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DIPOLE AND QUADRUPOLE SYNTHESIS OF ELECTRIC POTENTIAL FIELDS

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Dipole and Quadrupole Synthesis of Electric Potential Fields

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ABSTRACT

The accumulation of static charge on the surface of spacecraft can produce unknown potential fields that may cause error in the measurement of scientific data. Therefore methods for determining potential fields are being sought. This report describes a general technique for expanding an unknown potential field in terms of a linear summation of weighted dipole or quadrupole fields. Laplace's equation describes the unknown potential in regions that are devoid of electric charge when the nearby surface charge distribution is assumed to be independent of time. Therefore, general solutions to Laplace's equation can be generated by summing multipole fields (i.e., particular solutions to Laplace's equation) as long as the multipole locations where the multipole fields have singularities are not included in the spatial region of interest. The general solution approximating an unknown potential near a charged surface can be developed if suitable boundary conditions are available or if measurements of charged particle trajectories in the unknown potential can be obtained.

The classical boundary value problem in electrostatics can be described as finding the solution to Laplace's equation when the electric potential is known on a boundary enclosing a spatial domain of interest. In this report, computational methods are developed for the iterative addition of dipole fields until solutions to the classical boundary value problem can be obtained. Various solution potentials are compared inside the boundary with a more precise calculation of the potential to derive optimal schemes for locating the singularities of the dipole fields (e.g., dipoles should not be placed
too close to the boundary). Then, the problem of determining solutions to Laplace's equation on an unbounded domain as constrained by pertinent electron trajectory data is considered. The initial electron coordinates and velocities, as well as the final electron coordinates, comprise a set of constraints on the various schemes that are developed for dipole and quadrupole synthesis of approximations to the test potential. The various schemes are then compared in terms of convergence limits and rates and in terms of their accuracy on a finite domain. The report is concluded with a description of an electron-beam apparatus used for making trajectory measurements in a bell jar. The best computational schemes are used to synthesize approximations of a simulated test potential created in the bell jar.
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I. INTRODUCTION

The accumulation of electrostatic charge on the surface of a spacecraft produces an electric potential field in nearby spatial regions. Laplace's equation, $\nabla^2 \psi = 0$, defines a class of functions which may represent this potential field everywhere outside the spacecraft (i.e., spatial regions having little charge). If the surface charge is known, the potential field $\psi$ can be determined by an integration over the spacecraft of the field due to a point charge weighted with the known surface charge distribution. However, for many applications the surface charge is unknown and must be determined by solving Laplace's equation constrained by the pertinent boundary conditions that may be known. The potential fields of the ideal dipole, quadrupole, octapole, etc. are solutions of Laplace's equations in regions bounded away from the sites of these ideal poles. Therefore, the unknown potential $\psi$ can be synthesized as a linear sum of multipole potential fields, with all point multipoles located outside the region of interest (ref. 1).

The classical boundary value problem may be considered for Laplace's equation in terms of a potential $\phi$ unknown in region $R$, but known at a finite number of points on a simple closed curve containing the region $R$. In two dimensions, the method of finite differences could be employed to solve for the unknown potential $\phi$ at the nodes of a grid covering region $R$ subject to the known boundary values of the potential $\psi$ (e.g., the specification of the potential at $4N$ points on a square boundary determines an approximation of the potential at $N^2$ points on the interior of the square boundary). Consider instead,
an iterative solution technique. Suppose some potential $\phi$ exists as an estimate of $\psi$. Then a measure of the discrepancy $D$ between $\psi$ and $\phi$ can be defined as the sum of the absolute differences between $\psi$ and $\phi$ at the finite number of nodes on the boundary where $\psi$ is known. The estimated potential $\phi$ can be said to be converging toward $\psi$ if the discrepancy $D$ is reduced upon successive applications of the iterative scheme which may include operations of rotation, shifting or scaling. However, discrepancy reduction and the shaping of the potential approximation will be primarily accomplished by the summation of multipole fields successively centered at points outside the boundary near locations where the discrepancy $D$ is largest.

The expansion of potential fields is most often expressed as the linear sum of weighted multipole fields (e.g., one each of monopole, dipole, quadrupole, etc.) with all pole locations superimposed. This practice produces only one singularity in the spatial solution space. However, we have adopted the practice of allowing multipole locations along a line parallel to the surface and at a specified distance from it (i.e., usually all alike dipoles or quadrupoles). Thus, field shaping is facilitated but many field singularities are produced.

When considering the potential field existing near a charged surface, the potential may be unknown in an unbounded spatial region. Due to insufficient boundary conditions, additional constraints must be specified in order to approximate the potential on a finite spatial region. Actually, in mathematical terms, the region is bounded at infinity where the potential must equal zero. In order to obtain measurements of potential fields existing near charged surfaces,
consider an electron beam emanating from a source point (X₀,Y₀) on a zero potential reference plane. If the reference plane were located far from the charged surface, it would approximate the boundary condition at infinity. Practically speaking, the reference plane can be moved to a finite and measurable distance from the charged surface while producing only minimal distortions of the potential field. If the spacecraft surface is negatively charged the electron beam will be deflected back toward the reference plane where the coordinates of a sink point (Xₜ,Yₜ) can be recorded. As seen in Figure 1, a displacement vector measured from the beam source point to the beam sink point represents one constraint on the family of functions that satisfy Laplace's equation. Suppose that a potential field, partially bounded by a charged surface and the reference plane, is to be determined based on measurements of many such displacement vectors associated with electron beams with known energies and initial directions. Suppose further that some estimate \( \phi \) of the potential \( \psi \) exists on the region of interest. Estimated displacement vectors can be obtained from the initial electron beam data and the estimated potential by programming the laws of electron mechanics. These estimated displacements can be compared to those measured as data to define the error related to the electron trajectory constraints. A discrepancy \( D_D \) can be defined as the sum of the absolute differences between the estimated displacement vectors and the measured displacement vectors. The estimated potential \( \phi \) is said to be converging toward \( \psi \) if the discrepancy is reduced upon successive applications of some iterative scheme which may include scaling or rotating operations. However, major alterations in the estimated potential can be accomplished by adding the potential fields
Figure 1. The Orientation of Ideal Line Multipoles with Respect to a Charged Surface and Reference Plane
of pairs of ideal multipoles. One multipole can be located above the charged surface at a given distance from the reference plane, while another multipole is located an equal distance below the reference plane. This imaging technique maintains a potential of zero on the reference plane while allowing for the shaping of the estimated potential in the region between the reference plane and the charged surface.

The nature and substance of the iterative schemes referred to in the above discussion are defined depending arbitrarily on the data that are available and the ingenuity of the programmer. A particular iterative scheme may have different convergence properties when applied to a number of different problems. A particular iterative scheme may have better convergence properties than other schemes when applied to a common problem. Furthermore, the selection of data defining a given problem may influence the convergence of a particular iterative scheme and the selection of an initial approximation may also influence the convergence toward a potential solution. Therefore, there are a broad range of topics that can be investigated and discussed relative to the synthesis of electric potentials by the iterative addition of ideal multipole fields. For the purposes of this report, the effects of boundary geometry, data selection and initial potential estimation will not be considered. Attention will be focused on the development of optimal iterative schemes based upon a well-defined model problem.
II. THE BOUNDARY VALUE PROBLEM

The classical boundary value problem in electrostatics may be stated as the problem of calculating values of a potential field existing on a spatial region devoid of electric charge (i.e., where Laplace's equation is valid) when given the value of the potential on a simple boundary enclosing the region of interest. When the potential is specified as a continuous function on the boundary, the problem can be solved by a separation of variables technique and the expansion of the potential in terms of a complete set of orthogonal eigenfunctions (ref. 2). The geometric shape of the boundary influences the choice of an appropriate coordinate system and the particular set of eigenfunctions used for the expansion of the potential field. The solution is continuous inside the boundary and the error depends on the number of terms used in the eigenfunction expansion.

In a practical situation, the value of the potential may be known only at a finite number of points on a boundary. The separation of variables and eigenfunction expansion method can be applied to obtain a continuous solution, if some approximation is used to interpolate a continuous functional value of the potential on the boundary from the finite number of known boundary values. The error of the solution depends on the approximation of the potential on the boundary as well as the number of terms in the expansion of the potential. The method of finite differences may also be used to approximate the potential inside a closed boundary when only a finite number of boundary values are known (ref. 3). The solution is developed by using Taylor series expansions to derive a finite number of linear potential equations in
terms of an equal number of unknowns (e.g., the potential values at all points on the grid enclosed by the boundary) and a finite number of knowns (e.g., the potential values at all points on the grid coinciding with the boundary). A solution may then be obtained by using Gaussian elimination procedures to solve the system of linear equations. A finite difference solution is not continuous since it is defined only at nodal points on the grid. Therefore, interpolation errors will exist in addition to Taylor approximation errors, both of which depend on the spacing of the grid lines. Boundaries with irregular shapes will be problematic; since either a very fine grid will be required, leading to lengthy computer calculations, or some boundary approximations will be required to accommodate a larger grid spacing.

A. Dipole Synthesis of Potential Fields

Consider an iterative scheme for summing the potential fields of ideal electric dipoles to develop solutions to Laplace's equation on a domain enclosed by a boundary where the potential is known at a finite number of points (i.e., Dirichlet boundary condition). An initial estimate of the potential satisfying Laplace's equation is an average of the known boundary values plus any linear terms suggested by the boundary values (e.g., \( \phi_0(x, y) = a + bx + cy \)). This estimated potential can be improved by the iterative addition of an ideal dipole outside the boundary and nearest the location exhibiting the largest discrepancy between estimated and known boundary values. The potential field of an ideal dipole is itself a solution to Laplace's equation. Hence, the linear sum of the fields due to many such electric dipoles is also a solution to Laplace's equation. We will assume to have synthesized a
particular solution to the boundary value problem when the sum of all
the absolute discrepancies between estimated and known boundary poten-
tial values is reduced below some tolerable limit. Such a solution is
not unique, since the choice of dipole locations is arbitrary. If the
above procedure for the reduction of discrepancy is pursued indefinite-
ly, ultimately one dipole of optimal magnitude will be located outside
the boundary near each one of the points where a boundary value is
given. Thus, the iterative scheme expands the potential as a linear
sum of the fields associated with dipoles located outside the
boundary. If the dipoles are allowed infinitely close to the boundary
points, then each dipole magnitude will be chosen to reduce the dis-
crepancy existing at one boundary location and have no effect on the
discrepancies at the other boundary locations. Solutions developed in
this manner are unacceptable since the field singularities introduced
near the boundary produce only localized reduction of discrepancy. Con-
sequently, the continuity of the potential solution on the boundary is
jeopardized by placing dipoles too near the boundary itself. There-
fore, we shall expect to synthesize solutions to the boundary value
problem, if the dipoles are located well outside the bounded domain of
interest. Convergence (e.g., discrepancy reduction) will be slower as
dipoles are moved further from the boundary, but better approximations
will result on the boundary between the pointwise specified values.
Irregularly shaped boundaries can be easily accommodated by the iterative
procedure for dipole synthesis. In addition, the estimated potential
is defined in a continuous manner as the sum of dipole fields at all
points on the bounded domain and no interpolation from a coordinate
grid is required.
To define a specific class of boundary value problems, assume that the boundary \( S \) of region \( A \) in the two-dimensional \( r-\theta \) plane can be represented by \( N \) points distributed around the origin with no two points having the same \( \theta \) coordinate value. Assume also that the potential \( \psi(r, \theta) \) satisfies Laplace's equation (i.e., \( \nabla^2 \psi = 0 \)) and is given at the same \( N \) points on the boundary

\[
\psi(R_n, \theta_n) = \psi_n; \quad n = 1, 2, 3 \ldots N
\]  

(II-1)

In order to allow calculations of the potential on the interior region \( A \), as well as the boundary \( S \), construct a \( 2N \times N \) coordinate grid with its origin at the geometric center or centroid of region \( A \) and including the \( N \) points of equation (II-1). Define the \( 2N^2 \) points \( (r_m, \theta_n) \) letting \( r_m = R_m \) for \( m = 1, 2, 3 \ldots N \) and \( r_m = \frac{(m-N)R}{N} \) for \( n = N+1, N+2, \ldots 2N \);

where \( R \) is the radius of the smallest circle having its center at the origin of the coordinate grid which also encloses the region of interest.

In order to begin an iterative procedure for approximating the potential at points on the interior, given the potential at \( N \) points on the boundary, define an initial estimated potential \( \phi_0 \) existing as an approximation to \( \psi \).

\[
\phi_0(r_m, \theta_n) = 1/2[\psi_n^{\max} - \psi_n^{\min}] + 1/2[\psi_n^{\max} - \psi_n^{\min}] r_m \cos \theta_n
\]  

(II-2)

This choice is arbitrarily based on the maximum and minimum values of the \( N \) known boundary potentials; it satisfies Laplace's equation (i.e., \( \nabla^2 \phi_0 = 0 \)) and it guarantees that the difference between \( \psi \) and \( \phi_0 \) on the boundary is no larger than \( \psi_n^{\max} - \psi_n^{\min} \).

As a general procedure for improving the accuracy of the estimated potential on the boundary \( S \), consider the iterative addition of dipole
potential fields \( \phi_d(r_m, \theta_n) \) due to the ideal dipoles located at 
\((r_d, \theta_d)\) outside of the boundary \( S \) enclosing region \( A \).

\[
\phi_E(r_m, \theta_n) = \phi_o(r_m, \theta_n) + \sum_{d=1}^{E} \phi_d(r_m, \theta_n) \tag{II-3}
\]

The following algorithm is used for calculating in two dimensions
(i.e., the \( r-\theta \) coordinate system) the potential fields associated with
ideal electric dipoles at \((r_d, \theta_d)\),

\[
\phi_d(r_m, \theta_n) = \frac{A_d [r_d - r_m \cos(\theta - \theta_n)]}{r_d^2 + r_m^2 - 2r_d r_m \cos(\theta - \theta_n)} \tag{II-4}
\]
as derived in Appendix A. We will assume that the estimated potential
is a good approximation to \( \psi \) on the interior of region \( A \), as well as on
the boundary \( S \), when a sufficient number of dipole potentials \( \phi_d \) have
been added to \( \phi_o \) such that \( \phi_E \) matches the boundary potential \( \psi_n \) at all
points where the latter quantity is given.

\[
\phi_E(R_n, \theta_n) = \psi(R_n, \theta_n) \quad n = 1, 2, \ldots, N \tag{II-5}
\]
The arbitrary constant \( A_d = qA_r \) represents the charge magnitude \( q \) of an
ideal dipole having separation \( A_r \) as measured along a radius extending
from the centroid of region \( A \) to the location \((r_d, \theta_d)\) of the dipole.

For the purpose of calculating the error of the approximation of \( \psi \)
by \( \phi_E \), define the discrepancy function \( D_n \) as the difference between \( \psi \)
and \( \phi_E \) at each of the \( N \) boundary locations where \( \psi \) is known.

\[
D_n \equiv \psi(R_n, \theta_n) - \phi_E(R_n, \theta_n) \tag{II-6}
\]
The average discrepancy and average absolute discrepancy, as defined
by equations (II-7) and (II-8), can be used as measures of the accuracy
of the estimated potential.

\[ \bar{D} = \frac{1}{N} \sum_{n=1}^{N} D_n \]  

(II-7)

\[ |\bar{D}| = \frac{1}{N} \sum_{n=1}^{N} |D_n| \]  

(II-8)

We are interested in developing iterative procedures for reducing \( \bar{D} \) and \( |\bar{D}| \). A uniform shift of the estimated potential can be used to reduce \( \bar{D} \) to zero. Suppose the average discrepancy \( \bar{D} \) exists for estimated potential \( \phi_E \). If a new estimated potential \( \phi'_E \) is formed by adding the average discrepancy \( \bar{D} \) to \( \phi_E \), then we wish to show that the average discrepancy \( \bar{D}' \) for \( \phi'_E \) is equal to zero.

\[ \bar{D}' = \frac{1}{N} \sum_{n=1}^{N} D' = \frac{1}{N} \sum_{n=1}^{N} (\psi_n - \phi'_n) \]

\[ = \frac{1}{N} \sum_{n=1}^{N} (\psi_n - (\phi_n + \bar{D})) \]  

(II-9)

Then by rearranging terms and recalling the definition of \( \bar{D} \) expressed by equation (II-7), we derive the desired result.

\[ \bar{D}' = \frac{1}{N} \sum_{n=1}^{N} (\psi_n - \phi_n) - \frac{1}{N} \sum_{n=1}^{N} \bar{D} = \bar{D} - \bar{D} = 0 \]  

(II-10)

Thus, the estimated potential can be shifted so that its average discrepancy with known boundary values is reduced to zero.

The reduction of the average absolute discrepancy is not as easily accomplished. Consider the rotation of the estimated potential, with respect to the coordinate grid, as one operation for reducing \( |\bar{D}| \) on the boundary. Define \( N \) rotated potentials \( \phi_R(r_m, \theta_n) \); for \( R = 1, 2, \ldots N \) such that
for all \( m = 1,2, \cdots 2N \). Then one might choose, as the best estimated potential, the rotated potential yielding the lowest average absolute discrepancy and zero average discrepancy.

The average absolute discrepancy might also be reduced by scaling the estimated potential after the rotating and shifting operations. The following algorithm is proposed for scaling the estimated potential so as to reduce the average absolute discrepancy whenever the average discrepancy is zero.

\[
\phi(r_m, \theta_n) = \bar{\phi} + \frac{1}{N} \sum_{n=1}^{N} \frac{|\psi(R_n, \theta_n) - \bar{\psi}|}{\sum_{n=1}^{N} |\phi(R_n, \theta_n) - \bar{\phi}|} \quad [\phi(r_m, \theta_n) - \bar{\phi}] \\
\bar{\phi} = \frac{1}{N} \sum_{n=1}^{N} \phi(R_n, \theta_n) \\
\bar{\psi} = \frac{1}{N} \sum_{n=1}^{N} \psi(R_n, \theta_n)
\]

After the rotation, shifting and scaling operations, the addition of ideal dipole potentials can be used to further reduce \(|D|\) and improve the approximation of \( \psi \) by \( \phi \). The definition of a specific iterative scheme depends on the general criteria that can be established for determining optimal locations and magnitudes for each successive electric dipole that is added near the boundary enclosing the region of interest.
B. Iterative Schemes for the Dipole Synthesis of Solutions to the Boundary Value Problem

VMODGR.F4 is the name of the main Fortran routine used for solving boundary value problems. Initially, subroutine DATA is called to set up the problem in the X-Y coordinate system corresponding to the user's particular reference frame. The subroutine asks the user for up to forty points representing the boundary and for the potential value at each of the points which must be specified in a clockwise sequence around any plane boundary. The main routine generates an r-θ coordinate grid with origin at the centroid of the region enclosed by the boundary.

The determination of a lower bound on the distance between the boundary and the dipole locations is the primary objective of the study. It is assumed that all dipoles will be best aligned if their axes, extending in the direction of charge separation, pass through the centroid of the bounded domain. Subroutine ADPOLE locates a dipole at the angular coordinate \( \theta_d \) where the discrepancy \( D \) is greatest between the estimated potential \( \phi \) and the boundary potential \( \psi \). Then a parameter \( G \) is defined in subroutine ADPOLE so that all dipoles are located a fixed radial distance from the centroid. This distance

\[
r_d = R(1+G/N)
\]  

(II-16)

is calculated in terms of the number \( N \) of known boundary potentials and the largest radius \( R \) needed to construct a circle centered at the centroid and enclosing all of the region of interest. Dipoles will be allowed closer to the boundary as the value of the \( G \) parameter, specified by the user interactively with subroutine DATA, is allowed to decrease toward a limiting value of zero. Once a dipole location has
been determined, subroutine ADPOLE choses a dipole magnitude $A_d$ that reduces the discrepancy at the boundary location nearest the dipole to approximately zero. Alternatively, subroutine TADPOLE may be called by the main routine to select dipole locations and magnitudes. In this subroutine, dipoles are also located at the angular coordinate $\theta_d$ corresponding to the greatest boundary discrepancy $D_1$ and the dipole axis is aligned with the centroid of the region of interest. However, for the TADPOLE scheme, the G parameter does not fix the radial coordinate for dipole location but specifies a minimum value that can be allowed. Whenever possible, a dipole magnitude $A_d$ and dipole radius $r_d$ are chosen simultaneously so that

$$\phi_d(R_d, \theta_d) = -D_1$$  \hspace{1cm} (II-17)$$

and

$$\phi_d(R_c, \theta_c) = -\frac{D_1}{|D_1|} T \cdot |D|,$$  \hspace{1cm} (II-18)$$

where the coordinates $(R_c, \theta_c)$ represent the point on the boundary nearest the point $(R_d, \theta_d)$ where the discrepancy has a sign opposite from the sign of the largest discrepancy $D_1$. Whenever, the criterion expressed in equations (II-17) and (II-18) cannot be achieved without placing the dipole inside the minimum radius, as expressed by equation (II-16), $r_d$ is chosen equal to this minimum radius on the basis of the G parameter specified in subroutine DATA. Thus, the T parameter, also entered interactively with subroutine DATA, balances discrepancy reduction at the boundary point where the discrepancy is maximum with the accompanying discrepancy increase at a nearby boundary location.

The main routine VMODGR.F4 first defines an initial estimated potential according to equation (II-2) and then calls subroutine
ROTATE to achieve discrepancy reduction by the rotation of the initial estimated potential relative to the coordinate grid. An iterative scheme, calling subroutines SHIFT, SCALE and either ADPOLE or TADPOLE; is then applied to accomplish the reduction of boundary discrepancies. Subroutines SHIFT and SCALE, listed in Appendix B, accomplish discrepancy reduction according to equations (II-10) and (II-13) respectively. The other Fortran routines described in this section are also listed in Appendix B.

The particular problem of a square boundary with a potential of 10 volts specified on three sides and 0 volts specified on the fourth side has been chosen to demonstrate the method of dipole synthesis and to investigate various criteria for choosing dipole locations in an iterative scheme. Figure 2 shows how twelve boundary points were selected at uniform angular intervals and how one interior point at \((r = .471, \theta = 90^\circ)\) was selected for comparing synthesized potentials with a precise series expansion which yields a value of 9.59 volts just inside the unit square boundary. Tables 1 and 2 show the potential values synthesized at the test point with the main routine VMODGR using subroutines ADPOLE and TADPOLE, respectively. The fields of ideal dipoles were added at each boundary point and at the test point in an iterative process, one dipole per iteration, until the sum of the absolute discrepancies was reduced below \(10^{-5}\) volts. Note, in Table 1, that the estimated potential converges to its precise value at the test point as the dipoles are moved further from the boundary. However, the number of iterative computer calculations required is large. When dipoles are allowed closer to the boundary (i.e., smaller G parameter), convergence is much faster but errors in the estimated potential at the
Table 1

Potential Synthesis With Constant Dipole Radius

<table>
<thead>
<tr>
<th>G</th>
<th>Iterations</th>
<th>Test Potential (volts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>65</td>
<td>10.42</td>
</tr>
<tr>
<td>2</td>
<td>73</td>
<td>10.08</td>
</tr>
<tr>
<td>4</td>
<td>109</td>
<td>9.83</td>
</tr>
<tr>
<td>2\pi</td>
<td>157</td>
<td>9.67</td>
</tr>
<tr>
<td>10</td>
<td>354</td>
<td>9.61</td>
</tr>
<tr>
<td>15</td>
<td>791</td>
<td>9.59</td>
</tr>
<tr>
<td>20</td>
<td>1718</td>
<td>9.59</td>
</tr>
</tbody>
</table>

Table 2

Potential Synthesis With Variable Dipole Radius, G = 2\pi

<table>
<thead>
<tr>
<th>T</th>
<th>Iterations</th>
<th>Test Potential (volts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>157</td>
<td>9.67</td>
</tr>
<tr>
<td>0.30</td>
<td>157</td>
<td>9.67</td>
</tr>
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<td>0.50</td>
<td>161</td>
<td>9.64</td>
</tr>
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<td>0.70</td>
<td>169</td>
<td>9.59</td>
</tr>
<tr>
<td>0.90</td>
<td>214</td>
<td>9.59</td>
</tr>
</tbody>
</table>
Figure 2. Polar Coordinate Frame of Reference for The Boundary Value Problem
test point are substantial. Note, in Table 2, that the G parameter is set equal to $2\pi$ thereby establishing the minimum radial coordinate available for the location of point dipoles. The convergence of the estimated potential to the specified boundary values is fastest when the T parameter forces many dipoles close to the boundary (i.e., smaller values of T). The occasional placement of dipoles at greater distances from the boundary, using both the G and T parameters, provides a flexibility that allows compromise between the speed of the convergence and the accuracy of the estimated potential at interior locations. Thus, convergence with the required accuracy at the test point can be achieved with 791 iterations using the G factor alone and with 169 iterations using the G and T factors together.

Once convergence is achieved, VMODGR calls subroutine VGRAPH to plot equipotential lines as determined on the interior coordinate grid. Figures 3 and 4, corresponding to G parameter values of 2 and $2\pi$ respectively, show solutions to the boundary value problem for 20 points on the square boundary spaced at equal angular intervals. Figure 5 shows the potential solution for a G parameter of $2\pi$ and a T parameter value of 0.1. A comparison of Figures 4 and 5 reveals no major differences between the potential solutions generated by the two iterative schemes. However, the solution shown in Figure 5 was generated by using subroutine TADPOLE and required less than half the computer time needed when subroutine ADPOLE was used. Figure 6 shows the improvement in the solution potential resulting from the specification of 36 boundary values as compared to only 20 boundary values. It should be noted that the grid structure used inside the boundary results in linear interpolation errors in the drawing of equipotential lines. These
Figure 3. Equipotential Lines for Dipole Synthesis with $G=2$ and 20 angular grid lines
Figure 4. Equipotential Lines for Dipole Synthesis with $G=2\pi$ and 20 angular grid lines
Figure 5. Equipotential Lines for Dipole Synthesis with $G=2\pi$, $T=0.1$ and 20 angular grid lines
Figure 6. Equipotential Lines for Dipole Synthesis with $G=2\pi$, $T=0.1$ and 36 angular grid lines
errors are not a consequence of the procedure for potential field synthesis but result when an attempt is made to display a potential field based upon its value at a finite number of points. The potential display could be improved by a better interpolation procedure or by calculating the potential at more points. However, the accuracy of the potential estimate synthesized at a particular point of interest does not depend on the number of interior points considered. Therefore, potential estimates could be synthesized at as many, or as few, interior points as desired with accuracy depending on the number of boundary values specified.
III. THE DEFINITION OF A MODEL PROBLEM

The electric potential resulting from a distribution of surface charge can be shown to satisfy Laplace's equation everywhere except on the surface itself. Since the potential fields associated with ideal multipoles are also solutions of Laplace's equation, we assume that a linear sum of weighted multipole potentials can be used to construct a particular solution that approximates the potential field created by the surface charge distribution. This particular solution will have singularities at the spatial locations of the multipoles. Therefore, the spatial domain near a charged surface can be partitioned into two subspaces; one subspace defined as the region where a valid potential approximation is desired and another subspace reserved for the location of ideal multipoles. In the boundary value problem considered previously, the closed boundary separated the interior subspace (where solutions were desired) from the exterior subspace (where dipoles were located). In a general situation, the charged surface may only partially define the subspace where multipole location is allowed. In any event, an iterative scheme for summing multipole potentials to approximate the potential near a charged surface will depend on the geometry of the particular problem under consideration.

Furthermore, in contrast to the boundary value problem, the surface charge and surface potential may be unknown in the general problem. Trajectory data for electrons moving in an unknown potential field created by an unknown charge distribution can be used as constraints on the class of functions that approximate the potential field (ref. 4). Electron deflection measurements can be defined, in the absence of
boundary values, as a basis for calculating the discrepancy between the potential in the region probed by the electron trajectories and the multipole potential approximation generated by an iterative procedure. In such a case, the electron trajectories define a subspace where the potential approximation is desired and where ideal multipoles cannot be located. Therefore, a particular iterative scheme for determining the optimal locations of dipoles, quadrupoles or multipoles will be influenced by the electron trajectory data provided as part of a general problem.

