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TECHNICAL MEMORANDUM

SIGNATURE EXTENSION IN REMOTE SENSING

By

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SIGNATURE EXTENSION IN REMOTE SENSING*

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ABSTRACT

This paper considers the problem of signature extension in remote sensing. Signature extension is a process of increasing the spatial-temporal range over which a set of training statistics can be used to classify data without significant loss of recognition accuracy.

Methods are developed for the selection of segments for obtaining the training data. Selection of the number of segments is treated as the problem of expansion of rectangular matrix with basis matrices. Computational algorithms based on mean minimum square estimation error are developed for the selection of best segments. Furthermore, a combinatorial algorithm for generating all possible r combinations of S in Sc_r steps with a single change at each step is presented.

Key words:

Blocks
Combinatorial algorithm
Mean minimum square
  estimation error
Remote sensing
Segment selection
Signature extension
Spectral classes

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1. INTRODUCTION

In the application of remote sensing for large-area crop inventories and other, the multispectral Land Satellite data are processed in units called segments (a segment is an area 5 by 6 nautical miles). The processing of a segment necessitates the acquisition of labels of picture elements (pixels) to train the classifier. Obtaining the labels is costly, and training the classifier for every segment is time consuming.

To overcome the cost and time constraints, attempts have been made to solve the problem of signature extension; i.e., to train the classifier for classifying data acquired over large areas or many segments without significant loss of recognition accuracy. The goal of signature extension is then to minimize the requirements for obtaining the labels and extracting the training statistics.

Many current signature extension techniques (refs. 1 to 3) are based on a transformation of training statistics to compensate for changes in Sun angle, atmospheric and viewing conditions, etc., between the training area and the recognition area. The signature extension transformation of these techniques is both multiplicative and additive. Minter (ref. 4) reviews the techniques proposed for signature extension in the literature.

This paper considers an approach for signature extension (ref. 5) based on the assumptions that the data variations due to changes in Sun angle, atmospheric and viewing conditions, etc., can be significantly reduced by preprocessing and that the data are governed by a few inherent spectral classes related to the ground covers. With these assumptions, the training samples can be drawn from a few representative segments, and the classifier can be used to classify data acquired over large areas.

Let S be the total number of segments. Suppose that clustering these segments produces T spectral classes with a total of J blocks. The situation is illustrated in figure 1.

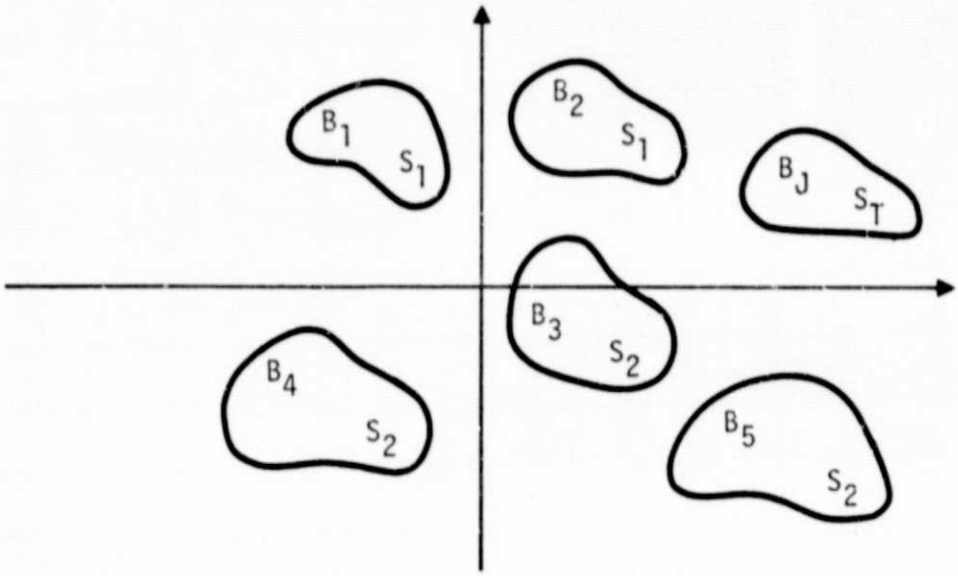

Figure 1.— Spectral classes and blocks in measurement space.

