# RELATIVISTIC INFORMATION OF THE GPS-NAVSTAR POSITION DETERMINATION ALGORITHM

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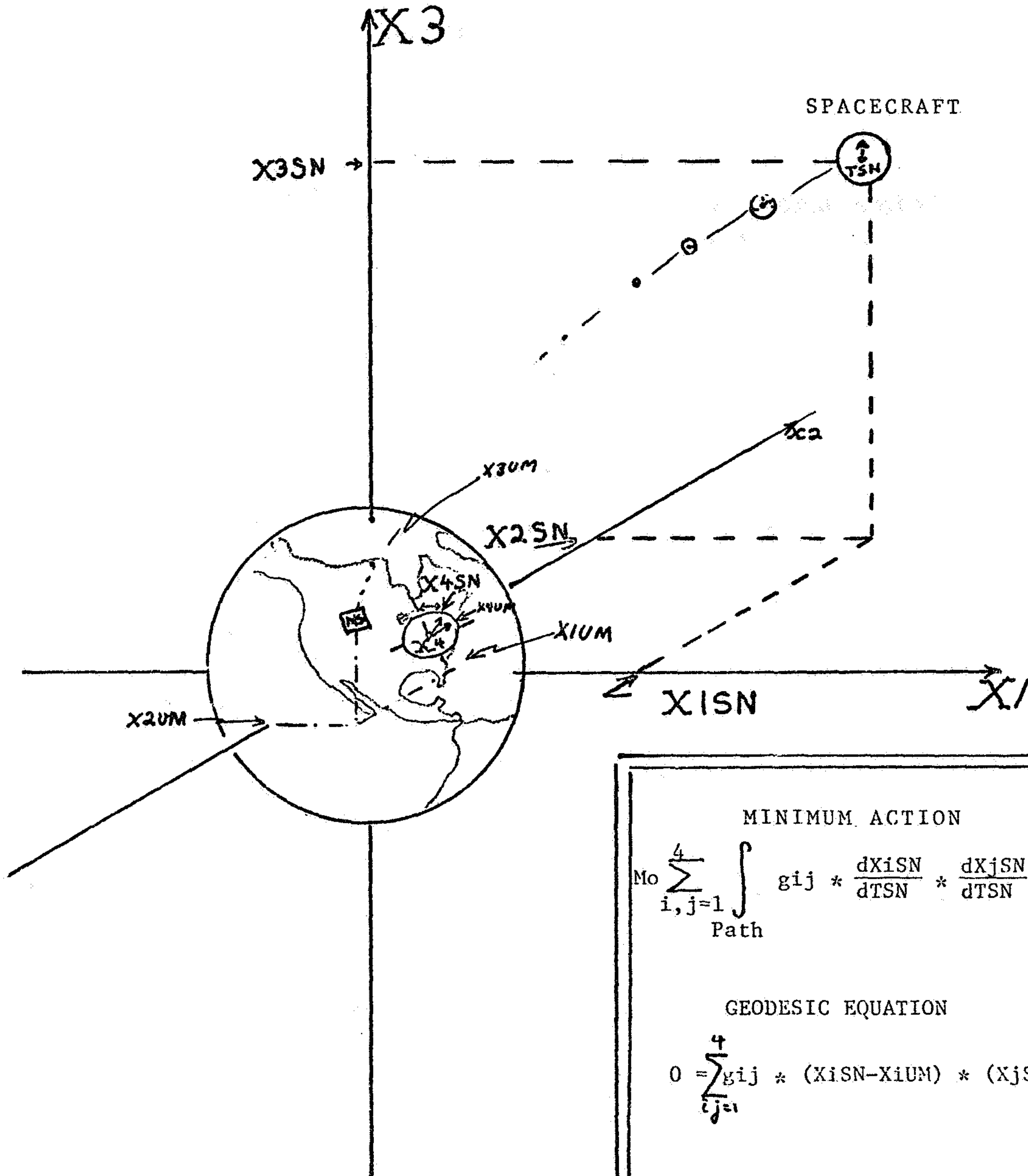
## ABSTRACT

A relativistic formulation of the timation navigation position algorithm is developed. The use of satellite proper time as integration variable and space-time correlation handling are examined for means to reduce computational requirements. Implementation architectures are discussed with emphasis on minimization of computer configuration hardware requirements. Selected benchmark experiment results are presented to size processor requirements for achieving GPS-NAVSTAR navigation accuracies in both the User and Control Segments.

