NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS
FOR LAMINAR, TRANSONIC FLOWS

BY

LOUIE TURNER, III

A Dissertation
Submitted to the Faculty of
Mississippi State University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
in the College of Engineering

Mississippi State, Mississippi
May 1979
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By
Louie Turner, III

Professor of Aerospace
Engineering (Chairman of
Committee and Dissertation
Director)

Professor of Aerospace
Engineering (Member of
Committee)

Professor of Electrical
Engineering (Member of
Committee)

Associate Professor of
Mechanical Engineering
(Member of Committee)

Associate Professor of
Computer Science (Member of
Committee)

Professor and Head of the
Department of Aerospace
Engineering

Dean of the College of
Engineering

Dean of the Graduate School

May, 1979
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L.T.

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ABSTRACT

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Abstract

An implicit finite difference solution of the Navier-Stokes equations was investigated. Time histories of the transonic laminar flow development about a circular cylinder and a NACA-0018 airfoil were obtained. Reynolds numbers ranged from those corresponding to purely laminar flow to Reynolds numbers corresponding to a significant region of turbulence in the boundary layer. Body thermal conditions of an adiabatic wall and a specified body temperature were considered.

Versatility in treating arbitrary bodies was incorporated in the solution approach by using numerically generated, body-fitted coordinate transformations. Arbitrarily shaped computational boundaries in the physical field were mapped into rectangular boundaries in the transformed field. Difference equations were obtained from the transformed differential equations and their boundary conditions.

Solution of the simultaneous difference equations for the dependent variables at each time step was obtained using an accelerated Gauss-Seidel iterative scheme. The solution was started from physically realistic initial conditions. Acceleration parameters for the iteration
were determined by numerical experimentation and, for simplicity, were maintained constant over the field and for all time steps. Computational results are presented in the form of velocity vector fields, Mach number contours, aerodynamic coefficients, heat transfer rates at the body surface and body temperature distributions.

Modeling the flow in the body leading edge region presented the most difficult challenge for the numerical method. Iterative convergence was slow there, particularly for the density solution, and a spatially oscillating flow solution was obtained. Modifying the iterative scheme and increasing the time difference order in the body continuity equation reduced the computer time requirements for time step iteration. Explicit diffusion and a filter were evaluated for controlling spatial oscillations in the solutions; a flux corrected transport filter worked best. In addition, available analytical tools were employed to describe the functional dependence of the iterative convergence rate and the oscillatory solutions.

Truncation analyses of first and second derivative difference approximations were performed. These analyses resulted in general criteria for numerically generated coordinates so that flow near a body is more accurately represented.
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\( \tau \) Unit Tangent Vector

\( T_c \) Specified Body Temperature

\( T_o \) Total Temperature

\( T_{i,j}^n \) Principal Truncation Error

\( u \) Non-dimensional Velocity Component Parallel to x Axis

\( u_m \) Boundary Layer Velocity for Attraction Model

\( v \) Non-dimensional Velocity Component Parallel to y Axis

\( \gamma \) Non-dimensional Velocity Vector with Magnitude V

\( x \) Physical Coordinate

\( x_d \) Abscissa of Wake Coordinate System Downstream Boundary

\( y \) Physical Coordinate

\( \alpha \) Coordinate Transformation Parameter, Equation (3.21)

\( \beta \) Coordinate Transformation Parameter, Equation (3.21)

\( \gamma \) Coordinate Transformation Parameter, Equation (3.21)

\( \Gamma_1, \Gamma_2, \ldots \) Physical Plane Contours

\( \Gamma^*_1, \Gamma^*_2, \ldots \) Transformed Plane Contours

\( \delta \) Boundary Layer Thickness

\( \varepsilon_f \) Vector of Convergence Tolerances for Iterative Solution

\( \eta \) Transformed Coordinate which Increases Outward from the Body

\( \eta_\delta \) \( \eta \) Value at Boundary Layer Edge

\( \theta \) Ratio of Specific Heat at Constant Pressure to Specific Heat at Constant Volume

