a_3^2b_3
\end{align*}
\]  
(5-37)

Notice that the 16 constraint equations on the feedback gains in Eq. (5-10) are conditions on decoupling of the three subsystems, whereas the 8 constraint equations in Eq. (5-37) are the conditions for pole placement. The 24 constraint equations contain 24 unknowns in the elements of the feedback matrix \(G\), \(g_{ij}\), \(i=1, 2, 3, j=1, 2, \ldots, 8\).

The 24 constraint equations in Eqs. (5-10) and (5-37) can be written as

\[ WG = C \]  
(5-38)

where

\[
W = \begin{bmatrix}
    b_{21} & b_{22} & b_{23} \\
    b_{51} & b_{52} & b_{53} \\
    b_{81} & b_{82} & b_{83}
\end{bmatrix}
\]

\(G\) is the feedback matrix given in Eq. (5-8), and \(C\) is a 3x8 matrix whose elements are composed of \(a_1, a_2, a_3, b_1, b_2, b_3,\) and \(a_{1j}, i=1, 2, \ldots, 8, j=1, 2, \ldots, 8\).

The feedback gain matrix \(G\) can be solved from Eq. (5-38) if \(W\) is nonsingular.
to be nonsingular, it is necessary and sufficient for B to have full rank. and the results are given as follows:

\[ G = W^{-1} C \quad (5-39) \]

or the system parameters given for the ASPS, the feedback matrix is solved q. (5-39), and the results are given as follows:

- \( g_{11} = 36.449 \)
- \( g_{12} = 212.4 \)
- \( g_{13} = -9.256981248 \times 10^8 \)
- \( g_{14} = -2.5471662624 \times 10^7 \)
- \( g_{15} = -3.3892568 \times 10^5 \)
- \( g_{16} = 0 \)
- \( g_{17} = 0 \)
- \( g_{18} = 0 \)
- \( g_{21} = -73.35 \)
- \( g_{22} = -415.4544 \)
- \( g_{23} = 2.2126125552 \times 10^9 \)
- \( g_{24} = 6.0882526771 \times 10^7 \)
- \( g_{25} = 8.1012732 \times 10^5 \)
- \( g_{26} = 3.96750304 \times 10^5 \)
- \( g_{27} = 1.091703152 \times 10^5 \)
- \( g_{28} = 1.452664 \times 10^4 \)
- \( g_{31} = 0 \)
- \( g_{32} = 0 \)
- \( g_{33} = 3.96750304 \times 10^8 \)
- \( g_{34} = 1.091703152 \times 10^7 \)
- \( g_{35} = 1.452664 \times 10^5 \)
- \( g_{36} = 3.96750304 \times 10^5 \)
For the same bandwidth requirements described in Chapter 5, the following eigenvalues are selected in the s-plane:

x component: (0.04 Hz bandwidth)

\[ p_1, p_2 = -0.178 \pm j0.178 \]

\[ \phi_1 \text{ component: (10 Hz bandwidth)} \]

\[ p_3 = -200 \]
\[ P_4, P_5 = -44.436 \pm j44.436 \]

\[ \phi_2 \text{ component: (1 Hz bandwidth)} \]

\[ p_6 = -20 \]

\[ p_7, p_8 = -4.4436 \pm j4.4436 \]

Using the transformation \( z = e^{Ts} \), these eigenvalues are transformed into the \( z \)-plane. The corresponding \( z \)-plane eigenvalues are:

\[ x: \quad z_1, z_2 = 0.9964457 \pm j0.0035417 \]

\[ \phi_1: \quad z_3 = 0.01831564 \]
\[ z_4, z_5 = 0.2592945 \pm j0.3191948 \]

\[ \phi_2: \quad z_6 = 0.67032 \]
\[ z_7, z_8 = 0.911366 \pm j0.096987 \]

The condition of decoupling of the closed-loop digital system given in Eq. (6-8) results in 42 constraint equations. This is due to the fact that Eq. (6-8) represents 64 scalar equations, but there are 42 zero elements in the matrix. The pole placements of the eighth-order system would produce 8 more constraint equations, so there are only 24 unknowns in the feedback matrix \( G \). Therefore, in general, there would not be a set of solutions with more equations than unknowns. However, a closer look at the elements of \( \phi(T) \) and \( \theta(T) \) reveals that some of the elements are extremely small so that a good approximation can be obtained by assuming that these small elements are zeros. In other words, the state transition matrix \( \phi(0.02) \) given in Eq. (6-12) can be approximated by the following matrix.
Similarly, we can approximate \( \theta(0.02) \) by the following matrix:

\[
\theta(0.02) \approx \begin{pmatrix}
0 & 0 & 0 & 0 \\
1.164 \times 10^{-2} & 5.937 \times 10^{-3} & -5.937 \times 10^{-3} \\
0 & 0 & 0 \\
5.937 \times 10^{-3} & 3.035 \times 10^{-3} & -3.035 \times 10^{-3} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-5.937 \times 10^{-3} & -3.035 \times 10^{-3} & 3.075 \times 10^{-3}
\end{pmatrix}
\]  

Similarly, we can approximate \( \phi(0.02) \) by the following matrix:

\[
\phi(0.02) \approx \begin{pmatrix}
9.998 \times 10^{-1} & 1.999 \times 10^{-2} & 0 & 0 & 0 & 0 \\
-1.224 \times 10^{-2} & 9.998 \times 10^{-1} & 0 & -2.968 \times 10^{-5} & -2.968 \times 10^{-7} & 0 & 2.968 \times 10^{-5} & 2.968 \times 10^{-7} \\
0 & 0 & 1 & 1.999 \times 10^{-2} & 1.999 \times 10^{-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 9.999 \times 10^{-1} & 1.999 \times 10^{-2} & 0 & 0 & 0 \\
-6.240 \times 10^{-5} & -4.160 \times 10^{-7} & 0 & -1.517 \times 10^{-5} & 9.999 \times 10^{-1} & 0 & 1.517 \times 10^{-5} & 1.517 \times 10^{-7} \\
0 & 0 & 0 & 0 & 0 & 1 & 1.999 \times 10^{-2} & 1.999 \times 10^{-4} \\
0 & 0 & 0 & 0 & 0 & 0 & 9.999 \times 10^{-1} & 1.999 \times 10^{-2} \\
6.240 \times 10^{-3} & 6.240 \times 10^{-5} & 0 & 1.517 \times 10^{-5} & 1.517 \times 10^{-7} & 0 & -1.537 \times 10^{-5} & 9.999 \times 10^{-1}
\end{pmatrix}
\]  

(6-14)  

Then, the matrix \( \theta(0.02)G \) becomes
Based on Eqs. (6-14), (6-15) and (6-16), the following 16 equations are obtained for decoupling the $\phi \cdot \theta G$ matrix:

\[
\begin{align*}
(\theta G)_{23} &= \theta_{21} g_{13} + \theta_{22} g_{23} + \theta_{23} g_{33} = \phi_{23} = 0 \\
(\theta G)_{24} &= \theta_{21} g_{14} + \theta_{22} g_{24} + \theta_{23} g_{34} = \phi_{24} = -2.968 \times 10^{-5} \\
(\theta G)_{25} &= \theta_{21} g_{15} + \theta_{22} g_{25} + \theta_{23} g_{35} = \phi_{25} = -2.968 \times 10^{-7} \\
(\theta G)_{26} &= \theta_{21} g_{16} + \theta_{22} g_{26} + \theta_{23} g_{36} = \phi_{26} = 0 \\
(\theta G)_{27} &= \theta_{21} g_{17} + \theta_{22} g_{27} + \theta_{23} g_{37} = \phi_{27} = 2.968 \times 10^{-5} \\
(\theta G)_{28} &= \theta_{21} g_{18} + \theta_{22} g_{28} + \theta_{23} g_{38} = \phi_{28} = 2.968 \times 10^{-7} \\
(\theta G)_{51} &= \theta_{51} g_{11} + \theta_{52} g_{21} + \theta_{53} g_{31} = \phi_{51} = -6.240 \times 10^{-5} \\
(\theta G)_{52} &= \theta_{51} g_{12} + \theta_{52} g_{22} + \theta_{53} g_{32} = \phi_{52} = -4.160 \times 10^{-7} \\
(\theta G)_{56} &= \theta_{51} g_{16} + \theta_{52} g_{26} + \theta_{53} g_{36} = \phi_{56} = 0 \\
(\theta G)_{57} &= \theta_{51} g_{17} + \theta_{52} g_{27} + \theta_{53} g_{37} = \phi_{57} = 1.517 \times 10^{-5} \\
(\theta G)_{58} &= \theta_{51} g_{18} + \theta_{52} g_{28} + \theta_{53} g_{38} = \phi_{58} = 1.517 \times 10^{-7} \\
(\theta G)_{81} &= \theta_{81} g_{11} + \theta_{82} g_{21} + \theta_{83} g_{32} = \phi_{81} = 6.240 \times 10^{-3}
\end{align*}
\]
The eight additional constraint equations for the solution of the 24 elements of the feedback gain matrix \( G \) come from the pole placement requirements on the submatrices \( A_1, A_2 \) and \( A_3 \).