To define a particular problem that can be used to compare the convergence properties of various iterative procedures, consider the potential field created by two uniformly biased fins in the vicinity of a zero potential ground plane. So that calculations may be performed in a two-dimensional space, assume that both the ground plane and the biased fins extend infinitely in the positive and negative z coordinate directions. Note, in Figure 7, that the fins begin at x-y coordinate values \((\frac{3DX}{4}, DY)\) and \((-\frac{3DX}{4}, DY)\) respectively and extend infinitely in the positive y dimension. Suppose the x-y coordinate plane can be mapped onto the \(x'-y'\) coordinate plane by shifting the origin by amounts \(DY\) and \(\frac{DX}{2}\), by scaling the axes by factors of \(\frac{DX}{\pi}\) and \(\frac{DY}{\pi}\), and by rotation of the axes through 90°.

\[
x' = \frac{\pi}{DY} (DY-y) \quad \text{and} \quad y' = \frac{\pi}{DX} (x-\frac{DX}{2})
\]  

(III-1)

The conformal mapping of the complex \(z\) plane \((z = x' + iy')\) onto the complex \(w\) plane \((w = u + iv)\) is

\[
w = \sin^{-1}(e^{z})
\]

(III-2)
Figure 7. The Reference Frames of the Biased Fins and the Conformal Mapping $w = \sin^{-1} \exp(z)$
which collapses the infinite fins onto a finite interval (i.e., [0',0']) along the u-coordinate axis (ref. 5). The value of this transformation lies in the fact that the potential field created by the biased fins in the x-y plane transforms to a potential with a uniform gradient in the u-v plane. Thus,

\[ \phi(z) - C = \nu(z) = \text{Im}[w] = \text{Im}[\sin^{-1}(e^z)] \]  

(III-3)

can be used with an appropriate expression for \( \sin^{-1}(e^z) \) to construct an algorithm

\[ \phi(x',y') - C = \text{Im}\{-i\ln[ie^{x'y'} + (1-e^{2x'y'})^{1/2}] \} \]  

(III-4)

that can be used to calculate the biased fin potential at selected points in the x-y plane. The constant C is required so that the potential can be shifted enough to simulate the zero potential line \( x' = Dy \) (i.e., \( y = 0 \)). In order to simplify the algebraic evaluation of the natural logarithm in equation (III-4), convert the complex argument of the square root to polar form.

\[ r \equiv (1+e^{-2x'} - 2e^{-x'}\cos 2y')^{1/2} \]  

(III-5)

\[ \theta \equiv \pi + \tan^{-1}\left[\frac{e^{2x'}\sin 2y'}{1-e^{2x'}\cos 2y'}\right]. \]  

(III-6)

These expressions lead to the form

\[ \phi(x',y',r,\theta) = \text{Im}\{-i\ln[ie^{x'y'} + \sqrt{r}e^{i\theta/2}]\} + C. \]  

(III-7)

To further simplify the calculations, let

\[ R \equiv [r+e^{2x'} - 2\sqrt{r}e^{x'}\sin(y'-\theta/2)]^{1/2} \]  

(III-8)
and
\[
\Omega = \pi + \tan^{-1} \left[ \frac{\rho x' \cos y' + \sqrt{r} \sin \theta/2}{\sqrt{r} \cos \theta/2 - e^{x'} \sin y'} \right]
\] (III-9)

to obtain
\[
\phi(r) - C = \text{Im}[-i \cdot \ln(\text{Re}^{i \Omega})] = \text{Im}[-i \cdot \ln(r) + i \Omega] = -\ln(r).
\] (III-10)

Therefore, to find the potential anywhere in the x-y plane; we first calculate values of x' and y', then find the corresponding values of r and \( \theta \), and finally determine a value of R. The potential at point (x,y) is then equal to the natural logarithm of R plus an arbitrary constant.

This algorithm for calculating the potential field in the x-y coordinate space of the biased fins is implemented by main routine TEST.F4 (ref. Appendix C). When the additive constant is properly chosen, the potential falls to zero along the x axis within a tolerance of 1% of the fin potential if the spacing of the fins is not greater than the distance from the fins to the x axis by more than a factor of 1.8. Thus, the ground reference plane could be moved to within a distance \( \Delta Y = \Delta X / 3.6 \), when the spacing of the fins is \( \Delta X / 2 \), without altering the potential field of the biased fins by significant proportions. TEST.F4 calculates potential values on an 80 x 27 coordinate grid corresponding to a region \( \Delta X \) by 1.4\( \Delta Y \) in the x-y plane. The I and J indices are used to identify points in a region with dimensions \( \Delta X \) by \( \Delta Y \) according to
\[
X(I) = (I-2)\Delta X / 80
\] (III-11)
and
\[
Y(J) = (J-2)\Delta Y / 20.
\] (III-12)
These values are written into a storage file POTL.DAT which may be accessed by other routines that require the test potential. Table 3 lists values of the biased fin potential at 110 points within the spatial region of interest. TEST.F4 also includes a graphic subroutine, VPlot, that utilizes a linear interpolation technique to plot equipotential lines in the space between the biased fins and the ground reference plane. Figure 8 shows the graphical plot generated by TEST.F4 with input parameters, DX = 3.6 and DY = 1.0.

Also consider six electron trajectories, emanating at six locations along the x axis, that probe the region between the fins before falling back to the reference plane. Assume that the initial coordinate values and the initial coordinate velocities are known as well as the final coordinate values of the six trajectories that serve as constraints placed upon approximations of the biased fin potential. In particular, let five trajectories emanate from locations (0.900,0), (1.305,0), (1.710,0), (2.105,0) and (2.520,0) along the x axis and with an initial direction parallel to the y axis and with initial kinetic energy equal to 85% of the fin bias. Also let a scaling trajectory emanate from the location (0.855,0) on the x axis with an initial velocity in the x direction equal to 1.1 times the velocity in the y direction. The main routine DATA.F4 is used to determine the final coordinates of the six trajectories defined by the initial electron parameters and the test potential. DATA.F4 calls subroutine DVOGEL in an iterative manner to calculate successive electron steps through a test potential specified by reading file POTL.DAT. DVOGEL solves the electron's second order equations of motion according to a method proposed by the mathematician DeVogelaire (ref. 6). Electric fields
Table 3

Values of the Test Potential at 110 Selected Points

<table>
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<tr>
<th>y-coordinate index (J)</th>
<th>21</th>
<th>19</th>
<th>17</th>
<th>15</th>
<th>13</th>
<th>11</th>
<th>9</th>
<th>7</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
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<td>1.82</td>
<td>1.67</td>
<td>1.57</td>
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<td>1.52</td>
<td>1.57</td>
<td>1.67</td>
<td>1.82</td>
<td>2.02</td>
</tr>
<tr>
<td></td>
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<td>1.61</td>
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<td>1.43</td>
<td>1.38</td>
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<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
</tr>
</tbody>
</table>

x-coordinate index (I)

| 22 | 26 | 30 | 34 | 38 | 42 | 46 | 50 | 54 | 58 | 62 |

Figure 8. Equipotential Lines for the Test-Biased Fin Potential on the Domain 22 < I < 62, 2 < J < 22
required by subroutine DVOGEL at points between the grid lines are obtained by a call to subroutine NEWTON which implements a divided difference interpolation (ref. 7) of the electric fields based upon the grid potential values provided by POTL.DAT. The electric fields at the 16 grid points (i.e., a 4 x 4 minigrid) closest to the point where an interpolated electric field is required, are used to determine a spatial step size (not in excess of 0.01 times a typical diagonal measurement of the region of interest) used for the piecewise development of electron trajectories.

The object of routine DATA.F4 is the determination of the final coordinate values (0.901,0), (1.848,0), (1.824,0), (1.709,0) (2.086,0) and (2.691,0) for the six constraining trajectories. These final coordinate values, in addition to the initial coordinate values and velocities are written into file TRAJ.DAT which can be referenced by routines that approximate the potential field of the biased fins. DATA.F4 includes a graphics routine for plotting the constraining electron trajectories and calls subroutine VPLOT to superimpose equipotential lines representing the test potential as shown in Figure 9.

Figure 9. Equipotential Lines for the Test-Biased Fin Potential and Constraining Electron Trajectories on the Domain 2< I <82, 2< J <22
IV. DIPOLE SYNTHESIS OF ELECTROSTATIC POTENTIALS

Preliminary investigation was begun with the intention of developing methods for dipole synthesis of three-dimensional electrostatic potentials assuming a uniform potential in one coordinate direction. The problem can be considered two dimensional if a line dipole is constructed by letting two infinite lines of opposite charge approach each other in the limit of superposition. The result is a line dipole potential which is uniform in the coordinate direction parallel to the line dipole. Furthermore, in a plane perpendicular to the line dipole; a difference results from integration along the line charges producing a $1/r$ variation of the potential with distance from the line dipole, rather than the $1/r^2$ dependence associated with a point dipole.

The line dipole described above is to be used as the basic building block for synthesizing unknown potentials in a two-dimensional space that can be probed by electron beams. This space is bounded above by the charged surface that produces the potential of interest and bounded below by the zero potential reference plane of the electron beam apparatus. Since the potential must be zero on this boundary, line dipoles can be placed in pairs, one on either side of the boundary so as to be electrostatic images of each other. Figure 1 shows the space of interest, the zero-potential boundary and a line dipole pair. Notice that the pair of line dipoles is located outside the space of interest so that Laplace's equation will hold on the interior.

The model problem under consideration requires the determination of the magnitude, location and number of line dipole pairs that best approximate the potential existing in the space of interest resulting
from an unknown surface charge distribution. The potential in the region of interest is described by electron trajectory data including initial electron coordinates and velocities and final electron coordinates. The basic method begins with a uniform field approximation of the potential which can be modified by the iterative addition of line dipole potentials until a potential is created that reproduces the electron trajectory data.

To formulate an iterative scheme for the dipole synthesis of solutions to the model problem, suppose that the potential $\psi$ exists as a result of the biasing of two fins relative to a zero potential reference plane. The last of the six trajectories listed in the statement of the model problem (i.e., the scaling trajectory) can be used for a uniform field approximation of $\psi$ in order to develop an initial estimated potential $\phi_0$. The equations of motion describing the scaling trajectory are

$$\frac{d^2x}{dt^2} = \frac{q}{m} \frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{q}{m} \frac{\partial \phi}{\partial y} = \frac{q}{m} A_o. \quad (IV-1)$$

A linear system of two equations

$$XT(6) = X0(6) + T \cdot VX0(6) \quad (IV-2)$$

and

$$YT(6) = Y0(6) + T \cdot VY0(6) + \frac{1}{2} \frac{q}{m} A_o \cdot T^2 \quad (IV-3)$$

in two unknowns (i.e., the constants $A_o$ and $T$) is developed by integrating equations (IV-1) twice and substituting the data for the scaling trajectory. When $YT(6) = Y0(6) = 0$, the result is
Thus, an initial estimated potential, \( \phi_0 = A_0 y \), can be calculated at all
points in the region of interest based upon the scaling trajectory data
and the electron charge to mass ratio \( \frac{q}{m} \). The estimated potential \( \phi \) can
be synthesized from \( \phi_0 \) and a number of dipole fields (see Appendix A),

\[
\phi_d(x,y) = A_d \left( \frac{y-y_d}{(x-x_d)^2 + (y-y_d)^2} + \frac{y+y_d}{(x-x_d)^2 + (y+y_d)^2} \right),
\]

to improve the estimation of the biased fin potential,

\[
\phi(x,y) = \phi_0(x,y) + \sum_d \phi_d(x,y),
\]

according to some iterative scheme that specifies optimal dipole param-
eters \( A_d, x_d, y_d \) for meeting the constraints imposed by the electron
trajectory data (i.e., trajectories 1-5). For each iteration of the
scheme, estimated electron trajectories are developed from the initial
trajectory data (i.e., XO(K), YO(K); K=1,2,...6) and the equations of
motion (IV-1) substituting, for \( \phi_0 \), the best estimate of the potential
field \( \phi(x,y) \) calculated from equation (IV-6). The coordinates XM(K)
and YM(K) at the highest points on the estimated trajectories are re-
corded as well as the exit coordinates XE(K) and YE(K); for K=1,2,...6.
A discrepancy is defined

\[
D(K) = \frac{[XT(K)-XE(K)]}{[XT(K)-XO(K)]}; \quad K=1,2,...5
\]

based upon estimated trajectory data XE(K) and actual trajectory data
XT(K) and XO(K). In order to reduce the discrepancies, dipoles should
be located between trajectories having discrepancies opposite in sign.
Therefore, the trajectory with the largest absolute discrepancy
(e.g., \( K = K_1 \)) is identified and the nearest trajectory having a discrepancy of opposite sign (e.g., \( K = K_2 \)) is identified. The dipole \( x_d \) parameter,

\[
x_d = \frac{1}{2} \left[ XM(K_1) + XM(K_2) \right],
\]

is chosen midway between these two trajectories. The dipole \( y_d \) parameter is fixed, \( y_d = 1.4 \cdot DY \), at a reasonable distance from the region of interest (e.g., as close to the region as possible without introducing local singularities, as discussed for the boundary value problem). Then a dipole magnitude \( A_d \) can be chosen which reduces the discrepancy of the \( K_1 \) trajectory to approximately zero.

Toward this end, consider the integration of the equations of motion (IV-1), with \( \phi \) replacing \( \phi_0 \), for electrons following the \( K_1 \) estimated trajectory through the estimated potential. If the time \( T \) represents the time required for electrons to traverse this trajectory, then the results of the integration are

\[
XE(K_1) = X_0(K_1) + T \cdot VX_0(K_1) + \frac{1}{2} \frac{q}{m} \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial x} \, dt \, dt'.
\]

and

\[
YE(K_1) = Y_0(K_1) + T \cdot VY_0(K_1) + \frac{1}{2} \frac{q}{m} \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial y} \, dt \, dt'.
\]

The \( K_1 \) trajectory consists of a number of points whose coordinates have been determined by a stepwise solution of the equations of motion. The electric fields, \( \frac{\partial \phi}{\partial x} \) and \( \frac{\partial \phi}{\partial y} \), at these points were calculated along with the time steps for electron motion between the points in the development of the \( K_1 \) trajectory. Therefore, the integrals in equations (IV-9) and (IV-10) can be replaced by a pointwise summation of the electric fields
over the \( N \) points of the \( K_1 \) trajectory using trapezoidal integration to develop

\[
XI(K_1) \equiv \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial x} \, dt \, dt' = \frac{1}{2} \sum_{n=1}^{N} (S_n - 1 + S_n) \Delta t_n
\]  

(IV-11)

where

\[
S_n = \frac{1}{2} \sum_{i=1}^{n} \left[ (\frac{\partial \phi}{\partial x})_{i-1} + (\frac{\partial \phi}{\partial x})_i \right] \Delta t_i
\]  

(IV-12)

and

\[
YI(K_1) \equiv \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial y} \, dt \, dt' = \frac{1}{2} \sum_{n=1}^{N} (R_{n-1} + R_n) \Delta t_n
\]  

(IV-13)

where

\[
R_n = \frac{1}{2} \sum_{i=1}^{n} \left[ (\frac{\partial \phi}{\partial y})_{i-1} + (\frac{\partial \phi}{\partial y})_i \right] \Delta t_i.
\]  

(IV-14)

Substituting the approximation of equation (IV-13) into equation (IV-10), we have for \( YE(K_1) = YO(K_1) = 0 \),

\[
T = -\frac{1}{2} \frac{q}{m} YI(K_1).
\]  

(IV-15)

Substituting the approximation of equation (IV-11) into equation (IV-9), we have

\[
XE(K_1) = XO(K_1) + T \cdot VXO(K_1) + \frac{1}{2} \frac{q}{m} XI(K_1).
\]  

(IV-16)

Now, we wish to add a dipole potential \( \phi_d \) to the estimated potential \( \phi \), altering the \( K_1 \) trajectory only slightly and shifting the exit
coordinate from $XE(K_1)$ to the data coordinate $XT(K_1)$). The equations of motion for this new estimated potential are

$$\frac{d^2 x}{dt^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi_d}{\partial x} \right) \quad \text{and} \quad \frac{d^2 y}{dt^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi_d}{\partial y} \right). \quad (IV-17)$$

Integration of the first of these equations (IV-17), yields

$$XT(K_1) = X0(K_1) + T \cdot VX0(K_1) + \frac{1}{2} \frac{q}{m} \int_0^T \int_0^{t'} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi_d}{\partial x} \right) dt \, dt'. \quad (IV-18)$$

When trajectory $K_1$ is shifted only slightly, the integral in equation (IV-18) can be replaced by the summation of electric fields over the same $N$ points used to generate equation (IV-16). Hence,

$$XT(K_1) = X0(K_1) + T \cdot VX0(K_1) + \frac{1}{2} \frac{q}{m} \left[ VI(K_1) + A_d \cdot VI(K_1) \right], \quad (IV-19)$$

where

$$VI(K_1) \equiv \frac{1}{2} \sum_{n=1}^{N} \left( V_{n-1} + V_n \right) \Delta t_n \quad (IV-20)$$

and where

$$V_n \equiv \frac{1}{2} \sum_{i=1}^{n} \frac{1}{A_d} \left[ \frac{\partial \phi_d}{\partial x_i} \right]_{i-1} + \left[ \frac{\partial \phi_d}{\partial x_i} \right]_{i} \Delta t_i. \quad (IV-21)$$

Subtracting equation (IV-16) from equation (IV-20), we may solve for the dipole magnitude

$$A_d = \frac{2m}{q} \frac{XT(K_1) - XE(K_1)}{VI(K_1)}. \quad (IV-22)$$

After the addition of a dipole with magnitude $A_d$ at coordinates $(x_d, y_d)$, one iteration of the scheme for discrepancy reduction is
completed by scaling the dipole modified estimated potential,

\[ \phi'(x,y) = SF \cdot \phi(x,y). \]  

(IV-23)

The scale factor,

\[ SF = \frac{XE(6) - X0(6)}{XT(6) - X0(6)}, \]  

(IV-24)

is calculated using the exit coordinate \( XE(6) \) obtained for the stepwise development of the scaling trajectory in the dipole modified estimated potential.

Appendix D lists the main Fortran routine PTENT.F4 which implements the iterative routine described by equations (IV-1) through (IV-24). The user has the option of entering electron trajectory data by reading from file TRAJ.DAT or by answering pertinent questions in an interactive user mode. Table 4 shows the reduction of discrepancy and of error estimates obtained for 15 iterations of the routine. Note that these quantities do not decrease monotonically and limits are reached after about 8 iterations of the routine. Figure 10 shows equipotential lines drawn with a linear interpolation subroutine of PTENT similar to the VPlot subroutine described for the boundary value problem. The solution to the model problem depicted in Figure 10 can be compared to the test potential in Figure 8.

Appendix E contains a listing of the main Fortran routine RTENT.F4 which is also an iterative routine for the dipole synthesis of solutions to the model problem. RTENT.F4 differs from PTENT.F4 only in the methods used for determining the dipole \( x_d, y_d \) and \( \Delta_d \) factors. For this new scheme, the \( x_d \) parameter can be calculated by equation (IV-8) when \( K_1 \) is identified as the trajectory having the largest positive discrepancy and \( K_2 \) is identified as the trajectory having the largest
Table 4

Error Estimates for Dipole Synthesis with PTENT.F4

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<th>Average Absolute Error (%)</th>
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Figure 10. Solution to the Model Problem Synthesized with PTENT.F4

Figure 11. Solution to the Model Problem Synthesized with RTENT.F4
negative discrepancy. The $y_d$ parameter is arrived at by a trial and error procedure designed to reduce the discrepancies of both the $K_1$ and $K_2$ trajectories. This is accomplished by finding a minimum value of

$$\frac{D(K_1)}{D(K_2)} = \frac{\frac{\partial \phi_d}{\partial x}(XM(K_1),YM(K_1))}{\frac{\partial \phi_d}{\partial x}(XM(K_2),YM(K_2))}$$

(IV-25)

as $y_d$ is allowed to run through a range of values. When the values of $x_d$ and $y_d$ have been chosen, a value of $A_d$ is chosen to reduce the discrepancy of the $K_1$ trajectory (or the $K_2$ trajectory if its discrepancy is larger in absolute magnitude than the $K_1$ trajectory) to approximately zero as described by equations (IV-9) through (IV-22). Table 5 lists the error estimates for 15 iterations of the RTENT.F4 routine as compared to Table 4. Note that error estimates are larger for RTENT.F4 than for PTENT.F4 and that the reduction of discrepancy is slower. However, RTENT.F4 consistently underestimates the biased fin potential while PTENT.F4 consistently yields an overestimation. Therefore, upper and lower bounds can be established for the solution of the model problem. Figure 11 shows the estimated potential generated by RTENT.F4 which can be compared to Figures 10 and 8 to appreciate the accuracy of these iterative dipole routines.

Tables 6, 7 and 8 show the potential error estimates at 110 selected locations in the region of interest for 5, 10 and 15 iterations of routine RTENT.F4; respectively. First note that the largest errors occur at the edges of the region of interest in spatial regions unprobed by the electron trajectories. Note also that the errors at
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Table 6
Error Estimates for Dipole Synthesis with Routine RTENT.F4

(5 dipole pairs)

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Table 7

Error Estimates for Dipole Synthesis with Routine RTENT.F4
(10 dipole pairs)

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X-COORDINATE INDEX (I)

22  26  30  34  38  42  46  50  54  58  62
Table 8

Error Estimates for Dipole Synthesis with Routine RTENT.F4

(15 dipole pairs)

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nearly all of the selected locations decrease monotonically from the fifth to the tenth to the fifteenth iteration.
V. QUADRUPOLE SYNTHESIS OF ELECTROSTATIC POTENTIALS

The two iterative routines for dipole field synthesis of electric potentials yield good approximations of the biased fin potential. However, the addition of dipole fields individually chosen to reduce trajectory discrepancies may result in slow convergence and lengthy computer runs (e.g., the 15 iterations appearing in Table 4 required 30' 38" of CPU time). Efforts to speed the convergence of the synthesis process by adding dipole arrays, rather than single dipoles, have been unsuccessful. Note, in Figure 12, that the x-component of the electric field associated with the line dipole is not symmetric about the axis of the line dipole. The five curves of Figure 12 represent the electric field in the x-y reference plane of the biased fins (see Figure 7) along a line midway between the reference plane (y=0) and the biased fins (y=\(D_Y\)), as the result of a line dipole and its image located at five perpendicular distances from the reference plane. Note also that the dipole has equal and opposite effects at two spatially distinct locations in the region of interest. These characteristics of the dipole field are problematic when considering the placement of several line dipoles simultaneously in an effort to reduce the discrepancies associated with several electron trajectories. Consequently, the optimal placements of dipole array elements are not easily deduced since it is difficult to associate discrepancy reduction for one trajectory with a particular element of the dipole array.

A. Quadrupole Synthesis of Solutions to the Model Problem

Although the difficulties associated with synthesizing potential fields with dipole arrays may not be insurmountable, we have chosen to investigate
Figure 12. Electric Fields for Five Dipole and Quadrupole Locations along a Line inside the Region of Interest
the quadrupole synthesis of electric potentials. In analogy to the ideal line dipole, consider the ideal line quadrupole consisting of four lines of alternating charge extending infinitely in a direction perpendicular to the two dimensional region of interest. Figure 12 shows these four lines of charge which are allowed to approach each other in the limit of superposition to derive the potential field of the ideal line quadrupole. Since, we are interested in the synthesis of potentials which fall to zero along a reference plane, an image line quadrupole is simultaneously located an equal distance from the reference plane but on the side opposite the location of the line quadrupole itself. The potential field, as derived in Appendix A, due to a line dipole at \((x_q', y_q')\) and its image at \((x_q', -y_q')\) is

\[
\phi_q(x,y) = \frac{A_q}{(x-x_q')^2 + (y+y_q')^2 + [(x-x_q')^2 + (y-y_q')^2]^2} \cdot (V-1)
\]

Notice that this potential vanishes along the reference plane \((y=0)\) regardless of the values of the quadrupole parameters \(A_q, x_q', \) and \(y_q'.\) In analogy with dipole synthesis, the line quadrupole potential can be used for the synthesis of unknown potentials existing in a two dimensional space of the biased fins (i.e., along a line, \(y=\)constant, midway between the fins and the reference plane) as the result of an ideal line quadrupole and its image located at the same five perpendicular distances that were considered for the ideal dipole field of Figure 12. In comparing these two diagrams, notice that the quadrupole field is symmetric about its axis. Furthermore, the quadrupole field is largest
on its axis, rather than at locations on either side of the axis as was found for the dipole field. These quadrupole field characteristics facilitate the definition of iterative schemes for the synthesis of unknown potentials. In particular, a number of ideal quadrupoles can be located simultaneously with that quadrupole which is placed directly over each trajectory being primarily responsible for the reduction of discrepancy associated with that trajectory. However, before considering the topic of potential synthesis using the fields of quadrupole arrays, the synthesis of electric potentials by the iterative addition of single quadrupole fields will be considered. This will provide a basis for comparing dipole and quadrupole methods for the synthesis of solutions to Laplace's equation subject to the constraints imposed by electron trajectory data.