The number of spectral points counted in each block result in a matrix of segment versus count number for each block:

$$
\begin{array}{cc}
 & \rightarrow \text{Blocks} \\
 & \begin{array}{cccc} 1 & 2 & \ldots & J \end{array} \\
A = \begin{array}{c} 1 \\ 2 \\ \vdots \\ S \end{array} & \begin{bmatrix} () & () & \ldots & () \\ () & () & \ldots & () \\ \vdots & \vdots & & \vdots \\ () & () & \ldots & () \end{bmatrix} \downarrow \text{Segments}
\end{array}
\tag{1}
$$

Based on matrix A, computational algorithms for the selection of training segments are developed. The segments selected are representative; i.e., all the blocks associated with the T spectral classes are present in the selected segments. Section 2 describes an orthonormal expansion for rectanular matrices. Section 3 describes methods for the selection of best segments. Section 4 develops computational algorithms for the selection of individual segments. Section 5 presents a combinatorial algorithm for generating all possible r combinations out of S in Sc_r steps with a single change at each step. Appendix A presents a statistical description of the signature extension model, and appendix B derives matrix relations used in the paper.

2. EXPANSION OF RECTANGULAR MATRIX WITH BASIS MATRICES

In this section, the $S \times J$ rectangular matrix A is expanded in terms of basis matrices, and the basis matrices are obtained. From equation (1), form an $S + J \times S + J$ symmetric matrix B.

$$
B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \tag{2}
$$

Let C and D be the eigenvector and eigenvalue matrices of B. It can easily be verified that

$$
BC = CD \tag{3}
$$

where

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}}\Phi & \frac{1}{\sqrt{2}}\Phi \\ \frac{1}{\sqrt{2}}\psi & -\frac{1}{\sqrt{2}}\psi \end{bmatrix} \tag{4a}$$

and

$$D = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \tag{4b}$$

Φ is an $S \times S$ orthonormal matrix, ψ is a $J \times S$ matrix with orthonormal columns, and Λ is an $S \times S$ diagonal matrix; that is,

$$\left.\begin{aligned} \Phi^T\Phi &= \Phi\Phi^T = I \\ \psi^T\Psi &= I \end{aligned}\right\} \tag{5}$$

Since C is an eigenvector matrix of a symmetric matrix,

$$C^TC = I \tag{6}$$

equation (6) can easily be verified using equations (4a) and (5). Inner-multiplying both sides of equation (3) by matrix B of equation (2),

$$BBC = BCD$$

or

$$B^2C = CD^2 \tag{7}$$

is obtained. Using equations (2), (4a), and (4b) in equation (7),

$$\begin{bmatrix} AA^T & 0 \\ 0 & A^TA \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}\Phi & \frac{1}{\sqrt{2}}\Phi \\ \frac{1}{\sqrt{2}}\psi & -\frac{1}{\sqrt{2}}\psi \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\Phi & \frac{1}{\sqrt{2}}\Phi \\ \frac{1}{\sqrt{2}}\psi & -\frac{1}{\sqrt{2}}\psi \end{bmatrix}\begin{bmatrix} \Lambda^2 & 0 \\ 0 & \Lambda^2 \end{bmatrix} \tag{8}$$

is obtained. Expanding equation (8) yields

$$AA^T\Phi = \Phi\Lambda^2 \tag{9}$$

and

$$A^TA\psi = \psi\Lambda^2 \tag{10}$$

Let $S < J$. Since the maximum rank of A^TA is S, it will have at most S non-zero eigenvalues. Multiplying equation (9) on the left by A^T and on the right by Λ^{-1} gives

$$A^TA(A^T\Phi\Lambda^{-1}) = (A^T\Phi\Lambda^{-1})\Lambda^2 \tag{11}$$

On the comparison of equation (10) and (11), the following is obtained.