After the matrix \( \phi - \Theta G \) has been decoupled, it can be written as

\[
\phi(0.02) - \Theta(0.02)G = \begin{bmatrix}
- \Lambda_1 & 0 & 0 & 0 \\
0 & \Lambda_2 & 0 & 0 \\
0 & 0 & \Lambda_3 \\
\end{bmatrix}
\]

where

\[
\Lambda_1 = \begin{bmatrix}
\phi_{11} & \phi_{12} \\
\phi_{21} & \phi_{22}
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\phi_{11} & \phi_{12} \\
\phi_{21} - (\Theta G)_{21} & \phi_{22} - (\Theta G)_{22}
\end{bmatrix}
\]

\[
\Lambda_2 = \begin{bmatrix}
\phi_{33} & \phi_{34} & \phi_{35} \\
\phi_{43} & \phi_{44} & \phi_{45} \\
\phi_{53} & \phi_{54} & \phi_{55}
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
\phi_{33} & \phi_{34} & \phi_{35} \\
\phi_{43} & \phi_{44} & \phi_{45} \\
\phi_{53} - (\Theta G)_{53} & \phi_{54} - (\Theta G)_{54} & \phi_{55} - (\Theta G)_{55}
\end{bmatrix}
\]
The characteristic equation of the decoupled $x$ dynamics is

$$z - A_1 = \begin{bmatrix} \phi_{11} & -\phi_{12} \\ -\phi_{21} + (\theta G)_{21} & z - \phi_{22} + (\theta G)_{22} \end{bmatrix}$$

$$= z^2 + (z(\theta G)_{22} - \phi_{11} - \phi_{22})z + \phi_{11}\phi_{22} - \phi_{11}(\theta G)_{22} + \phi_{12}((\theta G)_{21} - \phi_{21})$$

$$= (z + a_1 + jb_1)(z + a_1 - jb_1)$$

where $a_1 = -0.9964457$ and $b_1 = -0.0035417$.

Now solving for the unknowns in Eq. (6-22), we have

$$(\theta G)_{21} = \theta_{21}\gamma_{11} + \theta_{22}\gamma_{21} + \theta_{23}\gamma_{31}$$

$$= \frac{1}{\phi_{12}}(a_1^2 + b_1^2 + \phi_{11}^2 + 2a_1\phi_{11} + \phi_{12}\phi_{21})$$

$$(\theta G)_{22} = \theta_{21}\theta_{12} + \theta_{22}\theta_{22} + \theta_{23}\theta_{32} = \phi_{11} + \phi_{12} + 2a_1$$

For the $\phi_1$ dynamics, the characteristic equation of $A_2$ is

$$z - A_2 = \begin{bmatrix} \phi_{31} & \phi_{34} & -\phi_{35} \\ -\phi_{43} & z - \phi_{44} & -\phi_{45} \\ -\phi_{53} + (\theta G)_{53} & -\phi_{54} + (\theta G)_{54} & z - \phi_{55} + (\theta G)_{55} \end{bmatrix}$$

$$= z^3 + (z(\theta G)_{55} - \phi_{55} - \phi_{44} - \phi_{33})z^2 + ((\phi_{44} + \phi_{33})(\theta G)_{55} + \phi_{45}(\theta G)_{54}$$
The desired characteristic equation is written as

\[(z + a_2 + jb_2)(z + a_2 - jb_2)(z + c_2) = 0\]  

(6-26)

where

\[c_2 = -0.01831564\]
\[a_2 = -0.2592945\]
\[b_2 = -0.3191948\]

Equating Eqs. (6-25) and (6-26), we get

\[(\theta G)_{55} = \theta_{51}g_{15} + \theta_{52}g_{25} + \theta_{53}g_{35} = \phi_{55} + \phi_{44} + 1 + 2a_2 + c_2\]  

(6-27)

\[(\theta G)_{54} = \theta_{51}g_{14} + \theta_{52}g_{24} + \theta_{53}g_{34}\]
\[= \frac{-\phi_{35}(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{45}(\phi_{35} + \phi_{34}\phi_{45} - \phi_{35}\phi_{44})} + \frac{1}{\phi_{45}} ((\phi_{44} + 1)(\phi_{55} + \phi_{44})
+ 1 + 2a_2 + c_2) - \phi_{44}\phi_{45} + \phi_{45}\phi_{54} - \phi_{55} - \phi_{44} + 2a_2c_2 + a_2^2 + b_2^2)\]  

(6-28)

\[(\theta G)_{53} = \theta_{51}g_{13} + \theta_{52}g_{23} + \theta_{53}g_{33}\]
\[= \frac{(1 + 2a_2 + a_2^2 + b_2^2)(1 + c_2)}{\phi_{35} + \phi_{34}\phi_{45} - \phi_{35}\phi_{44}}\]  

(6-29)

For the \(\phi_2\) dynamics, the characteristic equation of \(\Lambda_3\) is

\[|z| - \Lambda_3| = \begin{vmatrix}
  z - \phi_{66} & -\phi_{67} & -\phi_{68} \\
  -\phi_{76} & -\phi_{77} & -\phi_{78} \\
  -\phi_{86} + (\theta G)_{86} & -\phi_{87} + (\theta G)_{87} & z - \phi_{88} + (\theta G)_{88}
\end{vmatrix} \]
= z^3 + ((\theta G)_{88} - \phi_{88} - \phi_{77} - \phi_{66})z^2 + (-\phi_{77} + \phi_{66})(\theta G)_{88} + \phi_{78}(\theta G)_{87} + \phi_{68}(\theta G)_{86} - \phi_{76}\phi_{78} + \phi_{66}\phi_{88} + \phi_{66}\phi_{77} - \phi_{76}\phi_{78} - \phi_{68}\phi_{86} + \phi_{77}\phi_{78})z
+ ((\phi_{66}\phi_{77} - \phi_{76}\phi_{67})(\theta G)_{88} + (\phi_{68}\phi_{76} - \phi_{66}\phi_{78})(\theta G)_{87} + (\phi_{67}\phi_{78} - \phi_{68}\phi_{77})(\theta G)_{86} - \phi_{66}\phi_{77}\phi_{88} + \phi_{66}\phi_{78}\phi_{78} + \phi_{76}\phi_{67}\phi_{88} - \phi_{76}\phi_{68}\phi_{86} - \phi_{67}\phi_{78}\phi_{86} + \phi_{68}\phi_{77}\phi_{86})

The desired characteristic equation is written as

\[(z + a_3 + jb_3)(z + a_3 - jb_3)(z + c_3) = 0\] (6-31)

where
\[c_3 = -0.67032\]
\[a_3 = -0.911366\]
\[b_3 = -0.096987\]

Equating Eqs. (6-30) and (6-31), we get

\[(\theta G)_{86} = \theta_{81}q_{16} + \theta_{82}q_{26} + \theta_{83}q_{36}
\frac{(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{68} + \phi_{67}\phi_{78} - \phi_{68}\phi_{77}}\] (6-32)
\[(\theta G)_{87} = \theta_{81}q_{17} + \theta_{82}q_{27} + \theta_{83}q_{37}
\frac{-\phi_{68}(1 + 2a_3 + a_3^2 + b_3^2)(1 + c_3)}{\phi_{78}(\phi_{68} + \phi_{67}\phi_{78} - \phi_{68}\phi_{77})} + \frac{1}{\phi_{78}}(\phi_{77} + 1)(\phi_{88} + \phi_{77} + 1 + 2a_3 + c_3)
- \phi_{77}\phi_{78} - \phi_{78}\phi_{87} - \phi_{88} - \phi_{77} + 2a_3c_3 + a_3^2 + b_3^2)\] (6-33)
\[(\theta G)_{88} = \theta_{81}q_{18} + \theta_{82}q_{28} + \theta_{83}q_{38}
\phi_{88} + \phi_{77} + 1 + 2a_3 + c_3\] (6-34)

Equations (6-23), (6-24), (6-27), (6-28), (6-29), (6-32), (6-33) and (6-34) form the additional eight constraint equations, together with the 16 equations in Eq. (6-17) are adequate for the solution of the 24 unknown feedback gains in
The solution of the 24 constraint equations gives the following results:

\[ g_{11} = 35.432513595904985 \]
\[ g_{12} = 209.57461847387285 \]
\[ g_{13} = -9.368484250209332 \times 10^7 \]
\[ g_{14} = -8.1139701621271032 \times 10^6 \]
\[ g_{15} = -1.4453443216549823 \times 10^5 \]
\[ g_{16} = g_{17} = g_{18} = 0 \]
\[ g_{21} = -71.361744259885901 \]
\[ g_{22} = -409.94846384061196 \]
\[ g_{23} = 2.2392641102247482 \times 10^8 \]
\[ g_{24} = 1.9394105849068884 \times 10^7 \]
\[ g_{25} = 3.454675889387745 \times 10^5 \]
\[ g_{26} = 3.578272520216161 \times 10^5 \]
\[ g_{27} = 1.3239693135355619 \times 10^6 \]
\[ g_{28} = 1.2749737170875715 \times 10^4 \]
\[ g_{31} = g_{32} = 0 \]
\[ g_{33} = 4.015292755500438 \times 10^7 \]
\[ g_{34} = 3.4776162570406019 \times 10^6 \]
\[ g_{35} = 6.19468467849835 \times 10^4 \]
\[ g_{36} = 3.578272527392904 \times 10^5 \]
\[ g_{37} = 1.323969309412944 \times 10^6 \]
\[ g_{38} = 1.2749737129324844 \times 10^4 \]
VII. COMPUTER SIMULATION AND DESIGN OF THE DIGITAL ASPS

7.1 Introduction

In the preceding chapter the digital ASPS was designed through decoupling and pole placement. In general, the decoupling problem does not have a unique solution. However, in the ASPS it turns out that some of the elements of the matrices \( \phi(T) \) and \( \Theta(T) \) are very small so that the matrix \( \Theta S \) can be approximated by a matrix which has only one nonzero row. We recall that the design objective is to realize simultaneously the following bandwidth requirements for the \( x, \phi_1 \) and \( \phi_2 \) components of the system:

- \( x \): bandwidth = 0.04 Hz
- \( \phi_1 \): bandwidth = 10 Hz
- \( \phi_2 \): bandwidth = 1 Hz

However, the "partial" decoupling of these components means that the desired bandwidths cannot be realized directly by the design method outlined in Chapter VI. A trial-and-error method is used to determine the feedback gains to meet the design requirements. The method involves first assuming a desired bandwidth for \( x, BW = 0.04 \) Hz, then we compute the required feedback gains, simulate the digital system, and observe the time responses to see if the individual bandwidth requirement is satisfied. If any one of the bandwidth is not satisfied, we adjust the bandwidth of \( x \) again, and repeat the process until we come close to the desired bandwidths. Only the bandwidth of \( x \) is varied in this case, since the bandwidths of \( \phi_1 \) and \( \phi_2 \) are not so sensitive to parameter variations.