To formulate an iterative scheme for quadrupole synthesis of solutions to the model problem, suppose that the potential \( \psi \) exists as the result of the biasing of the fins relative to a zero potential reference plane. An initial estimated potential \( \phi_0 \) can be defined as the uniform-field approximation of \( \psi \) which satisfies the data for the last of the six trajectories listed in the statement of the model problem (i.e., the scaling trajectory). In analogy with equations IV-1) to (IV-4), an initial estimated potential \( \phi_0 = A_0 y \), can be calculated at any point in the region of interest based upon the data for the scaling trajectory and the electron charge to mass ratio. After determining the uniform field approximation of the estimated potential, the estimated potential can be modified according to an iterative scheme that adds quadrupole fields \( \phi_q(x,y) \) to \( \phi_0(x,y) \) in order to improve the approximation of the biased fin potential \( \psi(x,y) \).
\[ \phi(x,y) = \phi_0(x,y) + \sum_{q=1}^{Q} \phi_q(x,y) \]  
\[ \text{(V-2)} \]

For each iteration of the scheme, estimated electron trajectories in the estimated potential field are developed from a knowledge of initial electron coordinates and velocities for the first five electron trajectories listed as data for the model problem. The coordinates \((X_M(K), Y_M(K))\) at the highest points on the estimated electron trajectories are recorded as well as the exit coordinates \((X_E(K), Y_E(K))\) for \(K = 1, 2, 3, 4, 5\). Then trajectory discrepancies,

\[ D(K) = X_T(K) - X_E(K), \]
\[ \text{(V-3)} \]

are defined for each trajectory as the difference between the final x-coordinate \(X_T(K)\) listed as data and the final x-coordinate \(X_E(K)\) found for the estimated trajectories in the estimated potential. The sum of the absolute discrepancy values

\[ DD = \sum_{K=1}^{5} |D(K)| \]
\[ \text{(V-4)} \]

is calculated in order to obtain a quantitative measure of the error in the approximation of \(\psi(x,y)\) by \(\phi(x,y)\). The object of each iteration is the reduction of \(DD\) by the addition of an ideal line quadrupole and its image with a magnitude and spatial location that is optimal in some regard. In particular, let the quadrupole be located a fixed distance from the region of interest (e.g., \(y_q = 1.4DY\)) and let the trajectory \(K_1\) having the largest absolute discrepancy, \(|D(K_1)|\), define the x-coordinate location of the quadrupole (e.g., \(x_q = X_M(K_1)\)). Then a quadrupole magnitude \(A_q\) can be chosen which reduces the largest absolute trajectory discrepancy to approximately zero. Toward this end, consider the equations of motion for electrons following the estimated trajectory \(K_1\) through the estimated potential \(\phi(x,y)\).
The integration of these equations over the time $T$ required for electrons traveling along trajectory $K_1$ yields

\[
\frac{\partial^2 x}{\partial t^2} = \frac{q}{m} \frac{\partial \phi}{\partial x} \quad \frac{\partial^2 y}{\partial t^2} = \frac{q}{m} \frac{\partial \phi}{\partial y}
\]

(V-5)

The integration of these equations over the time $T$ required for electrons traveling along trajectory $K_1$ yields

\[
XE(K_1) = X0(K_1) + T \cdot VX0(K_1) + \frac{1}{2} \frac{q}{m} \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial x} \, dt \, dt'.
\]

(V-6)

and

\[
YE(K_1) = Y0(K_1) + T \cdot VY0(K_1) + \frac{1}{2} \frac{q}{m} \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial y} \, dt \, dt'.
\]

(V-7)

Recall that the trajectory $K_1$ consists of a number of points whose coordinates were determined by a stepwise solution of the equations of motion. Furthermore, the electric fields $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ were determined at these points by interpolation from a grid in order to proceed with the stepwise development of the trajectory. Since these electric fields and corresponding coordinate values are available, the integrals in equations (V-6) and (V-7) may be replaced by a pointwise summation over the trajectory $K_1$. Using the trapezoidal rule for numerical integration, we obtain

\[
VXI(K_1) = \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial x} \, dt \, dt' = \frac{N}{2} \sum_{n=1}^{N} \frac{1}{2} (S_n + S_{n-1}) \Delta t_n
\]

(V-8)

where

\[
S_n = \sum_{i=1}^{n} \frac{1}{2} \left[ \frac{\partial \phi}{\partial x}_i + \frac{\partial \phi}{\partial x}_{i-1} \right] \Delta t_i.
\]

(V-9)

Similarly, we obtain
\[ V_{YI}(K_1) = \int_0^T \int_0^{t'} \frac{\partial \phi}{\partial y} \, dt \, dt' = \sum_{n=1}^N \frac{1}{2} \left( V_n + V_{n-1} \right) \Delta t_n \]  
(V-10)

where

\[ V_n = \frac{1}{2} \sum_{i=1}^n \left[ \frac{\partial \phi}{\partial y}_i + \frac{\partial \phi}{\partial y}_{i-1} \right] \Delta t_i \]  
(V-11)

and \( N \) represents the number of time steps \( \Delta t_i \) used in the development of trajectory \( K_1 \). Substituting these expressions for the integrals in equation (V-6) and (V-7), we obtain estimates of the time \( T \) required for electrons following trajectory \( K_1 \) and the electron exit coordinate \( XE(K_1) \). With \( YE(K_1) = Y0(K_1) = 0 \), we have

\[ T = -\frac{1}{2} \frac{q}{m} \frac{V_{YI}(K_1)}{V_{Y0}(K_1)} \]  
(V-12)

and

\[ XE(K_1) = X0(K_1) + T \cdot VX0(K_1) + \frac{1}{2} \frac{q}{m} V_{XI}(K_1). \]  
(V-13)

Notice that \( T > 0 \) for negatively charged electrons.

We wish to add a quadrupole field \( \phi(x,y) \) altering the trajectory \( K_1 \) only slightly and shifting the exit coordinate from its estimated value \( XE(K_1) \) to the data value \( XT(K_1) \). The equations of motion for electrons in this new potential field are

\[ \frac{\partial^2 x}{\partial t^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right) \]  
and

\[ \frac{\partial^2 y}{\partial t^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \right). \]  
(V-14)

The integration of the first of these equations yields

\[ XT(K_1^*) + X0(K_1^*) + T \cdot VX0(K_1^*) + \frac{1}{2} \frac{q}{m} \int_0^{T^*} \int_0^{t'} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right) dt \, dt'. \]  
(V-15)
When $K_1^*$ trajectory is not too different from the $K_1$ trajectory, the
time $T^*$ is not much different from the time $T$ and the integral ex-
pression in equation (V-15) can be evaluated by summation of electric
fields over the same $N$ points used for the evaluation of the integral
in equation (V-6). Therefore,

$$XT(K_1^*) = X0(K_1) + T \cdot VX0(K_1) + \frac{1}{2} \frac{q}{m} [\text{VXI}(K_1) + A_q \ \text{VI}(K_1)] \quad (V-16)$$

where

$$\text{VI}(K_1) = \sum_{n=1}^{N} \frac{1}{2} (V_{q_n} + V_{q_{n-1}}) \Delta t_n \quad (V-17)$$

and

$$V_{q_n} = \sum_{i=1}^{n} \frac{1}{2A_q} \left[ \frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial x_{i-1}} \right] \Delta t_i \quad (V-18)$$

Subtracting equation (V-13) from equation (V-16), and solving for $A_q$, we find

$$A_q = \frac{2m}{q} \left[ \frac{XT(K_1^*) - \text{XE}(K_1)}{\text{VI}(K_1)} \right]. \quad (V-19)$$

This expression for $A_q$ can be used to calculate the magnitude of the
ideal line quadrupole which, when located above the trajectory ex-
hibiting the largest discrepancy, reduces that discrepancy approximately
to zero.

After a quadrupole field with the appropriate magnitude is added
to the estimated potential, a scaling operation is executed to complete
one iteration of the scheme. The estimated potential is scaled by
an amount sufficient to reduce the discrepancy of the scaling trajectory
approximately to zero. To derive an algorithm for this purpose, we
assume that electrons following the scaling trajectory experience no net force in the x-direction. More specifically, this implies that

\[ X_E(6) = X_0(6) + T \cdot V_X(6) \quad (V-20) \]

with

\[ T = - \frac{1}{2} \frac{q}{m} V_Y(6) \cdot V_Y(6) \quad (V-21) \]

Then, scaling the potential by a factor \( SF \), we shift the exit coordinate to a value

\[ X_T(6) = X_0(6) + T^* \cdot V_X(6) \quad (V-22) \]

where

\[ T^* = - \frac{1}{2} \frac{q}{m} \frac{SF \cdot V_Y(6)}{V_Y(6)} = SF \cdot T \quad (V-23) \]

Therefore, using equations (V-20) to (V-22) to solve equation (V-23) for the scale factor, we obtain

\[ SF = \frac{T^* \cdot X_T(6) - X_0(6)}{X_E(6) - X_0(6)} \quad (V-24) \]

Recall that this same algorithm was used for scaling the estimated potential when solving the model problem by dipole synthesis.

The main routine QPOLE.F4 implements the iterative scheme described above for the quadrupole synthesis of solutions to the model problem. QPOLE.F4 is similar to the dipole routines PTENT.F4 and RTENT.F4, particularly in regard to the iterative procedures used for scaling the estimated potential and for determining an optimal pole magnitude after a pole location has been specified. The major difference between the dipole and quadrupole routines is the criteria used for selecting optimal pole locations. Table 9 lists error
Table 9

Error Estimates for Quadrupole Synthesis with Routine QPOLE.F4

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<th>Average Absolute Error (%)</th>
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estimates obtained after 15 iterations of the scheme for quadrupole synthesis which can be compared to the equivalent error estimates for dipole synthesis appearing in Tables 4 and 5. Note that in 15 iterations of the quadrupole routine, the discrepancy is reduced to a value considerably lower than the discrepancy values achieved with 15 iterations of the dipole schemes. This discrepancy reduction for the quadrupole scheme is accomplished in 30'32'' of CPU time representing only a small improvement in the amount of time expended per iteration. Although convergence limits are not reached in 15 iterations of the QPOLE.F4 routine, the indication is that the method would ultimately converge to a better approximation of the biased fin potential than solutions obtained with the dipole methods. Tables 10, 11 and 12 show the error percentages, after 5, 10 and 15 iterations respectively, associated with the estimated potential at 110 points chosen arbitrarily from a much finer grid (i.e., the interpolation of electric fields with subroutine NEWTON requires a finer grid spacing) in the reference frame of the biased fins. Notice that the error percentages for points near spatial regions probed by the electron trajectories (e.g., 30<\(I<54\) and \(5<J<17\)) decrease with successive iterations of the quadrupole scheme. The improvement of the estimated potential in these regions is accomplished while a degradation in the accuracy of the estimated potential results for points further from the space probed by the electron trajectories. This result for quadrupole synthesis of solutions to the model problem can be compared to similar results for dipole synthesis which have been discussed previously with references to Tables 4, 5 and 6. Figure 13 shows equipotential lines for the QPOLE.F4 routine which can be compared to Figures 8, 10 and 11 depicting the test
Table 10
Estimated Potential Error Percentages for Quadrupole Synthesis (5 Iterations of Routine QPOLE.F4)

| Y  | 21   | -16.5 - 6.8 | 0.9 | 5.0 | 5.5 | 2.9 | 0.4 | 1.1 | -0.4 | -1.6 - 5.7 |
| C  | 19   | -10.8 - 6.4 | -0.6 | 3.0 | 3.9 | 2.5 | 1.0 | 0.6 | -0.3 | -1.4 - 1.8 |
| O  | 17   | - 7.9 - 5.5 | 1.4 | 1.7 | 2.7 | 2.1 | 1.1 | 0.5 | -0.2 | -0.8 - 0.2 |
| R  | 15   | - 6.2 - 4.8 | 1.8 | 0.8 | 1.9 | 1.7 | 1.1 | 0.5 | -0.0 | -0.2 - 0.6 |
| D  | 13   | - 5.1 - 4.2 | 2.0 | 0.1 | 1.2 | 1.3 | 0.9 | 0.4 | 0.1 | 0.1 - 1.1 |
| I  | 11   | - 4.5 - 3.9 | 2.2 | -0.4 | 0.7 | 1.0 | 0.8 | 0.4 | 0.1 | 0.3 - 1.3 |
| N  | 9    | - 4.3 - 3.9 | 2.5 | -0.8 | 0.3 | 0.7 | 0.6 | 0.2 | -0.1 | 0.1 - 1.1 |
| A  | 7    | - 4.7 - 4.4 | 3.0 | -1.3 | -0.1 | 0.5 | 0.3 | -0.1 | -0.5 | -0.4 | 0.5 |
| T  | 5    | - 6.4 - 6.0 | 4.3 | -2.2 | -0.5 | 0.2 | -0.0 | -0.9 | -1.8 | -2.1 | -1.4 |
| E  | 3    | -15.3 -14.2 | -10.7 | -5.9 | -1.8 | -1.8 | -1.3 | -4.7 | -8.3 | -10.7 | -10.8 |

(j)  
22 26 30 34 38 42 46 50 54 58 62

X COORDINATE INDEX (I)
Table 11

Estimated Potential Error Percentages for Quadrupole Synthesis (10 Iterations of Routine QPOLE.F4)

<table>
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<tr>
<th>X COORDINATE INDEX (I)</th>
<th>Y 21</th>
<th>-16.8</th>
<th>-7.1</th>
<th>0.1</th>
<th>3.5</th>
<th>4.4</th>
<th>4.0</th>
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<th>2.9</th>
<th>0.7</th>
<th>-0.9</th>
<th>-5.3</th>
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<tbody>
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X COORDINATE INDEX (I)
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<th>54</th>
<th>58</th>
<th>62</th>
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X COORDINATE INDEX (I)
Figure 13. Solution to the Model Problem Synthesized with QPOLE.F4

Figure 14. Solution to the Model Problem Synthesized with QPOLES.F4
potential and the dipole synthesized solutions.

B. Potential Synthesis with Quadrupole Arrays

In order to speed the process of discrepancy reduction, consider the placement of five ideal line quadrupoles upon each iteration of a new scheme for quadrupole synthesis of solutions to the model problem. The procedure for calculating an initial estimated potential, \( \phi_0(x,y) \), based upon a uniform field approximation and the data for the scaling trajectory (i.e., \( K=6 \)) is unchanged and described by equations IV-1) through (IV-4). The estimated potential can then be modified by an iterative scheme that locates five quadrupoles simultaneously in order to achieve discrepancy reduction for all five of the electron trajectories. Equation (V-25) depicts this situation

\[
\phi(x,y) = \phi_0(x,y) + \sum_{i=1}^{N} \sum_{K=1}^{5} \phi_K(x,y)
\]

where \( \phi_K(x,y) \) is given by equation (V-1) and the \( i \) index runs up to the total number of iterations that are needed for adequate discrepancy reduction. In developing this scheme, we expect that each iteration will require more CPU time since five quadrupoles are located as compared to only one with the old scheme. However, we hope that fewer iterations will be required to achieve a given reduction in discrepancy and that the total CPU time expended will decrease. To define the new scheme, let all five quadrupoles be located with a \( y \)-coordinate value of 1.4DY. Let the \( x \)-coordinate values of the quadrupole locations be equal to the \( x \)-coordinates corresponding to the highest points on the five estimated trajectories that have been developed for the estimated potential.
Once the five quadrupole locations have been chosen, we seek five quadrupole magnitudes (i.e., $A^K_k; K = 1, 2, 3, 4, 5$) which reduce the discrepancies associated with all five trajectories to approximately zero. Toward this end, consider the equations of motion for electrons moving in the estimated potential $\phi(x,y)$. Then, in analogy with equations (V-5) through (V-13), an equation for each estimated trajectory can be determined.

\[
X_E(K) = X_0(K) + T \cdot VX_0(K) + \frac{1}{2} \frac{q}{m} VXI(K) ; K = 1, 2, 3, 4, 5 \tag{V-27}
\]

Now, we wish to add five quadrupole fields to the estimated potential so as to alter the five estimated trajectories only slightly and shifting the estimated exit coordinates $(X_E(K); K = 1, 2, 3, 4, 5)$ over to the exit coordinates $(X_T(K); K = 1, 2, 3, 4, 5)$ measured as data. The equations of motion in this new estimated potential are

\[
\frac{\partial^2 x}{\partial t^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial x} + \Sigma_{K=1}^{5} \frac{\partial \phi_K}{\partial x} \right) \tag{V-28}
\]

and

\[
\frac{\partial^2 y}{\partial t^2} = \frac{q}{m} \left( \frac{\partial \phi}{\partial y} + \Sigma_{K=1}^{5} \frac{\partial \phi_K}{\partial x} \right). \tag{V-29}
\]

An integration of the equations of motion subject to the initial and final conditions available as trajectory data results in five equations describing the electron trajectories in a new estimated potential. Thus,

\[
X_T(K) = X_0(K) + T \cdot VX_0(K) + \frac{1}{2} \frac{q}{m} \left[ VXI(K) + \sum_{K=1}^{5} A^K_k VI(K) \right], \tag{V-30}
\]

with $VI(K)$ defined by equations (V-17) and (V-18) and where
K = 1, 2, 3, 4, 5. The subtraction of equation (V-27) from equation (V-30) yields,

\[ X_T(K) - X_E(K) = \frac{1}{2} q \sum_{K=1}^{5} A_K V_I(K) ; K = 1, 2, 3, 4, 5. \]  

Equation (V-31) represents a linear system of five equations in five unknowns (i.e., \( A_K ; K = 1, 2, 3, 4, 5 \)) which can be solved by gaussian elimination procedures to determine the magnitude of the five ideal line quadrupoles. When the potential fields associated with these quadrupoles are added to the estimated potential, a new estimated potential is synthesized. Then the discrepancies associated with the new estimated trajectories, developed in the new estimated potential, should individually be approximately equal to zero. Thus, complete discrepancy reduction could be accomplished with a single application of the new scheme. However, the scheme involves several approximations which produce a less than total reduction of discrepancy. For example, the time \( T \) in equations (V-27) and (V-30), as well as the \( V_{XI}(K) \) terms, depend upon the old and new estimated potentials respectively. Therefore, the terms involving these expressions will not cancel exactly as indicated in the derivation of equation (V-31). The approximation as implied by equations (V-27) and (V-30) is that the five electron trajectories are not altered greatly by the addition of five quadrupoles for the reduction of discrepancy. Therefore, the errors introduced by these approximations will result in a less than total reduction of discrepancy for any one iteration of the scheme. However, on successive iterations, the approximations will produce lesser errors and for a suitable initial potential estimate \( \phi_0 \), the method of
successive approximation will result in a convergent iterative scheme.

The main routine QPOLES.F4 (Appendix E) implements the method described above for the synthesis of solutions to the model problem. Table 13 shows the error estimates generated at 110 points on a grid located between the biased fins and above the zero potential reference plane. These results can be compared to Tables 10, 11 and 12 which list the results of routine QPOLE.F4 also used for quadrupole synthesis of solutions to the model problem. Note that the single pole scheme results in solutions that are scaled better than the solutions synthesized by the addition of five poles simultaneously. However, discrepancy reduction is much more rapid for the latter method. A reduction in discrepancy from 1.5 to 0.4 can be accomplished with a single iteration of the QPOLES.F4 scheme requiring 3'24" of computer processing time (CPU time) as compared to 14 iterations of the QPOLE.F4 scheme requiring 28'29". Although 4 iterations of the method have been listed, the reduction of discrepancy and of approximation errors reaches a limit after only 2 iterations. This is in contrast to the single pole scheme where convergence limits had not been reached after 15 iterations. Figure 14 shows equipotential lines for the QPOLES.F4 routine.

Although convergence is rapid for the main routine QPOLES.F4, the synthesized solution to the model problem underestimates the biased fin potential at 109 of the 110 points considered. The scaling procedure itself is unchanged from the successful procedure used previously for the single quadrupole and dipole routines. The difficulty seems to be manifested in balancing the strategies of field scaling and field shaping depending on whether the addition of quadrupoles, reducing discrepancies for the first five trajectories, has a sufficiently detrimental effect on the discrepancy of the scaling trajectory. To possibly save
Table 13

Error Estimates for Routine QPOLES.F4 with Step Size \( d \) (0.013 < \( d < 0.025 \)) and Scale Tolerance of 1.0%

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.51</td>
<td>0.05</td>
<td>-4.6</td>
<td>5.8</td>
<td>20.0</td>
</tr>
<tr>
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<td>14.5</td>
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Y COORDINATE INDEX (J):

21 -12.4 - 2.2 1.3 -1.3 -1.5 -1.4 -1.3 -1.1 -2.5 -3.7 - 8.8
19 -7.3 - 3.3 - 0.6 -1.2 -1.5 -1.5 -1.4 -1.6 -2.5 -3.8 -4.7
17 -5.0 - 3.3 - 1.5 -1.4 -1.5 -1.6 -1.9 -2.5 -3.3 -3.0
15 -3.8 - 3.2 - 2.0 -1.7 -1.7 -1.8 -2.0 -2.5 -2.8 -2.0
13 -3.1 - 3.0 - 2.3 -1.9 -1.8 -2.1 -2.4 -2.4 -2.4 -1.4
11 -2.8 - 3.0 - 2.5 -2.1 -1.9 -2.0 -2.2 -2.4 -2.3 -1.3
 9 -2.8 - 3.1 - 2.8 -2.4 -2.1 -2.0 -2.2 -2.4 -2.6 -2.4 -1.4
 7 -3.4 - 3.6 - 3.3 -2.8 -2.3 -2.2 -2.4 -2.8 -3.1 -2.9 -2.0
 5 -5.1 - 5.4 - 4.7 -3.6 -2.7 -2.4 -2.7 -3.5 -4.3 -4.6 -3.9
 3 -14.2 -13.7 -11.0 -7.2 -3.9 -2.6 -3.9 -7.2 -10.7 -13.0 -13.0

X COORDINATE INDEX (I):

22 26 30 34 38 42 46 50 54 58 62
time, a decision to scale or not is made by comparing the scale error (i.e., discrepancy) of this trajectory with an arbitrary scale tolerance. Tables 14, 13 and 15 show error estimates for the synthesis of solutions to the model problem for scale tolerances of 5.0%, 1.0% and 0.1% respectively. Note that a scale tolerance of 0.1% is preferable. Decreasing the scale tolerance further will not improve the accuracy of the solution, since the routine already scales at every opportunity with a scale tolerance of 0.1%. Thus, invoking the scaling operation after each shaping operation facilitates the reduction of discrepancy and produces a better approximation of the biased fin potential. However, the synthesized solution to the model problem still underestimates the biased fin potential at 109 of the 110 points considered. Thus, field shaping by the addition of five quadrupoles simultaneously results in a rapid convergence at the expense of accurate scaling.

Recall that the terms $VI(K)$ and $VXI(K)$ in equations (V-17) and (V-8) are calculated by summing terms generated by the trapezoidal rule for integration over the estimated electron trajectories. The accuracy of this integration depends upon the number of trapezoidal terms as determined by the size of the time step used for tracking electrons through a potential field. Subroutine DVOGEL monitors the trajectory arc length corresponding to a particular time step to determine the acceptance or rejection of that time step. Whenever the arc length surpasses some upper limit, the time step is reduced and conversely whenever the arc length falls below some lower limit, the time step is increased. Table 16 shows x and y coordinate values at points along a particular electron trajectory as well as the electric field at those points for an ideal line quadrupole that will be added
Table 14

Error Estimates for Routine QPOLES.F4 with Scale Tolerance of 5.0%

<table>
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<tr>
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<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
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Y Coordinate Index (J)
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17 - 9.0 - 7.3 - 5.3 - 4.5 - 4.0 - 3.9 - 4.2 - 4.9 - 6.1 - 7.5 - 7.6
15 - 7.8 - 7.1 - 5.7 - 4.8 - 4.3 - 4.2 - 4.5 - 5.1 - 6.1 - 6.9 - 6.5
13 - 7.1 - 6.8 - 5.9 - 5.1 - 4.6 - 4.5 - 4.7 - 5.3 - 6.0 - 6.5 - 5.9
11 - 6.7 - 6.7 - 6.1 - 5.3 - 4.8 - 4.7 - 4.9 - 5.5 - 6.1 - 6.3 - 5.7
09 - 6.7 - 6.9 - 6.3 - 5.6 - 5.1 - 4.9 - 5.2 - 5.7 - 6.2 - 6.4 - 5.7
07 - 7.2 - 7.4 - 6.8 - 6.0 - 5.4 - 5.1 - 5.4 - 6.0 - 6.7 - 6.9 - 6.3
05 - 8.9 - 9.0 - 8.1 - 6.8 - 5.7 - 5.3 - 5.8 - 6.8 - 7.9 - 8.5 - 8.0
03 -17.6 -17.0 -14.2 -10.3 -6.9 -5.6 -6.9 -10.3 -14.0 -16.5 -16.8

X Coordinate Index (I)
22 26 30 34 38 42 46 50 54 58 62

Graphical representation of the data.
Table 15

Error Estimates for Routine QPOLES.F4 with Scale Tolerance of 0.1%

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<td>-3.0</td>
<td>3.0</td>
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</tr>
</tbody>
</table>

Y COORDINATE INDEX (I)

X COORDINATE INDEX (J)

[Diagram showing Y and X coordinate indices with error estimates graphed]
Table 16

Quadrupole Electric Fields at Points Defining Estimated Electron Trajectories in the Uniform Field Potential

<table>
<thead>
<tr>
<th>STEP SIZE</th>
<th>STEP SIZE</th>
<th>STEP SIZE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>E_x</td>
<td>Y</td>
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<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.03</td>
<td>-0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0.07</td>
<td>-0.11</td>
<td>0.10</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.16</td>
<td>0.15</td>
</tr>
<tr>
<td>0.13</td>
<td>-0.21</td>
<td>0.20</td>
</tr>
<tr>
<td>0.16</td>
<td>-0.27</td>
<td>0.24</td>
</tr>
<tr>
<td>0.20</td>
<td>-0.33</td>
<td>0.29</td>
</tr>
<tr>
<td>0.23</td>
<td>-0.40</td>
<td>0.34</td>
</tr>
<tr>
<td>0.26</td>
<td>-0.46</td>
<td>0.39</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.54</td>
<td>0.44</td>
</tr>
<tr>
<td>0.33</td>
<td>-0.62</td>
<td>0.49</td>
</tr>
<tr>
<td>0.36</td>
<td>-0.71</td>
<td>0.54</td>
</tr>
<tr>
<td>0.39</td>
<td>-0.81</td>
<td>0.59</td>
</tr>
<tr>
<td>0.42</td>
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<td>0.64</td>
</tr>
<tr>
<td>0.46</td>
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<td>0.69</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>0.56</td>
<td>-1.54</td>
<td>0.83</td>
</tr>
<tr>
<td>0.59</td>
<td>-1.76</td>
<td>0.88</td>
</tr>
<tr>
<td>0.62</td>
<td>-2.01</td>
<td>0.82</td>
</tr>
<tr>
<td>0.65</td>
<td>-2.31</td>
<td>0.73</td>
</tr>
<tr>
<td>0.69</td>
<td>-2.68</td>
<td>0.64</td>
</tr>
<tr>
<td>0.72</td>
<td>-3.08</td>
<td>0.55</td>
</tr>
<tr>
<td>0.75</td>
<td>-3.57</td>
<td>0.46</td>
</tr>
<tr>
<td>0.78</td>
<td>-4.17</td>
<td>0.38</td>
</tr>
<tr>
<td>0.82</td>
<td>-4.94</td>
<td>0.29</td>
</tr>
<tr>
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<td>0.20</td>
</tr>
<tr>
<td>0.88</td>
<td>-7.04</td>
<td>0.11</td>
</tr>
<tr>
<td>0.84</td>
<td>-5.73</td>
<td>0.02</td>
</tr>
<tr>
<td>0.75</td>
<td>-3.57</td>
<td>0.12</td>
</tr>
<tr>
<td>0.66</td>
<td>-2.38</td>
<td>0.04</td>
</tr>
<tr>
<td>0.57</td>
<td>-1.65</td>
<td>0.04</td>
</tr>
<tr>
<td>0.48</td>
<td>-1.14</td>
<td>0.04</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.82</td>
<td>0.04</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.36</td>
<td>0.04</td>
</tr>
<tr>
<td>0.21</td>
<td>-0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>0.12</td>
<td>-0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>0.04</td>
<td>-0.06</td>
<td>0.04</td>
</tr>
</tbody>
</table>
for field shaping. This trajectory shows no variation in the x coordinate since the estimated potential field used to generate the trajectory from the initial electron data is the uniform field approximation developed as the first operation in all of the iterative routines that have been discussed. Note that decreasing the lower limit on the step size results in more steps as the electron moves against the potential gradient (increasing y coordinate). Note also that increasing the upper limit on the step size results in fewer steps as the electron moves with the potential gradient (decreasing y coordinate). Tables 17 and 18 when compared to Table 15 show that the three different bounds on the step size result in synthesized solutions to the model problem which are only slightly different from each other.