$$\psi = A^T\Phi\Lambda^{-1} \tag{12}$$

Taking the transpose of ψ and innermultiplying the result by Λ and Φ yields

$$A = \Phi\Lambda\psi^T \tag{13}$$

Expanding equation (13), matrix A can now be represented as

$$A = \sum_{i=1}^{S} \lambda_i \Phi_i \psi_i^T \tag{14}$$

The following work shows that the relative importance of each term in equation (14) is proportional to λ_i. Let $\hat{A}$ be the approximation of A using $\ell(<S)$ terms in equation (14). Define the squared error as

$$\begin{aligned} e^2 &= tr[(A - \hat{A})(A - \hat{A})^T] \\ &= tr\left\{\left(\sum_{i=\ell+1}^{S} \lambda_i \Phi_i \psi_i^T\right)\left(\sum_{j=\ell+1}^{S} \lambda_j \Phi_j \psi_j^T\right)\right\} \\ &= tr\left(\sum_{i=\ell+1}^{S} \sum_{j=\ell+1}^{S} \lambda_i \lambda_j \Phi_i \psi_i^T \psi_j \Phi_j^T\right) \\ &= \sum_{i=\ell+1}^{S} \lambda_i^2 \end{aligned} \tag{15}$$

Equation (15) shows that if a term is dropped in equation (14), the representation error in the mean square error sense is equal to the square of the corresponding eigenvalue.

3. COMPUTATIONAL ALGORITHMS FOR THE SELECTION OF INDIVIDUAL SEGMENTS

Based on the theory developed in the last section, a number m that gives an acceptable representation error can be chosen. This section considers the problem of choosing a particular set of m rows of matrix A or m segments and develops computational algorithms for their identification.

3.1 MEAN SQUARE ERROR IN ESTIMATION

Let the rows of matrix A be f_i^T, i = 1, 2 ..., S. Arbitrarily let the first ℓ segments be chosen. Let

$$F_\ell = [f_1, f_2, \ldots, f_\ell] \tag{16}$$

Where F_ℓ is a $J \times \ell$ rectangular matrix. Let the row $f_{\ell+j}^T$ be estimated as a linear combination of the rows f_i^T, i = 1, 2, ..., ℓ. That is,

$$\hat{f}_{\ell+j} = F_\ell \beta \tag{17}$$

where β is a vector of parameters. The estimation error between $\hat{f}_{\ell+j}$ and $f_{\ell+j}$ can be written as

$$\begin{aligned} \varepsilon_{\ell+j}^2 &= \left(\hat{f}_{\ell+j} - f_{\ell+j}\right)^T\left(\hat{f}_{\ell+j} - f_{\ell+j}\right) \\ &= \beta_\ell^T F_\ell^T F_\ell \beta + f_{\ell+j}^T f_{\ell+j} - 2\beta^T F_\ell^T f_{\ell+j} \end{aligned} \tag{18}$$

Differentiating equation (18) with respect to β and equating to zero yields the β, which minimizes $\varepsilon_{\ell+j}^2$ as

$$\beta = (F_\ell^T F_\ell)^{-1} F_\ell^T f_{\ell+j} \tag{19}$$

The substitution of equation (19) into (18) results in the minimum error as

$$\varepsilon^2_{\ell+j} = f^T_{\ell+j}\left[I - F_\ell\left(F^T_\ell F_\ell\right)^{-1}F^T_\ell\right]f_{\ell+j}$$

$$= \mathrm{tr}\left\{\left[I - F_\ell\left(F^T_\ell F_\ell\right)^{-1}F^T_\ell\right]f_{\ell+j}f^T_{\ell+j}\right\} \tag{20}$$