## 1) An introduction to Timation Navigation in Relativistic Notation.

Relativistic Timation Navigation is a technique for calculating the position of an event by measuring it's distance from at least four events of known location. A set of repetitive events, such as the completion of a clock cycle, can mark the trajectory of an object in which those events occur. Figure 1 shows the trajectory of a spacecraft marked by constant time ticks as measured by a standard on-board clock located in an earth-centered coordinate system X. Each clock tick (TSN) is located by three positions (X1SN, X2SN, X3SN) and a time coordinate (X4SN).

To show how relativistic notation can be used in the NAVSTAR Program, the following sets of equations are introduced and summarized in the Figure 1 insert.[1]




MINIMUM ACTION

$$M_o \sum_{i,j=1}^{4} \int_{Path} g_{ij} * \frac{dXiSN}{dTSN} * \frac{dXjSN}{dTSN} \; dTSN$$

GEODESIC EQUATION

$$0 = \sum_{i,j=1}^{4} g_{ij} * (XiSN-XiUM) * (XjSN-XjSM)$$

a) The solution of motion of satellite clock N in the NAVSTAR constellation:

X1SN = F1SN (TSN)

X2SN = F2SN (TSN)

X2SN = F3SN (TSN)

X4SN = F4SN (TSN)

Where X4SN = coordinate time

TSN = proper time of the Nth Satellite Clock

F2SN (TSN) = a function of TSN

This can also be written as:

XiSN = FiSN (TSN) i = 1,4

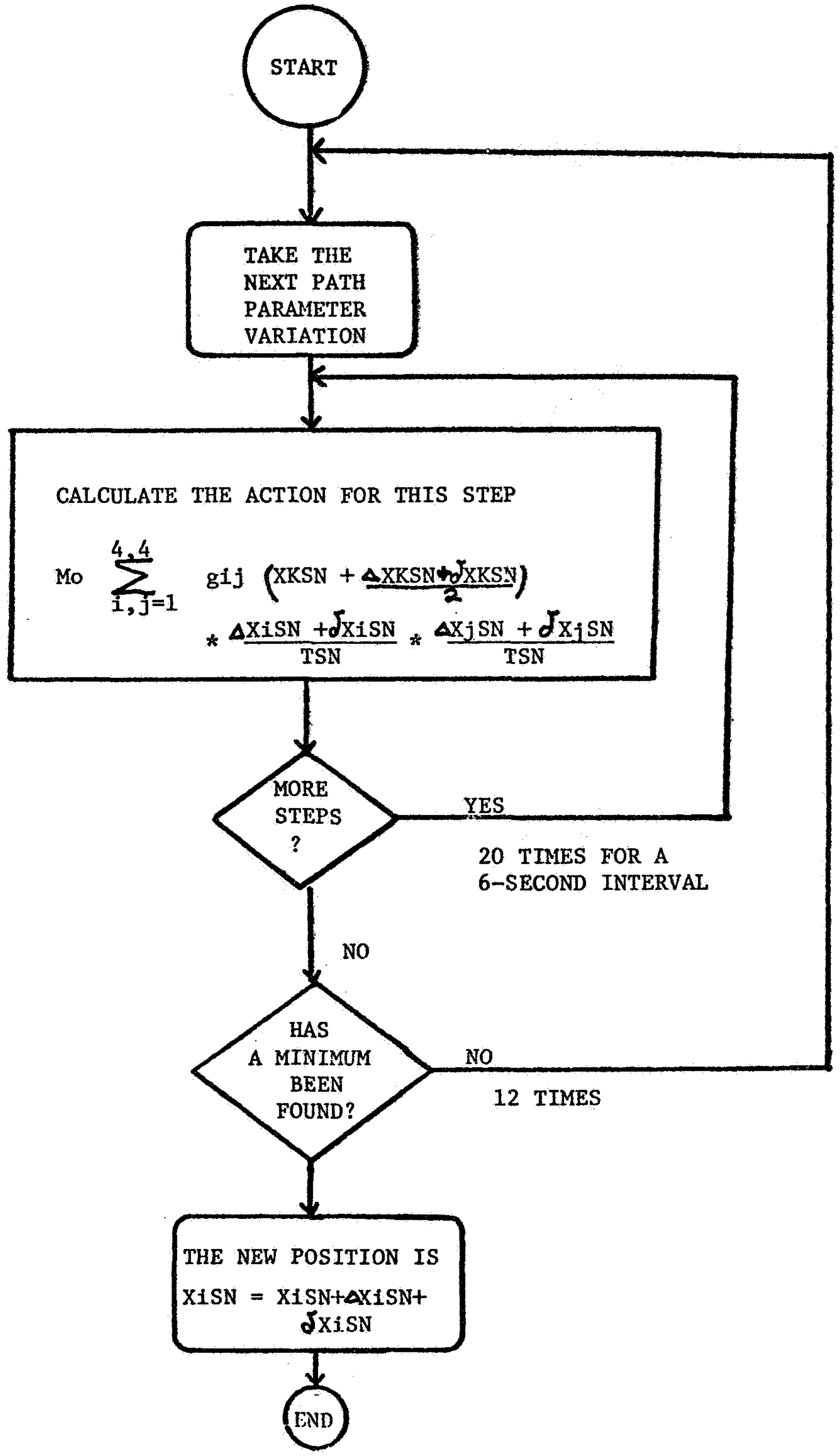
b) The solution of motion for a satellite clock can be found by applying a four-dimensional minimum action principle, i.e.,