It is interesting to consider the effects of erroneous measurements of initial electron energy or initial electron direction. Tables 19 and 20 show error estimates for the approximations of the biased fin potential which result from erroneous measurements of electron initial energies of 3\% and 6\% respectively. Note that this type of error affects the field scaling operation but has little effect on the field shaping operations. Tables 21 and 22 show error estimates for approximations of the biased fin potential resulting from erroneous measurements of electron initial directions of 1° and 3° respectively. Note that this type of error severely affects field shaping operations as well as the scaling operations.
Table 17

Error Estimates for Routine QPOLES.F4 with Step Size \(d\) 

\((0.008 < d < 0.025)\)

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.51</td>
<td>0.05</td>
<td>-4.6</td>
<td>5.8</td>
<td>20.0</td>
</tr>
<tr>
<td>1</td>
<td>0.38</td>
<td>0.02</td>
<td>-3.2</td>
<td>3.3</td>
<td>14.1</td>
</tr>
<tr>
<td>2</td>
<td>0.09</td>
<td>0.22</td>
<td>-3.4</td>
<td>3.5</td>
<td>13.9</td>
</tr>
<tr>
<td>3</td>
<td>0.04</td>
<td>0.18</td>
<td>-3.3</td>
<td>3.4</td>
<td>14.1</td>
</tr>
<tr>
<td>4</td>
<td>0.02</td>
<td>0.21</td>
<td>-3.3</td>
<td>3.4</td>
<td>14.3</td>
</tr>
</tbody>
</table>

X COORDINATE INDEX (I)

<table>
<thead>
<tr>
<th>Y</th>
<th>C O O R D I N A T E I N D E X (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>-12.7 - 2.1 1.5 -1.1 -1.4 -1.3 -1.3 -1.1 -2.6 -3.8 -8.9</td>
</tr>
<tr>
<td>19</td>
<td>-7.6 - 3.3 -0.5 -1.0 -1.4 -1.4 -1.4 -1.6 -2.6 -3.8 -4.8</td>
</tr>
<tr>
<td>17</td>
<td>-5.3 - 3.4 -1.5 -1.3 -1.4 -1.5 -1.5 -1.6 -1.9 -2.6 -3.3 -3.0</td>
</tr>
<tr>
<td>15</td>
<td>-4.0 - 3.2 -2.0 -1.6 -1.6 -1.6 -1.7 -2.0 -2.5 -2.8 -2.1</td>
</tr>
<tr>
<td>13</td>
<td>-3.3 - 3.1 -2.3 -1.8 -1.7 -1.7 -1.8 -1.9 -2.1 -2.5 -2.5 -1.6</td>
</tr>
<tr>
<td>11</td>
<td>-3.0 - 3.1 -2.5 -2.1 -1.9 -1.9 -2.0 -2.2 -2.5 -2.5 -2.3 -1.4</td>
</tr>
<tr>
<td>9</td>
<td>-3.0 - 3.2 -2.8 -2.4 -2.1 -2.0 -2.1 -2.4 -2.6 -2.4 -1.5</td>
</tr>
<tr>
<td>7</td>
<td>-3.6 - 3.8 -3.4 -2.8 -2.3 -2.2 -2.4 -2.8 -3.1 -3.0 -2.1</td>
</tr>
<tr>
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</tr>
<tr>
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<td>-14.4 -13.8 -11.0 -7.2 -3.9 -2.5 -3.9 -7.2 -10.7 -13.0 -13.1</td>
</tr>
</tbody>
</table>

X COORDINATE INDEX (I)
### Table 18

Error Estimates for Routine QPOLES.F4 with Step Size $d$

$(0.013 < d < 0.050)$

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>6.9</td>
<td>17.4</td>
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<tr>
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<td>0.01</td>
<td>-3.2</td>
<td>3.3</td>
<td>14.2</td>
</tr>
<tr>
<td>2</td>
<td>0.85</td>
<td>0.02</td>
<td>-3.9</td>
<td>4.3</td>
<td>17.3</td>
</tr>
<tr>
<td>3</td>
<td>0.28</td>
<td>0.03</td>
<td>-3.1</td>
<td>3.3</td>
<td>14.8</td>
</tr>
<tr>
<td>4</td>
<td>0.08</td>
<td>0.37</td>
<td>-3.3</td>
<td>3.4</td>
<td>15.1</td>
</tr>
</tbody>
</table>

Y Coordinates (J)

- 21: $-13.7 - 3.3 - 0.2 - 2.0 - 1.9 - 0.6 - 0.5 - 1.1 - 1.9 - 4.2 - 7.8$
- 19: $-8.6 - 4.4 - 1.5 - 1.8 - 1.8 - 0.9 - 0.6 - 0.2 - 1.9 - 3.6 - 3.9$
- 17: $-6.3 - 4.4 - 2.4 - 2.0 - 1.8 - 1.2 - 0.9 - 0.9 - 1.8 - 2.8 - 2.2$
- 15: $-5.0 - 4.2 - 2.9 - 2.2 - 1.9 - 1.4 - 1.2 - 1.2 - 1.8 - 2.2 - 1.3$
- 13: $-4.2 - 4.0 - 3.1 - 2.4 - 2.0 - 1.6 - 1.4 - 1.4 - 1.7 - 1.8 - 0.8$
- 11: $-3.9 - 3.9 - 3.3 - 2.6 - 2.1 - 1.8 - 1.6 - 1.6 - 1.8 - 1.6 - 0.6$
- 09: $-3.9 - 4.1 - 3.6 - 2.9 - 2.3 - 1.9 - 1.8 - 1.8 - 2.0 - 1.7 - 0.7$
- 07: $-4.4 - 4.6 - 4.1 - 3.3 - 2.5 - 2.1 - 2.0 - 2.2 - 2.4 - 2.3 - 1.8$
- 05: $-6.2 - 6.3 - 5.4 - 4.1 - 2.9 - 2.3 - 2.3 - 3.0 - 3.7 - 3.9 - 3.2$
- 03: $-15.1 - 14.5 - 11.7 - 7.7 - 4.1 - 2.5 - 3.6 - 6.6 - 10.1 - 12.4 - 12.4$

X Coordinate Index (I)

- 22: 26 30 34 38 42 46 50 54 58 62
- 21
- 17
- 13
- 09
- 05
- 03

Graph showing trends in error estimates.
Table 19

Error Estimates for Routine QPOLES.F4 with 3% Error in Beam Energy

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.51</td>
<td>0.05</td>
<td>-1.7</td>
<td>5.0</td>
<td>17.6</td>
</tr>
<tr>
<td>1</td>
<td>0.36</td>
<td>0.02</td>
<td>-0.1</td>
<td>1.8</td>
<td>10.9</td>
</tr>
<tr>
<td>2</td>
<td>0.09</td>
<td>0.18</td>
<td>-0.2</td>
<td>1.6</td>
<td>10.7</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
<td>0.17</td>
<td>-0.2</td>
<td>1.6</td>
<td>11.0</td>
</tr>
<tr>
<td>4</td>
<td>0.01</td>
<td>0.18</td>
<td>-0.2</td>
<td>1.6</td>
<td>11.1</td>
</tr>
</tbody>
</table>

Y COORDINATE INDEX (I)

<table>
<thead>
<tr>
<th></th>
<th>21</th>
<th>19</th>
<th>17</th>
<th>15</th>
<th>13</th>
<th>11</th>
<th>9</th>
<th>7</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21</td>
<td>3.6</td>
<td>1.3</td>
<td>0.1</td>
<td>0.5</td>
<td>0.8</td>
<td>0.7</td>
<td>0.1</td>
<td>-1.8</td>
</tr>
<tr>
<td>1</td>
<td>17</td>
<td>2.7</td>
<td>1.6</td>
<td>1.5</td>
<td>1.2</td>
<td>1.0</td>
<td>0.7</td>
<td>0.4</td>
<td>-2.0</td>
</tr>
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<td>5.7</td>
<td>2.2</td>
<td>1.7</td>
<td>1.0</td>
<td>0.7</td>
<td>0.6</td>
<td>0.7</td>
<td>-2.7</td>
</tr>
<tr>
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<td>0.6</td>
<td>0.5</td>
<td>0.6</td>
<td>0.6</td>
<td>0.1</td>
<td>-2.0</td>
</tr>
<tr>
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<td>0.9</td>
<td>0.9</td>
<td>0.8</td>
<td>0.8</td>
<td>0.6</td>
<td>0.4</td>
<td>-1.4</td>
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<tr>
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<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.5</td>
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<td>-1.2</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>-1.0</td>
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</table>

X COORDINATE INDEX (J)

<table>
<thead>
<tr>
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<th>22</th>
<th>26</th>
<th>30</th>
<th>34</th>
<th>38</th>
<th>42</th>
<th>46</th>
<th>50</th>
<th>52</th>
<th>58</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21</td>
<td>30</td>
<td>42</td>
<td>54</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Graph showing Y and X coordinates with error bars.
Table 20

Error Estimates for Routine QPOLES.F4 with 6% Error in Beam Energy

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.51</td>
<td>0.05</td>
<td>1.2</td>
<td>5.2</td>
<td>15.1</td>
</tr>
<tr>
<td>1</td>
<td>0.38</td>
<td>0.02</td>
<td>2.7</td>
<td>3.8</td>
<td>8.8</td>
</tr>
<tr>
<td>2</td>
<td>0.07</td>
<td>0.16</td>
<td>2.6</td>
<td>3.6</td>
<td>8.5</td>
</tr>
<tr>
<td>3</td>
<td>0.04</td>
<td>0.14</td>
<td>2.7</td>
<td>3.7</td>
<td>8.8</td>
</tr>
<tr>
<td>4</td>
<td>0.02</td>
<td>0.15</td>
<td>2.7</td>
<td>3.7</td>
<td>9.0</td>
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</tbody>
</table>

Y Coordinate Index (J)

<table>
<thead>
<tr>
<th>X Coordinate Index (I)</th>
</tr>
</thead>
</table>

\[
\begin{align*}
Y &\quad 21 & -7.2 & 4.1 & 7.8 & 4.9 & 4.6 & 4.7 & 4.9 & 5.1 & 3.6 & 2.3 & -3.2 \\
&\quad 19 & -1.7 & 2.8 & 5.7 & 5.0 & 4.7 & 4.7 & 4.7 & 4.5 & 3.5 & 2.2 & 1.2 \\
&\quad 17 & 0.7 & 2.6 & 4.6 & 4.8 & 4.6 & 4.6 & 4.5 & 4.2 & 3.5 & 2.7 & 3.0 \\
&\quad 15 & 2.0 & 2.8 & 4.0 & 4.5 & 4.5 & 4.5 & 4.4 & 4.1 & 3.6 & 3.3 & 4.0 \\
&\quad 13 & 2.7 & 3.0 & 3.8 & 4.2 & 4.4 & 4.4 & 4.2 & 4.0 & 3.6 & 3.6 & 4.6 \\
&\quad 11 & 3.1 & 3.0 & 3.5 & 4.0 & 4.2 & 4.2 & 4.1 & 3.8 & 3.6 & 3.8 & 4.8 \\
&\quad 9  & 3.0 & 2.8 & 3.2 & 3.7 & 4.0 & 4.1 & 3.9 & 3.7 & 3.4 & 3.6 & 4.7 \\
&\quad 7  & 2.5 & 2.2 & 2.6 & 3.3 & 3.7 & 3.9 & 3.7 & 3.3 & 3.0 & 3.1 & 4.0 \\
&\quad 5  & 0.6 & 0.4 & 1.2 & 2.4 & 3.1 & 3.7 & 3.1 & 2.5 & 1.6 & 1.3 & 2.9 \\
&\quad 3  & -9.0 & -8.9 & -5.5 & -1.5 & 2.1 & 3.5 & 2.1 & -1.4 & -5.1 & -7.6 & -7.7 \\
\end{align*}
\]

\[
\begin{align*}
X &\quad 22 & 26 & 30 & 34 & 38 & 42 & 46 & 50 & 54 & 58 & 62 \\
\end{align*}
\]
Table 21

Error Estimates for Routine QPOLES.F4 with 1% Error in Initial Electron Direction

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy</th>
<th>Scale Error</th>
<th>Average Error</th>
<th>Average Absolute Error</th>
<th>Maximum Absolute Error</th>
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<tbody>
<tr>
<td></td>
<td>(DD)</td>
<td>(%)</td>
<td>(%)</td>
<td>(%)</td>
<td>(%)</td>
</tr>
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<td>-4.9</td>
<td>6.0</td>
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<td>0.01</td>
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<td>3.8</td>
<td>-15.0</td>
</tr>
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<td>0.37</td>
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<td>4.0</td>
<td>-15.7</td>
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<td>3.9</td>
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<td>0.44</td>
<td>-4.0</td>
<td>4.0</td>
<td>-16.5</td>
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</table>

Y

X COORDINATE INDEX (I)

21 -16.1 - 4.7  0.1 -1.5 -1.9 -1.7 -1.9 -1.4 - 2.9 - 3.4 - 6.3
19 -10.8 - 5.8 - 2.0 -1.8 -2.0 -1.9 -1.9 - 2.6 - 3.1 - 2.7
17 - 8.3 - 5.8 - 3.1 -2.3 -2.1 -2.0 -2.0 - 2.4 - 2.4 - 1.3
15 - 6.8 - 5.6 - 3.7 -2.7 -2.3 -2.1 -2.1 - 2.2 - 1.9 - 0.6
13 - 6.0 - 5.4 - 4.0 -3.0 -2.5 -2.3 -2.2 -2.1 - 2.1 - 1.6 - 0.2
11 - 5.6 - 5.3 - 4.3 -3.3 -2.7 -2.4 -2.3 -2.2 -2.1 - 1.5 - 0.1
 9 - 5.5 - 5.4 - 4.6 -3.6 -3.0 -2.6 -2.4 -2.4 - 2.2 - 1.6 - 0.3
 7 - 6.0 - 5.9 - 5.1 -4.1 -3.2 -2.7 -2.6 -2.7 - 2.7 - 2.2 - 1.0
 5 - 7.7 - 7.5 - 6.4 -4.9 -3.6 -2.9 -2.9 - 3.5 - 3.9 - 3.9 - 2.9
 3 -16.5 -15.6 -12.6 -8.5 -4.8 -3.1 -4.1 -7.1 -10.3 -12.3 -12.2

X COORDINATE INDEX (J)

22 26 30 34 38 42 46 50 54 58 62
Table 22

Error Estimates for Routine QPOLES.F4
with 3% Error in Initial Electron Direction

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Discrepancy (DD)</th>
<th>Scale Error (%)</th>
<th>Average Error (%)</th>
<th>Average Absolute Error (%)</th>
<th>Maximum Absolute Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.83</td>
<td>0.05</td>
<td>-6.0</td>
<td>6.7</td>
<td>21.2</td>
</tr>
<tr>
<td>1</td>
<td>0.58</td>
<td>0.04</td>
<td>-4.9</td>
<td>5.0</td>
<td>20.1</td>
</tr>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.80</td>
<td>-5.6</td>
<td>5.6</td>
<td>21.9</td>
</tr>
<tr>
<td>3</td>
<td>0.13</td>
<td>0.05</td>
<td>-4.3</td>
<td>4.7</td>
<td>21.6</td>
</tr>
<tr>
<td>4</td>
<td>0.07</td>
<td>0.31</td>
<td>-4.5</td>
<td>5.1</td>
<td>-22.3</td>
</tr>
</tbody>
</table>

Y COORDINATE INDEX (I)  
21 -22.3 - 9.6 - 2.3 - 1.6 - 2.4 - 2.6 - 2.7 - 2.6 - 2.9 - 2.1 - 0.1
19 -16.2 -10.3 - 4.6 - 2.8 - 2.7 - 2.7 - 2.6 - 2.4 - 2.1 - 0.8 2.7
17 -13.1 -10.0 - 5.8 - 3.7 - 3.1 - 2.8 - 2.5 - 2.1 - 1.4 0.2 3.5
15 -11.3 - 9.4 - 6.5 - 4.4 - 3.4 - 2.9 - 2.5 - 1.9 - 1.0 0.8 3.8
13 -10.1 - 9.0 - 6.8 - 4.9 - 3.7 - 3.0 - 2.4 - 1.7 - 0.6 1.2 3.9
11 - 9.5 - 8.7 - 7.0 - 5.3 - 4.0 - 3.1 - 2.5 - 1.7 - 0.5 1.3 3.8
09 - 9.2 - 8.8 - 7.3 - 5.6 - 4.3 - 3.3 - 2.5 - 1.7 - 0.6 1.1 3.4
07 - 9.6 - 9.2 - 7.8 - 6.0 - 4.5 - 3.4 - 2.7 - 2.0 - 1.0 0.5 2.6
05 -11.1 -10.7 - 9.0 - 6.9 - 4.9 - 3.6 - 3.0 - 2.7 - 2.3 - 1.2 0.6
03 -19.5 -18.5 -15.1 -10.4 - 6.1 - 3.8 - 4.2 - 6.3 - 8.7 - 9.9 - 9.0

22 26 30 34 38 42 46 50 54 58 62

X COORDINATE INDEX (J)
VI. EXPERIMENTAL RESULTS

The biased fin potential referred to in sections III, IV and V of this report has been a conceptual entity rather than a physical entity. The same can be said for the electron trajectories that are used to measure and probe the biased fin potential. In this section, however, we will describe a laboratory experiment that was conducted to test the computer codes that have been developed for the synthesis of approximations to the biased fin potential.

The potential developed in section III was proposed to approximate the field existing near an infinite number of parallel-planar fins, all of which extended infinitely in both parallel directions. Thus, the conceptual system extended infinitely in three spatial directions. However, the physical system was constructed inside a cylindrical bell jar and consisted of six parallel plates approximately five inches high and were biased with -100 volts at a distance of 1.7 inches from a zero potential ground plane. Due to the finite size of the system, we expect that the physical potential field existing inside the evacuated bell jar will be somewhat different from the biased fin potential as derived in section III. In spite of this dimensional limitation, we hoped to use the computer codes described in section V to process electron trajectory data and develop a synthesized potential field for comparison with the biased fin potential plotted in Figure 8.

An electron beam was generated by an electron gun consisting of a wire filament inside a cylindrical aluminum can having a pinhole at the center of one end. The electron gun was positioned below a grounded wire mesh (i.e., the reference plane) so that the electron beam emanating
from the can would pass through the wire mesh on a trajectory that ultimately returned to the wire mesh due to the deflection of the electron beam by the negatively charged fins. The source and sink points in the reference plane were recorded for six trajectories as listed in Table 23. Although the dimensions of the physical system (e.g., 3.17 inches by 1.7 inches) are different from those of the conceptual system (e.g., 1.8 units by 1 unit), it is apparent that the trajectories listed as physical data do not correspond to those trajectories defined conceptually in the statement of the model problem. Thus, it is interesting to speculate what the electron trajectories might have looked like in the conceptual biased fin potential given the initial electron parameters measured as physical data. Figure 15 shows projected trajectories developed for an assumption of the biased fin potential as expressed in equation (III-10) and for the initial electron velocities listed in Table 23. Note that the sink points along the reference plane do not correspond to the sink points measured as physical data. Therefore, we conclude that the physical potential field of the biased fins must be somewhat different from the conceptualized potential. With the knowledge that the unknown potential field in the bell jar is similar to, but not identical to, the conceptualized biased fin potential; it is appropriate to apply the computerized methods described in section V in order to synthesize approximations of the unknown potential. Figure 16 shows the results obtained with five iterations of the QPOLE.F4 routine using all the data that appears in Table 23. Note that this solution indicated a lower potential midway between the fins when compared to the potential field of Figure 8. This may well be the result of approximating an infinite fin height (measured
Figure 15. Trial Electron Trajectories for Experimental Electron Data and the Ideal Biased Fin Potential

Figure 16. Approximation of the Experimental Potential Synthesized with QPOLE.F4

Figure 17. Approximation of the Experimental Potential Synthesized with QPOLES.F4
perpendicular to the reference plane) with a fin only 5 inches high.

Figure 17 shows the results obtained when only four of the trajectories (i.e., K = 2, 3, 4 and 6) are used with the QPOLES.F4 routine. Attempts to use more of the data in Table 23 caused instabilities in the method preventing the synthesis of approximations to the potential field existing in the bell jar. Thus, by concentrating only on the interior trajectories, a potential field can be created which reproduces the sink points of those interior trajectories. However, as seen in Figure 17, the sink points for the K=1 and K=5 trajectories are in error since the data for these trajectories is not used.

Thus, we have demonstrated that the quadrupole methods developed for an idealized biased fin potential and a particular set of trajectory data can be used to synthesize approximations of the similar potential field existing in the bell jar by using a different set of trajectory data. We see that the choice of data for a particular scheme does affect the convergence properties of that scheme, to the extent that no solution is a possibility. Therefore, we conclude that data selection is important for the application of these iterative routines. Conversely, the definition of a particular iterative scheme may depend heavily on the data that can be obtained.
<table>
<thead>
<tr>
<th>Trajectory</th>
<th>Source Points</th>
<th>Source Velocities</th>
<th>Sink Points</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>XO(K)</td>
<td>YO(K)</td>
<td>VX0(K)</td>
</tr>
<tr>
<td></td>
<td>(inches)</td>
<td>(inches)</td>
<td>(m/sec)</td>
</tr>
<tr>
<td>1</td>
<td>0.73</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>1.25</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>2.27</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>2.54</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>2.91</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>0.12</td>
<td>0.0</td>
<td>3.4\times10^6</td>
</tr>
</tbody>
</table>
The synthesis of electric potentials from the fields of electric dipoles and quadrupoles has been considered as a method for approximating unknown potential fields in two dimensions. Although we were most interested in determining potential fields on an unbounded spatial domain; computational methods were developed first for the classical boundary value problem in electrostatics, since other well-known methods exist for its solution.

Several iterative schemes for the dipole synthesis of solutions to a square boundary value problem were developed and their accuracy was compared to a series expansion solution. We found that as the dipole singularities were moved farther away from the boundary, the synthesized solutions became more accurate at the expense of more computation (i.e., more iterations of a scheme were required for convergence to a desired limit). Additionally, procedures for scaling, shifting and rotating the estimated potential were developed to assist in the solution of the boundary value problem. The rotational procedure was found to be of assistance only in determining an initial estimated potential that best matched the known boundary values. The scaling and shifting procedures were found to be helpful immediately after each successive addition of a dipole field. The dipole fields were chosen to reduce the discrepancy between a known potential value and its estimated value at the point on the boundary where the discrepancy was largest. This scheme showed good convergence properties. We found that convergence could be hastened by considering potential discrepancy reduction at two boundary points simultaneously and then allowing the
location of dipole field singularities at variable distances from the boundary. Extensions of this work on the boundary value problem could be interesting if quadrupole synthesis is explored and if other boundary shapes are considered.

Both dipole and quadrupole methods have been developed for the synthesis of unknown potential fields in two dimensions of an unbounded spatial domain. Our first step was to introduce the concept of a reference plane existing near the surface charge distribution responsible for the potential field existing in some spatial region of interest. The reference plane represents an artificial boundary condition, since we have required a potential of zero on it in order to allow the presence of a measuring device (i.e., an electron beam probe) in the vicinity of the charged surface. Charged particles emanate from the reference plane, they are deflected by the potential field of a charge distribution and fall back to the reference plane. The energy of the charged particle and its initial and final coordinate values have been used as constraints upon acceptable dipole and quadrupole synthesized approximations to the potential resulting from the charge distribution. Dipole and quadrupole fields have been added as image pairs to insure a zero potential on the reference plane.

Basic schemes have been developed for the dipole synthesis of electric potentials as constrained by electron trajectory data. A uniform field approximation is determined so as to meet the constraining data for a particular trajectory called the scaling trajectory. Then a dipole field is added in order to meet the constraints of two other trajectories while keeping the dipole field singularity at a fixed distance from the reference plane. A potential shifting operation has
not been required since the dipole fields have been added in a manner that kept the potential at zero on the reference plane. However, potential scaling has been found to be beneficial after the addition of each dipole field in order to meet the constraints of the scaling trajectory. Convergence limits for this scheme have been adequate, although for the model problem, the synthesized solution overestimated the test potential. We found that an underestimation of the test potential could be obtained by altering the criterion used for determining which two sets of trajectory constraints are satisfied when adding a dipole field. Therefore, we were able to establish bounds on the unknown potential created by the surface charge distribution of the model problem. We also found that convergence to a desired accuracy could be hastened by allowing the location of the dipole field singularities at a variable distance from the reference plane.

Quadrupole fields were also used for synthesis of solutions to the model problem. When used in the same way as the dipole fields, we found that better convergence limits were achieved with about the same amount of computation time. In addition, we were able to develop procedures for the addition of one quadrupole field for each trajectory during a single iteration of a scheme designed to meet the constraints of all the trajectories, excepting the scaling trajectory, simultaneously. This scheme reduced by a factor of 5 (i.e., the number of trajectories considered), the number of iterations and the amount of computational time required to obtain a desired accuracy. Future possibilities for extending this work include the development of schemes that combine the scaling operation and the quadrupole synthesis operation into a single operation which meets the constraints for all of the trajectories.
This would not alter the speed of the procedures appreciably but might improve the scale accuracy. Also of interest are schemes which combine the dipole and quadrupole methods for potential field synthesis.

Finally, a test potential was created in the laboratory to test the computer codes developed for the synthesis of unknown potentials. An electron-beam probe was used to collect constraining electron trajectory data. Potentials were synthesized which seemed like reasonable approximations considering that the biased-fin potential that we attempted to create in the laboratory was undoubtably altered somewhat by the finite size of the physical apparatus. We learned that data selection is important for the best determination of unknown potentials when applying the particular codes that we have developed. Therefore, the challenge in the area of dipole and quadrupole field synthesis lies in the determination of optimal schemes depending on whatever data are available.
References


APPENDIX A

Calculating Potential Fields for
Ideal Line Dipoles and Quadrupoles
To determine the potential field \( \phi_d \) existing near an ideal line dipole located at point \((x_d, y_d)\), consider a line of point dipoles extending to infinity in the positive and negative \(z\)-directions as in Figure A1. At any given distance \(z_1\) from the \(x-y\) plane, there exists an electric dipole contributing a potential field,

\[
\phi_d(x,y,z_1) = A_d \frac{\cos \beta(x,y)}{b^2(x,y,z_1)},
\]

at all points in the \(x-y\) plane (ref. 1). The electric potential at an arbitrary point \((x,y)\) due to the ideal line dipole

\[
\phi_d(x,y) = \int_{-\infty}^{\infty} \phi_d(x,y,z_1)dz_1
\]

is found by integrating the potential field due to an electric dipole at distance \(z_1\) from the \(x-y\) plane over all values of \(z_1\) from \(-\infty\) to \(+\infty\). Note that

\[
b^2(x,y,z_1) = z_1^2 + (x-x_d)^2 + (y-y_d)^2
\]

and

\[
\cos \beta(x,y) = \frac{(y-y_d)}{[z_1^2 + (x-x_d)^2 + (y-y_d)^2]^{1/2}}.
\]

Therefore,

\[
\phi_d(x,y) = A_d(y-y_d) \int_{-\infty}^{\infty} \frac{dz_1}{[z_1^2 + (x-x_d)^2 + (y-y_d)^2]^{3/2}}.
\]

Evaluating the integral, we find

\[
\phi_d(x,y) = \frac{A_d(y-y_d)}{(x-x_d)^2 + (y-y_d)^2}.
\]
Figure A1. X-Y Reference Frame for the Ideal Line Dipole Consisting of an Infinite Number of Ideal Point Dipoles
with the aid of CRC Integral Tables (ref. 8). Then, if an image line
dipole is also present at point \((x_d, -y_d)\), the combined electric
potential at point \((x, y)\) is

\[
\phi_d(x, y) = A_d \left[ \frac{(y-y_d)}{(x-x_d)^2 + (y-y_d)^2} + \frac{(y+y_d)}{(x-x_d)^2 + (y+y_d)^2} \right]. \tag{A-7}
\]

In a similar manner, the potential \(\phi_q\) existing near an ideal line
quadrupole may be considered as an integral of the field due to a
point quadrupole.

\[
\phi_q(x, y, z_1) = a q \frac{\sin \beta(x,y) \cos \beta(x,y)}{b^3(x,y,z_1)} \tag{A-8}
\]

located a distance \(z_1\) from the x-y plane, over all values from \(-\infty\) to
\(\infty\).

\[
\phi_q(x, y) = \int_{-\infty}^{\infty} \phi_q(x, y, z_1) dz_1 \tag{A-9}
\]

Using equations (A-3) and (A-4), along with

\[
\sin \beta(x,y) = \frac{(x-x_d)}{\sqrt{z_1^2 + (x-x_d)^2 + (y-y_d)^2}^{1/2}}, \tag{A-10}
\]

we obtain the integral expression;

\[
\phi_q(x, y) = a q (x-x_d)(y-y_d) \int_{-\infty}^{\infty} \frac{dz_1}{\sqrt{z_1^2 + (x-x_d)^2 + (y-y_d)^2}^{5/2}}. \tag{A-11}
\]

Evaluating the integral, we find

\[
\phi_q(x, y) = \frac{A_q (x-x_d)(y-y_d)}{[(x-x_d)^2 + (y-y_d)^2]^{2/3}}. \tag{A-12}
\]
where $A_q = \frac{2}{3} a_q$. Then, if an image line dipole is also present at point $(x_d', -y_d')$, the combined electric quadrupole potential at point $(x,y)$ is

$$\phi_q(x,y) = A_q (x-x_d') \left[ \frac{(y-y_d')}{[(x-x_d')^2 + (y-y_d')^2]^2} + \frac{(y+y_d')}{[(x-x_d')^2 + (y+y_d')^2]^2} \right]. \quad (A-13)$$

For an $r-\theta$ coordinate system, consider the potential field due to an ideal line dipole intersecting the $r-\theta$ plane at the point $(r_d, \theta_d)$ and oriented so that the axes of all the point dipoles comprising the line dipole extend through points on a perpendicular line at the origin of the $r-\theta$ plane. This situation is shown in Figure A2. The point dipole located a distance $z_1$ from the $r-\theta$ plane contributes a potential field

$$\phi_d(r,\theta,z_1) = A_d \frac{\cos \theta(r,\theta)}{b^2(r,\theta,z_1)}, \quad (A-14)$$

at all points $(r,\theta)$ in the $r-\theta$ plane. The electric potential due to the entire line dipole

$$\phi_d(r,\theta) = \int_{-\infty}^{\infty} \phi_d(r,\theta,z_1) dz_1 \quad (A-15)$$

is found by integrating equation (A-14) for all point dipoles located along the line (i.e. $-\infty < z_1 < \infty$).