The mean minimum square error in the estimation of last $(S - \ell)$ rows in terms of the first ℓ rows of matrix A can be written as

$$\varepsilon^2(\ell) = \frac{1}{(S - \ell)}\sum_{j=\ell+1}^{S}\varepsilon^2_j$$

$$= \mathrm{tr}\left\{\left[I - F_\ell(F^T_\ell F_\ell)^{-1}F^T_\ell\right]\left[\frac{1}{(S - \ell)}\sum_{j=\ell+1}^{S} f_j f^T_j\right]\right\} \tag{21}$$

3.2 SELECTION OF THE BEST m SEGMENTS

The quantity $\varepsilon^2(\ell)$ derived in section 3.1 measures the effectiveness of the selected ℓ rows in estimating the remaining $S - \ell$ rows. The computation can be organized in two ways in finding the best m rows.

3.2.1 FORWARD SEQUENTIAL SEARCH

The computation involves finding each additional row, one at a time. After selection of the r rows, the $r + 1th$ row is selected among the remaining $S - r$ rows (by checking one at a time). This is a suboptimal procedure; it involves much less computation compared to an exhaustive search. Section 4 gives the recursive expressions for reducing the computation.

3.2.2 EXHAUSTIVE SEARCH

A method of selecting m optimal rows for mean minimum square estimation error is accomplished by forming all possible Sc_m combinations and evaluating $\varepsilon^2(m)$ for each combination and then selecting the best combination. Section 4 gives recursive expressions for $\varepsilon^2(m)$ for reducing the amount of

computation. These results, coupled with the results of section 5, provides an efficient algorithm for the implementation of exhaustive search in segment selection.

4. RECURSIVE EXPRESSIONS FOR COMPUTATION OF MEAN MINIMUM SQUARE ESTIMATION ERROR

In this section, recursive expressions are developed for the computation of mean minimum square estimation error when a row is added to the selected segment set and when a row is deleted from a selected segment set.

4.1 CHANGE IN THE CRITERION WHEN A PARTICULAR ROW IS ADDED

Let f_i^T , $i = 1, 2, \ldots, r$ be the rows of matrix A selected at the r*th* step. From the r*th* step to the $r + 1$*th* step, a row that reduces the estimation error most is added. Let f_{r+1}^T be the row that is added. Then,

$$F_r = [f_1, f_2, \ldots, f_r] \tag{22}$$

$$F_{r+1} = [f_1, f_2, \ldots, f_r, f_{r+1}] = [F_r f_{r+1}] \tag{23}$$

In the r*th* step, the mean minimum square estimation error $\varepsilon^2(r)$ is expressed as

$$\varepsilon^2(r) = \mathrm{tr}\left\{\left[I - F_r\left(F_r^T F_r\right)^{-1} F_r^T\right]\left[\frac{1}{(S - r)}\sum_{j=r+1}^{S} f_j f_j^T\right]\right\} \tag{24}$$

Similarly, $\varepsilon^2(r + 1)$ at the $r + 1$*th* step is

$$\varepsilon^2(r + 1) = \mathrm{tr}\left\{\left[I - F_{r+1}(F_{r+1}^T F_{r+1})^{-1} F_{r+1}^T\right]\left[\frac{1}{S - r - 1}\sum_{j=r+2}^{S} f_j f_j^T\right]\right\} \tag{25}$$

Consider

$$F_{r+1}^T F_{r+1} = \begin{bmatrix} F_r^T \\ f_{r+1}^T \end{bmatrix} \begin{bmatrix} F_r & f_{r+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_r^T F_r & F_r^T f_{r+1} \\ f_{r+1}^T F_r & f_{r+1}^T f_{r+1} \end{bmatrix} = \begin{bmatrix} B_r & b_r \\ b_r^T & \beta \end{bmatrix} \tag{26}$$

Let

$$\left(F_{r+1}^T F_{r+1}\right)^{-1} = \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} \tag{27}$$