In Chapter VI the digital ASPS was designed using the sampling period of 0.02 seconds. In this chapter larger sampling periods are considered. It is shown that the digital ASPS with state feedback can be stabilized for
a sampling period as large as 0.1 second.

7.2 Computer Simulation Results, $T = 0.02$ sec

In Chapter VI, we set the sampling period at $T = 0.02$ sec, and required that the bandwidths are 0.04 Hz, 10 Hz and 1 Hz, respectively for the $x$, $\phi_1$ and $\phi_2$ components. These bandwidth requirements correspond to a set of $z$-plane eigenvalues which the closed-loop system must realize. The corresponding feedback gains are obtained as in Eq. (6-35). The computer program listing for the design and simulation of the digital ASPS is given in Table 7-1. Figure 7-1 shows the time response of $x(t)$ when the feedback gains are as given in Eq. (7-35). Notice that the response is very slow which corresponds to a system with a very narrow bandwidth, certainly much smaller than the desired value of 0.04 Hz. By reassigning the bandwidth of $x$, it is possible to fine tune the dynamics of $x$, $\phi_1$ and $\phi_2$. However, due to the coupling of the three components, it is extremely difficult to obtain the exact responses desired, depending on how the bandwidth is defined. In any case, the closed-loop eigenvalues of the system are placed at the desired location by the state feedback design. The desired eigenvalues are now specified as follows:

\[
\begin{align*}
\lambda_1, \lambda_2 &= 0.964471639922 \pm j0.00342947 \\
\lambda_3 &= 0.01831564 \\
\lambda_4, \lambda_5 &= 0.2592945 \pm j0.3191948 \\
\lambda_6 &= 0.67032 \\
\lambda_7, \lambda_8 &= 0.911366 \pm j0.096987
\end{align*}
\]

Note that only $\lambda_1$ and $\lambda_2$ are changed, but the other six roots are still the same as those specified in Chapter VI. The pole-placement design now yields the following feedback gains:

\[ g_{11} = 3643.66148126889 \]
\[ g_{12} = 2128.0189640019839 \]
\[ g_{13} = -9.3684842502093316 \times 10^7 \]
\[ g_{14} = -8.11397701621271032 \times 10^6 \]
\[ g_{15} = -1.4453443216549823 \times 10^5 \]
\[ g_{16} = g_{17} = g_{18} = 0 \]
\[ g_{21} = -42330.651505108181 \]
\[ g_{22} = -10394.903057183335 \]
\[ g_{23} = 2.2392641102247482 \times 10^8 \]
\[ g_{24} = 1.9394105849068884 \times 10^7 \]
\[ g_{25} = 3.454675889387745 \times 10^5 \]
\[ g_{26} = 3.5782725250216161 \times 10^5 \]
\[ g_{27} = 1.3239693135355619 \times 10^6 \]
\[ g_{28} = 1.2749737170875715 \times 10^4 \]
\[ g_{31} = g_{32} = 0 \]
\[ g_{33} = 4.0152927555004380 \times 10^7 \]
\[ g_{34} = 3.4776162570406019 \times 10^6 \]
\[ g_{35} = 6.194684677849835 \times 10^4 \]
\[ g_{36} = 3.5782725273929039 \times 10^5 \]
\[ g_{37} = 1.323969309412944 \times 10^6 \]
\[ g_{38} = 1.2749737129324844 \times 10^4 \]

Note that except for \( g_{11}, \ g_{12}, \ g_{21} \) and \( g_{22} \) all the other feedback gains are identical to those given in Eq. (6-35). The time responses of \( x, \phi_1 \) and \( \phi_2 \) for these feedback gains are shown in Figs. 7-2, 7-3 and 7-4, respectively. These responses are very close to having the desired bandwidths.
### Table 7-1. Computer Program For the Design and Simulation of the Digital ASPS

```plaintext
IMPLICIT REAL*8 (A-H,O-Z)
C
C CALCULATION OF GWH AND GWH FOR THE POINT & POLLED METHOD
C CALCULATION BY SERIES SUMMATION
DIMENSION A(8,8),B(8,8),DO,3,2,DM(3,8),TO(1,1),CUL(1,1)
DIMENSION STMTS(9,8),II(3,8),THF(5,5),HILT(8,8),N(8,8)
DIMENSION STHHT(8,8),LW(3),H(3),WORK(3,3),WORK(3,3),WORK(3,3)
DIMENSION WORK4(3,3),WORK5(3,8),WORK3(3,3),WORK7(3,3),WORK7(3,3)
C
DIMENSION WI(3,3),WO(3,8)

DET=0.
KMAX=2000
N=8
M=3
DO 18 I=1,N
DO 17 J=1,M
R(I,J)=0,
H(J,J)=C.
17 G0(J,T)=0.
DO 18 J=1,N
A(T,J)=0.

C SYSTEM CONSTANTS
C
AMI=600.
AKSX=1.051
AKSF=0.005
AKSS=0.005
AJ=503.
A33=2605.15
R8=1.956
AO=R8
BO=1./AMI
CO=AKSX*R8/A33
DO=AJ/A33
E=1./A33
F=1./AJ
DELTA=1.-AO*CO-DO

C NON-ZERO ENTRIES OF THE MATRICES
C
A(1,2)=1.
A(2,1)=-BO*AKSX*(1.-DO)/DELTA
A(2,4)=-E*AKSF*AO/DELTA
A(2,7)=AKSS*F*DO*AO/DELTA
A(3,4)=1.
A(4,5)=1.
A(5,1)=-AKSX*BO*CO/DELTA
A(5,4)=-AKSF*E/DELTA
A(5,7)=AKSS*DO*F/DELTA
A(6,7)=1.
```

*ORIGINAL PAGE IS OF POOR QUALITY*
A(7,3) = 1.
A(8,1) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}
A(8,4) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}

A(8,7) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}
A(\theta, 1) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}
B(2, 1) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}
B(2, 2) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}

R(3, 1) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}
R(3, 2) = \frac{6 \times \sin(x) \times \cos(2x)}{\Delta\theta}

\text{GO(1, 1) = 3.6447E1}
\text{GO(1, 2) = 2.124E1}
\text{GO(1, 3) = 2.3563E1248E0}
\text{GO(1, 4) = 2.5176E126624E7}
\text{GO(1, 5) = 2.39874548E5}
\text{GO(1, 6) = 0.6E}
\text{GO(1, 7) = 7.355E1}
\text{GO(2, 2) = 4.154144E1}
\text{GO(2, 3) = 2.2126125525E9}
\text{GO(2, 4) = 6.0882562624E7}
\text{GO(2, 5) = 8.10127328E5}
\text{GO(2, 6) = 3.7675304E5}
\text{GO(2, 7) = 1.0917031532E5}
\text{GO(2, 8) = 1.4522640E4}
\text{GO(3, 3) = 1.56750304E8}
\text{GO(3, 4) = 1.0917031532E7}
\text{GO(3, 5) = 1.4522640E5}
\text{GO(3, 6) = 3.96750304E5}
\text{GO(3, 7) = 1.0917031532E5}
\text{GO(3, 8) = 1.4522640E4}

\text{H(1, 3) = 0.99}
\text{H(1, 4) = 0.01}
\text{H(2, 3) = 0.91}
\text{H(2, 4) = 0.98}
\text{H(3, 6) = 0.01}

\text{H(3, 7) = 0.98}
\text{H(3, 8) = 0.01}

\text{TYPE 107}
\text{DD 201 J=1, N}

\text{201 TYPE 110, (A, (2, 1), 1=1, N)}
\text{TYPE 108}
\text{DD 202 J=1, N}

\text{202 TYPE 210, (B, (1, 1), 1=1, M)}
\text{TYPE 109}
\text{DD 203 J=1, M}

\text{ORIGINAL PAGE IS OF POOR QUALITY}
-03  TYPE I10F(RO(JT),I=1,N)

   C   CALL PRINT PERIOD
   I=0
   TYPE 100
   TYPE 501

301  FORMAT (4X,'FIRST PASS UI STRMAT')
      CALL STRMAT(A,N,T,STMTS,WORK3,K1,WORK4,WORK5)
      IF(K1-KMAX) 30,30,20

30  CALL BMFRD(WORK3,8,THETAS,N,N)
      TYPE 401
      FORMAT (4X,'THE EXP(AAT) MATRIX')
501  TYPE 110(I=1,N)
      FORMAT (4X,'THE THERA MATRIX')
      DO 404 I=1,N
      P1=3.1415926536
      DSO2=1.57079632679
      FAC1=1.0
      FAC2=0.8
      FAC3=0.13