$$Mo \sum_{i,j=1}^{4} \int_{Path} \sqrt{gij * \frac{dXiSN}{dTSN} * \frac{dXjSN}{dTSN}} \, dTSN = \text{a minimum along the actual path}$$

Where Mo = the rest mass of the satellite clock.

gij = the metric tensor or gravitational potential

The action integral can be minimized in a search algorithm of the type presented in Figure 3. The basic clock cycle in a NAVSTAR satellite is the repetition of the navigation message occurring every 6 seconds measured in satellite time. For a


START
TAKE THE NEXT PATH PARAMETER VARIATION
CALCULATE THE ACTION FOR THIS STEP
Mo $\sum_{i,j=1}^{4,4}$ gij (XKSN + $\frac{\Delta XKSN + \delta XKSN}{2}$) * $\frac{\Delta XiSN + \delta XiSN}{TSN}$ * $\frac{\Delta XjSN + \delta XjSN}{TSN}$
MORE STEPS ?
YES
20 TIMES FOR A 6-SECOND INTERVAL
NO
HAS A MINIMUM BEEN FOUND?
NO
12 TIMES
THE NEW POSITION IS
XiSN = XiSN+ΔXiSN+ δXiSN
END

12-hour synchronous satellite traveling at 4 km per second it is estimated that approximately 20 steps are required to avoid granularity introduced by numerical integration. Assuming that the gravitational potentials (gij) are known,(2) the action contribution for each step requires the execution of approximately 20 additions and 20 multiplications and must be performed 240 times for a total 9.6K instructions to calculate the time and position of the next NAVSTAR message cycle.

c) The solution of motion for a user or monitor station clock M in the NAVSTAR system:

X1UM = F1UM (TUM)

X2UM = F2UM (TUM)

X3UM = F3UM (TUM)

X4UM = F4UM (TUM)

or

eq. 1b: XiUM = FiUM (TUM) i=1,4

Where TUM - the proper time of a user or monitor station M.

For the purposes of this paper, monitor station solutions of motion are assumed to be known quantities derived from site surveys and the rotational motion of the earth derived from the pole wander service.

d) The geodesic equations defining the square of the distance between two events in an arbitrary coordinate system X with a metric tensor of gij are:

$$(SSNUM)^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} gij * (XiSN-XiUM) * (XjSN-XjSM)$$

Where: SSNUM = the distance between the two events SN, UM

XiSN = coordinates of the event SN i = 1,4

XjUM = coordinates of the event UM j = 1,4

To first approximation, the metric tensor of an earth-centered initial coordinate system is:

$$gij = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -c^2 \end{bmatrix}$$

Where C = the speed of light in the vicinity of the earth in vacuum.

Also, the first approximation the distance between any two events connected by a light pulse is zero, or stated another way, light travels along null geodesics.