From appendix B, the relationships between B's and A's can be written as

$$\frac{1}{\alpha} = \beta - b_r^T B_r^{-1} b_r \tag{28}$$

$$a_r = -B_r^{-1} b_r \alpha \tag{29}$$

$$a_r^T = -\alpha b_r^T B_r^{-1} \tag{30}$$

$$A_r = B_r^{-1} + B_r^{-1} b_r b_r^T B_r^{-1} \alpha \tag{31}$$

Let

$$\left.\begin{aligned} C_1 &= \left(F_r^T F_r\right)^{-1} \\ C_2 &= F_r \left(F_r^T F_r\right)^{-1} F_r^T \\ C_3 &= C_2 f_{r+1} \end{aligned}\right\} \tag{32}$$

Now consider

$$F_{r+1}\left(F_{r+1}^T F_{r+1}\right)^{-1} F_{r+1}^T = \begin{bmatrix} F_r f_{r+1} \end{bmatrix} \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} \begin{bmatrix} F_r^T \\ f_{r+1}^T \end{bmatrix}$$

$$= F_r A_r F_r^T + f_{r+1} a_r^T F_r^T + F_r a_r f_{r+1}^T + \alpha f_{r+1} f_{r+1}^T \tag{33}$$

However,

$$F_r A_r F_r^T = F_r\left(B_r^{-1} + B_r^{-1} b_r b_r^T B_r^{-1} \alpha\right) F_r^T$$

$$= F_r B_r^{-1} F_r^T + F_r B_r^{-1} F_r^T f_{r+1} f_{r+1}^T F_r B_r^{-1} F_r^T \alpha$$

$$= C_2 + C_2 f_{r+1} f_{r+1}^T C_2 \alpha \tag{34}$$

$$f_{r+1}\ a_r^T F_r^T = -f_{r+1} b_r^T B_r^{-1} F_r^T \alpha$$

$$= -f_{r+1} f_{r+1}^T C_2 \alpha \tag{35}$$

$$\frac{1}{\alpha} = f_{r+1}^T f_{r+1} - f_{r+1}^T F_r B_r^{-1} F_r^T f_{r+1}$$

$$= f_{r+1}^T f_{r+1} - f_{r+1}^T C_2 f_{r+1}$$

$$= \mathrm{tr}\left(f_{r+1} f_{r+1}^T\right) - \mathrm{tr}\left(C_2 f_{r+1} f_{r+1}^T\right) \tag{36}$$

From equations (33) through (36),

$$F_{r+1}\left(F_{r+1}^T F_{r+1}\right)^{-1} F_{r+1}^T = C_2 + C_2 f_{r+1} f_{r+1}^T C_2 \alpha - f_{r+1} f_{r+1}^T C_2 \alpha$$

$$- C_2 f_{r+1} f_{r+1}^T \alpha + \alpha f_{r+1} f_{r+1}^T \tag{37}$$

is obtained, where

$$\left.\begin{aligned} \alpha &= \frac{1}{tr\left(f_{r+1}f_{r+1}^T\right) - tr\left(C_2 f_{r+1}f_{r+1}^T\right)} \\ C_2 &= F_r\left(F_r^T F_r\right)^{-1} F_r^T \end{aligned}\right\} \tag{38}$$

Consider

$$\frac{1}{(S - r - 1)}\left(\sum_{j=r+2}^{S} f_j f_j^T\right) = \frac{S - r}{(S - r - 1)}\left[\frac{1}{(S - r)}\sum_{j=r+1}^{S} f_j f_j^T - \frac{1}{(S - r)} f_{r+1}f_{r+1}^T\right] \tag{39}$$

With equations (37), (38), and (39) in equation (25), (25) can be recursively computed from equation (24).

4.2 CHANGE IN THE CRITERION WHEN A PARTICULAR ROW IS DELETED

Let f_i^T, $i = 1, 2, \ldots, r$ be the rows of matrix A selected at the *rth* step. Let a row f_r^T be deleted from this set. This section presents the expressions derived for the computation of mean minimum square estimation error with this reduced set.