      AL=-DEXP(-FAC1)*COS(FAC1)
      UI=-DEXP(-FAC2)*SIN(FAC2)
      C1=-DEXP(-200.)*T
      A2=-DEXP(-FAC2)*COS(FAC2)
      C2=-DEXP(-20.4)*T
      U3=-DEXP(-FAC3)*SIN(FAC3)
      TYPE 502
      FORMAT (4X,'####POLES OF THE DISCRETE SYSTEM')
   TYPE 502,A1,B1,C1
   TYPE 502,A2,B2,C2
   TYPE 502,A3,B3

502  FORMAT (4X,'I0P3E21.12')
      WI(1,J)=THETAS(2,J)
      WI(1,2)=THETAS(2,2)
      WI(1,3)=THETAS(2,3)
      WI(2,J)=THETAS(5,J)
      WI(2,1)=THETAS(5,1)
      WI(2,2)=THETAS(5,2)
      WI(2,3)=THETAS(5,3)
      WI(3,1)=THETAS(8,1)
      WI(3,2)=THETAS(8,2)
      WI(3,3)=THETAS(8,3)
      W0(1,J)=(1.*STMTS(1,J))*A3*A3*83*83+STMTS(1,1)*STMTS(1,J)
      & 12.*83+STMTS(1,J)*STMTS(J,2)*STMTS(2,J)
      W0(2,J)=STMTS(5,J)
      W0(3,J)=STMTS(8,J)
      W0(1,2)=STMTS(1,1)+STMTS(2,2)+2.*S
<table>
<thead>
<tr>
<th>Row</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>W0(2,3)=STMTS(3,4)</td>
</tr>
<tr>
<td>2</td>
<td>W0(3,2)=STMTS(4,5)</td>
</tr>
<tr>
<td>3</td>
<td>W0(1,3)=STMTS(2,3)</td>
</tr>
<tr>
<td>4</td>
<td>W0(7,5)={1+y^2+3y+1}ALW1+2y+1}*(1+y^2)/STMTS(5,5)</td>
</tr>
<tr>
<td>5</td>
<td>W0(2,4)=(1/STMTS(4,5))*{(STMTS(4,5)+1)*STMTS(5,5)}</td>
</tr>
<tr>
<td>6</td>
<td>+STMTS(4,4)+1+2,A1(1)+STMTS(4,4)*STMTS(5,5)</td>
</tr>
<tr>
<td>7</td>
<td>=STMTS(3,5)*STMTS(4,5)-STMTS(4,4)*1+2,A1(1)</td>
</tr>
<tr>
<td>8</td>
<td>W0(2,5)=STMTS(5,5)</td>
</tr>
<tr>
<td>9</td>
<td>W0(3,6)=STMTS(5,6)</td>
</tr>
<tr>
<td>10</td>
<td>W0(1,7)=STMTS(2,7)</td>
</tr>
<tr>
<td>11</td>
<td>W0(2,7)=STMTS(3,7)</td>
</tr>
<tr>
<td>12</td>
<td>W0(3,7)={(1.+STMTS(7,8))*{(STMTS(7,8)+1)*STMTS(8,0)}}</td>
</tr>
<tr>
<td>13</td>
<td>=STMTS(7,1)*STMTS(7,1)+STMTS(7,1)*2,A2(1),C2</td>
</tr>
<tr>
<td>14</td>
<td>+STMTS(7,2)*STMTS(8,2)-STMTS(8,2)+STMTS(8,2)+2,A2(1),C2</td>
</tr>
<tr>
<td>15</td>
<td>+STMTS(7,3)*STMTS(7,3)+STMTS(7,3)+2,A2(1),C2</td>
</tr>
<tr>
<td>16</td>
<td>+STMTS(7,4)*STMTS(7,4)+STMTS(7,4)+2,A2(1),C2</td>
</tr>
<tr>
<td>17</td>
<td>W0(2,8)=STMTS(5,8)</td>
</tr>
<tr>
<td>18</td>
<td>W0(3,8)=STMTS(7,8)+STMTS(7,8)+1+2,A2(1),C2</td>
</tr>
<tr>
<td>19</td>
<td>CALL ONPRI(WU,WW,UM,M,M,N)</td>
</tr>
<tr>
<td>20</td>
<td>CALL ONPRD(STMTS,GW,WORK7,N,M,N)</td>
</tr>
<tr>
<td>21</td>
<td>CALL ONSUB(STMTS,WORK7=WORK6,+N,N)</td>
</tr>
</tbody>
</table>

**Type 503**

<table>
<thead>
<tr>
<th>Format</th>
<th>THE FEEDBACK COEFFICIENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>504 I=1,N</td>
</tr>
</tbody>
</table>

**Type 504**

<table>
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<tr>
<th>Format</th>
<th>THE FEEDBACK COEFFICIENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>505 I=1,N</td>
</tr>
</tbody>
</table>

**Type 505**

<table>
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<tr>
<th>Format</th>
<th>THE PHI-THETA*G MATRIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>506 I=1,N</td>
</tr>
</tbody>
</table>

**Type 506**

<table>
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<tr>
<th>Format</th>
<th>THE PHI-THETA*G MATRIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>507 I=1,N</td>
</tr>
</tbody>
</table>

**Type 507**

<table>
<thead>
<tr>
<th>Format</th>
<th>THE PHI-THETA*G MATRIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>508 I=1,N</td>
</tr>
</tbody>
</table>

**Type 508**
CALL EXIT

100 FORMAT(/4X • 4HOET= • IPE= 120.4)
101 FORMAT(/4X • 1B8WEIGHTED MATRIX H.
102 FORMAT(/4X • 4SHRTAT, TRANSITION MATRIX FOR CONTINUOUS SYSTEM 10
t12HE= 1.0).
103 FORMAT(/4X • 4SHRTAT, TRANSITION MATRIX FOR SAMPLE, DATA SYSTEM L1
t21HE= 1.0).
104 FORMAT(/4X • 2HRTW)
105 FORMAT(/4X • 2HRTW)
106 FORMAT(/4X • 2HRTW)
107 FORMAT(/4X • 2HRTW)
108 FORMAT(/4X • 2HRTW)
109 FORMAT(/4X • 2HRTW)
110 FORMAT(/4X • 2HRTW)
111 FORMAT(/4X • 2HRTW)
210 FORMAT(/2X • F17.12)

END

SUBROUTINE IMPRU(A=R+R*N+I)
IMPLICIT REAL*8(A-N+1)
DIMENSION A(N+1),B(N+1),R(N+1)

DO 1 I=1,N
1 R(I,J)=R(I,J)+A(I,K)*B(K,J)
RETURN

END

SUBROUTINE GMUL(A,B=R+R*N+I)
IMPLICIT REAL*8(A-N+1)
DIMENSION A(N+1),B(N+1),R(N+1)

TR=0
JK=-M
DO 10 K=1,M
TR=TR+K
10 TR=TR+1
1 J=1,N
10 J=1,N
1 R=TR+1

J1=J1+M
TJ=TJ+K
R(JTR)=0
10 R(JTR)=R(JTR)+A(J1)•B(J1)
RETURN

END

SUBROUTINE UNSUR(A,B=R+N+I)
IMPLICIT REAL*8(A-N+1)
DIMENSION A(N+1),B(N+1),K(N+1)

DO 1 I=1,N
1 DO 1 J=1,M
1 R(J,J)=A(J,J)-B(J,J)
RETURN

END

SUBROUTINE ONPSC(A,B=R+N+1)
IMPLICIT REAL*8(A-N+1)
DIMENSION A(N+1),K(N+1)
BO 1 I=1,N
C
C
130

IMPLICIT REAL*(N-L-0-Z)

DIMENSION A(N,N),PHI(N,N),THETA(N,N),W1(N,N),W2(N,N)

N2=N*N
ALPHA=1.E-5
BETA=1.E-40
KMIN=2000
LIN=0

C
C

C

INITIALIZATION
C
C

ON 3 I=1,N
GO TO 3
IF(I,J) LT 1

1

PHI(I,J)=0.
W1(I,J)=0.
THETA(I,J)=0.

GO TO 3

2

PHI(I,J)=1.
W1(I,J)=T
THETA(I,J)=T

CONTINUE
C

NORM OF A
C

TYPE 601

601 FORMATT(/4X,'THE PASSED A MATRIX')
GO TO 600 I=1,N

600 TYPE 203*(A(I,J),J=1,N)

TYPE 602

602 FORMATT(4X,'**')

ANORM=0.