Hence the geodesic equation when the two events in question are the transmission of an electromagnetic wave feature by a satellite and the reception of the same feature by a user or monitor station reduces to:

eq. 1c: $$0 = (X1SN - X1UM)^2 + (X2SN - X2UM)^2 + (X3SN - X3UM)^2 - c^2 (X4SN - X4UM)^2$$

which is the standard equation connecting range to time intervals and also known as the user equation.[3]

## 2) Clock Ephemeris Determination

The determination of clock ephemeridies requires a measurement of clock position over an arc. The measurement instrument used in the NAVSTAR Program is a set of up to eight monitor stations on the semi-rigid rotating earth crust.

Single Point Clock Tick Position Measurement

The emission of the wave feature, representing a clock tick, from a satellite antenna is a single event which can be labelled by the proper time (TSN) of emission and occurs at unique space time coordinate values XiSN for i=1 to 4.

The arrival time of the wave feature at a monitor station M is tagged with the monitor station proper time TAM. Since we have assumed knowledge of monitor station motion (eq. 1b) as a function of TAM, for each monitor station satellite pair the geodesic equation (eq. 1c) applies.

$$0 = \sum_{i=1}^{4} \sum_{j=1}^{4} gij * (XiSN - XiUM) * (XjSN - XjSM)$$

If four monitor stations receive the signal, the four geodesic equations can be constructed and solved for the four unknown c-ordinates of the satellite clock tick event (XiSN for i=1,4) for an initial estimate.

Filter Techniques

Typically many measurements using more than four monitor stations and multiple clock ticks will be used to construct a clock ephemeris. Consequently filter techniques are applicable. If we assume to have a guess of an initial satellite position (XISN1) and a good four-dimensional stepper, then one can find the best satellite position by minimizing the sum total satellite weighted error calculated over all measurements as shown in Figure 4.

$$\sum_{N}^{24} \text{Single Satellite Weighted Error} = \sum_{N}^{24}\sum_{M}^{8}\sum_{TSN}^{14400}\sum_{i}^{4}\sum_{j}^{4} W(TS,N,M)*g_{ij}(TS,N,M)*\left(X_i SN(TS,N)-X_j UM\right)*\left(X_i SN(TS,N)-X_j UM\right)$$

- $i$, $j$ — Dimension Variables
- W(TS,N,M) — Weighting Factor
- $g_{ij}$(TS,N,M) — Gravity Tensor of Transmission Medium
- $X_i$SN(TS,N) — Satellite N Position Coordinates
- $X_j$UM — Monitor Position Coordinates M
- $\sum_{TSN}^{14400}$ — The Sum Over All Satellite Clock Ticks Considered in this Estimation
- $\sum_{M}^{8}$ — The Sum Over All Monitor Stations (M)
- $\sum_{N}^{24}$ — The Sum Over All Satellites (N)

| Add/Sub | Mult/Div |
| --- | --- |
| $36 \times 10^6$ | $36 \times 10^6$ |

To achieve a 300-fold reduction in the single point estimation as anticipated for NAVSTAR III, a day's measurements might be required. Then each satellite weighted error calculation would require 36 mega add/subtract, multiply/divide, and trajectory steps to perform - a sizable computer load to use in a search loop.