Let

$$\begin{aligned} F_r &= \left[f_1, f_2, \ldots, f_{r-1} f_r\right] \\ &= \left[F_{r-1} f_r\right] \end{aligned} \tag{40}$$

The estimation error $\varepsilon^2(r - 1)$, with the selected rows f_i^T, $i = 1, 2, \ldots, r - 1$, can be written as

$$\varepsilon^2(r - 1) = tr\left\{\left[I - F_{r-1}\left(F_{r-1}^T F_{r-1}\right)^{-1} F_{r-1}^T\right]\left[\frac{1}{S - r + 1}\sum_{j=r}^{S} f_j f_j^T\right]\right\} \tag{41}$$

Consider

$$F_r^T F_r = \begin{bmatrix} F_{r-1}^T \\ f_r^T \end{bmatrix} \begin{bmatrix} F_{r-1} f_r \end{bmatrix}$$

$$= \begin{bmatrix} F_{r-1}^T F_{r-1} & F_{r-1}^T f_r \\ f_r^T F_{r-1} & f_r^T f_r \end{bmatrix} = \begin{bmatrix} B_r & b_r \\ b_r^T & \beta \end{bmatrix} \tag{42}$$

Let

$$\left(F_r^T F_r\right)^{-1} = \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} \tag{43}$$

The relationship between A_r and B_r can be written as

$$A_r = B_r^{-1} + \frac{a_r a_r^T}{\alpha} \tag{44}$$

Consider

$$F_r\left(F_r^T F_r\right)^{-1} F_r^T = \begin{bmatrix} F_{r-1} f_r \end{bmatrix} \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} \begin{bmatrix} F_{r-1}^T \\ f_r^T \end{bmatrix}$$

$$= F_{r-1} A_r F_{r-1}^T + f_r a_r^T F_{r-1}^T + F_{r-1} a_r f_r^T + f_r f_r^T \alpha \tag{45}$$

From equations (44) and (45),

$$F_{r-1}\left(F_{r-1}^T F_{r-1}\right)^{-1} F_{r-1}^T = F_r\left(F_r^T F_r\right)^{-1} F_r^T - \frac{(F_{r-1} a_r)(R_{r-1} a_r)^T}{\alpha}$$

$$- f_r\left(F_{r-1} a_r\right)^T - F_{r-1} a_r f_r^T - f_r f_r^T \alpha \tag{46}$$

is obtained.

Now consider

$$\mathrm{tr}\left(\frac{1}{S - r + 1}\sum_{j=r}^{S} f_j f_j^T\right) = \mathrm{tr}\left[\frac{1}{S - r + 1}\sum_{j=r+1}^{S} f_j f_j^T + \frac{1}{S - r + 1} f_r f_r^T\right]$$

$$= \mathrm{tr}\left[\frac{S - r}{S - r + 1}\left(\frac{1}{S - r}\sum_{j=r+1}^{S} f_j f_j^T\right) + \frac{1}{(S - r + 1)} f_r f_r^T\right] \quad (47)$$

With equations (43), (46), and (47) in equation (41), (41) can be recursively computed from equation (24).

5. A COMBINATORIAL ALGORITHM FOR GENERATING ALL POSSIBLE COMBINATIONS

This section describes an algorithm for generating all possible r combinations out of S in Sc_r steps (ref. 6). At each step, a single change is made; i.e., one row is deleted and one is added. The recursive relations developed in section 4, coupled with this algorithm, can be used to search for r best segments out of all possible Sc_r combinations.

The initial combination may be any combination in which all the r-selected numbers are consecutive. In the binary representation, it means that all the r 1's are in one run in a vector of length S. For example, if r = 3 and S = 5, it may be started with 11100 or 01110 or 00111. The binary vector is denoted by A, and its *ith* component is A(i). Initially, all the components of A, except those of the last run, are marked. For example, if A = 00111000 (for r = 3 and S = 8), then it is marked as $\overline{00111}000$.