GO 4 I=1,N
GO 4 J=1,N

4 ANORM=ANORM+ABS(A(I,J))

K=2
C
C

START OF LOOP
CALL BHPSC(W2,W1,C,N,N)
CALL BMAUU(W1,HETAT,HETB,N,N)
FMT(16)

IF(ANORM*T) 51,51,7
LIN=1
GO TO 5

7 B NORM=0.
DO 8 I=1,N
DO 8 J=1,N

8 B NORM=BNORM+DABS(W2(I,J))
EPSLON =ANORM*T/(AK+?)
TEST=BNORM*ANORM*T/(AK+1.)*K(1.-EPSLON))
DO 9 I=1,N
DO 9 J=1,N
IF(DABS(PHI(I,J))>BETA) 9,9,10

9 IF(T E S T - A L P H A * D A B S ( P H I ( I , J ) ) ) 4,4,52
LIN=2
GO TO 5

9 CONTINUE
RETURN

5 IF(K-KMAX) 11,12,12

11 K=K+1
LIN=0
GO TO 13

12 TYPE 100,K
   TYPE 207,LIN

20 FORMAT(4X,STS A T U S H 1 S ,2)
   TYPE 500,ANORM,K

500 FORMAT(4X,ANORM=,1PE14.7,,K=,15)
RETURN

100 FORMAT(20H K GREATER THAN KMAX,15/)
CND SUBROUTINE MINV(A,N,0,L,Y)
CND IMPLICIT REAL(A-H,U-Z)
CND DIMENSION A(1),L(1),N(1)

C SEARCH FOR LARGEST ELEMENT

C

D=1.0
NK=N
DO 80 K=1,N
NK=NK+1

80 L(K)=K
M(K)=K
KK=NK+K
BIGA=A(KK)
B0 20 J=K,N
LJ=1N(J-1)
B0 20 I=K+N
LI=1N(I)
10 IF(|ABS(BIGA)|ABS(A(IJ)))<5,0,0
15 BIGA=A(IJ)
     L(K):=I
     M(K):=J
20 CONTINUE

C INTERCHANGE ROWS
C
J=L(K)
LJ=1N(J-K)
UJ=1N
25 KI=K-N
DO 30 I=1,N
       KI=KI+N
       HOLD=A(KI)
       JI=KI+KJ
       A(KI)=A(JI)
30 A(JI):=HOLD

C INTERCHANGE COLUMNS
C
35 J=M(K)
    IF(J=1-N)55,55,58
38 JF=M(J-1)
DO 40 J=1,N
       JK=NH+1
       JI=JF+1
       HOLD=-A(JK)
       A(JK)=A(JI)
40 A(JI):=HOLD

C DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C CONTAINED IN BIGA)
C
45 IF(BIGA)48,46,49
48 D=-0.0
RETURN
48 30 55 I=1,N
60 IF(I=1-K)50,55,50
50 IK=NK+I
     A(IK)=A(IK)/(-BIGA)
55 CONTINUE

C REDUCE MATRIX
C
DO 65 J=1,N
     IK=NK+I
HOLD = H(1K)
1J = J - 1
IF (J < N) T,J = J - 1
70 A(J,J-K) = A(J,J-K) + A(J,J)
70 CONTINUE

C: UTILITY ROW BY PIVOT
C
KJ = K - 1
70 DO 7J J = 1, N
KJ = KJ + 1
IF (J < K) 70, 75 = 70
70 A(KJ) = A(KJ) / B(JJ)
75 CONTINUE

C: PRODUCT OF PIVOTS
C
P = 0
C: REPLACE PIVOT BY RECIPROCAL
C
A(KK) = 1.0 / P
80 CONTINUE

C: FINAL ROW AND COLUMN INTERCHANGE
C
K = N
100 K = K - 1
IF (K < 1) 100, 105
105 I = I(K).
108 J0 = (K - J)
100 JR = N - J0
110 DO 110 J = 1, N
110 A(JK) = -A(JK)
120 J = M(K)
125 IF (J < K) 125, 130
130 A(JJ) = HOLD
120 J = M(K)
125 KI = K - N
130 DO 130 I = 1, N
KI = KI + 1
HOLD = A(KI)
JI = KI - J
A(KI) = -A(JI)
130 CONTINUE
150 RETURN
Figure 7-1. Time response of $x$, $T = 0.02$ sec, with feedback gains given in Eq. (6-35).
Figure 7-1. Time response of $\phi$; $\tau = 0.02$ sec, with feedback gains.
given in Eq. (7-11).
Fig. 7-4: Time response of $y_2(t)$ with $T = 0.02$ sec and the feedback gains given in Eq. (7-1).
7.3 Computer Simulation Results, $T = 0.04$ sec

It is of interest to investigate if the sampling period can be increased and still be able to realize state feedback control to achieve the desired bandwidths for the three system components. The sampling period is now increased to 0.04 sec. The matrices $\phi(T)$ and $\theta(T)$ for $T = 0.04$ sec are given as follows:

\[
\phi(0.04) =
\begin{bmatrix}
9.995 \times 10^{-1} & 3.999 \times 10^{-2} & 0 & -1.187 \times 10^{-6} & -1.583 \times 10^{-8} & 0 & 1.187 \times 10^{-6} & 1.583 \times 10^{-8} \\
-2.447 \times 10^{-2} & 9.995 \times 10^{-1} & 0 & -5.936 \times 10^{-5} & -1.187 \times 10^{-6} & 0 & 5.936 \times 10^{-5} & 1.187 \times 10^{-6} \\
-3.328 \times 10^{-6} & -3.33 \times 10^{-8} & 1 & 3.999 \times 10^{-2} & 7.999 \times 10^{-4} & 0 & 8.094 \times 10^{-9} & 8.094 \times 10^{-11} \\
-2.496 \times 10^{-4} & -3.33 \times 10^{-6} & 0 & 9.999 \times 10^{-1} & 3.999 \times 10^{-2} & 0 & 6.070 \times 10^{-7} & 8.094 \times 10^{-9} \\
-1.247 \times 10^{-2} & -2.49 \times 10^{-4} & 0 & -3.035 \times 10^{-5} & 9.999 \times 10^{-1} & 0 & 3.035 \times 10^{-5} & 6.070 \times 10^{-7} \\
3.328 \times 10^{-6} & 3.328 \times 10^{-8} & 0 & 8.094 \times 10^{-9} & 8.094 \times 10^{-11} & 1 & 3.999 \times 10^{-2} & 7.999 \times 10^{-4} \\
-3.296 \times 10^{-4} & 3.328 \times 10^{-6} & 0 & 6.070 \times 10^{-7} & 8.094 \times 10^{-9} & 0 & 9.999 \times 10^{-1} & 3.999 \times 10^{-2} \\
1.247 \times 10^{-2} & 2.496 \times 10^{-4} & 0 & 3.035 \times 10^{-5} & 8.094 \times 10^{-7} & 0 & -3.074 \times 10^{-5} & 9.999 \times 10^{-1}
\end{bmatrix}
\]

(7-2)

\[
\theta(0.04) =
\begin{bmatrix}
4.658 \times 10^{-4} & 2.374 \times 10^{-4} & -2.347 \times 10^{-4} \\
2.329 \times 10^{-2} & 1.187 \times 10^{-2} & -1.187 \times 10^{-2} \\
3.166 \times 10^{-6} & 1.618 \times 10^{-6} & -1.618 \times 10^{-6} \\
2.374 \times 10^{-4} & 1.214 \times 10^{-4} & -1.214 \times 10^{-4} \\
1.187 \times 10^{-2} & 6.070 \times 10^{-3} & -6.070 \times 10^{-3} \\
-3.166 \times 10^{-6} & -1.618 \times 10^{-6} & 1.640 \times 10^{-6} \\
-2.374 \times 10^{-4} & -1.214 \times 10^{-4} & 1.230 \times 10^{-4} \\
-1.187 \times 10^{-2} & -6.070 \times 10^{-3} & 6.149 \times 10^{-3}
\end{bmatrix}
\]

(7-3)
The desired eigenvalues of the closed-loop ASPS are set at:

\[ x: \quad z_1, z_2 = 0.976901072005 + j0.00225733531031 \]
\[ \phi_1: \quad z_3 = 0.0003354626270 \]
\[ \phi_2: \quad z_4, z_5 = 0.0028068097799 + j0.003939345 \]
\[ z_6 = 0.4493289641172 \]
\[ z_7, z_8 = 0.8587194942988 + j0.16334107966 \]

The feedback gains are determined and are tabulated as follows:

\[ g_{11} = 3.8178805894610170 \times 10^2 \]
\[ g_{12} = 6.8560283114373904 \times 10^2 \]
\[ g_{13} = -1.8228873974050188 \times 10^7 \]
\[ g_{14} = -2.5313721087607277 \times 10^6 \]
\[ g_{15} = -8.7845436116937588 \times 10^4 \]
\[ g_{16} = g_{17} = g_{18} = 0 \]
\[ g_{21} = -7.4488328494704123 \times 10^2 \]
\[ g_{22} = -1.3410796296921878 \times 10^3 \]
\[ g_{23} = 4.3570829799433742 \times 10^7 \]
\[ g_{24} = 6.0505099423975705 \times 10^6 \]
\[ g_{25} = 2.0996900572869749 \times 10^5 \]
\[ g_{26} = 2.0185680288217810 \times 10^5 \]
\[ g_{27} = 3.6134117793464126 \times 10^5 \]
\[ g_{28} = 1.0477885260603344 \times 10^4 \]
\[ g_{31} = g_{32} = 0 \]
\[ g_{33} = 7.8128183443659252 \times 10^6 \]
\[ g_{34} = 1.0849353874460890 \times 10^6 \]
\[ g_{35} = 3.7650182657481631 \times 10^4 \]
\[ g_{36} = 2.0185680341725262 \times 10^5 \]
\[ g_{37} = 3.6134117389247105 \times 10^5 \]
The time responses of $x(t)$, $\phi_1(t)$ and $\phi_2(t)$ for $T = 0.04$ and the feedback gains as listed above are shown in Figs. 7-5, 7-6 and 7-7, respectively.

7.4 Computer Simulation Results, $T = 0.1$ sec

Since the on-board computer has a limited capacity and processing speed, it is important that the sampling period of the ASPS should be kept as large as possible. It is well known that in general large sampling periods are detrimental to the stability of digital control systems. In the preceding sections we have shown the design of the feedback gains for the ASPS to realize the bandwidth requirements with sampling periods of 0.02 and 0.04 seconds. In this section we shall show that the ASPS can be operated with a sampling period as large as 0.1 second.