\( \kappa \) Thermal Conductivity

\( \mu \) Non-dimensional Shear Viscosity

\( \mu' \) Non-dimensional Bulk Viscosity

\( \nu \) Non-dimensional Kinematic Viscosity
$\xi$  Transformed Coordinate which Increases Along the Body

$\xi_{T1}$  Minimum $\xi$ Coordinate for the Trailing Edge

$\xi_{T2}$  Maximum $\xi$ Coordinate for the Trailing Edge

$\rho$  Non-dimensional Density

$\rho_o$  Stagnation Density

$\rho_J$  Jacobi Iteration Spectral Radius

$\rho_{AGS}^*$  Accelerated Gauss-Sidel Iteration Spectral Radius Using Optimum Acceleration Parameter

$\sigma$  Non-dimensional Stress Tensor

$\sigma_{xx}, \sigma_{yy}$  Normal Stress Elements of the Stress Tensor

$\tau_{xy}, \tau_{yx}$  Shear Stress Elements of the Stress Tensor

$\phi$  Chord Line Angle of Attack

$\omega_f$  Vector of Acceleration Parameters

$\nabla$  Gradient Vector Operator, Equation (B.4)

$(\_)$  Dimensional Variable

$(\hat{\_})$  Transformed Velocities

$(\_)^*$  Value of Dependent Variable Computed from its Difference Equation

**Superscripts**

n  Discrete Time Index

k  Iteration Counter

$'$  Point of Taylor Series Expansion

**Subscripts**

B  Body Surface in the Physical Field

i,j  Point in the Transform Plane

k  Index for $\Phi$traction Function Summation, Equation (3.38)
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I. INTRODUCTION

This report details the development of an implicit finite difference solution of the Navier-Stokes equations for transonic flow. This research focused on the laminar flow about two-dimensional bodies. With appropriate modifications, the method is extendable to turbulent flows and to three-dimensional geometries.

Complexity of the governing equations has dictated the development of numerical methods to compliment experimental techniques in fluid dynamic design and analysis. Numerical methods become especially attractive in the transonic flow regime because even the potential flow equations are complicated by nonlinearity.

At the present time solving the complete Navier-Stokes equations is demanding on computational resources, regardless of the technique employed. However, it is the only method which inherently describes the elliptic and hyperbolic nature of the potential field and, at the same time, the complex shock wave-boundary layer interaction that is common in transonic flows.

The most widely used Navier-Stokes solver is the explicit finite difference method described in Reference [8] and a more efficient variation reported in Reference [9]. This improved variation was used to determine the transonic flow field about airfoils, and it is discussed further in the next chapter. Nevertheless, explicit methods have stability and computational accuracy limitations that limit the allowable time step in transonic flow computations. These limitations have a direct impact on computer time required for a solution.
Based on linear analyses, implicit methods do not have the time step limitations of explicit methods. Consequently, they offer a potential reduction in computational time for transonic flow solutions. However, realization of these potential computer time reductions can be difficult. Since implicit methods require the simultaneous solution of large systems of difference equations, an efficient algorithm is necessary for solving the system of equations at each time step.

In this research, an accelerated Gauss-Seidel method was used to solve the system of difference equations. This algorithm had proved effective in solving the incompressible viscous flow equations as reported in Reference [13]. In the results presented here, the computations were started from what was thought to be a physically realistic initial solution and continued toward steady state. Hence, the time history of flow development as well as the steady state solution is available.

The capability to treat arbitrary bodies was incorporated into the technique by using the numerically generated grids described in Reference [14]. A coordinate transformation from the physical field to a transformed plane is computed, and the lines of constant transformed coordinates form the grid system on the physical field for the finite difference solution. These numerically generated coordinates have the characteristic that each boundary of the physical field is a constant coordinate line in the transformed plane. The other transformed coordinate can be distributed as desired along the physical field boundaries. Consequently, obtaining the difference equations from the Navier-Stokes equations in the transformed field permitted
accurate and flexible discretation of boundary conditions. Only a minimum of changes to the Navier-Stokes solution algorithm is required for analyzing various bodies.