If a is a symbol, then a^m stands for $\underbrace{aa \ldots a}$, m times. Let i be the highest index so that $A(i)$ is marked. A vector $T(1), T(2), \ldots, T(S)$ of integers that satisfy the condition $|T(j)| \leq j$ for $j = 1, 2, \ldots, S$ is defined. Initially, $T(1) = 0$. If the initial combination is $(\overline{0})^p(\overline{1})^r 0^{S-r-p}$, where $S > r + p$, then $T(p + r) = -1$ and all the rest are immaterial. If the initial combination is $(\overline{0})^{S-r}1^r$, then $T(S - r) = -1$ and all the rest are immaterial. The changes that T must undergo in each combination generation are described by subroutines α and β as follows:

α: (i) If $T(k) = 0$, then output A and halt.

(ii) If $T(k) > 0$, then $i \leftarrow T(k)$, output A, and go to step (i) of the procedure.

(iii) $i \leftarrow k - 1$. If $T(k) > -(k - 1)$, then $T(k - 1) \leftarrow T(k)$.

(iv) Output A and go to step (i) of the procedure.

β: (i) $T(i) \leftarrow -(k + 1)$. If $T(k) \geq 0$, then $T(k + 1) \leftarrow T(k)$, output A, and go to step (i) of the procedure.

(ii) $T(k + 1) \leftarrow k - 1$. If $T(k) > -(k - 1)$, then $T(k - 1) \leftarrow T(k)$.

(iii) Output A and go to step (i) of the procedure.

Now the vector $F(0), F(1), \ldots, F(S)$ is introduced as follows. If $A(m) = 1$ and it is the rightmost element in a run of 1's, then $F(m)$ is the index of the first 1 of this run. If not, $F(m)$ is immaterial. Let ℓ be the index of the rightmost 1; that is, $\ell = \max_{A(m)=1} m$

An algorithm for generating all possible combinations with a single change at each step can now be described. The initial conditions of the algorithm are illustrated as follows. Let $r = 3$, $S = 8$ with an initial $A = 01110000$. Then $i = 4$.

(1) $k \leftarrow i$. If $A(i) = 1$, go to step (8).

(2) $j \leftarrow F(\ell)$.

(3) $A(i) \leftarrow 1$, $A(j) \leftarrow 0$. $F(k) \leftarrow k$. If $A(k - 1) = 1$ and $k > 1$, then $F(k) \leftarrow F(k - 1)$. $F(\ell) \leftarrow j + 1$; if $j < \ell$, go to step (5).

(4) $\ell \leftarrow i$. Perform α.

(5) If $\ell < S$, go to step (7).

(6) $i \leftarrow j$. Perform β.

(7) $i \leftarrow \ell$. Perform β.

(8) $F(i - 1) \leftarrow F(i)$. If $\ell > i$, go to step (12).

(9) $A(i) \leftarrow 0$, $A(S) \leftarrow 1$, $F(S) \leftarrow S$, $\ell \leftarrow S$. If $i < S - 1$, go to step (11).

(10) Perform α.

(11) $i \leftarrow S - 1$ and perform β.

(12) $J \leftarrow F(\ell)$

(13) $A(i) \leftarrow 0$, $A(j - 1) \leftarrow 1$, $F(\ell) \leftarrow j - 1$. If $\ell < S$, go to step (17).

(14) If $\ell + 1 \leftarrow j - 1$, go to step (16).

(15) Perform α.

(16) $i \leftarrow j - 2$. Perform β.

(17) $i \leftarrow \ell$. Perform β.

6. REFERENCES

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APPENDIX A

STATISTICAL INTERPRETATION OF THE SIGNATURE EXTENSION MODEL

In this appendix, a statistical interpretation of the signature extension model considered in the paper is given. Let i, t, and j respectively be the segment index, the spectral class index, and the block index.