Applying the Pythagorean Theorem, $b^2 = z_1^2 + d^2$, and the law of cosines,

$$d^2 = r^2 + r_d^2 - 2rr_d \cos(\theta_d - \theta), \quad (A-16)$$

and recognizing in Figure A2 that
Figure A2. R-θ Reference Frame for The Ideal Line Dipole Consisting of an Infinite Number of Ideal Point Dipoles
we derive

$$\cos \beta = \frac{r_d - r \cos(\theta_d - \theta)}{b}; \quad (A-17)$$

Then the integral may be evaluated with the result

$$\phi_d (r, \theta) = \frac{A_d [r_d - r \cos(\theta_d - \theta)]}{\sqrt{r_d^2 + r^2 - 2rr_d \cos(\theta_d - \theta)}}. \quad (A-19)$$
APPENDIX B

Fortran Routines for Dipole Synthesis of Solutions to The Boundary Value Problem
cROUTINE VMODGR.F4

IMPLICIT REAL *8(A-K,0-Z)
COMMON/HVD/XB(40),YB(40),PHI(40),IB,F,S
COMMON/ALL/PSI(80,40),PHI(40),M,N
COMMON/HYPA/R(80),THETA(40)

IT=0
CALL DATA
XMAX=-1.0D38
XMIN=1.0D38
YMAX=-1.0D38
YMIN=1.0D38
XB(IB+1)=XB(1)
YB(IB+1)=YB(1)
XL=0.0
YL=0.0
AL=0.0
DO 120 K=1,IB
  IF(XB(K).GT.XMAX) XMAX=XB(K)
  IF(XB(K).LT.XMIN) XMIN=XB(K)
  IF(YB(K).GT.YMAX) YMAX=YB(K)
  IF(YB(K).LT.YMIN) YMIN=YB(K)
  IF(XB(K+1).LT.XB(K)) GO TO 100
  XL=XL+XB(K)*DABS(YB(K+1)-YB(K))
  YL=YL+(YB(K+1)+YB(K))*0.5*DABS(YB(K+1)-YB(K))
  AL=AL+DABS(YB(K+1)-YB(K))
GO TO 120
100 IF(YB(K+1).LT.YB(K)) GO TO 110
  XL=XL+(XB(K+1)+XB(K))*0.5*DABS(XB(K+1)-XB(K))
  YL=YL+YB(K)*DABS(XB(K+1)-XB(K))
  AL=AL+DABS(XB(K+1)-XB(K))
GO TO 120
110 TANB=(YB(K+1)-YB(K))/(XB(K+1)-XB(K))
  XL=XL+(1+TANB**2)*0.5*0.5*(XB(K+1)**2-XB(K)**2)
  YL=YL+(1+TANB**2)*0.5*0.5*(YB(K+1)**2-YB(K)**2)
  AL=AL+((XB(K+1)-XB(K))**2+(YB(K+1)-YB(K))**2)**0.5
GO TO 120
120 CONTINUE
X0=XL/AL
Y0=YL/AL
TYPE 125,X0,Y0
125 FORMAT(’X0=’,D,’Y0=’,D)
PHIMAX=-1.0D38
PHIMIN=1.0D38
RMAX=0.0
PI=4*DATAN(1.DO)
DO 290 J=1,N
  JX=0
  THETA(J)=2*PI*J/N
  IF(DABS(DSIN(THETA(J))).LE.1.D-10) GO TO 170
  IF(DABS(DCOS(THETA(J))).LE.1.D-10) GO TO 180
  TANJ=DSIN(THETA(J))/DCOS(THETA(J))
GO TO 190
170 JX=1
GO TO 190
180 JK=-1
190 DO 280 K=1,IB
THETA1=DATA2(YB(K+1)-YO,XB(K+1)-XO)
THETA2=DATA2(YB(K)-YO,XB(K)-XO)
IF(THETA1.GE.0.) GO TO 195
THETA1=THETA1+2*PI
195 IF(THETA2.GE.0.) GO TO 197
THETA2=THETA2+2*PI
197 IF(THETA1.GT.THETA2) GO TO 198
IF(THETA(J).LE.THETA1) GO TO 280
IF(THETA(J).GT.THETA2) GO TO 280
GO TO 888
198 IF(THETA(J).LT.PI) GO TO 199
IF(THETA(J).LE.THETA1) GO TO 280
GO TO 888
199 IF(THETA(J).GT.THETA2) GO TO 280
888 IF(DABS(XB(K)-XB(K+1)).LE.1.D-10) GO TO 230
IF(DABS(YB(K)-YB(K+1)).LE.1.D-10) GO TO 240
IF(JK) 210,220,200
200 Y=YO
X=XB(K)-(YB(K)-YO)*(XB(K)-XB(K+1))/(YB(K)-YB(K+1))
GO TO 270
210 X=XO
Y=YB(K)-(XB(K)-XO)*(YB(K)-YB(K+1))/(XB(K)-XB(K+1))
GO TO 270
220 YN=YB(K)*(XB(K)-XO)/(YB(K)-YB(K+1))-XB(K)+XO+YO/TANJ
YD=(XB(K)-XB(K+1))/(YB(K)-YB(K+1))
Y=YN/YD
X=X0+(Y-YO)/TANJ
GO TO 270
230 X=XB(K)
IF(JK) 280,236,232
232 Y=YO
GO TO 270
236 Y=YO+(X-XO)*TANJ
GO TO 270
240 Y=YB(K)
IF(JK) 244,246,280
244 X=XO
GO TO 270
246 X=XO+(Y-YO)/TANJ
270 R(J)=(X**2+Y**2)**0.5
IF(R(J).GT.RMAX) RMAX=R(J)
PHI(J)=PHI(K)
IF(PHI(J).LT.PHMIN) PHMIN=PHI(J)
IF(PHI(J).GT.PHIMAX) PHIMAX=PHI(J)
230 CONTINUE
290 CONTINUE
M=2*N
IN=N+1
DO 291 I=IN,N
R(I)=(I-N)*RMAX/(N-I)
291 CONTINUE
CONTINUE
DO 295 J=1,N
DO 294 I=1,M
PSI(I,J)=(PHIMAX-PHIMIN)*R(I)*DCOS(THETA(J))/(2*RMIN)+(PHIMAX+PHMIN)/2
CONTINUE
294 CONTINUE
CONTINUE
CALL ROTATE
GO TO 301
CALL SHIFT
CALL SCALE
DAN=0.0
DO 310 J=1,N
DAN=DAN+DABS(PHI(J)-PSI(J,J))
CONTINUE
IF(DAN.LE.0.00001) GO TO 320
CALL TADPOLE(F,S)
IT=IT+1
GO TO 300
PAUSE
CALL INIT
CALL PHINDO(SNGL(XMIN),SNGL(YMIN),SNGL(YMAX))
CALL TWINDO(0,1023,0,780)
CALL ERASE
CALL VGRAPH(XO,YO,PHIMIN,PHIAX)
PAUSE
CALL PRINT(IT)
STOP
TYPE 340
FORMAT(' THE POINT XO,YO LIES ON THE BOUNDARY')
CALL FINIT
STOP
END
SUBROUTINE DATA
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/MVD/XB(40),YB(40),PPHI(40),IB,F,S
COMMON/ALL/PSI(80,40),PHI(40),M,N

TYPE 1
1 FORMAT( ' ENTER NUMBER OF ANGULAR GRID LINES' )
   ACCEPT 2,N
2 FORMAT(I)
   TYPE 3
3 FORMAT( ' ENTER NUMBER OF STRAIGHT BOUNDARY SEGMENTS' )
   ACCEPT 2,IB
   DO 8 K=1,IB
   TYPE 4,K
4 FORMAT( ' ENTER X-COORD. AT BEGINNING OF SEGMENT ',13)
   ACCEPT 5,XB(K)
5 FORMAT(D)
   TYPE 6,K
6 FORMAT( ' ENTER Y-COORD. AT BEGINNING OF SEGMENT ',13)
   ACCEPT 5,YB(K)
   TYPE 7,K
7 FORMAT( ' ENTER POTENTIAL ON BOUNDARY SEGMENT ',13)
   ACCEPT 5,PPHI(K)
   CONTINUE
   TYPE 28
8 FORMAT( ' ENTER DIPOLE G PARAMETER' )
   ACCEPT 5,F
9 FORMAT( ' ENTER DIPOLE T PARAMETER' )
   ACCEPT 5,S
RETURN
END
SUBROUTINE ADPOLE(T)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/ALL/PSI(80,40), PHI(40), N
COMMON/MVPA/R(80), THETA(40)
DIMENSION D(45)
DAN=0.0
DNAX=0.0
DO 41 J=1,N
D(J)=PHI(J)-PSI(J,J)
DAN=DAN+DABS(D(J))
IF(DABS(D(J)).LE.DMAX) GO TO 41
DMAX=DABS(D(J))
JD=J
41 CONTINUE
AVDAN=DAN/N
KL=0
JN=JD+N
DO 44 J=JD,JN
IF(J.GT.N) GO TO 42
JL=J
GO TO 43
42 JL=J-N
43 IF(D(JL)/D(JD).GT.0.0) GO TO 44
KL=KL+1
IF(KL.EQ.1) L=JL
LL=JL
44 CONTINUE
IF(L.GT.JD) GO TO 45
IL=JD-L
GO TO 46
45 IL=L-JD
46 IF(IL.GT.JD) GO TO 805
ILL=JD-IL
GO TO 806
805 ILL=LL-JD
806 IF(IL.GT.ILL) GO TO 807
JC=L
GO TO 47
807 JC=LL
47 A=DABS(D(JD))-.T*AVDAN
B=R(JC)*DCOS(THETA(JC)-THETA(JD))*(2*T*AVDAN-DABS(D(JD)))-1
R(JD)*DABS(D(JD))
C=R(JC)*R(JD)*DCOS(THETA(JC)-THETA(JD))*DABS(D(JD))-1
R(JC)*2*T*AVDAN
IF(E**2-4*A*C.LT.0.0) GO TO 60
IF(B.LT.0.0) GO TO 48
RD1=(2*A)/(B-(B**2-4*A*C)**0.5)
RD2=(-B-(B**2-4*A*C)**0.5)/(2*A)
GO TO 49
48 RD1=(-3+(B**2-4*A*C)**0.5)/(2*A)
RD2=(2*A)/(-B+(B**2-4*A*C)**0.5)
49 IF(RD1.LT.R(JD)) GO TO 50
RD=RD1
GO TO 51
50 IF(RD2.LE.R(JD)) GO TO 61
   RD=RD2
51 IF(RD.LT.R(JD)*6.2835/10) RD=R(JD)*(6.2835/10+1.0)
   AD=(RD-R(JD))*D(JD)
   TYPE 515, RD, THETA(JD)
515 FORMAT(‘RD=’,D,’ THETA=’,D)
   M=2*N
   DO 53 J=1,N
      DO 52 I=1,M
         PSI(I,J)=PSI(I,J)+AD*(RD-R(I)*DCOS(THETA(J)-THETA(JD)))/(RD**2
                     +R(I)**2-2*RD*R(I)*DCOS(THETA(J)-THETA(JD)))
52      CONTINUE
53      CONTINUE
   RETURN
60 TYPE 62
62 FORMAT(‘DIPOLE RADIUS INCLUDES IMAGINARY ROOT’)
   RETURN
61 TYPE 63
63 FORMAT(‘DIPOLE LOCATED INSIDE BOUNDARY’)
   RETURN
END
SUBROUTINE TADPOLE(G,T)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/ALL/PSI(80,40),PHI(40),M,N
COMMON/MVPA/R(80),THETA(40)
DIMENSION D(45)
DAN=0.0
DMAX=0.0
DO 41 J=1,N
D(J)=PHI(J)-PSI(J,J)
DAN=DAN+DABS(D(J))
IF(DABS(D(J)).LE.DMAX) GO TO 41
DMAX=DABS(D(J))
JD=J
41 CONTINUE
AVDAN=DAN/N
KL=0
JN=JD+N
DO 44 J=JD,JN
IF(J.GT.N) GO TO 42
JL=J
GO TO 43
42 JL=J-N
43 IF(D(J)/D(JD).GT.0.0) GO TO 44
KL=KL+1
IF(KL.EQ.1) L=JL
LL=JL
44 CONTINUE
IF(L.GT.JD) GO TO 45
IL=JD-L
GO TO 46
45 IL=L-JD
46 IF(LL.GT.JD) GO TO 805
ILL=JD-LL
GO TO 806
805 ILL=LL-JD
806 IF(IL.GT.I) GO TO 807
JC=L
GO TO 47
807 JC=LL
47 A=DABS(D(JD))-T*AVDAN
B=R(JC)*DCOS(THETA(JC)-THETA(JD))*(2*T*AVDAN-DABS(D(JD)))-
R(JD)*DABS(D(JD))
C=R(JD)*R(JC)*DCOS(THETA(JC)-THETA(JD))*DABS(D(JD))-
R(JC)**2*T*AVDAN
IF(B**2-4*A*C.LT.0.0) GO TO 60
IF(B.LT.0.0) GO TO 48
RD1=(2*C)/(B-(B**2-4*A*C)**0.5)
RD2=(-B-(B**2-4*A*C)**0.5)/(2*A)
GO TO 49
48 RD1=(-B+(B**2-4*A*C)**0.5)/(2*A)
RD2=(2*C)/(-B+(B**2-4*A*C)**0.5)
49 IF(RD1.LE.R(JD)) GO TO 50
RD=RD1
GO TO 51
50 IF(RD2.LE.R(JD)) GO TO 61
RD=RD2
51 IF(RD.LT.R(JD)*(1.0+G/N)) RD=R(JD)*(G/N+1.0)
AD=(RD-R(JD))*D(JD)
TYPE 515,RD,THETA(JD)
515 FORMAT(’RD=',D,’ THETA=',D)
M=2*N
DO 53 J=1,N
DO 52 I=1,M
PSI(I,J)=PSI(I,J)+AD*(RD-R(I)*DCOS(THETA(J)-THETA(JD)))/(RD**2
1 +R(I)**2-2*RD*R(I)*DCOS(THETA(J)-THETA(JD)))
52 CONTINUE
53 CONTINUE
RETURN
60 TYPE 62
62 FORMAT(’DIPOLE RADIUS INCLUDES IMAGINARY ROOT’)
RETURN
61 TYPE 63
63 FORMAT(’DIPOLE LOCATED INSIDE BOUNDARY’)
RETURN
END
SUBROUTINE SHIFT
IMPLICIT REAL*8(A-H,O-Z)
COMMON/ALL/PSI(80,40),PHI(40),M,N
DIMENSION QPSI(40)
DON=0.0
DO 85 J=1,N
DON=DON+PHI(J)-PSI(J,J)
85 CONTINUE
ADON=DON/N
DO 88 I=1,M
DO 86 J=1,M
QPSI(J)=PSI(I,J)
86 CONTINUE
DO 87 J=1,N
PSI(I,J)=QPSI(J)+ADON
87 CONTINUE
88 CONTINUE
RETURN
END

SUBROUTINE VGRAPH(XO, YO, PHI1-1II-I, PHIMAX)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/MVD/XB(40),YB(40),PPHI(40),IB,S
COMMON/ALL/PSI(80,40),PHI(40),M,N
COMMON/MVPA/R(30),THETA(40)
DIMENSION X(190),Y(190)
CALL MOVEA(SNGL(XB(I3)) ,SNGL(YB(IB) ))
DO 500 K=1,IB
CALL DRA',YA( SNGL (X3 (K) ) , SNGL (YB ( K) ) )
500 CONTINUE
PI=4*DATAN(1.0D0)
MR=N+1
MN=N+2
DO 595 K=1,39
V=PHIMIN+K*(PHIMAX-PHIMIN)/40
L=0
DO 515 J=1,N
DO 510 I=MN,1,-1
IF(R(I).GT.R(J)) GO TO 515
IF((PSI(I-1,J)-V)*(PSI(I,J)-V).GT.0.0) GO TO 510
L=L+1
RR=R(I)+(V-PSI(I,J))*(R(I)-R(I-1))/(PSI(I,J)-PSI(I-1,J))
X(L)=RR*DCOS(THETA(J))
Y(L)=RR*DSIN(THETA(J))
510 CONTINUE
515 CONTINUE
THETA(N+1)=THETA(1)
DO 525 I=MR,1,-1
PSI(I,N+1)=PSI(I,1)
DO 520 J=2,MR
JJ=J
IF(JJ.EQ.N+1) JJ=1
IF(R(I).GT.R(JJ)) GO TO 520
IF(R(I).LT.R(J-1)) GO TO 520
IF((PSI(I,J-1)-V)*(PSI(I,J)-V).GT.0.0) GO TO 520
L=L+1
TT=THETA(J)+(V-PSI(I,J))*2*PI/((PSI(I,J)-PSI(I,J-1)))
X(L)=R(I)*DCOS(TT)
Y(L)=R(I)*DSIN(TT)
520 CONTINUE
525 CONTINUE
LMAX=L
DG=0.0
DO 570 L=1,LMAX
IF((XO-X(L))**2+(YO-Y(L))**2.LT.DG) GO TO 570
DG=(XO-X(L))**2+(YO-Y(L))**2
LY=L
570 CONTINUE
XX=X(1)
YY=Y(1)
X(1)=X(LY)
Y(1)=Y(LY)
X(LY)=XX
Y(LY)=YY
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
535 LL=LL+1
DSQ=1.0D38
DO 540 L=1,LMAX
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2
IF(DSQL.GT.DSQ) GO TO 540
DSQ=DSQL
LDMIN=L
540 CONTINUE
XX=X(LL)
YY=Y(LL)
X(LL)=X(LDMIN)
Y(LL)=Y(LDMIN)
X(LDMIN)=XX
Y(LDMIN)=YY
CALL DRAWA(SNGL(X(LL)),SNGL(Y(LL)))
IF(LL.EQ.LMAX) GO TO 595
GO TO 535

595 CONTINUE
RETURN
END
SUBROUTINE SCALE
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/ALL/PSI(80,40),PHI(40),M,N
AVPHI=0.0
AVPSI=0.0
DO 80 J=1,N
AVPHI=AVPHI+PHI(J)
AVPSI=AVPSI+PSI(J,J)
80 CONTINUE
AVPHI=AVPHI/N
AVPSI=AVPSI/N
ABPHI=0.0
ABPSI=0.0
DO 81 J=1,N
ABPHI=ABPHI+DABS(PHI(J)-AVPHI)
ABPSI=ABPSI+DABS(PSI(J,J)-AVPSI)
81 CONTINUE
M=2*N
DO 83 J=1,N
DO 82 I=1,M
IF(ABPSI.EQ.0.0) GO TO 32
PSI(I,J)=(PSI(I,J)-AVPSI)*ABPHI/ABPSI+AVPSI
82 CONTINUE
83 CONTINUE
RETURN
END

SUBROUTINE VGRAPI!(X0,Y0,PHIMIN,Phiphi,AX)
IMPLICIT REAL *3 (A-H,O-Z)
COMMON/IVD/XB(40),YB(40),PPHI(40),IB,S
49 IF(RD1.LE.R(JD)) GO TO 50
RD=RD1
GO TO 51
50 IF(RD2.LE.R(JD)) GO TO 61
RD=RD2
51 IF(RD.LT.R(JD)*(1.0+6.2835/N)) RD=R(JD)*(6.2835/N+1.0)
AD=(RD-R(JD))*D(JD)
TYPE 515,RD,THETA(JD)
515 FORMAT(’ RD=’,D,’ THETA=’,D)
M=2*N
DO 53 J=1,N
DO 52 I=1,M
PSI(I,J)=PSI(I,J)+AD*(RD-R(I)*DCOS(THETA(J)-THETA(JD)))/(RD**2
+R(I)**2-2*RD*R(I)*DCOS(THETA(J)-THETA(JD)))
52 CONTINUE
53 CONTINUE
RETURN
TYPE 62
62 FORMAT(’ DIPOLE RADIUS INCLUDES IMAGINARY ROOT’)
RETURN
TYPE 63
63 FORMAT(’ DIPOLE LOCATED INSIDE BOUNDARY’)
RETURN
END
COMMON/ALL/PSI(80,40),PHI(40),M,N
COMMON/MVPA/R(80),THETA(40)
DIMENSION X(190),Y(190)
CALL MOVEA(SNGL(XB(IB)),SNGL(YB(IB)))
DO 500 K=1,IB
CALL DRAUA(SNGL(XB(K)),SNGL(YB(K)))
500 CONTINUE
PI=4*DATAN(1.DO)
MR=N+1
NR=N+2
DO 595 K=1,39
V=PHIMIN+K*(PHIMAX-PHIMIN)/40
L=0
DO 515 J=1,N
DO 510 I=MR,N
IF(R(I).GT.R(J)) GO TO 515
IF((PSI(I-1,J)-V)*(PSI(I,J)-V).GT.0.DO) GO TO 510
L=L+1
RR=R(I)+(V-PSI(I,J))*(R(I)-R(I-1))/(PSI(I,J)-PSI(I-1,J))
X(L)=RR*DCOS(THETA(J))
Y(L)=RR*DSIN(THETA(J))
510 CONTINUE
THETA(N+1)=THETA(1)
DO 525 I=MR,N
PSI(I,N+1)=PSI(I,1)
DO 520 J=2,N
JJ=J
IF(JJ.EQ.M+1) JJ=1
IF(R(I).GT.R(JJ)) GO TO 520
IF(R(I).GT.R(J-1)) GO TO 520
IF((PSI(I,J-1)-V)*(PSI(I,J)-V).GT.0.DO) GO TO 520
L=L+1
TT=THETA(J)+(V-PSI(I,J))*2*PI/((PSI(I,J)-PSI(I,J-1))*M)
X(L)=R(I)*DCOS(TT)
Y(L)=R(I)*DSIN(TT)
520 CONTINUE
LMAX=L
DG=0.0
DO 570 L=1,LMAX
IF((XO-X(L))*2+(YO-Y(L))*2.LT.DG) GO TO 570
DG=(XO-X(L))*2+(YO-Y(L))*2
LY=L
570 CONTINUE
XX=X(1)
YY=Y(1)
X(1)=X(LY)
Y(1)=Y(LY)
X(LY)=XX
Y(LY)=YY
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
LL=LL+1
DSQ=1.0D33
DO 540 L=LL,LMAX
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2
IF(DSQL.GT.DSQ) GO TO 540
DSQ=DSQL
LDMIN=L
540 CONTINUE
XX=X(LL)
YY=Y(LL)
X(LL)=X(LDMIN)
Y(LL)=Y(LDMIN)
X(LDMIN)=XX
Y(LDMIN)=YY
CALL DRAUA(SNGL(X(LL)),SNGL(Y(LL)))
IF(LL.EQ.LMAX) GO TO 595
GO TO 535
595 CONTINUE
RETURN
END
APPENDIX C

Fortran Routines for Generating the Biased Fin
Potential and the Associated Electron Trajectory Data
PROGRAM TEST.F4
IMPLICIT REAL *8 (A-H,O-Z)
COMPLEX Z,ZZ
DIMENSION PHI(82,22),XX(200),YY(200),X(82),Y(22)

PI=4*DATAH(1.0DO)

TYPE 10
FORMAT(‘ ENTER X DIMENSION=’,$)
ACCEPT 20,DX

TYPE 30
FORMAT(‘ ENTER Y DIMENSION=’,$)
ACCEPT 20,DY
X(1)=-DX/30
DO 40 I=2,82
X(I)=(I-2)*DX/30
40 CONTINUE
Y(1)=-DY/20
DO 50 J=2,22
Y(J)=(J-2)*DY/20
50 CONTINUE
S=PI/1.8
PSI=DLOG(-DEXP(S)+DSQRT(1+DEXP(2*S)))

DO 70 I=2,41
DO 60 J=2,22
S=PI*(1-Y(J)/DY)/1.8
T=PI*(2*X(I)/DX-0.5)
Z=CMPLX(SNGL(S),SNGL(T))
ZZ=CSQRT(1-CEXP(2*Z))
D=REAL(ZZ)
E=AIMAG(ZZ)
ZZ=CEXP(Z)
DD=-AIMAG(ZZ)
EE=REAL(ZZ)
ZZ=CMPLX(SNGL(D+DD),SNGL(E+EE))
ZZ=CLOG(ZZ)
PHI(I,J)=-REAL(ZZ)-PSI
60 CONTINUE
70 CONTINUE

DO 75 J=2,22
S=PI*(1-Y(J)/DY)/1.8
PHI(42,J)=DLOG(-DEXP(S)+DSQRT(1+DEXP(2*S)))-PSI
75 CONTINUE

DO 90 K=2,21
DO 80 J=2,22
PHI(K,J)=PHI(44-K,J)
80 CONTINUE
90 CONTINUE

DO 110 L=3,42
DO 100 J=2,22
PHI(40+L,J)=PHI(L,J)
100 CONTINUE
110 CONTINUE
CONTINUE

CALL INITT
CALL DWINDO(SNGL(X(2)), SNGL(X(82)), SNGL(Y(2)), SNGL(Y(22)))
II=INT(SNGL(780*DY/DX))
CALL TWINDO(0, 780, 0, II)
CALL MOVEA(SNGL(X(2)), SNGL(Y(2)))
CALL DRAWA(SNGL(X(82)), SNGL(Y(2)))

PHIMIN=1.0D38
PHIMAX=-1.0D38
DO 140 I=2, 82
  DO 130 J=2, 22
    IF(PHI(I,J).GT.PHIMIN) GO TO 125
    PHIMIN=PHI(I,J)
  125 IF(PHI(I,J).LT.PHIMAX) GO TO 130
    PHIMAX=PHI(I,J)
  130 CONTINUE
  140 CONTINUE

TYPE 290, PHIMIN, PHIMAX

290 FORMAT(' PHIMIN=',D12.6,' PHIMAX=',D12.6)

K=0

142 K=K+1
V=PHIMIN+K*0.2
L=0
DO 160 J=2, 22
  DO 150 I=3, 82
    IF((PHI(I-1,J)-V)*(PHI(I,J)-V).GE.0.0) GO TO 150
    L=L+1
  150 XV(L)=X(I-1)+(X(I)-X(I-1))*(V-PHI(I-1,J))/(PHI(I,J)-PHI(I-1,J))
  160 CONTINUE