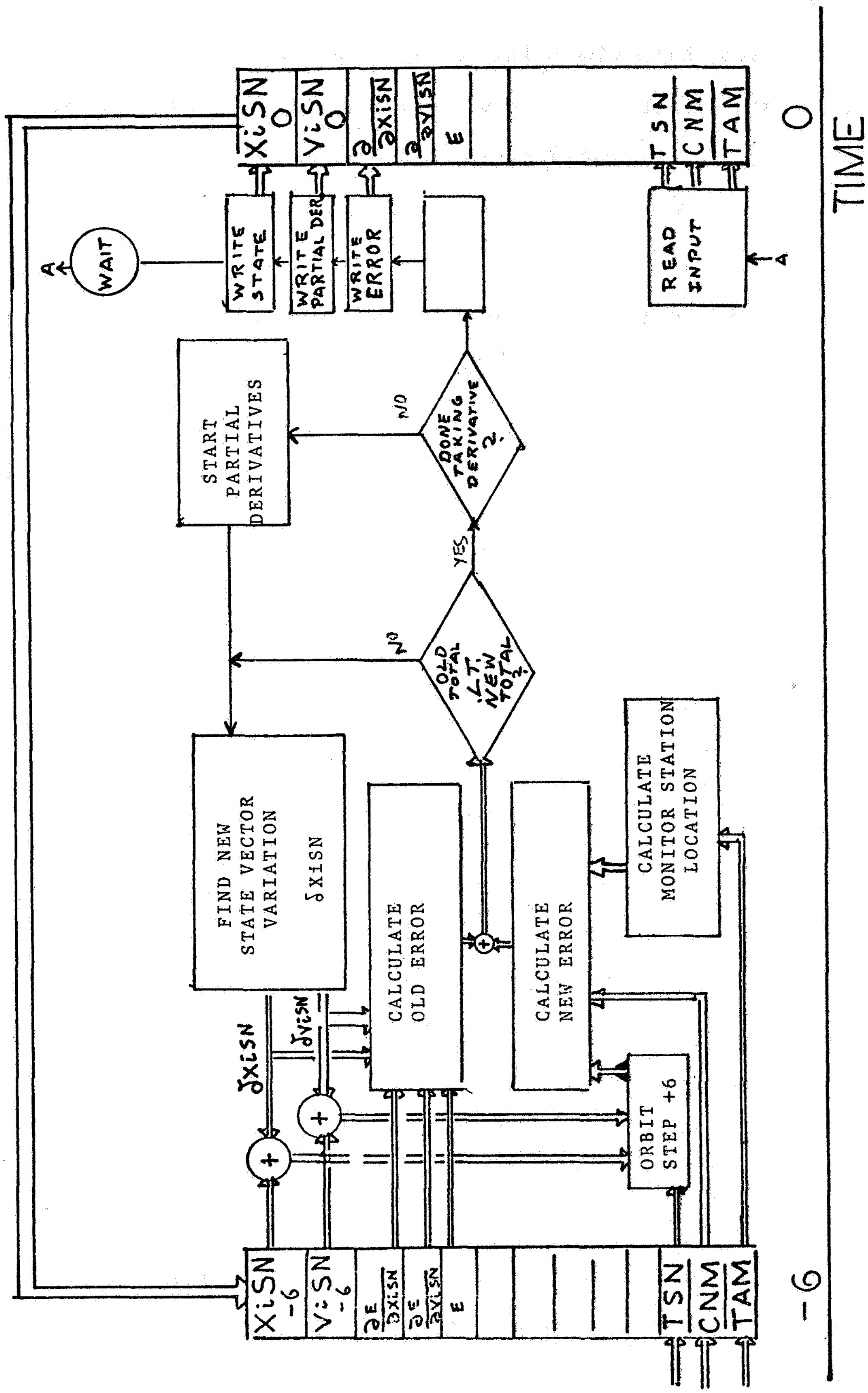
The same minimization can be performed directly using a filter algorithm as shown in Figure 5, in which the total error is minimized recursively.

3) Features of the Relativistic Formulation

The advantage of using relativistic notation is that the satellite proper time TSN is the independent variable physically available to the NAVSTAR Spacecraft. Proper time is measured by counting the output of precision frequency standard (rubidium or cesium) which is a physically independent device. All on-board processing, transmission frequencies, bit rates, etc. can be derived from this device. Hence, real time schedules and navigation processing loops are naturally indexed with the TSN variable.

A second advantage is derived form the property that the relativistic formulation is coordinate independent, hence solutions of motion can be obtained directly in any coordinate system of interest.

The relativistic formulation also allows clock rates and clock drifts, as well as space time correlations to be automatically included. A fourth component of force can be added to account for clock rate