For $T = 0.1$ second, the $\phi(T)$ and $\theta(T)$ matrices are,

$$\phi(0.1) = \begin{bmatrix}
9.969 \times 10^{-1} & 9.999 \times 10^{-2} & 0 & -7.418 \times 10^{-6} & -2.473 \times 10^{-7} & 0 & 7.418 \times 10^{-6} & 2.473 \times 10^{-7} \\
-6.114 \times 10^{-2} & 9.969 \times 10^{-1} & 0 & -1.483 \times 10^{-4} & -7.418 \times 10^{-6} & 0 & 1.483 \times 10^{-4} & 7.418 \times 10^{-6} \\
-5.199 \times 10^{-5} & -1.299 \times 10^{-6} & 1 & 9.999 \times 10^{-2} & 4.999 \times 10^{-3} & 0 & 1.264 \times 10^{-7} & 3.161 \times 10^{-9} \\
-1.559 \times 10^{-3} & -5.198 \times 10^{-5} & 0 & 9.999 \times 10^{-1} & 9.999 \times 10^{-2} & 0 & 3.792 \times 10^{-6} & 1.264 \times 10^{-7} \\
-3.117 \times 10^{-2} & -1.559 \times 10^{-3} & 0 & -7.581 \times 10^{-5} & 9.999 \times 10^{-1} & 0 & 7.581 \times 10^{-5} & 3.792 \times 10^{-6} \\
5.199 \times 10^{-5} & 1.299 \times 10^{-6} & 0 & 1.264 \times 10^{-7} & 3.161 \times 10^{-9} & 1 & 9.999 \times 10^{-2} & 4.999 \times 10^{-3} \\
1.559 \times 10^{-3} & 5.198 \times 10^{-5} & 0 & 3.792 \times 10^{-6} & 1.264 \times 10^{-7} & 0 & 9.999 \times 10^{-1} & 9.999 \times 10^{-2} \\
3.117 \times 10^{-2} & 1.559 \times 10^{-3} & 0 & 7.581 \times 10^{-5} & 3.793 \times 10^{-6} & 0 & -7.681 \times 10^{-5} & 9.999 \times 10^{-1}
\end{bmatrix}$$

(7-5)
Fig. 7.5. The response of $x(t)$ for $t > 0$ sec, with the feedback given in Eq. (7.4).
Corresponding to the desired bandwidths of the three system components, the eigenvalues of the closed-loop digital ASPS are set at:

\[ x: \quad z_1, z_2 = 0.9423049137666 + j 0.05448575545211 \]

\[ \phi_1: \quad z_3 = 2.06 \times 10^{-9} \]

\[ \phi_2: \quad z_4, z_5 = 0.5790229524118 + j 0.27563222675412 \]

\[ z_6 = 0.1353352832366 \]

\[ z_7, z_8 = 0.6570647872999 + j 0.4965162436514 \]

The feedback gains are determined and are tabulated as follows:

\[ g_{11} = 356.54391190867254 \]

\[ g_{12} = 673.937543782499 \]

\[ g_{13} = -2.971496035004556 \times 10^5 \]

\[ g_{14} = -2.1929182981938171 \times 10^5 \]

\[ g_{15} = -2.1617128726694806 \times 10^4 \]

\[ g_{16} = g_{17} = g_{18} = 0 \]

\[ g_{21} = -699.45360421649164 \]

\[ g_{22} = -1318.3208244459652 \]

\[ g_{23} = 7.102498404377198 \times 10^5 \]

\[ g_{24} = 5.2415343590564353 \times 10^4 \]
The time responses of $x$, $\phi_1$ and $\phi_2$ of the digital ASPS with $T = 0.1$ and the feedback gains in Eq. (7-7) are shown in Figs. 7-8, 7-9, and 7-10, respectively. These time responses show that with the proper design, the digital ASPS can operate with a relatively large sampling time without any difficulties.
Figure 7-6: Time response of $x; T = 0.1$ sec. with the feedback gains given in Eq. (7.7).
Figure 7-10. Time response of $\phi_2(t)$, $T = 0.1$ sec, with the feedback gains given in Eq. (7-7).
VIII. QUANTIZATION EFFECTS IN THE DIGITAL ASPS

8.1 Introduction

In the preceding chapters the digital ASPS was designed through decoupling and pole-placement techniques, and the designed system was simulated on the digital computer. It was shown that the bandwidth requirements were approximately satisfied by digital feedback, and the computer simulation results showed that the responses of the \( x, \phi_1 \) and \( \phi_2 \) dynamics were quite good. With proper design, the digital ASPS can operate with a relatively large sampling period of \( T = 0.1 \) sec without any difficulty.

In reality, because of the finite bit length used in a digital computer, the state-feedback gains designed for the decoupled digital ASPS must be quantized. It should also be noted that the quantization effect due to operations in the digital computer is not the same as that introduced by the A/D converter.

In this chapter we shall investigate the effects of amplitude quantization in the digital ASPS with state feedback. In general, it is well-known that quantization can lead to two possible difficulties in system performance. One form of difficulty is sustained oscillations or instability due to amplitude quantization. The second type of performance degradation due to quantization is in the form of steady-state errors.

Although closed-form analytical methods are available for the studies of sustained oscillations and steady-state error due to quantization in digital control systems, the complexity and high-order nature of the closed-loop digital ASPS precludes the effective use of these methods. The discrete describing function method can be applied to the study of sustained oscillations in digital control systems due to one quantizer. However, the closed-loop
digital ASPS has many A/D converters and many quantizers representing the
digital computer operation of the feedback gains. The complexity of the
system also does not allow an analytical solution to the steady-state error
by the maximum-upper-bound method, treating all quantizers as noise sources.

The method adopted in this chapter is computer simulation. It seems
that considering all difficulties, computer simulation still resulted in
useful conclusions to the problem of quantization.

8.2 Quantization Considerations in the ASPS

In a digital computer a real number is usually represented by two parts:
the mantissa and the exponent. Figure 8-1 illustrates the floating-point
representation of the number X by an N-bit mantissa M and an exponent part.
Thus, X can be expressed as

\[ X = M \times B^e \]  \hspace{1cm} (8-1)

where \( M \) is the mantissa, \( B \) is the base number, 2 in general, and \( e \) is an
integer which represents the order of the exponent. The N-bit floating-point
number \( M \) is generally scaled to be

\[ B^{-1} < M < 1 \]  \hspace{1cm} (8-2)

and two's complement arithmetic is used for denoting negative numbers.

In numerical computation, amplitude quantization is introduced by the
finite length of the mantissa. However, the truncation error introduced by
the quantization effect is not a fixed value; it depends on the exponential
scaling; i.e., it is proportional to \( B^e \). Therefore, if the bit length of
the mantissa is \( N \), the quantization level is \( 2^{-N} \times B^e \). The quantization of
a given number should begin with the calculation of the mantissa \( M \) and the
exponent \( e \). The flow chart given in Fig. 8-2 shows how the quantization
operation is implemented.

<table>
<thead>
<tr>
<th>X:</th>
<th>mantessa</th>
<th>exponent</th>
</tr>
</thead>
</table>

Figure 8-1. Representation of a real number

Table 8-1 shows the subroutine FUNCTION QT(X) which is used for the simulation of the 16-bit quantization of the feedback gains for the digital ASPS.

Table 8-1

```
FUNCTION QT(X)
H=2.**-16
IF(X.NE.0.) GO TO 1
QT=0.
RETURN
1
X=ABS(X)
SIGN=X/X
IF(X.EQ.0.) 2,3,4
3
QT=X
RETURN
2
K=0
22 IF(X.GT.10.**K) GO TO 21
K=K-1
GO TO 22
1
K=1
42 IF(X.LT.10.**K) GO TO 21
K=K+1
GO TO 42
21 BX=X/10.**K
IV=(BX/H)+Q...
QT=SIGN*FLOAT(IV)*H*10.**K
RETURN
END
```
START WITH REAL NUMBER $X$

$H = 2^{-N}$

Quantization level on mantessa

$X = 0$ ?

YES

$QT = 0$

RETURN

NO

$|X| < 1$ ?

YES

$|X| = 1$

YES

$QT = X$

RETURN

NO

$K = 1$

$|X| < B^K$

YES

$K = K + 1$

NO

$K = 0$

$|X| > B^{K-1}$

YES

$M = \frac{|X|}{B^K}$

$IV = \frac{M}{H} + 0.5$

$M = IV \ast H \ast \text{SIGN}(X)$ Mantessa

$QT = M \ast B^K$ Quantized value

RETURN

Figure 8-2. Quantization operation.
8.3 Scaling Effects on the A/D Converters

The block diagram of the closed-loop digital ASPS with state feedback is shown in Fig. 8-3. The digital computer which implements the digital controller is interfaced with the ASPS through the A/D and D/A converters.

The A/D converters serve as sample-and-hold devices and quantizers because of their finite bit lengths. If the A/D converter has \( N \) bits in the output and is biased with voltage \( V \), then the full-scale input \( V \) volts will result in a digital number \( 1 - 2^{-N} \), which when taken by the digital controller will be interpreted as \( V(1 - 2^{-N}) \) which is the quantized value of \( V \). Therefore, the A/D converter, together with the necessary I/O port in the computer, can be considered as an A/D converter with normalized 1 volt bias, i.e., simply a quantizer. The functional block diagram is shown in Fig. 8-4.

Since the steady-state error due to quantization is directly related to the quantization level, the accuracy in the responses of the ASPS will depend on the quantization of the A/D converters. The most straightforward method of increasing the accuracy of the system is to increase the bit length of the A/D converter. However, this is an expensive proposition, and the cost can be saved by making use of the scaling technique in the A/D converter.