150 CONTINUE

160 CONTINUE
DO 180 I=2, 82
  DO 170 J=3, 22
    IF((PHI(I,J-1)-V)*(PHI(I,J)-V).GE.0.0) GO TO 170
    L=L+1
    XV(L)=X(I)
  170 XV(L)=Y(J-1)+(Y(J)-Y(J-1))*(V-PHI(I,J-1))/(PHI(I,J)-PHI(I,J-1))
  180 CONTINUE

170 CONTINUE

180 CONTINUE
LM=L
XXMIN=X(82)
DO 190 L=1, LM
  IF(XXMIN.GT.XMIN) GO TO 190
  XXMIN=XX(L)
LDMIN=L

190 CONTINUE
XS=XX(1)
YS=YY(1)
XX(1)=XX(LMIN)
YY(1)=YY(LMIN)
XX(LMIN)=XS
YY(LMIN)=YS
CALL MOVEA(SNGL(XX(1)),SNGL(YY(1)))
LL=1
195
LL=LL+1
DSQ=1.0D38
DO 210 L=LL,LM
  DSQ=(XX(L)-XX(LL-1))**2+(YY(L)-YY(LL-1))**2
  IF(DSQ.GT.DSQ) GO TO 210
  DSQ=DSQ
  LDM=L
210
CONTINUE
XS=XX(LL)
YS=YY(LL)
XX(LL)=XX(LDM)
YY(LL)=YY(LDM)
XX(LDM)=XS
YY(LDM)=YS
CALL DRAWA(SNGL(XX(LL)),SNGL(YY(LL)))
IF(LL.EQ.LM) GO TO 220
GO TO 195
220
IF(V+0.2.LE.PHIMAX) GO TO 142
PAUSE
OPEN(UNIT=1,FILE='POTL.DAT')
WRITE (1,20) X(I)
WRITE (1,20) Y(I)
DO 240 I=2,82
  WRITE(1,20) X(I)
  DO 230 J=2,22
    IF(I.GT.2) GO TO 222
    WRITE(1,20) Y(J)
  222 WRITE(1,20) PHI(I,J)
  225 FORMAT(D12.8)
  230 CONTINUE
240 CONTINUE
STOP
END
cROUTINE DATA.F4
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,22),XX(82),YY(22),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(250),Y(250),FX(250),FY(250),DT(250),KT
OPEN(UNIT=1,FILE='POTL.DAT')
READ(1,50) XX(1)
READ(1,50) YY(1)
DO 200 I=2,82
READ (1,50) XX(I)
50 FORMAT(D)
DO 100 J=2,22
IF(I.GT.2) GO TO 80
READ(1,50) YY(J)
80 READ(1,50) PHI(I,J)
100 CONTINUE
200 CONTINUE
CALL INITT
CALL DWINDO(SNGL(XX(1)),SNGL(XX(82)),SNGL(YY(1)),SNGL(YY(22)))
DX=XX(82)-XX(2)
DY=YY(22)-YY(2)
II=INT(SNGL(780*DY/DX))
CALL TWINDO(0,780,0,II)
K=0
300 K=K+1
XO(K)=XX(13+K*9)
YO(K)=0.0
CALL MOVEA(SNGL(XO(K)),SNGL(YO(K))
VXO(K)=1.0
VYO(K)=7.35D5
KT=0
DT(1)=DY/(100*VYO(K))
400 KT=KT+1
CALL DVOGEL(K)
CALL DRAUA(SNGL(X(KT+1)),SNGL(Y(KT+1)))
IF(Y(KT+1).LE.0.0) GO TO 500
GO TO 400
500 XT(K)=X(KT+1)
YT(K)=0.0
IF(K.EQ.5) GO TO 600
GO TO 300
600 K=K+1
XO(K)=XX(21)
YO(K)=0.0
CALL MOVEA(SNGL(XO(K)),SNGL(YO(K))
VXO(K)=5.5D5
VYO(K)=5.0D5
KT=0
DT(1)=DY/(100*VYO(K))
700 KT=KT+1
CALL DVOGEL(K)
CALL DRAUA(SNGL(X(KT+1)),SNGL(Y(KT+1)))
IF(Y(KT+1).LE.0.0) GO TO 800
GO TO 700
800     XT(K)=X(KT+1)
     YT(K)=0.0
OPEN(UNIT=1, FILE='TRAJ.DAT')
IL=6
WRITE(1,1500) IL
DO 1400 K=1,6
    WRITE(1,1600) XO(K),VX0(K)
    WRITE(1,1600) YO(K),VYO(K)
    WRITE(1,1600) XT(K),YT(K)
1400 CONTINUE
1500 FORMAT(1)
1600 FORMAT(2D)
CALL VPLOT
STOP
END

SUBROUTINE NEWTON(X,Y,EX,EY,PSI,IK)
IMPLICIT REAL *8(A-H, O-Z)
COMMON/COORD/PHI(82,22),XX(82),YY(22),DX,DY
DIMENSION FM(4,4,4),FMN(4,4,4),XM(4),XM1(4),YN(4),Y1U(4)
IF(X.GT.XX(82)) GO TO 220
IF(X.LT.XX(1)) GO TO 220
IF(Y.GT.YY(22)) GO TO 220
IF(Y.LT.YY(1)) GO TO 220
HX=DX/80.0
HY=DY/20.0
I=0
10     I=I+1
     IF(X.GT.XX(I)) GO TO 10
     IF(I.EQ.2) GO TO 12
     IB=I-2
     GO TO 14
12     IB=I-1
14     IF(I.EQ.82) GO TO 16
     IE=I+1
     GO TO 18
16     IE=I
18     J=0
20     J=J+1
     IF(Y.GT.YY(J)) GO TO 20
     IF(J.EQ.2) GO TO 25
     JB=J-2
     GO TO 30
25     JB=J-1
30     IF(J.EQ.22) GO TO 35
     JE=J+1
     GO TO 40
35     JE=J
40     M=M-IB+1
     XM(1)=1.0
     XM1(1)=0.0
     DO 70 M=2,IM
         LM=IB+M-2
         DO 70 K=1,IM
             XM(K)=XM(K-1)+XM1(K-1)
             XM1(K)=XM(K+1)-XM(K-1)
70         CONTINUE
     CALL VPLOT
     STOP
END
XM(M) = XM(M-1) * (X - XX(I!))

XM1(M) = 0.0

DO 60 I = 2, 11
  XI = 1.0
  DO 50 K = 2, M
    IF (K .EQ. I) GO TO 50
    XI = XI * (X - XX(IB+K-2))
  50 CONTINUE

XM1(M) = XM1(M) + XI

60 CONTINUE

70 CONTINUE

M = JE - JB + 1
YN(1) = 1.0
YN1(1) = 0.0
DO 100 N = 2, NN
  JN = JB + N - 2
  YN(N) = YN(N-1) * (Y - YY(JN))
  YN1(N) = 0.0
  DO 90 J = 2, H
    Y1 = 1.0
    DO 80 L = 2, N
      IF (L .EQ. J) GO TO 80
      Y1 = Y1 * (Y - YY(JB+L-2))
    80 CONTINUE
    Y1 = Y1 * (Y - YY(JB+L-2))
  90 CONTINUE

100 CONTINUE

110 CONTINUE

FM(1, M, N) = PHI(IB+M-1, JB+N-1)

105 CONTINUE

110 CONTINUE

DO 140 L = 2, NN
  DO 130 N = L, NN
    DO 120 M = 1, MM
      FM(L, M, N) = (FM(L-1, M, N-1) - FM(L-1, M, N)) / (HY*(L-1))
    120 CONTINUE
  130 CONTINUE

140 CONTINUE

DO 160 M = 1, MM
  DO 150 N = 1, NN
    FNM(1, M, N) = FM(M, M, N)
  150 CONTINUE

160 CONTINUE

DO 190 K = 2, NM
  DO 180 M = K, NM
    DO 170 N = 1, NN
      FHN(K, M, N) = (FNM(K-1, M-1, N) - FNM(K-1, M, N)) / (HY*(K-1))
    170 CONTINUE
  180 CONTINUE

190 CONTINUE

EX = 0.0
EY = 0.0
DO 210 M=1,NN
  EX=EX-XM1(M)*YN(N)*FMN(M,N)
  EY=EY-YN1(N)*XM(M)*FMN(M,N)
  PSI=PSI+XM(M)*YN(N)*FMN(M,N)
  200 CONTINUE
  210 CONTINUE
GO TO 230
  220 IK=-1
  230 RETURN

SUBROUTINE DVOGEL(K)
  IMPLICIT REAL *8(A-H,O-Z)
  COMMON/COORD/P1II(82,22),XX(82),YY(22),DX,DY
  COMMON/DATA/X0(50),Y0(50),VX0(50),VY0(50),XT(50),YT(50)
  COMMON/SAVE/X(250),Y(250),FX(250),FY(250),DT(250),KT
  QM=1.76D11
  DIM=DTSQRT(DX**2+DY**2)
  IK=0
  IF(KT.GT.1) GO TO 200
  VX=VX0(K)
  VY=VY0(K)
  X(1)=X0(K)
  Y(1)=Y0(K)
  100 CALL NEWTON(X(KT),Y(KT),EX,EY,PSI,IK)
  IF(IK.NE.O) GO TO 400
  AX=QM*EX
  AY=QM*EY
  XH=X(KT)-(0.5*VX-0.125*AX*DT(KT))*DT(KT)
  YH=Y(KT)-(0.5*VY-0.125*AY*DT(KT))*DT(KT)
  CALL NEWTON(XH,YH,EX,EY,PSI,IK)
  IF(IK.NE.O) GO TO 400
  AXH=QM*EX
  AYH=QM*EY
  200 X=X(KT)+(0.5*VX+(4.0*AX-AKH)*DT(KT)/24.0)*DT(KT)
  Y=Y(KT)+(0.5*VY+(4.0*AY-AYH)*DT(KT)/24.0)*DT(KT)
  CALL NEWTON(X,YH,EX,EY,PSI,IK)
  IF(IK.NE.O) GO TO 400
  AXH=QM*EX
  AYH=QM*EY
  X(KT+1)=X(KT)+(VX+(AXH+2.0*AXH)*DT(KT)/6.0)*DT(KT)
  Y(KT+1)=Y(KT)+(VY+(AYH+2.0*AYH)*DT(KT)/6.0)*DT(KT)
  AAX=AX
  AAY=AY
  CALL NEWTON(X(KT+1),Y(KT+1),EX,EY,PSI,IK)
  IF(IK.NE.O) GO TO 400
  AX=QM*EX
  AY=QM*EY
  FX(KT)=EX
  FY(KT)=EY
  A1SQ=AAX**2+AAY**2
  DELD=DTSQRT((X(KT+1)-X(KT))**2+(Y(KT+1)-Y(KT))**2)
IF(DELD.GT.DIM/80) GO TO 320
IF(DELD.LT.DIM/400) GO TO 330
IF(AlSQ.LT.1.D-6) GO TO 300
280
VX=VX+DT(KT)*(AX+4.0*AXH+AAX)/6.0
VY=VY+DT(KT)*(AY+4.0*AYH+AAY)/6.0
DT(KT+1)=DT(KT)
GO TO 440
300
VISQ=VX**2+VY**2
DVSQ=AlSQ*DT(KT)**2
IF(VISQ.GT.DVSQ*1.D4) GO TO 280
IK=1
GO TO 410
320
DT(KT)=DT(KT)/1.5
GO TO 100
330
DT(KT)=DT(KT)*1.5
GO TO 100
400
TYPE 420
GO TO 440
410
TYPE 430
420 FORMA(' ELECTRON HAS ESCAPED POTENTIAL REGION
1 INCLUDED IN X-Y COORDINATE GRID')
430 FORMA(' PARTICLE VELOCITY AND ACCELERATION BOTH
1 EQUAL ZERO')
440 RETURN
END
SUBROUTINE VPLOT
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/COORD/PHI(82,22),XX(32),YY(22),DX,DY
DIMENSION X(180),Y(180)
CALL INITT
CALL DUINDO(SNGL(XX(1)),SNGL(XX(82)),SNGL(YY(1)),SNGL(YY(22)))
II=INT(SNGL(750*DY/DX))
CALL WRINDO(0,780,0,11)
CALL MOVEA(SNGL(XX(2)),SNGL(YY(2)))
CALL DRAWA(SNGL(XX(82)),SNGL(YY(2)))
PHIMIN=1.0D38
PHIMAX=-1.0D38
DO 10 1=2,82
DO 5 J=2,22
IF((PHI(I,J).GT.PHIMIN) GO TO 2
PHIMIN=PHI(I,J)
2 IF((PHI(I,J).LT.PHIMAX) GO TO 5
PHIMAX=PHI(I,J)
5 CONTINUE
10 CONTINUE
K=0
12 K=K+1
V=PHIMIN+K*0.2
L=0
DO 20 J=2,22
DO 15 I=3,82
IF((PHI(I-1,J)-V)*(PHI(I,J)-V).*GE.0.0) GO TO 15
L=L+1
20 CONTINUE
X(L) = XX(I-1) + (XX(I) - XX(I-1)) * (V - PHI(I-1,J)) / (PHI(I,J) - PHII(I-1,J))
Y(L) = YY(J)

CONTINUE

DO 30 I = 2, 82
DO 25 J = 3, 22
IF((PHI(I,J-1) - V) * (PHI(I,J) - V) .GE. 0.0) GO TO 25
L = L + 1
X(L) = XX(I)
Y(L) = YY(J-1) + (YY(J) - YY(J-1)) * (V - PHI(I,J-1)) / (PHI(I,J) - PHII(I,J-1))

CONTINUE

LMAX = L
XMIN = XX(82)
DO 35 L = 1, LMAX
IF(X(L) .GT. XMIN) GO TO 35
XMIN = X(L)
LMIN = L

CONTINUE

XS = X(1)
YS = Y(1)
X(1) = X(LMIN)
Y(1) = Y(LMIN)
X(LMIN) = XS
Y(LMIN) = YS
CALL MOVEA(SNGL(X(1)), SNGL(Y(1)))

LL = 1

DSQ = 1.0E38
DO 45 L = LL, LMAX
DSQ = (X(L) - X(LL-1))**2 + (Y(L) - Y(LL-1))**2
IF(DSQ .GT. DSQ) GO TO 45
DSQ = DSQ
LDMIN = L

CONTINUE

XS = X(LL)
YS = Y(LL)
X(LL) = X(LDMIN)
Y(LL) = Y(LDMIN)
X(LDMIN) = XS
Y(LDMIN) = YS
CALL DRAUA(SNGL(X(LL)), SNGL(Y(LL)))
IF(LL .EQ. LMAX) GO TO 50
GO TO 40

CONTINUE

IF(V + 0.2 .LE. PHIMAX) GO TO 12
CALL FINITT(800, 500)
RETURN
END
APPENDIX D

Fortran Routines for Dipole Synthesis of Solutions
to The Model Problem
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(S2,32),XX(32),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIMENSION XS(110),YS(110),SX(110),SY(110),TD(110),DDX(50)
EXTERNAL TEKHAN,PLTHAN,SCPHAN
CALL INITG

TYPE 1
1 FORMAT(' ENTER X AND Y DIMENSIONS OF POTENTIAL REGION',$)
ACCEPT 2,DX,DY
2 FORMAT(2D)

TYPE 19
19 FORMAT(' ENTER 1 TO INPUT DATA FROM FILE TRAJ.DAT, ENTER 0
TO INPUT DATA MANUALLY',$)
ACCEPT 4, NT
IF(NT.EQ.1) GO TO 140

TYPE 3
3 FORMAT(' ENTER NUMBER OF ELECTRON TRAJECTORIES',$)
ACCEPT 4, KK
4 FORMAT(I)

DO 3 K=1, KK
5 FORMAT(' ENTER INITIAL X COORDINATE AND VELOCITY',$)
ACCEPT 2, XO(K), VXO(K)
6 FORMAT(' ENTER INITIAL Y COORDINATE AND VELOCITY',$)
ACCEPT 2, YO(K), VYO(K)

DO 32 1=1,82
7 FORMAT(' ENTER FINAL X AND Y COORDINATES',$)
ACCEPT 2, XT(K), YT(K)
8 CONTINUE
9 EQ=1.60210D-19
EM=9.1091D-31
QM=EQ/EM
AO=2*VXO(KK)*VYO(KK)/(QM*(XT(KK)-XO(KK)))

10 FORMAT(' AO=',D)

XX(1)=-DX/80
XX(2)=0.0
DO 20 1=3,82
XX(I)=XX(I-1)+DX/80
20 CONTINUE
YY(1)=-DY/20
YY(2)=0.0
DO 30 J=3,32
YY(J)=YY(J-1)+DY/20
30 CONTINUE

DO 50 1=1,82
DO 40 J=1,32
\begin{verbatim}
PHI(I,J)=AO*YY(J)

40 CONTINUE
50 CONTINUE
PAUSE
IT=0
60 DD=0.0
D=0.0
TYPE 65, IT
65 FORMAT( ' ITERATION=', 14)
IT=IT+1
DO 90 K=1, KK-1
YN(K)=0.0
KT=0
TYPE 82, XO(K), YO(K)
DT(1)=DY/(100*VY0(K))
70 KT=KT+1
CALL DVOCEL(K)
IF(Y(KT+1).LT.YN(K)) GO TO 75
YN(K)=Y(KT+1)
XN(K)=X(KT+1)
75 IF(Y(KT+1).GT.YO(K)) GO TO 70
80 TYPE 82, X(KT+1), Y(KT+1)
DDX(K)=(XT(K)-X(KT+1))/(XT(K)-XO(K))
82 FORMAT( ' X=', D12.4, ' Y=', D12.4)
DD=DD+DABS(XT(K)-X(KT+1))
IF(D.GT.DA3S(DDX(K))) GO TO 90
D=DABS(DDX(K))
XM=X(KT+1)
KM=K
DO 85 KI=1, KT+1
XS(KI)=X(KI)
YS(KI)=Y(KI)
SX(KI)=FX(KI)
SY(KI)=FY(KI)
TD(KI)=DT(KI)
85 CONTINUE
90 CONTINUE
TYPE 95, D
TYPE 95, DD
95 FORMAT(D)
IF(DD.LT.0.01*DX) GO TO 130
K=KM
180 K=K+1
IF(K.EQ.KK) GO TO 190
IF(DDX(K)*DDX(K+1).GT.0.0) GO TO 180
LP=K-KM
GO TO 200
190 LP=KK-KM
200 K=KM
210 K=K-1
IF(K.EQ.0) GO TO 220
IF((DDX(K)*DDX(K+1).GT.0.0) GO TO 210
L1=K-KM
\end{verbatim}
GO TO 230
LN=-KM
230 IF(LP.LT.-LN) GO TO 240
IF(LP.GT.-LN) GO TO 235
IF(DABS(DDX(KM+LP)).GT.DABS(DDX(KM+LN))) GO TO 240
235 L=LN
GO TO 250
240 L=LP
250 IF(L.EQ.-KM) GO TO 260
IF(L.EQ.KK-KM) GO TO 270
XL=0.5*(XN(KH+L)+XN(KH+L-1))
GO TO 300
260 XL=1.5*XN(1)-0.5*XN(2)
GO TO 300
270 XL=1.5*XN(KK-1)-0.5*XN(KK-2)
300 YL=1.4*DY
VXI1=0.0
DVXI1=0.0
VXXI=0.0
DO 98 I=1,KT
DVXI=DVXI1
DVXI1=2*(XS(I)-XL)*((YL-YS(I))/((XS(I)-XL)**2
1 + (YS(I)+YL)/(XS(I)-XL)**2
2 + (YS(I)+YL)**2)**2)
VXI=VXII
VXII=VXI+0.5*(DVXI1+DVXI)*TD(I)
VXXI=VXXI1+0.5*(DVXII+DVXI)*TD(I)
98 CONTINUE
AL=(XI-KT(KM))/(QM*VXXI)
TYPE 95,AL
IF(AL.LT.-0.1) GO TO 100
IF(AL.GT.0.1) GO TO 99
GO TO 101
99 AL=0.1
GO TO 101
100 AL=-0.1
101 TYPE 105,AL,XL,YL
105 FORMAT(‘ AL=’,D12.5,’ XL=’,D12.5,’ YL=’,D12.5)
PAUSE
DO 120 I=2,82
DO 110 J=2,32
PHI(I,J)=PHI(I,J)+AL*((YY(J)+YL)/((XX(I)-XL)**2
1 +(YY(J)+YL)/(XX(I)-XL)**2
2 + (YY(J)+YL)**2)**2)
110 CONTINUE
120 CONTINUE
KT=0
DT(I)=DY/(100*VYO(KK))
121 KT=KT+1
CALL DVOGEL(KK)
IF(Y(KT+1).GT.YO(KK)) GO TO 121
SF=(X(KT+1)-XO(KK))/(XT(KK)-XO(KK))
TYPE 122, SF
FORMAT(' SF=',D12.6)
DO 125 I=2,82
DO 124 J=2,32
PHI(I,J)=SF*PHI(I,J)
CONTINUE
GO TO 60
DO 134 I=10,70
TYPE 132, PHI(I,3),PHI(I,5),PHI(I,7),PHI(I,9),PHI(I,11),
PHI(I,13),PHI(I,15),PHI(I,17),PHI(I,19),PHI(I,21)
CONTINUE
CALL VPLOT
GO TO 160
OPEN(UNIT=1,FILE='TRAJ.DAT')
READ (1,4) KK
DO 150 K=1,KK
READ(1,2) XO(K),VXO(K)
READ(1,2) YO(K),VYO(K)
READ(1,2) XT(K),YT(K)
type 155,xO(k),xt(k)
CONTINUE
GO TO 9
STOP
END
SUBROUTINE HEWTON(X,Y,EX,EY,PSI,IK)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
DIMENSION FM(4,4,4),FMN(4,4,4),XM(4),XM1(4),YN(4),YN1(4)
FORMAT(2D12.6)
IF(X.GT.XX(82)) GO TO 220
IF(X.LT.XX(1)) GO TO 220
IF(Y.GT.YY(32)) GO TO 220
IF(Y.LT.YY(1)) GO TO 225
HX=DX/80.0
HY=DY/20.0
I=0
I=I+1
IF(X.GT.XX(I)) GO TO 10
IF(I.LE.3) GO TO 12
IB=I-2
GO TO 14
IB=2
IF(I.EQ.82) GO TO 16
IE=I+1
GO TO 18
IE=1
J=0
J=J+1
IF(Y.GT.YY(J)) GO TO 20
IF(J.LE.3) GO TO 25
JB=J-2
GO TO 30

25 JB = 2

30 IF(J.LE.32) GO TO 35

JE = J + 1

GO TO 40

35 JE = 32

40 MM = IE - IB + 1

XM(1) = 1.0

XM1(1) = 0.0

DO 70 M = 2, MM

IM = IB + M - 2

XM(N) = XM(M-1) * (X-XX(IM))

XM1(M) = 0.0

DO 60 I = 2, M

X1 = 1.0

DO 50 K = 2, M

IF(K.EQ.I) GO TO 50

X1 = X1 * (X-XX(IB+K-2))

50 CONTINUE

XM1(N) = XM1(M) + X1

60 CONTINUE

70 CONTINUE

NN = JE - JB + 1

YN(1) = 1.0

YN1(1) = 0.0

DO 100 N = 2, NN

JN = JB + N - 2

YN(N) = YN(N-1) * (Y-YY(JN))

YN1(N) = 0.0

DO 90 J = 2, N

Y1 = 1.0

DO 80 L = 2, N

IF(L.EQ.J) GO TO 80

Y1 = Y1 * (Y-YY(IB+L-2))

80 CONTINUE

YN1(N) = YN1(N) + Y1

90 CONTINUE

100 CONTINUE

DO 110 M = 1, MM

DO 105 N = 1, NN

FM(1, M, N) = PHI(IB + M - 1, JB + N - 1)

105 CONTINUE

110 CONTINUE

DO 140 L = 2, NN

DO 130 M = L, NN

DO 120 J = 1, MM

FM(L, M, N) = (FM(L-1, M, N-1) - FM(L-1, M, N)) / (HY * (1-L))

120 CONTINUE

130 CONTINUE

140 CONTINUE

DO 160 M = 1, MM

DO 150 N = 1, NN

FMN(1, M, N) = FM(N, M, N)
CONTINUE
DO 190 K=2,MM
DO 180 M=K,MM
DO 170 N=1,NN
FMN(K,M,N) = (FMN(K-1,M-1,N)-FMN(K-1,M,N))/(HX*(1-K))
CONTINUE
DO 210 N=1,NN
DO 200 M=1,MM
EX=EX-XM1(M)*YN(N)*FMN(M,M,N)
EY=EY-YN1(N)*XM(M)*FMN(M,M,N)
PSI=PSI+XM(M)*YN(N)*FMN(M,M,N)
CONTINUE
GO TO 230
IK=-1
GO TO 230
IK=2
RETURN
END
SUBROUTINE DVOGEL(K)
IMPLICIT REAL *8(A-H,0-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIM=DSQRT(DX**2+DY**2)
QM=1.76D11
IK=0
IF(KT.GT.1) GO TO 200
VX=VXO(K)
VY=VYO(K)
X(1)=XO(K)
Y(1)=YO(K)
100 CALL NEWTON(X(KT),Y(KT),EX,EY,PSI,IK)
101 FORMAT(’ EX=’,D12.4,’EY=’,D12.4)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
XH=X(KT)-(0.5*VX-0.125*AX*DT(KT))*DT(KT)
YH=Y(KT)-(0.5*VY-0.125*AY*DT(KT))*DT(KT)
CALL NEWTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY
200 XH=X(KT)+(0.5*VX+(4.0*AX-AXH)*DT(KT)/24.0)*DT(KT)
YH=Y(KT)+(0.5*VY+(4.0*AY-AYH)*DT(KT)/24.0)*DT(KT)
CALL NEWTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
\[ AXH = QM \cdot EX \]
\[ AYH = QM \cdot EY \]
\[ X(KT + 1) = X(KT) + (VX + (AX + 2.0 \cdot AXH) \cdot DT(KT) / 6.0) \cdot DT(KT) \]
\[ Y(KT + 1) = Y(KT) + (VY + (AY + 2.0 \cdot AYH) \cdot DT(KT) / 6.0) \cdot DT(KT) \]

\[ 260 \quad \text{FORMAT}(2I) \]
\[ AAX = AX \]
\[ AAY = AY \]
\[ \text{CALL NEWTON}(X(KT + 1), Y(KT + 1), EX, EY, PSI, IK) \]
\[ \text{IF}(IK \cdot NE.0) \text{ GO TO 400} \]
\[ AX = QM \cdot EX \]
\[ AY = QM \cdot EY \]
\[ A1SQ = AAX**2 + AAY**2 \]
\[ \text{IF}(A1SQ.LT.1.D-6) \text{ GO TO 300} \]
\[ \text{DELD} = \text{DSQRT}((X(KT + 1) - X(KT))**2 + (Y(KT + 1) - Y(KT))**2) \]

\[ 275 \quad \text{FORMAT}(' \text{DELD} = ', D12.6, ', \text{DIM} = ', D12.6) \]
\[ \text{IF}(\text{DELD.GT.DIM}/40) \text{ GO TO 320} \]
\[ \text{IF}(\text{DELD.LT.DIM}/120) \text{ GO TO 330} \]
\[ VX = VX + DT(KT) \cdot (AX + 4.0 \cdot AXH + AAX)/6.0 \]
\[ VY = VY + DT(KT) \cdot (AY + 4.0 \cdot AYH + AAY)/6.0 \]
\[ DT(KT + 1) = DT(KT) \]
\[ \text{GO TO 440} \]
\[ 300 \quad V1SQ = VX**2 + VY**2 \]
\[ DVSQ = A1SQ \cdot DT(KT)**2 \]
\[ \text{IF}(V1SQ.GT.DVSQ*1.D4) \text{ GO TO 280} \]
\[ IK = 1 \]
\[ \text{GO TO 410} \]
\[ 320 \quad DT(KT) = DT(KT)/1.1 \]
\[ \text{GO TO 100} \]
\[ 330 \quad DT(KT) = DT(KT)*1.1 \]
\[ \text{GO TO 100} \]
\[ 400 \quad \text{IF}(IK.EQ.-1) \text{ TYPE 420} \]
\[ \text{GO TO 440} \]
\[ 410 \quad \text{TYPE 430} \]
\[ 420 \quad \text{FORMAT(' ELECTRON HAS ESCAPED POTENTIAL REGION} \]
\[ 1 \quad \text{INCLUDED IN X-Y COORDINATE GRID')} \]
\[ 430 \quad \text{FORMAT(' PARTICLE VELOCITY AND ACCELERATION BOTH} \]
\[ 1 \quad \text{EQUAL ZERO'})} \]
\[ 440 \quad \text{RETURN} \]
\[ \text{END} \]

\[ \text{SUBROUTINE VPlot} \]
\[ \text{IMPLICIT REAL *8 (A-H,O-Z)} \]
\[ \text{COMMON/COORD/PHI(82,32),XX(32),YY(32),DX,DY} \]
\[ \text{DIMENSION X(250),Y(250)} \]
\[ \text{CALL INITT} \]
\[ \text{CALL SELINI} \]
\[ \text{CALL DWINDO(SNGL(XX(1)),SNGL(XX(82)),SNGL(YY(1)),SNGL(YY(22)))} \]
\[ \text{II=INT(SNGL(780*DY/DX))} \]
\[ \text{CALL TWINDO(0,780,0,II)} \]
\[ \text{CALL MOVEA(SNGL(XX(2)),SNGL(YY(2)))} \]
\[ \text{CALL DRAUA(SNGL(XX(82)),SNGL(YY(2)))} \]
\[ \text{PHININ}=1.0D38 \]
\[ \text{PHIMAX}=-1.0D38 \]
\[ \text{DO 10 I=2,82} \]
DO 5 J=2,22
IF(PHI(I,J).GT.PHIMIN) GO TO 2
PHIMIN=PHI(I,J)
2 IF(PHI(I,J).LT.PHIMAX) GO TO 5
PHIMAX=PHI(I,J)
5 CONTINUE
10 CONTINUE
K=0
12 K=K+1
V=PHIMIN+K*0.2
L=0
DO 20 J=2,22
DO 15 I=3,82
IF((PHI(I-1,J)-V)*(PHI(I,J)-V).GE.0.0) GO TO 15
L=L+1
X(L)=XX(I-1)+(XX(I)-XX(I-1))*(V-PHI(I-1,J))/(PHI(I,J)-
1 PHI(I-1,J))
Y(L)=YY(J)
15 CONTINUE
20 CONTINUE
DO 30 I=2,82
DO 25 J=3,22
IF((PHI(I,J-1)-V)*(PHI(I,J)-V).GE.0.0) GO TO 25
L=L+1
X(L)=XX(I)
Y(L)=YY(J-1)+(YY(J)-YY(J-1))*(V-PHI(I,J-1))/(PHI(I,J)-
1 PHI(I,J-1))
25 CONTINUE
30 CONTINUE
LMAX=L
XMIN=XX(82)
DO 35 L=1,LMAX
IF(X(L).GT.XMIN) GO TO 35
XMIN=X(L)
LMIN=L
35 CONTINUE
XS=X(1)
YS=Y(1)
X(1)=X(LMIN)
Y(1)=Y(LMIN)
X(LMIN)=XS
Y(LMIN)=YS
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
40 LL=LL+1
DSQ=1.0D38
DO 45 L=LL,LMAX
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2
IF(DSQL.GT.DSQ) GO TO 45
DSQ=DSQL
LDMIN=L
45 CONTINUE
XS=X(LL)
YS = Y(LL)
X(LL) = X(LDMIN)
Y(LL) = Y(LDMIN)
X(LDMIN) = XS
Y(LDMIN) = YS
CALL DRAUA(SNGL(X(LL)), SNGL(Y(LL)))
IF(LL.EQ.LMAX) GO TO 50
GO TO 40