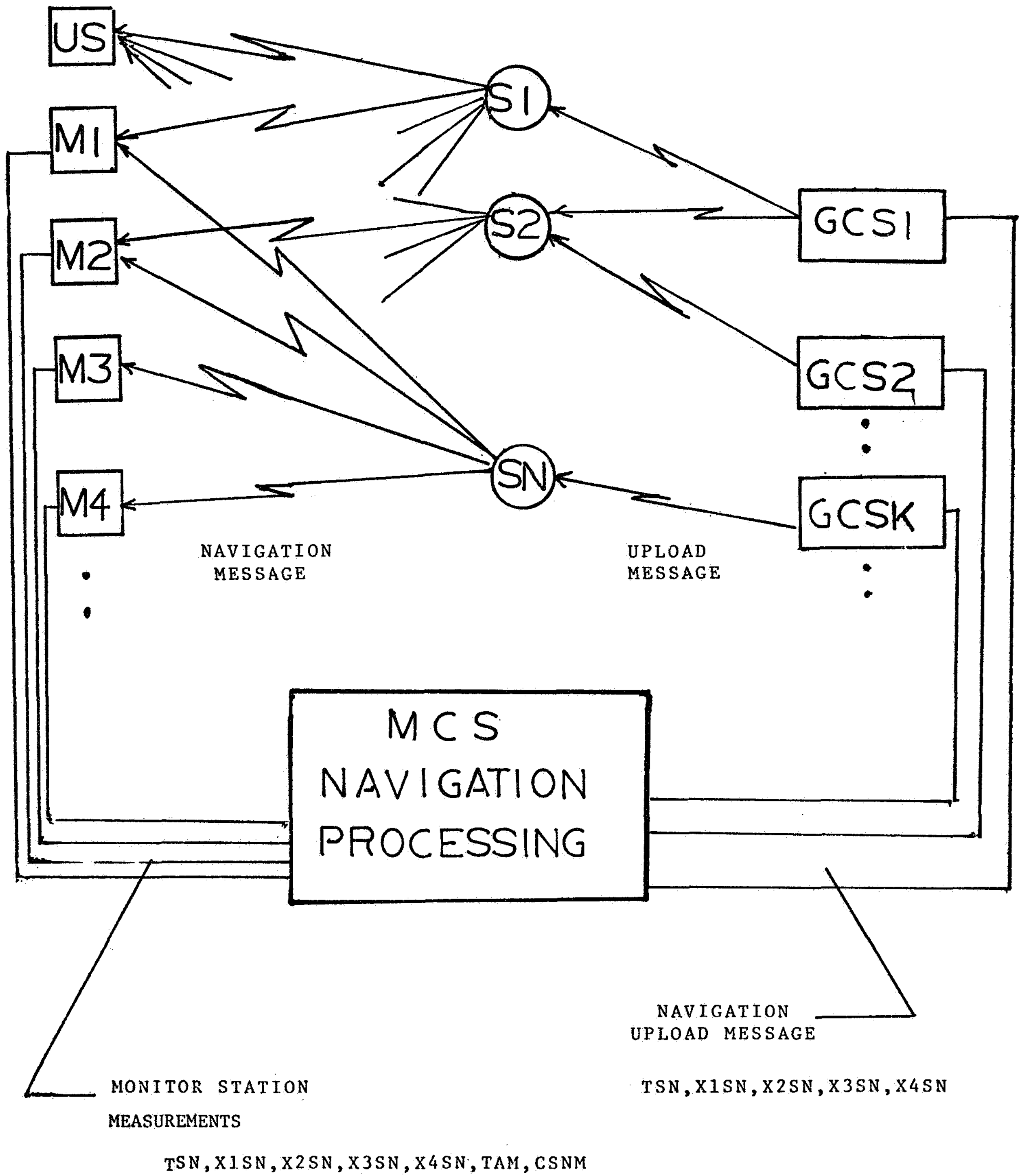

XiSN -6
ViSN -6
∂E/∂XiSN
∂E/∂YiSN
E
TSN
CNM
TAM
-6
δXiSN
δYiSN
FIND NEW STATE VECTOR VARIATION δXiSN
CALCULATE OLD ERROR
CALCULATE NEW ERROR
ORBIT STEP +6
CALCULATE MONITOR STATION LOCATION
OLD TOTAL L.T. NEW TOTAL?
NO
YES
START PARTIAL DERIVATIVES
DONE TAKING DERIVATIVE?
NO
WAIT
A
WRITE STATE
WRITE PARTIAL DER
WRITE ERROR
READ INPUT
A
XiSN 0
YiSN 0
∂/∂XiSN
∂/∂YiSN
E
TSN
CNM
TAM
0
TIME

variations much as the three space components of solar and thruster pressure determine mass point velocity variations.