A significant portion of the research was directed along two lines. Much emphasis was placed on obtaining physically realistic solutions of density, velocity, and total energy fields throughout the time histories. Consequently, for the higher Reynolds number solutions, explicit diffusion terms and numerical filters were considered to retain the qualitative nature of the flow field and not to reduce the computer time required. Quantitative verification of the method was reserved for future computations incorporating turbulence models since turbulent boundary layers significantly effect the aerodynamic forces in realistic transonic airfoil flows. Much emphasis was also placed on applying limited theoretical methods in evaluating the iterative convergence characteristics and the difference equation solutions. This effort was felt to be beneficial to present and future research since it provided qualitative relationships for the effect of the grid system and time step on the iterative convergence and the numerical accuracy.

In the following chapters, a review of related research is presented, the governing equations in the physical and transformed planes are discussed, and the finite difference solution is described. Results consisting of velocity fields, Mach contours, and aerodynamic coefficients are followed by the conclusions.
II. LITERATURE SURVEY

In this chapter previous approaches for obtaining viscous solutions of transonic flow about airfoils are discussed. Only a brief overview is provided for these solutions.

The most widely referenced techniques for solving the compressible Navier-Stokes equations are variations of the method first reported in Reference [7]. In this introductory paper, the method consisted of an explicit, two step procedure with each step based on the complete differential equations. By assuming local linearization of the potential flow terms, the method was shown to be second order accurate in time and stable if the Courant-Fredrick-Levy (CFL) condition is satisfied. The CFL condition requires the numerical domain of dependence to exceed the physical domain of dependence of the hyperbolic equations. This condition becomes very restrictive on the computational time step because of small mesh spacings near the body.

A variation of the solution approach of Reference [7] was reported in Reference [8]. In Reference [8] time split equations were solved. The time split procedure entailed successively applying the difference operator approximations of basic equation derivatives to previous time level data. Difference operators for x derivatives were grouped together and difference operators for y derivatives were grouped together. The sets of operator computations were performed according to the method of Reference [7]. Each operator was limited by the CFL condition; however, only the part of the computations with the smallest permissible time step were repeated successively. Thus, computational
time was reduced since the number of arithmetic operations was less than when the complete equation operators were computed successively.

Further development of the time splitting approach led to the algorithm referred to as the rapid solver in Reference [9]. Difference approximations were split into a parabolic operator which accounted for viscous terms and a hyperbolic operator that accounted for the potential flow terms. The hyperbolic operator predicted the potential flow field using characteristic relations, and the parabolic terms were solved using a tridiagonal inversion routine. Stability restrictions were significantly less stringent than the CFL condition.

The rapid solver was used in Reference [4]. Solutions were presented for a shock free airfoil and a NACA 0012 airfoil. Both solutions included turbulence models. There was no indication of the computational time required for these calculations.

The work of Reference [12] is mentioned since the approach was somewhat similar to the present method. Numerically generated meshes were used, and the governing equations were solved in the transformed plane. However, some viscous terms were dropped from the governing equations. The implicit approximate factorization solution described in Reference [1] was used to solve the difference equations.

Various other techniques for solving the compressible Navier-Stokes equations have appeared in the literature, e.g., Reference [11]. However, they are not discussed further because they were not used for transonic solutions, or they did not address flows past arbitrarily shaped boundaries.

In remaining chapters, the aspects of this research are discussed.
III. GOVERNING PARTIAL DIFFERENTIAL EQUATIONS

This approach to solving the Navier-Stokes equations utilized a transformed coordinate plane as discussed in Chapter I. In this chapter, the governing partial differential equations for laminar, transonic flow in the physical and transformed planes are presented. The procedure for generating the transformed plane is also discussed in the last section. The finite difference solution of the transformed partial differential equations is presented in the next chapter.