50 IF(V+0.2.LE.PHIMAX) GO TO 12
CALL FINIT(800, 500)
RETURN
END
cROUTINE RTENT.F4
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/X0(50),Y0(50),VX0(50),VY0(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIMENSION XS(110),YS(110),SX(110),SY(110),TD(110),DDX(50),XM(50)

EXTERNAL TEKHAN, PLTHAN, SCPHAN
CALL INIITG

TYPE 1
1 FORMAT( 'ENTER X AND Y DIMENSIONS OF POTENTIAL REGION', $)
ACCEPT 2, DX, DY
2 FORMAT(2D)

TYPE 19
19 FORMAT( 'ENTER 1 TO INPUT DATA FROM FILE TRAJ.DAT, ENTER 0 TO INPUT DATA MANUALLY', $)
ACCEPT 4, NT
IF(NT.EQ.1) GO TO 140

TYPE 3
3 FORMAT( 'ENTER NUMBER OF ELECTRON TRAJECTORIES', $)
ACCEPT 4, KK

TYPE 5
5 FORMAT( 'ENTER INITIAL X COORDINATE AND VELOCITY', $)
ACCEPT 2, X0(K), VX0(K)

TYPE 6
6 FORMAT( 'ENTER INITIAL Y COORDINATE AND VELOCITY', $)
ACCEPT 2, Y0(K), VY0(K)

TYPE 7
7 FORMAT( 'ENTER FINAL X AND Y COORDINATES', $)
ACCEPT 2, XT(K), YT(K)

TYPE 15
15 FORMAT( 'AO=' , D)
XX(I)=-DX/80
XX(2)=0.0
DO 20 I=3,82
XX(I)=XX(I-1)+DX/80
CONTINUE

YY(I)=-DY/20
YY(2)=0.0
DO 30 J=3,32
YY(J)=YY(J-1)+DY/20
CONTINUE

DO 50 I=1,82
DO 40 J=1,32
PHI(I,J)=A0*YY(J)
40 CONTINUE
50 CONTINUE
PAUSE
IT=0
60 DD=0.0
D=0.0
TYPE 65,IT
65 FORMAT(‘ ITERATION=’,I4)
   IT=IT+1
   DO 90 K=1,KK-1
   YN(K)=0.0
   KT=0
   TYPE 82,XO(K),YO(K)
   DT(1)=DY/(100*VY0(K))
70   KT=KT+1
   CALL DVOGEL(K)
   IF(Y(KT+1).LT.YN(K)) GO TO 75
   YN(K)=Y(KT+1)
   XN(K)=X(KT+1)
   IF(Y(KT+1).GT.YO(K)) GO TO 70
75   TYPE 82,X(KT+1),Y(KT+1)
   DDX(K)=(XT(K)-X(KT+1))/DABS(XT(K)-XO(K))
   82 FORMAT(X=’,D12.4,’ Y=’,D12.4)
   DD=D+DABS(XT(K)-X(KT+1))
   IF(D.GT.DABS(DDX(K))) GO TO 90
   D=DABS(DDX(K))
   XM(K)=X(KT+1)
   KM=K
   DO 85 KI=1,KT+1
   XS(KI)=X(KI)
   YS(KI)=Y(KI)
   SX(KI)=FX(KI)
   SY(KI)=FY(KI)
   TD(KI)=TD(KI)
85   CONTINUE
90 CONTINUE
TYPE 95,D
TYPE 95,DD
95 FORMAT(D)
   IF(DD.LT.0.01*DX) GO TO 130
   k=km
180   k=k+1
   IF(k.EQ.kk) GO TO 220
   IF(ddx(k)*ddx(km).GT.0.0) GO TO 180
190   k=k+1
   IF(k.EQ.kk) GO TO 200
   IF(ddx(k)*ddx(km).GT.0.0) GO TO 200
   IF(dabs(ddx(k)).LT.dabs(ddx(km))) GO TO 200
   GO TO 190
200   lp=k-1-km
   GO TO 230
220   lp=kk
\[ k = km \]
\[ k = k - 1 \]
if \( k \equiv 0 \) go to 270
if \( ddx(k) \cdot ddx(km) \cdot gt.0.0 \) go to 240
\[ k = k - 1 \]
if \( k \equiv 0 \) go to 260
if \( ddx(k) \cdot ddx(km) \cdot gt.0.0 \) go to 260
if \( dabs(ddx(k)) \cdot lt. dabs(ddx(k+1)) \) go to 260
go to 250
\[ ln = k + l - km \]
go to 280
\[ ln = -kk \]
if \( lp \cdot ne. \cdot kk \) go to 340
if \( ln \cdot ne. \cdot kk \) go to 340
\[ DDN = 0.0 \]
DO 285 \( K = 1, KM \]
\[ DDN = DDN + DDX(K) \]
\[ CONTINUE \]
\[ DDP = 0.0 \]
DO 290 \( K = KM, KK - 1 \]
\[ DDP = DDP + DDX(K) \]
\[ CONTINUE \]
if \( dabs(DDN) \cdot GT. \cdot dabs(DDP) \) GO TO 300
\[ XL = XN(1) - 0.5*(YN(2) - XN(1)) \]
GO TO 365
\[ XL = XN(KK - 1) + 0.5*(XN(KK - 1) - XN(KK - 2)) \]
GO TO 365
\[ IF(LP.GT.-LN) \]
GO TO 350
\[ L = LP \]
GO TO 360
\[ L = LN \]
\[ XL = XN(KM) + 0.5*(XN(KM + L) - XN(KM)) \]
\[ YL = 1.0 + 0.1)*DY \]
\[ DDE = 1.0*DY \]
\[ DDE = YYI + 0.05*DY \]
\[ D1 = ((YN(KM) - YL)**2 - 3*(YN(KM) - YL)*(XN(KM) - XL)**2) / \]
\[ (XN(KM) - XL)**2 + (YN(KM) - YL)**2)**3 \]
\[ D2 = ((YN(KM) + YL)**2 - 3*(YN(KM) + YL)*(XN(KM) - XL)**2) / \]
\[ (XN(KM) - XL)**2 + (YN(KM) + YL)**2)**3 \]
\[ DDA = DDE \]
\[ DDE = DABS(D1 - D2) \]
\[ IF(DDE.LT.DDA) \]
GO TO 370
\[ VXII = 0.0 \]
\[ DVXII = 0.0 \]
\[ VXIII = 0.0 \]
DO 98 \( I = 1, KT \)
\[ DVXI = DVXII \]
\[ DVXII = 2*(XS(I) - XL)*((YL-YS(I))/((XS(I) - XL)**2 \]
\[ 1 + (YL-YS(I)**2)**2 - (YS(I) + YL)/((XS(I) - XL)**2 \]
\[ 2 + (YS(I) + YL)**2)**2 \]
\[ VXI = VXII \]
\[ VXII = VXI + 0.5*(DVXII + DVXI)*TD(I) \]
\[ VXIII = VXIII + 0.5*(VXII + VXI)*TD(I) \]
CONTINUE

AL=(XM(KM)-XT(KM))/(QM*VXXI)

TYPE 95,AL
ifi(al.lt.-0.1) go to 100
ifi(al.gt.0.1) go to 99
GOTO 101

AL=0.1
GOTO 101

AL=-0.1

GOTO 101

DO 120 I=2,82
DO 110 J=2,32
PHI(I,J)=PHI(I,J)+AL*((YY(J)+YL)/((XX(I)-XL)**2
1 +(YY(J)+YL)**2)-(YL-YY(J))/((XX(I)-XL)**2
2 +(YL-YY(J))**2))

CONTINUE

CONTINUE

KT=0

DT(I)=DY/(100*VYO(KK))

KT=KT+1

CALL DVOGEL(KK)

IF(Y(KT+1).GT.YO(KK)) GO TO 121

SF=(X(KT+1)-XO(KK))/(XT(KK)-XO(KX))

TYPE 122,SF

CONTINUE

CONTINUE

GO TO 60

DO 134 I=10,70

CONTINUE

CALL VPLOT

GO TO 160

OPEN(UNIT=1,FILE='TRAJ.DAT')

READ (1,4) KK

READ(1,2) XO(K),VXO(K)

READ(1,2) YO(K),VYO(K)

READ(1,2) XT(K),YT(K)

WRITE(1,55) X0,XT

CONTINUE

GO TO 9

STOP

END

SUBROUTINE NEWTON(X,Y,EX,EY,PSI,IK)

IMPLICIT REAL *8(A-H,O-Z)

CONTINUE

AL=(XM(KM)-XT(KM))/(QM*VXXI)

TYPE 95,AL
ifi(al.lt.-0.1) go to 100
ifi(al.gt.0.1) go to 99
GOTO 101

AL=0.1
GOTO 101

AL=-0.1

GOTO 101

DO 120 I=2,82
DO 110 J=2,32
PHI(I,J)=PHI(I,J)+AL*((YY(J)+YL)/((XX(I)-XL)**2
1 +(YY(J)+YL)**2)-(YL-YY(J))/((XX(I)-XL)**2
2 +(YL-YY(J))**2))

CONTINUE

CONTINUE

KT=0

DT(I)=DY/(100*VYO(KK))

KT=KT+1

CALL DVOGEL(KK)

IF(Y(KT+1).GT.YO(KK)) GO TO 121

SF=(X(KT+1)-XO(KK))/(XT(KK)-XO(KX))

TYPE 122,SF

CONTINUE

CONTINUE

GO TO 60

DO 134 I=10,70

CONTINUE

CALL VPLOT

GO TO 160

OPEN(UNIT=1,FILE='TRAJ.DAT')

READ (1,4) KK

READ(1,2) XO(K),VXO(K)

READ(1,2) YO(K),VYO(K)

READ(1,2) XT(K),YT(K)

WRITE(1,55) X0,XT

CONTINUE

GO TO 9

STOP

END

SUBROUTINE NEWTON(X,Y,EX,EY,PSI,IK)

IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/
PHI(82,32),XX(82),YY(32),DX,DY
DIMENSION FM(4,4,4),FMN(4,4,4),XM(4),XM1(4),YN(4),YN1(4)

FORMAT(2D12.6)
IF(X.GT.XX(82)) GO TO 220
IF(X.LT.XX(1)) GO TO 220
IF(Y.GT.YY(32)) GO TO 220
IF(Y.LT.YY(1)) GO TO 225
HX=DX/80.0
HY=DY/20.0
I=0

10 I=I+1
IF(X.GT.XX(I)) GO TO 10
IF(I.LE.3) GO TO 12
IB=I-2
GO TO 14

12 IB=2

14 IF(I.EQ.82) GO TO 16
IE=I+1
GO TO 18

16 IE=I

18 J=0

20 J=J+1
IF(Y.GT.YY(J)) GO TO 20
IF(J.LE.3) GO TO 25
JB=J-2
GO TO 30

25 JB=2

30 IF(J.EQ.32) GO TO 35
JE=J+1
GO TO 40

35 JE=J

40 MM=IE-IB+1
XM(1)=1.0
XM1(1)=0.0
DO 70 N=2,MM
IM=IB+M-2
XM(M)=XM(M-1)*(X-XX(IM))
XM1(M)=0.0
DO 60 I=2,N
XI=1.0
DO 50 K=2,N
IF(K.EQ.I) GO TO 50
XI=XI*(X-XX(IB+K-2))
CONTINUE
XM1(M)=XM1(N)+XI

60 CONTINUE

70 CONTINUE
NN=JE-JB+1
YN(1)=1.0
YN1(1)=0.0
DO 100 N=2,NN
JN=JB+N-2
YN(N)=YN(N-1)*(Y-YY(JN))
YN1(N)=0.0
DO 90 J=2,N
  Y1=1.0
  DO 80 L=2,N
    IF(L.EQ.J) GO TO 80
    Y1=Y1*(Y-YY(JB+L-2))
  80 CONTINUE
  YN1(N)=YN1(N)+Y1
90 CONTINUE
100 CONTINUE
  DO 110 M=1,MM
    DO 105 N=1,NN
      FM(1,M,N)=PHI(IB+M-1,JB+N-1)
    105 CONTINUE
110 CONTINUE
  DO 140 L=2,NN
    DO 130 N=L,NN
      DO 120 M=1,MM
        FM(L,M,N)=(FM(L-1,M,N-1)-FM(L-1,M,N))/(HY*(1-L))
      120 CONTINUE
130 CONTINUE
140 CONTINUE
  DO 160 M=1,MM
    DO 150 N=1,NN
      FMN(1,M,N)=FM(M,M,N)
    150 CONTINUE
160 CONTINUE
  DO 190 K=2,MM
    DO 180 M=K,MM
      DO 170 N=1,NN
        FMN(K,M,M)=(F1-IM(K-1,M-1,N)-F1IN(K-1,M,N))/(HX*(1-K))
      170 CONTINUE
180 CONTINUE
190 CONTINUE
  EX=0.0
  EY=0.0
  PSI=0.0
  DO 210 N=1,NN
    DO 200 M=1,MM
      EX=EX-XM1(M)*YN(N)*FMN(M,M,N)
      EY=EY-YN1(N)*XM(M)*FMN(M,M,N)
      PSI=PSI+XM(M)*Y1I(N)*FMN(M,M,N)
    200 CONTINUE
210 CONTINUE
220 CONTINUE
GO TO 230
225 IK=-1
GO TO 230
230 CONTINUE
END
SUBROUTINE DVOGEL(K)
IMPLICIT REAL *(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(32),YY(32),DX,DY
COMMON/DATA/XO(50),Y0(50),VX0(50),VY0(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIM=DSQRT(DX**2+DY**2)
QM=1.76D11
IK=0
IF(KT.GT.1) GO TO 200
VX=VX0(K)
VY=VY0(K)
X(1)=X0(K)
Y(1)=Y0(K)
100 CALL NEHTON(X(KT),Y(KT),EX,EY,PSI,IK)
101 FORMAT(' EX=',D12.4,'EY=',D12.4)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
XH=X(KT)-(0.5*VX-0.125*AX*DT(KT))*DT(KT)
YH=Y(KT)-(0.5*VY-0.125*AY*DT(KT))*DT(KT)
CALL NEHTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY
200 XH=X(KT)+(0.5*VX+(4.0*AX-AXH)*DT(KT)/24.0)*DT(KT)
YH=Y(KT)+(0.5*VY+(4.0*AY-AYH)*DT(KT)/24.0)*DT(KT)
CALL NEHTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY
AAX=AX
AAY=AY
CALL NEHTON(X(KT+1),Y(KT+1),EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
A1SQ=AAX**2+AAY**2
IF(A1SQ.LT.1.D-6) GO TO 300
DELD=DSQRT((X(KT+1)-X(KT))**2+(Y(KT+1)-Y(KT))**2)
275 FORMAT(’ DELD=',D12.6,’ DIM=',D12.6)
IF(DELD.GT.DIM/40) GO TO 320
IF(DELD.LT.DIM/120) GO TO 330
280 VX=VX+DT(KT)*(AX+4.0*AAX+AAX)/6.0
VY=VY+DT(KT)*(AY+4.0*AYH+AAY)/6.0
DT(KT+1)=DT(KT)
GO TO 440
300 V1SQ=VX**2+VY**2
DVSQ=A1SQ*DT(KT)**2
IF(V1SQ.GT.DVSQ*1.D4) GO TO 280
IK=1
GO TO 410
320 DT(KT)=DT(KT)/1.1
GO TO 100
137

330  DT(KT)=DT(KT)*1.1

GO TO 100

400  IF(IK.EQ.-1) TYPE 420

GO TO 440

410  TYPE 430

420  FORMAT(' ELECTRON HAS ESCAPED POTENTIAL REGION
1 INCLUDED IN X-Y COORDINATE GRID')

430  FORMAT(' PARTICLE VELOCITY AND ACCELERATION BOTH
1 EQUAL ZERO')

440  RETURN

END

SUBROUTINE VPLOT
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
DIMENSION X(250),Y(250)
CALL INITT
CALL SELINI
CALL DWINDO(SNGL(XX(1)),SNGL(XX(82)),SNGL(YY(1)),SNGL(YY(22)))
II=INT(SNGL(780*DY/DX))
CALL TWINDO(0,780,0,II)
CALL MOVEA(SNGL(XX(2)),SNGL(YY(2)))
CALL DRAUA(SNGL(XX(82)),SNGL(YY(2)))
PHIMIN=1.0D38
PHIMAX=-1.0D38
DO 10  I=2,82
     DO 5  J=2,22
        IF(PIIII(I,J).GT.PHIMIN) GO TO 2
        PHIMIN=PHI(I,J)
2   IF(PHI(I,J).LT.PHIHAX) CO TO 5
4   PHIMAX=PHI(I,J)
5   CONTINUE
10  CONTINUE
K=0
12  K=K+1
V=PHIMIN+K*.2
L=0
DO 20  J=2,22
     DO 15  I=3,82
        IF((PHI(I-1,J)-V)*(PHI(I,J)-V).GE.0.0) GO TO 15
        L=L+1
15   X(L)=XX(I-1)+(XX(I)-XX(I-1))*(V-PHI(I,J-1))/(PHI(I,J)-
1   PHI(I-1,J))
Y(L)=YY(J)
20  CONTINUE
DO 30  I=2,82
     DO 25  J=3,22
        IF((PHI(I,J-1)-V)*(PHI(I,J)-V).GE.0.0) GO TO 25
        L=L+1
25   X(L)=XX(I)
     Y(L)=YY(J-1)+(YY(J)-YY(J-1))*(V-PHI(I,J-1))/(PHI(I,J)-
1   PHI(I,J-1))
25  CONTINUE
CONTINUE
LMAX=L
XMIN=XX(82)
DO 35 L=1,LMAX
IF(X(L).GT.XMIN) GO TO 35
XMIN=X(L)
LMIN=L
35 CONTINUE
XS=X(1)
YS=Y(1)
X(1)=X(LMIN)
Y(1)=Y(LMIN)
X(LMIN)=XS
Y(LMIN)=YS
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
40 LL=LL+1
DSQ=1.0D38
DO 45 L=LL,LMAX
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2
IF(DSQL.GT.DSQ) GO TO 45
DSQ=DSQL
LDHIN=L
45 CONTINUE
XS=X(LL)
YS=Y(LL)
X(LL)=X(LDMIN)
Y(LL)=Y(LDMIN)
X(LDMIN)=XS
Y(LDMIN)=YS
CALL DRAWA(SNGL(X(LL)),SNGL(Y(LL)))
IF(LL.EQ.LMAX) GO TO 50
GO TO 40
50 IF(V+0.2.LE.PHIMAX) GO TO 12
CALL FINITT(800,500)
RETURN
END
APPENDIX E

Fortran Routines for Quadrupole Synthesis of Solutions to The Model Problem
cROUTINE QPOLE.F4

IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIMENSION XS(110),YS(110),TD(110),DDX(50)

EXTERNAL TEKHAN,PLTHAN,SCPFAII
CALL INITTG

TYPE 1
1 FORMAT( ' ENTER X AND Y DIMENSIONS OF POTENTIAL REGION', $)
ACCEPT 2,DX,DY

TYPE 19
19 FORMAT( ' ENTER 1 TO INPUT DATA FROM FILE TRAJ.DAT, ENTER 0
1 TO INPUT DATA MANUALLY', $)
ACCEPT 4, NT
IF(NT.EQ.1) GO TO 140

TYPE 3
3 FORMAT( ' ENTER NUMBER OF ELECTRON TRAJECTORIES', $
ACCEPT 4, KK

TYPE 5
5 FORMAT( ' ENTER INITIAL X COORDINATE AND VELOCITY',$
ACCEPT 2,XO(K),VXO(K)

TYPE 6
6 FORMAT( ' ENTER INITIAL Y COORDINATE AND VELOCITY',$
ACCEPT 2,YO(K),VYO(K)

TYPE 7
7 FORMAT( ' ENTER FINAL X AND Y COORDINATES',$
ACCEPT 2,XT(K),YT(K)

AQ=2*VXO(KK)*VYO(KK)/(QM*(XT(KK)-XO(KK)))

TYPE 15,A0

TYPE 15,A0=' ',D
XX(1)=DX/80
XX(2)=0.0
DO 20 I=3,32
XX(I)=XX(I-1)+DX/80

CONTINUE
YY(1)=DY/20
YY(2)=0.0
DO 30 J=3,32
YY(J)=YY(J-1)+DY/20

CONTINUE
DO 50 I=1,32
DO 40 J=1,32
PHI(I,J)=AO*YY(J)

CONTINUE

CONTINUE

PAUSE

IT=0

DD=0.0

D=0.0

TYPE 65,IT

FORMAT(’ ITERATION=’,I4)

IT=IT+1

DO 90 K=1,XX-1

YN(K)=0.0

KT=0

TYPE 82,X0(K),YO(K)

DT(1)=DY/(100*VY0(K))

KT=KT+1

CALL DVOGEL(K)

IF(Y(KT+1).LT.YN(K)) GO TO 75

YN(K)=Y(KT+1)

XN(K)=X(KT+1)

75 IF(Y(KT+1).GT.YO(K)) GO TO 70

TYPE 82,X(KT+1),Y(KT+1)

DDX(K)=(XT(K)-X(KT+1))/(XT(K)-X0(K))

82 FORMAT(’ X=’,D12.4,’ Y=’,D12.4)

DD=DD+DABS(XT(K)-X(KT+1))

IF(D.GT.DABS(DDX(K))) GO TO 90

D=DABS(DDX(K))

XM=X(KT+1)

E=K

DO 85 KI=1,KT+1

XS(KI)=X(KI)

YS(KI)=Y(KI)

TD(KI)=DT(KI)

85 CONTINUE

CONTINUE

CONTINUE

CONTINUE

TYPE 95,D

TYPE 95,DD

95 FORMAT(D)

IF(DD.LT.0.01*DX) GO TO 126

XL=XN(KM)

YL=1.4*DY

VXI=0.0

DVXI=0.0

VXXI=0.0

DO 98 I=1,KT

DVXI=DVXI

DVXII=((YS(I)+YL)**3-3*(YS(I)+YL)*(XS(I)-XL)**2)/(XS(I)-XL)**3

1 +((YS(I)-YL)**3-3*(YS(I)-YL)*(XS(I)-XL)**2)/(XS(I)-XL)**3

2 )/(XS(I)-XL)**2+((YS(I)-YL)**2)

VXI=VXII

VXII=VXI+0.5*(DVXI+DVXI)*TD(I)