4) Implementation and Computer Sizing Considerations

A simplified data flow diagram of the NAVSTAR Control System is shown in Figure 6. For purposes of this paper, the satellite is assumed to radiate a navigation message which is simply a table of four XiSN values plus information such as satellite clock offset ionospheric corrertion coefficients, etc. In practice, polynomial approximations to the satellite clock motion will be used to accommodate operational requirements. Nevertheless a simple table of XiSN position and time coordinates could be stored in a 32-bit 128K core memory and clocked out every 6 seconds for the first week to satisfy position accuracy requirements of better than 1 meter.

The radiated message is received at monitor stations in view of the satellite (SN). The signal is tagged with the Time of Arrival at each monitor station (TAM), the best speed of light estimate (CSNM) calculated and communicated to the Master Control Station (MCS). Messages from each monitor station are multiplexed into the MCS Navigation Process (Figure 2). The four coordinates are used for performance evaluation, while the values TSN, TAM and CSNM are used as measurements for position determination.

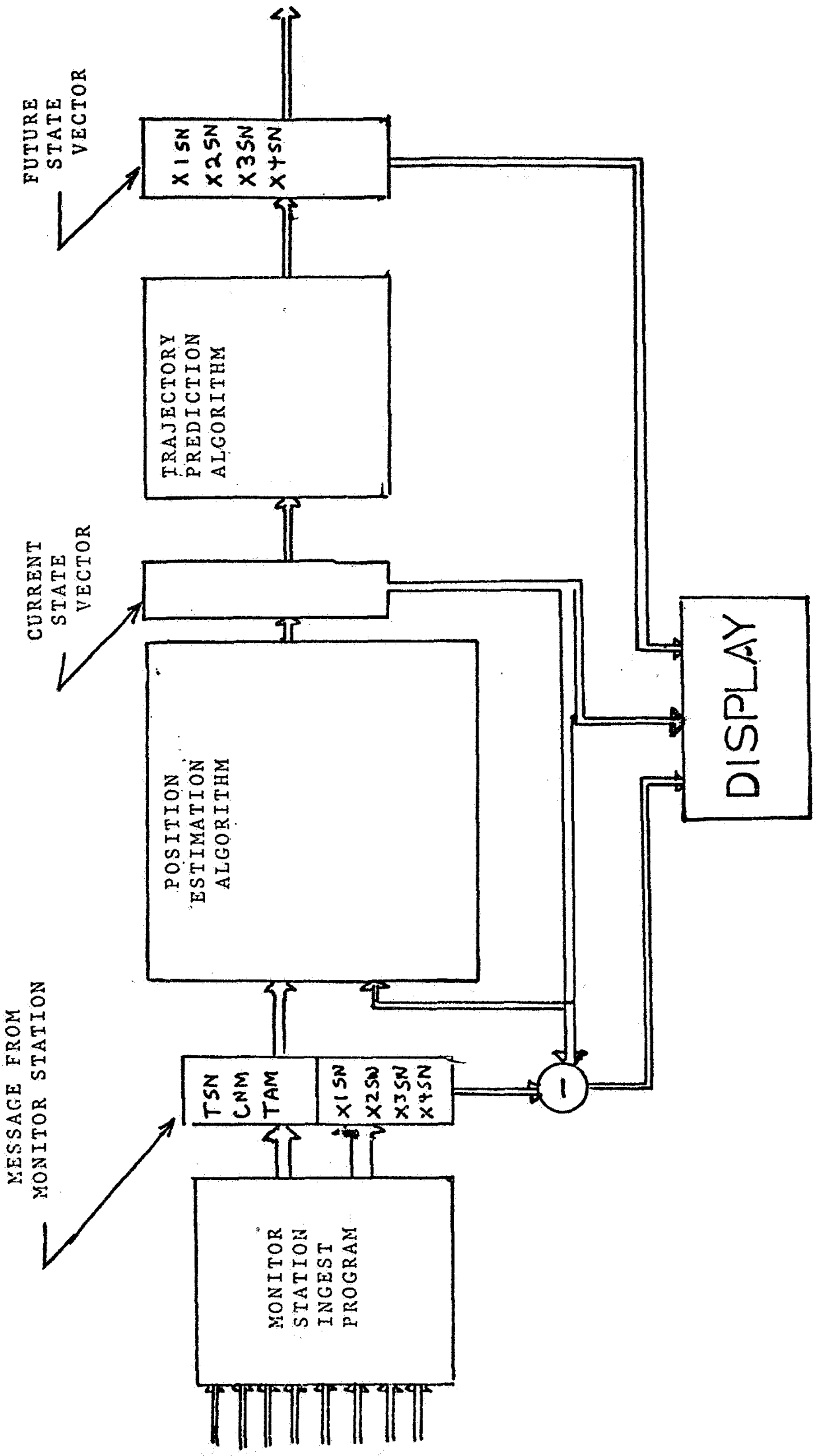

US
M1
M2
M3
M4
S1
S2
SN
GCS1
GCS2
GCSK
NAVIGATION MESSAGE
UPLOAD MESSAGE
MCS NAVIGATION PROCESSING
NAVIGATION UPLOAD MESSAGE
TSN,X1SN,X2SN,X3SN,X4SN
MONITOR STATION MEASUREMENTS
TSN,X1SN,X2SN,X3SN,X4SN,TAM,CSNM

The ingest data rate from eight monitor stations and twenty-four satellites is sixteen measurements per second since on the average only half the satellites will be visible from any one station.

The Navigation Processing Block Diagram (Figure 7) shows two major programs, the Position Update Algorithm and the Position Prediction Algorithm. Figure 8 gives a summary of their software characteristics. In calculating transaction rates, it was assumed that each space vehicle will require a maximum of two upload messages per day and predictions to one week. The instruction mix is derived from the average operation count for the relativistic formulations presented above.