In the ASPS controller designed in the preceding chapters, the sensor gains were assumed to be unity. For example, if the error in the x component is \( X \) meter, we assume that the sensor sensed \( X \) volts, and the controller received \( Q_N(X) \) as a number for data processing. Because of the quantization effect in the controller, the steady-state error will be confined to be within \( 2^{-N} \) meter, where \( N \) is the number of bits in the A/D output. In general, \( N \) can be 8 or 12 ( \( N = 12 \) is used here), and the error is apparently not satisfactory.
Figure 8-3. Block diagram of the ASPS with the state-feedback digital controller and quantizers.
If a sensor gain of 1000 is introduced before the sample-and-hold, as shown in Fig. 8-4, then the X meter error in the system will be sensed as $1000X$ mm. After the quantization, the error will be confined to be within $2^{-N}$ mm. The digital computer takes $Q_n(1000X)$ as an input number, and performs a division-by-1000 to convert it back to the number $Q_n(1000X)/1000 = X$ which is of the appropriate dimension. Since there are reciprocal gain factors at both sides of the quantizer, as shown in Fig. 8-5, the transfer function between points A and B as seen from the quantizer Q is the same as that between points C and D. Therefore, the control scheme thus far designed will not be affected by the application of a sensor gain.

For the eight state variables in the ASPS, it can be seen that the scaling factors for each of these variables will not be the same for best system performance. As a matter of fact, the sensor gain should not be too low so as to have larger quantization error. On the other hand, the sensor gain should not be too high so as to cause saturation in the system.

In actual implementation, it is suggested that the possible maximum error be sensed (with a certain sensor gain) as the bias voltage of the A/D converter and the quantized value be divided by that certain sensor gain before it is stored in the certain memory location in the computer. All the feedback gains are retained to be the same as they are designed without the scaling factor.
8.4 Computer Simulation of the ASPS With Quantization

In this section the responses of the digital closed-loop ASPS with quantization are obtained through computer simulation. The block diagram of the system is shown in Fig. 8-3. The sampling period of the system was chosen to be 0.1 sec. The feedback gains of the system are tabulated in Eq. (7-7) and are not repeated here. Eight-bit and 12-bit A/D and D/A converters were used. Since the 16-bit wordlength used in the digital computer is longer than the bit length of the A/D and D/A converters, quantization should be considered when computing the control signals of the ASPS. This process is represented by the subroutine program listed in Table 8-2.

Table 8-2

<table>
<thead>
<tr>
<th>DATA.SAMPLING AND A/D CONVERSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 = 0.35853685234375E2 \times Q(x) + 6.739349365234375E2 \times Q(xd) )</td>
</tr>
<tr>
<td>( v_2 = -2.9714966015625E5 \times Q(\phi_1 \phi_2) - 2.1929931640625E5 \times Q(\phi_1) )</td>
</tr>
<tr>
<td>( v_3 = -2.1617126484375E4 \times Q(\phi_1 \phi_2) )</td>
</tr>
</tbody>
</table>

\[ v_1 = Q(v_1) \]
\[ v_2 = Q(v_2) \]
\[ v_3 = Q(v_3) \]
FUNCTION O(XX)
COMMON X,XDOT,PHI1,PHI1DT,PHI1IN,PHI2,PHI2DT,PHI2IN,T,
& V1,V2,V3,AJ,AMI,AKS,AKSP,AKSS,TA3,T,PRT,PRB,DELT,
& A0,B0,C0,D0,E,F,TEND,H,
& NPRT,IV1,TV2,TV3,LINE
IF(XX,LT,0,0) GO TO 30
IV=(XX/H)+0.5
GO TO 30
20 IV=(XX/H)-0.5
30 Q=FLOAT(IV)*H
RETURN
END

Figure 8-6 shows the responses of the x-dynamics of the ASPS with an initial condition of $2 \times 10^{-3}$ for the first 60 seconds, with and without quantization. The A/D and D/A converters have 12 bits. Notice that the quantization effect on the transient response is to cause an overshoot in $x$ at the beginning. Figure 8-7 illustrates the response of the x-dynamics beyond 100 seconds (near steady state). Notice that the response cannot be classified as "sustained oscillation", since it is not periodic. However, it does appear that the amplitude of the response has reached a maximum saturated level. In this case, the maximum error due to quantization is less than $1.5 \times 2^{-12}$.

Figure 8-8 illustrates the response of the $\phi_1$-dynamics with the same quantization level as described above. In this case it is seen that the quantization effect was to cause high overshoots in the transient response, but the response quickly reached a bounded steady-state. Figure 8-9 shows the same $\phi_1$ response beyond 140 seconds. We see that the maximum steady-state error due to the 12-bit quantization is less than $1.5 \times 10^{-4}$. Again, the steady-state response cannot be described as a sustained oscillation.
Response with quantization, 12 bits

Response without quantization

Figure 8-6. Responses of the x dynamics with and without quantization; 12-bit A/D and D/A converters
Figure 8-7. Response of the $x$ dynamics near the steady-state with 12-bit A/D and D/A converters.
Figure 8-8. Time response of $\phi_1$ dynamics with 12-bit A/D and D/A converters.
Figure 8-9. Time response near steady state of φ₁ dynamics with 12-bit A/D and D/A converters.
Figure 8-10. Time response of $\phi_2$ dynamics with 12-bit A/D and D/A converters.
In a similar fashion the time response of the $\Phi_2$ dynamics with the 12-bit A/D and D/A converters in the ASPS system was obtained and plotted as shown in Fig. 8-10. Again, the steady-state error of $\Phi_2$ due to quantization is very close to that of $\Phi_1$.

Figure 8-11 illustrates the time response near the steady state of $\Phi_1$ under the same condition. In these cases, the steady-state errors due to quantization are all less than one quantization level, $2^{-8}$. As expected, the steady-state error due to quantization is directly related to the quantization level. In general, large quantization levels will result in large steady-state errors.
Figure 8-11. Time response of x near steady state with 8-bit A/D and D/A converters.
Figure 8-12. Time response of $\phi_1$ with 8-bit A/D and D/A converters.
IX. NONLINEAR SPRING EFFECTS ON THE DIGITAL ASPS

9.1 Introduction

The analysis and design of the digital ASPS conducted in the preceding chapters were based on the system model that the spring characteristics of the wire-cable torque were linear. On the ASPS a wire cable is attached to the center of the payload mounting surface with the other end of the cable attached to the shuttle frame. Similarly, there is another cable attached between the pallet and the shuttle (see Figure 1-2.).

For the linear analysis and design, the spring constant of the payload is designated by the constant $K_s$ ($z$ and $x$ dynamics), and the spring constant of the $\phi_1$ and $\phi_2$ dynamics are designated by $K_{s1}$ and $K_{s2}$, respectively.

The nonlinear spring characteristic of the wire-cable torque is shown in Fig. 9-1. The spring parameters are given by $H_{WF}$ and $K_s$, as shown in the figure. The abscissa is shown as $z_1$, but the same characteristics are assumed for $x$, $\phi_1$ and $\phi_2$.

The objective of this chapter is to investigate the effects of the nonlinear spring characteristics on the performance of the digital ASPS with quantization. Since the overall ASPS with the nonlinear spring and quantizers is a high-order digital control system with multiple nonlinearities, no analytical methods are available to conduct a closed-form study of the system. Thus, computer simulations are conducted for the studies. The listings of the simulation program are given in Table 9-1.

9.2 Effects of the Nonlinear Spring on the X-Dynamics

Since the nonlinear spring characteristic shown in Fig. 9-1 can be interpreted as the superposition of a velocity-dependent force (or torque) to the
Table 9-1. Computer listings of Digital ASPS with Wire-Cable Nonlinearity and Quantization.

```
TYPE SING,T,F
DIMENSION PRIT(5),Y(8),DERY(8),AUX(8,6)
EXTERNAL FCT,OUTF
.COMMON X,XDOT,PHII,PHI1DT,PHI2T,PHI2IN,PHI1N,T,
  V1,V2,V3,AJ,AMJ,AKSA,AKSP,AKSS,AJ3,TFR,T,RB,BELTA,
  A0,B0,CO,BO,E,F,TEND,H
& NPRT=IV1,IV2+1V3,LINE
  OPEN UNIT=6,DEVICE='DSK',FILE='FOR20.DAT',ACCESS='SERNOUT'

_LINE=0
TSP=0.1
H=2.*XX-12
TEND=200.,
T=RT=0.2
TINT=1.0E-2
C----SET INITIAL CONDITION
  X=-1.639E-5
  XDOT=-5.302E-6
  PHI1=6.756E-7
  PHI1DT=5.090E-7
  PHI1IN=3.549E-7
  PHI2=1.774E-6
  PHI2DT=6.753E-6
  PHI2IN=-1.211E-
  AMJ=600.,
  AKSA=X1.051
  AKSP=0.005
  AKSS=0.005
  AJ=503.,
  A3J=2805.15
  RA=1.956
  AO=RB
  BO=1./AMJ
  CO=AN1*RB/A3J
  SO=AJ/AN32
  C=1./A3J
  F=1./AMJ
  DELTA1=-A0*CO-DO
  T=9.7801E1
  NPRT=0
  PRMT(3)=TINT
  PRMT(4)=0.01*2,E-3
  DD 99 I=1+8.
  99 DERY(I)=0.125
  NDIM=8
  IF(T.GE.TEND) CALL 'EXIT

C-------DATA SAMPLING AND A/D CONVERSION
  GX=100.
```
GPI = 50.