Let $h(x|L = t)$ be the density function of the patterns in the t*th* spectral class. Let $f(x|s = i)$ be the density function of the patterns in the segments. Let T be the total number of spectral classes, J the maximum number of blocks, and S the total number of segments. Consider

$$f(x|s = i) = \sum_{t=1}^{T} f(x, L = t|s = i)$$

$$= \sum_{t=1}^{T} f(x|L = t, s = i)P(L = t|s = i)$$

$$= \sum_{t=1}^{T} h(x|L = t)P(L = t|s = i) \qquad \text{(A-1)}$$

where it is assumed that $h(x|L = t) = f(x|L = t, s = i)$. Let the euclidean space be partitioned into a set of blocks $B_1, B_2, \ldots, B_J$ and let

$$P(x \epsilon B_j|L = t) = \int_{B_j} h(x|L = t)dx$$

and

$$P(x\varepsilon B_j|s = i) = \int_{B_j} f(x|s = i)\, dx$$

$$= \sum_{t=1}^{T} \int_{B_j} P(L = t|s = i)h(x|L = t)dx$$

$$= \sum_{t=1}^{T} P(L = t|s = i)P(x\varepsilon B_j|L = t) \qquad \text{(A-2)}$$

Introducing a matrix notation, one can write equation (A-2) as

1, 2, ···, J blocks 1, 2, ···, T spectral classes 1, 2, ···, J blocks

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ S \\ \text{segments} \end{array} \begin{array}{c} \rightarrow j \\ \downarrow i \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} \quad \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ S \\ \text{segments} \end{array} \begin{array}{c} \rightarrow t \\ \downarrow i \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} \quad \end{bmatrix} \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ T \\ \text{spectral classes} \end{array} \begin{array}{c} J \rightarrow \\ \downarrow t \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} \quad \end{bmatrix}$$

$$P = AB \qquad \text{(A-3)}$$

where P is an $S \times J$ matrix with elements $p_{ij} = P(x\varepsilon B_j|s = i)$, A is an $S \times T$ matrix with elements $a_{it} = P(L = t|s = i)$, and B is an $T \times J$ matrix with elements $b_{tj} = P(x\varepsilon B_j|L = t)$. Equation (A-3) describes the probabilistic relationship among segments, spectral classes, and blocks.

APPENDIX B

MATRIX RELATIONSHIPS

This appendix derives the matrix relationships used in section 4. Let A and B be the inverse matrices of each other and are as shown below:

$$A = \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} B_r & b_r \\ b_r^T & \beta \end{bmatrix} \tag{B-1}$$

Since A is the inverse of B, BA = I. That is

$$BA = \begin{bmatrix} B_r & b_r \\ b_r^T & \beta \end{bmatrix} \begin{bmatrix} A_r & a_r \\ a_r^T & \alpha \end{bmatrix} = I \tag{B-2}$$

Expansion of equation (B-2) gives

$$B_r A_r + b_r a_r^T = I \tag{B-3}$$

$$B_r a_r + \alpha b_r = 0 \tag{B-4}$$

$$b_r^T A_r + \beta a_r^T = 0 \tag{B-5}$$

$$b_r^T a_r + \alpha\beta = 1 \tag{B-6}$$

From equation (B-3),

$$\left(B_r A_r\right)^{-1} = A_r^{-1} B_r^{-1} = \left(I - b_r a_r^T\right)^{-1}$$

$$= I + \frac{b_r a_r^T}{1 - a_r^T b_r} \tag{B-7}$$

is obtained. From equations (B-5), (B-6), and (B-7),

$$B_r^{-1} = A_r - \frac{a_r a_r^T}{\alpha} \tag{B-8}$$

is obtained. Equation (B-8) is used in section 4.2. From equation (B-4),

$$a_r = -B_r^{-1} b_r \alpha \tag{B-9}$$

is obtained. Substitution of equation (B-9) into (B-6) yields

$$\alpha = \frac{1}{\beta - b_r^T B_r^{-1} b_r} \tag{B-10}$$

From equations (B-3) and (B-9),

$$A_r = B_r^{-1} + B_r^{-1} b_r b_r^T B_r^{-1} \alpha \tag{B-11}$$

is derived. Equations (B-9), B-10), and (B-11) are used in section 4.1.