A summary of computer resource requirements is shown in Figure 9. The kip rate is typical of mini computers generally available in today's market.

An operational NAVSTAR Timation Navigation Control Center would require substantial increases in computing power to handle error checking, reporting, initialization, look up table calculations, satellite control, display and general site utility functions. The detail in this paper is not intended to provide hardware recommendations for an operational system, but present a theoretical method. However, the essential parts of a simplified navigation subsystem have been studied sufficiently to suggest that mini-computer-based relativistic software could be considered for advanced systems, simulations and validations or commercial piggy-back packages.


MONITOR STATION INGEST PROGRAM
MESSAGE FROM MONITOR STATION
TSN
CNM
TAM
X1SN
X2SN
X3SN
X4SN
POSITION ESTIMATION ALGORITHM
CURRENT STATE VECTOR
TRAJECTORY PREDICTION ALGORITHM
FUTURE STATE VECTOR
X1SN
X2SN
X3SN
X4SN
DISPLAY

| TRANSACTION NAME | NO. OF EXECUTIONS 1000/HR | NO. OF INSTRUCTIONS | NO. OF INSTRUCTIONS PER LOOP | NO. OF LOOPS EXECUTED PER TRANSACTION | I/O ACCESS | I/O BUFF SIZE |
|---|---|---|---|---|---|---|
| POSITION PREDICTION | .002 | 300 | 40 | 240/STEP<br>100K STEPS/WEEK | 1/STEP | 84 |
| POSITION DETERMINATION | 57.6 | 700 | 100 | 80 | 1 | 164 |

NOTE: INSTRUCTIONS ARE 50% ADD/SUB, 50% MULTIPLY

FIGURE 8 - ESTIMATED NAVIGATION PROCESSING SOFTWARE CHARACTERISTICS FOR 24 SATELLITES, EXCLUDING ELECTROMAGNETIC TERMS.

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REQUIRED ESTIMATE COMPUTER LOAD 128 KIPS

TYPICAL MINICOMPUTER CAPABILITIES WITH SINGLE PRECISION FLOATING POINT HARDWARE:

DATA GENERAL ECLIPSE
DEC PDP 11/45
HP 1000 SERIES

} ~ 150 KIPS

# REFERENCES


1) The Theory of Relativity, C. Moller; Clarenton Press 1972 2nd Ed. pg. 259, pg. 276

2) Relativistic Time for Terrestrial Circumnavigations, Y. C. Hafele, AJP Volume 40, Jan. 1972 pg. 81

3) The GPS Control Segment and its Service to the GPS USER, M. Y. Hurley, Y. L. Kramer, D. D. Thornburg IEEE 1976 Position Location and Navigation Symposium pg. 197