V1 = 3.5853986525 * (37.452% \(\times\) GPI) + 1,732.39 * 5.38734756.3

\(\sum (x_{D}T \times y)\)

-2.9714 * 6.6015625 \(\times\) GPI + 1,712.9316 + 403256.3

\(\sum (\phi1T \times y)\)

-2.1617 * 12.6448 * 37.56487555 \(\times\) GPI

V2 = -6.994476318357556 \(\times\) \(\sum (x_{D}T \times y)\) - 1.31859375 \(\times\) \(\sum (x_{D}T \times y)\)

+ 7.102508515625 \(\times\) \(\sum (x_{D}T \times y)\) (PHI1T \times GPI1) + 1.5415466464625665

\(\sum (\phi1T \times y)\)

+ 9.02430624375 \(\times\) \(\sum (x_{D}T \times y)\) \(\phi12 \times\) GPI2

\(\sum (\phi1T \times y)\)

+ 10.54762514054375 \(\times\) \(\sum (x_{D}T \times y)\) \(\phi12 \times\) GPI2 + 10.4279248531562543

\(\sum (\phi1T \times y)\)

+ 1.2134953515625 \(\times\) \(\sum (x_{D}T \times y)\) \(\phi11 \times\) GPI1

\(\sum (\phi1T \times y)\)

+ 9.364984130854375 \(\times\) \(\sum (x_{D}T \times y)\) (PHI1T \times GPI1) + 8.387074725562543

\(\sum (\phi1T \times y)\)

+ 10.54762514054375 \(\times\) \(\sum (x_{D}T \times y)\) \(\phi12 \times\) GPI2 + 10.4279248531562543

\(\sum (\phi1T \times y)\)

GU1 = 1.0 / ABS(V1).

GU2 = 1.0 / ABS(V2).

GU3 = 1.0 / ABS(V3).

V1 = Q(V1 \times GU1).

V2 = Q(V2 \times GU2).

V3 = Q(V3 \times GU3).

PRMT(1) = T

PRMT(2) = Y + TSP

Y(1) = X

Y(2) = X + TDP

Y(3) = PHI1N

Y(4) = PHI1T

Y(5) = PHI1D

Y(6) = PHI2N

Y(7) = PHI2T

Y(8) = PHI2DT

CALL RLGS(PRMT, Y(1), DERY, M01M, INH1, Y(1), Y(1), AUX)

GO TO 10
END

SUBROUTINE RLGS(YA, YB, DERY)

DIMENSION Y(B), DERY(B)

COMMON X, XDOT, PHI1, PHI1DT, PHI1N, PHI2, PHI2DT, PHI2N

V1, V2, V3, A1, A2, A3, AKS, AKSF, AKSS, A33, TPR, RB, DELTA

X0, BO, CO, DO, ER, FR, TND, H, TND

NPRT, IV1, IV2, IV3, LINE

HX = 0.00279

HP1 = 0.00279
HP2=0.0027
IF(Y(2).LT.0.)  HX=-HX
IF(Y(5).LT.0.)  HP1=-HP1
IF(Y(9).LT.0.)  HP2=-HP2
DERY(1)=Y(2)  
  Y(4)-HP1)*E+((V3+AKSS*Y(7)+HP2)*00*F)/DELTAY
  DER Y(3) = Y(4) 
  DERY(4) = Y(5) 
  DERY(5) = ((-V2-AKSP*Y(4)-HP1)*E+(-V1+AKSS*Y(1)+HX)*BO*C0
  #  +(V3+AKSS*Y(7)+HP2)*00*F)/DELTAY
  DERY(6) = Y(7) 
  DERY(7) = Y(8) 
  DERY(8) = ((-V3-AKSS*Y(7)-HP2)*F*(1-00*C0)+(-V2-AKSP*Y(4)+HP1)*E
  #  +(V1+AKSS*Y(1)+HX)*BO*C0)/DELTAY

RETURN
END

SUBROUTINE OUTP(XX,Y,DERY,HLF,NOIH,PRNT)
DIMENSION Y(8), DERY(8), FRNT(5)
COMMON X,XDOT,PHI1,PHI1DT,PHI1IN,PHI2,PHI2DT,PHI2IN,Y
#  V1,V2,V3,AMT,AKSS,AKSP,AKSS,A33,TPRT,RB,DELTAY,
#  A0,B0,C0,D0,E,F,TEND,H,
#  NPRT=IV1, IV2, IV3, LINE
IF(XX.T.E*TEND) PRNT(5)=1.
  T=XX
  IF(XX.LT.TPRT*FLOAT(NPRT)) RETURN.
  LINL=LNE+1
  WRITE(6,100) HLF,XX,(Y(I),I=1,8),V1,V2
100 FORMAT(1X,I2,3X,1PE10.4,1PE11.3)
  X=Y(1)
  XD O T=Y(2)
  PHI1IN=Y(3)
  PHI1=Y(4)
  PHI1DT=Y(5)
  PHI2IN=Y(6)
  PHI2=Y(7)
  PHI2DT=Y(8)
  NPRT=NPRT+1
  IF(LINE.LT.5) GO TO 200
  WRITE(5,150) THI,F,XX,(Y(I),I=1,5),V1,V2
  LINE=0
200  RETURN
END

FUNCTION Q(XX,G)
COMMON X,XDOT,PHI1,PHI1DT,PHI1IN,PHI2,PHI2DT,PHI2IN,Y
#  V1,V2,V3,AMT,AKSS,AKSP,AKSS,A33,TPRT,RB,DELTAY,
#  A0,B0,C0,D0,E,F,TEND,H,
#  NPRT=IV1, IV2, IV3, LINE
  HG=H/0
  IF(XX.LT.0.) GO TO 20
  IV=(XX/HG)*0.5
20  GO TO 30
30  O=FLOAT(IV)*HG:
  RETURN
END
linear characteristic of the ideal linear spring, it is expected that the gross effect is some damping effects on the system dynamics. The computer simulation results show that there are no sustained oscillations as a result of the nonlinear spring characteristics.

Figure 9-2 shows the transient responses of the x-dynamics of the digital ASPS with feedback control as designed in the preceding chapters. A 12-bit quantization is used. The initial value of $x$ is $2 \times 10^{-3}$, and $H_{WF}$ is varied from 0 to 0.008. Notice that when $H = 0$, the springs are ideal, and the response of $x$ is oscillatory. The initial overshoot of the x-dynamics responses is due to the quantization effects. The final steady-state error of $x$ when $H_{WF} = 0$ is small with a non-zero mean, but the response is relatively slow. As shown in Fig. 9-2, when the value of $H_{WF}$ is increased, the time response becomes faster in reaching its steady state. In fact, when $H_{WF} = 0.0028$, the response of $x$ resembles that of a deadbeat response, and the steady-state is nearly zero. However, as the value of $H_{WF}$ increases further, the steady-state error increases. When $H_{WF} = 0.008$, the steady-state error is very large, and the response is also very slow. Thus, the transient responses in
Figure 9.2: Transient responses of x-dynamics with various values of $H_{WF}$. 
Figure 9-2 clearly show that the nonlinear spring parameter $H_{WF}$ can be used to improve the x-dynamic response if the value of $H_{WF}$ can be controlled to some extent.

9.3 Effects of the Nonlinear Spring on the $\phi_1$ and $\phi_2$ Dynamics

The computer simulation results show that the nonlinear spring has little effects on the responses of the $\phi_1$ and $\phi_2$ dynamics. Since these two response components are relatively fast, the above-mentioned results were expected. However, different values of $H_{WF}$ did give slight difference on the steady-state error of the responses. These error trajectories tend to have zero mean as expected due to the effect of the integral feedback in the system. Figures 9-3 and 9-4 show the transient responses of $\phi_1$ and $\phi_2$, respectively when $H_{WF} = 0$. Figures 9-6 and 9-7 illustrate the steady-state responses of $\phi_1$ and $\phi_2$, respectively, as compared with the steady-state response of $x$ shown in Figure 9-5.

Figures 9-8, 9-9 and 9-10 illustrate the time responses of $x$, $\phi_1$ and $\phi_2$ for $H_{WF} = 0.001$. When $H_{WF} = 0.0028$ it was shown in Figure 9-2 that the $x$-component has a deadbeat response. However, this value of $H_{WF}$ did not improve the $\phi_1$ and $\phi_2$ responses, as shown in Figures 9-11 and 9-12.

The conclusions on the investigations in this chapter is that the nonlinear spring effect due to the wire-cables on the ASPS does not appreciably harm the responses of the system. Except for the steady-state error when the value of $H_{WF}$ is large, the transient responses are not much affected. There would be no sustained oscillations generated due to the nonlinear spring characteristic.
Figure 9.3. Transient response of dynamics with $H_f = 0$.
Figure 9-4: Transient response of $\phi_2$.
Figure 9-6. Steady-state response of &delta;l-dynamics with \( h_{\zeta} = 0 \) and sensor gain \( G_{s1} = 50 \).
Figure 9-7: Steady-state response of $\omega_n$ dynamics with $H_s=0$.

(sound gain $8_{op}=50$)
Figure 9-9: Time response of a dynamics with $H_NF = 0.00$.
(sensor gain $G_{s} = 50$).
Figure 9-10. Time response of $j_2$ dynamics with $K/F = 0.001$.
(sensor gain $B_{02} = 50$).
Figure 9-1. Steady-state response of $f_i$ with $\hat{R}_F = 0.0028$. (Sensor gain $G_{\alpha} = 50$).
Figure 9-12: Steady-state response of $e_2$ with $K_p = 0.0028$.
(sensor gain $g_{p2} = -50$).
References