VXXI=VXII+0.5*(VXII+VXI)*TD(I)
CONTINUE
AL=(XM-XT(KK))/(QM*VXXI)
TYPE 95, AL
IF(AL.LT.-0.1) GO TO 100
IF(AL.GT.0.1) GO TO 99.
GO TO 101
AL=0.1
GO TO 101
AL=-0.1
TYPE 105, AL, XL, YL
DO 120 I=2,82
DO 110 J=2,32
110 CONTINUE
105 FOR11ATC AL=’,D12.5,’ XL=’,D12.5,’ YL=’,D12.5)
120 CONTINUE
KT=0
DT(1)=DY/(100*VYO(KK))
TYPE 82, XO(KK), YO(KK)
KT=KT+1
CALL DVOGEL(KK)
IF(Y(KT+1)>YO(KK)) GO TO 121
TYPE 82, X(KT+1), Y(KT+1)
SF=(X(KT+1)-XO(KK))/(XT(KK)-XO(KK))
TYPE 122, SF
121 FOR11ATC SF=’,D12.6)
DO 125 I=2,82
DO 124 J=2,32
PHI(I,J)=SF*PHI(I,J)
124 CONTINUE
125 CONTINUE
GO TO 60
126 DO 128 I=10,70
TYPE 129, PHI(I,3), PHI(I,5), PHI(I,7), PHI(I,9), PHI(I,11)
1, PHI(I,13), PHI(I,15), PHI(I,17), PHI(I,19), PHI(I,21)
128 CONTINUE
129 FORMAT(10D12.4)
130 CALL VPLOT
GO TO 160
140 OPEN(UNIT=1,FILE=’TRAJ.DAT’)
READ (1,4) KK
DO 150 K=1,KK
READ(1,2) XO(K),VXO(K)
READ(1,2) YO(K),VYO(K)
READ(1,2) XT(K), YT(K)
type 155, XO(K), XT(K)
155 FORMAT(’ xO=’,D12.6,’ xt=’,D12.6)
150 CONTINUE
GO TO 9
160 STOP
END
SUBROUTINE NEWTON(X, Y, EX, EY, PSI, IK)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32), XX(82), YY(32), DX, DY
DIMENSION FM(4,4,4), FMN(4,4,4), XM(4), XM1(4), YN(4), YN1(4)
5 FORMAT(2D12.6)
IF(X.GT.XX(82)) GO TO 220
IF(X.LT.XX(1)) GO TO 220
IF(Y.GT.YY(22)) GO TO 220
IF(Y.LT.YY(1)) GO TO 225
HX=DX/80.0
HY=DY/20.0
I=0
10 I=I+1
IF(X.GT.XX(I)) GO TO 10
IF(I.LE.3) GO TO 12
IB=I-2
GO TO 14
12 IB=2
14 IF(I.EQ.82) GO TO 16
IE=I+1
GO TO 18
16 IE=I
18 J=0
20 J=J+1
IF(Y.GT.YY(J)) GO TO 20
IF(J.LE.3) GO TO 25
JB=J-2
GO TO 30
25 JB=2
30 IF(J.EQ.32) GO TO 35
JE=J+1
GO TO 40
35 JE=J
40 NM=IE-IB+1
XM(1)=1.0
YM1(1)=0.0
DO 70 N=2,NM
IN=IB+M-2
XY(N)=XM(M-1)*(X-XX(IN))
YM1(N)=0.0
DO 60 I=2,M
X1=1.0
DO 50 K=2,M
IF(K.EQ.I) GO TO 50
X1=X1*(X-XX(IN+K-2))
50 CONTINUE
XM1(N)=XM1(M)+X1
60 CONTINUE
70 CONTINUE
N=JE-JB+1
YN(1)=1.0
YN1(1)=0.0
DO 100 N=2,NM
SUBROUTINE DVOGEL(K)

JN=JB+N-2
YN(JN)=YN(JN-1)*(Y-YY(JN))
YN1(N)=0.0
DO 90 J=2,N
Y1=1.0
DO 80 L=2,N
IF(L.EQ.J) GO TO 80
Y1=Y1*(Y-YY(JB+L-2))

CONTINUE

YN1(N)=YN1(N)+Y1
CONTINUE

DO 110 M=1,MM
DO 105 N=1,NN
FM(1,M,N)=PHI(IB+M-1,JB+N-1)
CONTINUE

110 CONTINUE

DO 140 L=2,NN
DO 130 N=L,NN
DO 120 M=1,MM
FM(L,M,N)=(FM(L-1,M,N-1)-FM(L-1,M,M))/(HY*(1-L))
CONTINUE

120 CONTINUE

130 CONTINUE

140 CONTINUE

FMN(1,M,N)=FM(N,M,N)
CONTINUE

150 CONTINUE

DO 190 K=2,MM
DO 180 M=K,MM
DO 170 N=1,HN
RM(N,M,N)=(FMN(K-1,M-1,N)-FMN(K-1,M,N))/(HX*(1-K))
CONTINUE

170 CONTINUE

180 CONTINUE

190 CONTINUE

EX=0.0
EY=0.0
PSI=0.0
DO 210 N=1,NN
DO 200 M=1,NN
EX=EX-XM(M)*YN(N)*FMN(M,M,N)
EY=EY-YN1(N)*XM(M)*FMN(M,M,N)
PSI=PSI+XM(M)*YN(N)*FMN(M,M,N)
CONTINUE

200 CONTINUE

210 CONTINUE
GO TO 230

220 IK=-1
GO TO 230

225 IK=2

230 RETURN

END

SUBROUTINE DVOGEL(K)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(250),KT
DIM=DSQRT(DX**2+DY**2)
QM=1.76D11
IK=0
IF(KT.GT.1) GO TO 200
VX=VXO(K)
VY=VYO(K)
X(1)=XO(K)
Y(1)=YO(K)
100 CALL NEWTON(X(KT),Y(KT),EX,EY,PSI,IK)
101 FORMAT(‘EX=’,D12.4,’EY=’,D12.4)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
XH=X(KT)-(0.5*VX-0.125*AX*DT(KT))*DT(KT)
YH=Y(KT)-(0.5*VY-0.125*AY*DT(KT))*DT(KT)
CALL NEWTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY
200 XI=X(KT)+(0.5*VX+(4.0*AX-AXH)*DT(KT)/24.0)*DT(KT)
YH=Y(KT)+(0.5*VY+(4.0*AY-AYH)*DT(KT)/24.0)*DT(KT)
CALL NEWTON(XH,YH,EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY
X(KT+1)=X(KT)+(VX+(AX+2.0*AXH)*DT(KT)/6.0)*DT(KT)
Y(KT+1)=Y(KT)+(VY+(AY+2.0*AYH)*DT(KT)/6.0)*DT(KT)
260 FORMAT(2I)
AAX=AX
AAY=AY
CALL NEWTON(X(KT+1),Y(KT+1),EX,EY,PSI,IK)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
A1SQ=AAX**2+AAY**2
IF(A1SQ.LT.1.D-6) GO TO 300
DELD=DSQRT((X(KT+1)-X(KT))**2+(Y(KT+1)-Y(KT))**2)
275 FORMAT(‘DELD=’,D12.6,’DIM=’,D12.6)
IF(DELD.GT.DIM/40) GO TO 320
IF(DELD.LT.DIM/120) GO TO 330
280 VX=VX+DT(KT)*(AX+4.0*AXH+AAX)/6.0
VY=VY+DT(KT)*(AY+4.0*AYH+AAY)/6.0
DT(KT+1)=DT(KT)
GO TO 440
300 V1SQ=VX**2+VY**2
DVSQ=A1SQ**2+V1SQ
IF(V1SQ.GT.DVSQ*1.D4) GO TO 280
IK=1
GO TO 410
320 DT(KT) = DT(KT) / 1.1
GO TO 100
330 DT(KT) = DT(KT) * 1.1
GO TO 100
400 IF(IK.EQ.-1) TYPE 420
GO TO 440
410 TYPE 430
420 FORMAT(' ELECTRON HAS ESCAPED POTENTIAL REGION
1 INCLUDED IN X-Y COORDINATE GRID')
430 FORMAT(' PARTICLE VELOCITY AND ACCELERATION BOTH
1 EQUAL ZERO')
440 RETURN
END
SUBROUTINE VPLOT
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
DIMENSION X(250),Y(250)
CALL INITT
CALL SELINI
CALL DWINDO(XX(1),XX(82),YY(1),YY(22))
II=INT(780*DY/DX)
CALL TWINDO(0,780,0,II)
CALL MOVEA(XX(2),YY(2))
CALL DRAWA(XX(82),YY(2))
PHIMIN=1.0D38
PHIMAX=-1.0D38
DO 10 I=2,82
DO 5 J=2,22
IF(PHI(I,J).GT.PHIMIN) GO TO 2
PHIMIN=PHI(I,J)
2 IF(PHI(I,J).LT.PHIMAX) GO TO 5
PHIMAX=PHI(I,J)
5 CONTINUE
10 CONTINUE
K=0
12 K=K+1
V=PHIMIN+K*0.2
L=0
DO 20 J=2,22
DO 15 I=3,82
IF((PHI(I-1,J)-V)*(PHI(I,J)-V).GE.0.0) GO TO 15
L=L+1
X(L)=XX(I-1)+(XX(I)-XX(I-1))*(V-PHI(I-1,J))/(PHI(I,J)-
1 PHI(I-1,J))
Y(L)=YY(J)
15 CONTINUE
20 CONTINUE
DO 30 I=2,82
DO 25 J=3,22
IF((PHI(I,J-1)-V)*(PHI(I,J)-V).GE.0.0) GO TO 25
L=L+1
X(L)=XX(I)
Y(L)=YY(J-1)+(YY(J)-YY(J-1))*(V-PHI(I,J-1))/(PHI(I,J)-
PHI(I,J-1)
CONTINUE
CONTINUE
LMAX=L
XMIN=XX(82)
DO 35 L=1,LMAX
IF(X(L).GT.XMIN) GO TO 35
XMIN=X(L)
LMIN=L
CONTINUE
XS=X(1)
YS=Y(1)
X(1)=X(LMIN)
Y(1)=Y(LMIN)
X(LMIN)=XS
Y(LMIN)=YS
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
LL=LL+1
DSQ=1.0D38
DO 45 L=LL,LMAX
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2
IF(DSQL.GT.DSQ) GO TO 45
DSQ=DSQL
LDMIN=L
CONTINUE
XS=X(LL)
YS=Y(LL)
X(LL)=X(LDMIN)
Y(LL)=Y(LDMIN)
X(LDMIN)=XS
Y(LDMIN)=YS
CALL DRAWA(SNGL(X(LL)),SNGL(Y(LL)))
IF(LL.EQ.LMAX) GO TO 50
GO TO 40
IF(V+0.2.LE.PHIMAX) GO TO 12
CALL FINITT(800,500)
RETURN
END
cROUTINE QPOLES.F4

IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(110),KI(50)
COMMON/COEFF/VI(5,6),AL(5)
DIMENSION XN(6)
EXTERNAL TEKHan,PLTHAN,SCPHAN
CALL INITTG

TYPE 1
1 FORMAT(' ENTER X AND Y DIMENSIONS OF POTENTIAL REGION',$)
ACCEPT 2,DX,DY
2 FORMAT(2D)

TYPE 19
19 FORMAT(' ENTER 1 TO INPUT DATA FROM FILE TRAJ.DAT, ENTER 0
TO INPUT DATA MANUALLY',$)
ACCEPT 4, NT
IF(NT.EQ.1) GO TO 140

TYPE 3
3 FORMAT(' ENTER NUMBER OF ELECTRON TRAJECTORIES',$)
ACCEPT 4, KK

TYPE 5
5 FORMAT(' ENTER INITIAL X COORDINATE AND VELOCITY',$)
ACCEPT 2, XO(K), VXO(K)

TYPE 6
6 FORMAT(' ENTER INITIAL Y COORDINATE AND VELOCITY',$)
ACCEPT 2, YO(K), VYO(K)

TYPE 7
7 FORMAT(' ENTER FINAL X AND Y COORDINATES',$)
ACCEPT 2, XT(K), YT(K)

670 FORMAT(I2, 2D12.6)

8 CONTINUE

9 EQ=1.60210D-19
EM=9.1091D-31
QM=EQ/EM
AO=2*VXO(KK)*VYO(KK)/(QM*(XT(KK)-XO(KK)))
15 FORMAT(' AO=',D)

XX(1)=-DX/80
XX(2)=0.0
DO 20 I=3,82
XX(I)=XX(I-1)+DX/80
20 CONTINUE

YY(1)=-DY/20
YY(2)=0.0
DO 30 J=3,32
YY(J)=YY(J-1)+DY/20
30 CONTINUE

32 DO 50 I=1,82
DO 40 J=1,32
50 CONTINUE

END
\[ \text{PHI}(I,J) = A_0 \times YY(J) \]

40 CONTINUE

50 CONTINUE

IT = 0
ITT = 0
DD = 1.0D30

69 D = DD
DD = 0.0
RD = 0.0
DO 90 K = 1, KK
KT = 0
YMAX = 0.0
DT(1) = DY/(100*VYO(K))

70 KT = KT + 1
CALL DVOGEL(K, KT, KIK)
IF(Y(KT + 1) .LT. YMAX) GO TO 75
YMAX = Y(KT + 1)
XN(K) = X(KT + 1)

75 IF(Y(KT + 1) .GT. YO(K)) GO TO 70
TYPE 80, X(KT + 1), Y(KT + 1)
IF(K.EQ.KK) GO TO 90

80 FORMAT( ' X=', D12.6, ' Y=', D12.6)
DD = DD + DABS(XT(K) - X(KT + 1))
RD = RD + (X(KT + 1) - XO(K))/(XT(K) - XO(K))

85 FORMAT( ' IT=', I3, ' DD=', D12.6, ' RD=', D12.6)
YL = 1.4*DY
DO 87 L = 1, KK - 1
XL = XN(L)
VII = 0.0
VI(K, L) = 0.0
DO 86 I = 1, KT
DVII = DVII
DVII = ((Y(I) + YL)**3 - 3*(Y(I) + YL)*(X(I) - XL)**2)/(X(I) - XL)**2
1 + (Y(I) + YL)**2)*3 + ((Y(I) - YL)**3 - 3*(Y(I) - YL)*(X(I) - XL)**2)/(X(I) - XL)**2
2 + (Y(I) - YL)**3
VVI = VI(K, L) = VI(K, L) + 0.5*(VII + VVI)*DT(I)

86 CONTINUE
VI(K, L) = VI(K, L)*QM

87 CONTINUE

88 FORMAT( ' K=', I3, ' L=', I3, ' VI(K, L)=', D12.6)
VI(K, KK) = X(KT + 1) - XT(K)

TYPE 89, K, VI(K, KK)

89 FORMAT( ' K=', I3, ' DISCREPANCY=', D12.6)

90 CONTINUE

TYPE 85, IT, DD, RD
IF(D.LT.DD) ITT = ITT + 1
IF(ITT.GT.3) GO TO 125
IF(DD.LT.01*DX) GO TO 126
ERR = DABS((X(KT+1)-XT(KK))/XT(KK))
IF(ERR.LT.0.03) GO TO 905
SF = (X(KT+1)-XO(KK))/(XT(KK)-XO(KK))
TYPE 901, SF
901 FORMAT(' SCALE FACTOR=', D12.6)
DO 903 I = 2, 82
DO 902 J = 2, 32
PHI(I,J) = SF*PHI(I,J)
902 CONTINUE
903 CONTINUE
GO TO 69
905 CALL GAUSS(KK-1)
AA = .0
DO 906 K = 1, KK-1
IF(DABS(AL(K)).LT.AA) GO TO 906
AA = DABS(AL(K))
906 CONTINUE
DO 908 K = 1, KK-1
IF(AA.LT.0.1) GO TO 908
AL(K) = 0.1*AL(K)/AA
908 CONTINUE
IT = IT+1
YL = 1.4*DY
DO 95 L = 1, KK-1
XL = XN(L)
TYPE 91, AL(L), XL, YL
91 FORMAT(' AL=', D12.6, ' XL=', D12.6, ' YL=', D12.6)
DO 93 I = 2, 82
DO 92 J = 2, 32
PHI(I,J) = PHI(I,J) + AL(L)*((XX(I)-XL)*((YY(J)+YL)/((XX(I)-XL)**2
1 + (YY(J)+YL)**2)**2 + (YY(J)-YL)/((XX(I)-XL)**2
2 + (YY(J)-YL)**2)**2)
92 CONTINUE
93 CONTINUE
95 CONTINUE
GO TO 69
124 FORMAT(' FOUR DEPARTURES FROM MONOTONE DECREASING
1 SUM OF ABSOLUTE DISCREPANCIES HAVE OCCURED')
125 TYPE 124
126 DO 128 I = 10, 70
129 CONTINUE
128 CONTINUE
129 FORMAT(10D12.6)
130 CALL VPLOT
GO TO 160
140 OPEN(UNIT=1, FILE='TRAJ.DAT')
READ (1,4) KK
DO 150 K = 1, KK
READ(1,2) XO(K), VXO(K)
READ(1,2) YO(K), VYO(K)
READ(1,2) XT(K), YT(K)
type 155, xo(k), xt(k)
SUBROUTINE GAUSS(N)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COEFF/VI(5,6),AL(5)
DO 20 K=1,N-1
B=VI(K,K)
DO 5 J=K,N+1
VI(K,J)=VI(K,J)/B
5 CONTINUE
DO 15 I=K+1,N
B=VI(I,K)
DO 10 J=K,N+1
VI(I,J)=VI(I,J)-B*VI(K,J)
10 CONTINUE
15 CONTINUE
20 CONTINUE
AL(N)=VI(N,N+1)/VI(N,N)
K=N
25 K=K-1
AL(K)=VI(K,N+1)
DO 30 J=K+1,N
AL(K)=AL(K)-VI(K,J)*AL(J)
30 CONTINUE
IF(K.GT.1) GO TO 25
RETURN
END
SUBROUTINE NEWTON(X,Y,EX,EY,PSI,IK)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
DIMENSION FM(4,4,4),FMN(4,4,4),XM(4),XM1(4),YN(4),YN1(4)
FORMAT(2D12.6)
IF(X.GT.XX(82)) GO TO 220
IF(X.LT.XX(1)) GO TO 220
IF(Y.GT.YY(32)) GO TO 225
IF(Y.LT.YY(1)) GO TO 220
HX=DX/80.0
HY=DY/20.0
I=0
10 I=I+1
IF(X.GT.XX(I)) GO TO 10
IF(I.LE.3) GO TO 12
IB=I-2
GO TO 14
12 IB=2
14 IF(I.EQ.82) GO TO 16
IE=I+1
GO TO 18
16 IE=I
18 J=0
J=J+1
IF(Y.GT.YY(J)) GO TO 20
IF(J.LE.3) GO TO 25
JB=J-2
GO TO 30
JB=2
IF(J.EQ.32) GO TO 35
JE=J+1
GO TO 40
JE=J
II=IE-IB+1
XM(1)=1.0
XM1(1)=0.0
DO 70 M=2,II
IM=IB+M-2
XM(M)=XM(M-1)*(X-XX(IN))
XM1(M)=0.0
DO 60 I=2,M
XI=1.0
DO 50 K=2,M
IF(K.EQ.I) GO TO 50
XI=X1*(X-XX(IB+K-2))
CONTINUE
XM1(M)=XM1(M)+XI
CONTINUE
II=JE-JB+1
YN(1)=1.0
YN1(1)=0.0
DO 100 N=2,II
JN=JB+N-2
YN(N)=YN(N-1)*(Y-YY(JN))
YN1(N)=0.0
DO 90 J=2,N
YL=1.0
DO 80 L=2,N
IF(L.EQ.J) GO TO 80
YL=YL*(Y-YY(JB+L-2))
CONTINUE
YN1(N)=YN1(N)+YL
CONTINUE
DO 110 M=1,II
DO 105 N=1,II
FM(1,M,N)=PHI(IB+M-1,JB+N-1)
CONTINUE
105 CONTINUE
DO 140 L=2,II
DO 130 N=L,II
DO 120 M=1,II
FM(L,M,N)=(FM(L-1,M,N-1)-FM(L-1,M,N))/(HY*(1-L))
CONTINUE
120 CONTINUE
CONTINUE
130 CONTINUE
CONTINUE
DO 160 M=1,MM
DO 150 N=1,NN
FMN(1,M,N)=FM(N,M,N)
150 CONTINUE
CONTINUE
DO 190 K=2,MM
DO 180 M=K,MM
DO 170 N=1,NN
FMN(K,M,N)=(FMN(K-1,M-1,N)-FMN(K-1,M,N))/(HX*(1-K))
170 CONTINUE
CONTINUE
CONTINUE
CONTINUE
EX=0.0
EY=0.0
PSI=0.0
DO 210 N=1,NN
DO 200 M=1,MM
EX=EX-XM1(M)*YN(N)*FMN(M,M,N)
EY=EY-YN1(N)*XM(M)*FMN(M,M,N)
PSI=PSI+XM(M)*Y1T(N)*FMN(M,M,N)
200 CONTINUE
210 CONTINUE
GO TO 230
220 IK=-1
GO TO 230
225 IK=2
230 RETURN
END
SUBROUTINE DVOGEL(K,KT,KIK)
IMPLICIT REAL *8(A-H,O-Z)
COMMON/COORD/PHI(82,32),XX(82),YY(32),DX,DY
COMMON/DATA/XO(50),YO(50),VXO(50),VYO(50),XT(50),YT(50)
COMMON/SAVE/X(110),Y(110),DT(110),KI(50)
DIM=DSQRT(DX**2+DY**2)
QM=1.76D11
IK=0
IF(KT.GT.1) GO TO 200
VX=VXO(K)
VY=VYO(K)
X(1)=XO(K)
Y(1)=YO(K)
100 CALL NEWTON(X(KT),Y(KT),EX,EY,PSI,IK)
101 FORMAT(' EX=',D12.4,'EY=',D12.4)
IF(IK.NE.0) GO TO 400
AX=QM*EX
AY=QM*EY
XH=X(KT)-(0.5*VX-0.125*AX*DT(KT))*DT(KT)
YH=Y(KT)-(0.5*VY-0.125*AY*DT(KT))*DT(KT)
CALL NEWTON(XH,YH,EX,EY,PS1,IK)
IF(IK.NE.0) GO TO 400
AXH=QM*EX
AYH=QM*EY


200 \( \text{XH} = \text{X}(\text{KT}) + (0.5 \times \text{VX} + (4.0 \times \text{AX} - \text{AXH}) \times \text{DT}(\text{KT})/24.0) \times \text{DT}(\text{KT}) \)

\( \text{YH} = \text{Y}(\text{KT}) + (0.5 \times \text{VY} + (4.0 \times \text{AY} - \text{AYH}) \times \text{DT}(\text{KT})/24.0) \times \text{DT}(\text{KT}) \)

CALL \text{NEWTON}(\text{XH}, \text{YH}, \text{EX}, \text{EY}, \text{PSI}, \text{IK})

IF(\text{IK}.NE.0) GO TO 400

\( \text{AXH} = \text{QM} \times \text{EX} \)

\( \text{AYH} = \text{QM} \times \text{EY} \)

\( \text{X}(\text{KT}+1) = \text{X}(\text{KT}) + (\text{VX} + (\text{AX} + 2.0 \times \text{AXH}) \times \text{DT}(\text{KT})/6.0) \times \text{DT}(\text{KT}) \)

\( \text{Y}(\text{KT}+1) = \text{Y}(\text{KT}) + (\text{VY} + (\text{AY} + 2.0 \times \text{AYH}) \times \text{DT}(\text{KT})/6.0) \times \text{DT}(\text{KT}) \)

260 FORMAT(2I)

\( \text{AAX} = \text{AX} \)

\( \text{AAY} = \text{AY} \)

CALL \text{NEWTON}(\text{X}(\text{KT}+1), \text{Y}(\text{KT}+1), \text{EX}, \text{EY}, \text{PSI}, \text{IK})

IF(\text{IK}.NE.0) GO TO 400

\( \text{AX} = \text{QM} \times \text{EX} \)

\( \text{AY} = \text{QM} \times \text{EY} \)

\( \text{A1SQ} = \text{AAX}^2 + \text{AAY}^2 \)

IF(\text{A1SQ}.LT.1.D-6) GO TO 300

\( \text{DELD} = \text{DSQRT}((\text{X}(\text{KT}+1) - \text{X}(\text{KT}))^2 + (\text{Y}(\text{KT}+1) - \text{Y}(\text{KT}))^2) \)

275 FORMAT(\' \text{DELD} = ', D12.6, \' KT=', 13)

IF(\text{DELD}.GT.\text{DIM}/40) GO TO 320

IF(\text{DELD}.LT.\text{DIM}/120) GO TO 330

280 \( \text{VX} = \text{VX} + \text{DT}(\text{KT}) \times (\text{AX} + 4.0 \times \text{AXH} + \text{AAX})/6.0 \)

\( \text{VY} = \text{VY} + \text{DT}(\text{KT}) \times (\text{AY} + 4.0 \times \text{AYH} + \text{AAY})/6.0 \)

\( \text{DT}(\text{KT}+1) = \text{DT}(\text{KT}) \)

GO TO 440

300 \( \text{VISQ} = \text{VX}^2 + \text{VY}^2 \)

\( \text{DVSQ} = \text{A1SQ} \times \text{DT}(\text{KT})^2 \)

IF(\text{VISQ}.GT.\text{DVSQ} \times 1.D4) GO TO 280

\( \text{IK} = 1 \)

GO TO 410

320 \( \text{DT}(\text{KT}) = \text{DT}(\text{KT})/1.1 \)

GO TO 100

330 \( \text{DT}(\text{KT}) = \text{DT}(\text{KT}) \times 1.1 \)

GO TO 100

400 IF(\text{IK}.EQ.-1) TYPE 420

IF(\text{IK}.EQ.2) GO TO 402

GO TO 440

402 \( \text{KIK} = 1 \)

GO TO 440

410 TYPE 430

420 FORMAT(\' \text{ELECTRON HAS ESCAPED COORDINATE GRID}\')

425 FORMAT(\' \text{ELECTRON HAS ESCAPED POTENTIAL REGION}\')

430 FORMAT(\' \text{PARTICLE VELOCITY AND ACCELERATION BOTH}\n\text{EQUAL ZERO}\')

440 RETURN
END

SUBROUTINE \text{VPLOT}
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/COORD/\text{PHI}(82,32),XX(82),YY(32),\text{DX},\text{DY}
DIMENSION X(250),Y(250)
CALL \text{INITT}
CALL SELINI
CALL DWINDO(SNGL(XX(1)),SNGL(XX(82)),SNGL(YY(1)),SNGL(YY(22)))
II=INT(SNGL(780*DY/DX))
CALL TWINDO(0,780,0,II)
CALL MOVEA(SNGL(XX(2)),SNGL(YY(2)))
CALL DRAWA(SNGL(XX(82)),SNGL(YY(2)))
PHIMIN=1.0D38
PHIMAX=-1.0D38
DO 10 I=2,82
   DO 5 J=2,22
      IF(PHI(I,J).GT.PHIMIN) GO TO 2
      PHIMIN=PHI(I,J)
  2   IF(PHI(I,J).LT.PHIMAX) GO TO 5
      PHIMAX=PHI(I,J)
  5 CONTINUE
10 CONTINUE

K=0

12  K=K+1
    V=PHIMIN+K*0.2
  L=0
    DO 20 J=2,22
       DO 15 I=3,82
          IF((PHI(I-1,J)-V)*(PHI(I,J)-V).GE.0.0) GO TO 15
          L=L+1
    1      X(L)=XX(I-1)+(XX(I)-XX(I-1))*(V-PHI(I-1,J))/(PHI(I,J)-
    1      PHI(I-1,J))
     Y(L)=YY(J)
   15 CONTINUE

20 CONTINUE
    DO 30 I=2,82
       DO 25 J=3,22
          IF((PHI(I,J-1)-V)*(PHI(I,J)-V).GE.0.0) GO TO 25
          L=L+1
    1      X(L)=XX(I)
     Y(L)=YY(J-1)+(YY(J)-YY(J-1))*(V-PHI(I,J-1))/(PHI(I,J)-
    1      PHI(I,J-1))
   25 CONTINUE

30 CONTINUE

LMAX=L

XMIN=XX(82)
    DO 35 L=1,LMAX
       IF(X(L).GT.XMIN) GO TO 35
    XMIN=X(L)
       LMIN=L
   35 CONTINUE

XS=X(1)
YS=Y(1)
X(1)=X(LMIN)
Y(1)=Y(LMIN)
X(LMIN)=XS
Y(LMIN)=YS
CALL MOVEA(SNGL(X(1)),SNGL(Y(1)))
LL=1
LL=LL+1  
DSQ=1.0D38  
DO 45 L=LL,LMAX  
DSQL=(X(L)-X(LL-1))**2+(Y(L)-Y(LL-1))**2  
IF(DSQL.GT.DSQ) GO TO 45  
DSQ=DSQL  
LDNMN=L  
45 CONTINUE  
XS=X(LL)  
YS=Y(LL)  
X(LL)=X(LDNMN)  
Y(LL)=Y(LDNMN)  
X(LDNMN)=XS  
Y(LDNMN)=YS  
CALL DRAUA(SNGL(X(LL)),SNGL(Y(LL)))  
IF(LL.EQ.LMAX) GO TO 50  
GO TO 40  
50 IF(V+0.2.LE.PHIMAX) GO TO 12  
CALL FINITT(800,500)  
RETURN  
END