This solution technique was used to solve the Navier-Stokes equations for a two-dimensional body immersed in a stream. Both physical and transformed coordinate systems were inertial, and the transformed coordinate system was independent of time. Dependent variables were density (ρ), two velocity components, (u,v) and total energy per unit volume (E). The total energy is defined in (3.1), where ε denotes the specific internal energy.

\[ \bar{E} = \bar{\rho} \epsilon + \bar{\rho} (\bar{u}^2 + \bar{v}^2) / 2 \]  

(3.1a)

Overbars indicate dimensional variables.

A. Problem Description in the Physical Plane

The flow equations in the physical field are discussed in this section. These equations, including the initial and boundary conditions, are presented for a cartesian coordinate system.

Governing equations were the continuity, two momentum equations, and the energy equation; they are presented in vector form as Equations (3.2). A calorically perfect fluid with constant specific
heats and constant Prandtl number was assumed. Fourier's law of heat conduction was used to describe the heat transfer rate in the fluid. All variables were non-dimensionalized.

Dependent variables were non-dimensionalized using freestream conditions. Velocity components \((u,v)\) in the coordinate directions \((x,y)\), respectively, were non-dimensionalized by freestream velocity, \(V_\infty\). Freestream density was used to non-dimensionalize the field density, \(\rho\). Using the product of \(\rho_\infty\) and the freestream static enthalpy, \(h_\infty\), to non-dimensionalize total energy per unit volume gave

\[
E = \rho e + (\theta - 1)M_\infty^2 \rho (u^2 + v^2)/2
\]

where \(e\) equals \(\bar{e}/h_\infty\).

The spatial coordinates \(x\) and \(y\) were non-dimensionalized by the body chord, \(c\). Time, \(t\), was non-dimensionalized with respect to \(c/V_\infty\).

Additional equations to describe the fluid are given as (3.3) - (3.5). The viscous stresses, non-dimensionalized with respect to \(\rho_\infty V_\infty^2\), are defined by (3.3). Viscosities were non-dimensionalized by the freestream viscosity, \(\nu_\infty\). Non-dimensionalizing pressure by \(\rho_\infty h_\infty\) resulted in the equation of state denoted as Equation (3.4). The viscosity was described by Sutherland's law (3.5) as a function of non-dimensional static enthalpy, i.e., \(\bar{h}/h_\infty\). Bulk viscosity, \(\mu^*\), was assumed to be zero.

Parameters resulting from the non-dimensionalization were \(\theta\), the ratio of specific heats; \(M_\infty\), the freestream Mach number; and the Reynolds number \((R)\), based on freestream conditions and the body chord. Another parameter, the Prandtl number \((Pr)\), was introduced when the
thermal conductivity ($\kappa$) was written as $\mu c_p/Pr$, $c_p$ being the specific heat at constant pressure.

In the remainder of this section, the appropriate initial and boundary conditions for solving Equations (3.1) - (3.5) are discussed. Boundary conditions are discussed first.

On the body surface, velocity and thermal boundary conditions were specified. The velocity components were forced to satisfy the no-slip condition (3.6). Two thermal conditions were considered in the course of this research. Requiring the wall temperature to be a constant, $T_e$, was the simplest (3.7a). An alternative thermal condition was the specification of an adiabatic wall (3.7b). There was no boundary condition on density. These boundary conditions for a body at rest are listed on a subsequent page. The coordinates of points on the body are denoted $(x_s, y_s)$, and $\hat{n}_s$ is a unit vector normal to the body surface.

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 - \sigma_{xx} \\ \rho v u - \tau_{xy} \\ E u - (\theta - 1) \kappa^2 \left[ u \sigma_{xx} + v \sigma_{xy} \right] - \frac{\theta}{Pr \cdot R} \left[ \mu e_x \right] \end{bmatrix} = 0 \\
+ \frac{\partial}{\partial y} \begin{bmatrix} \rho v \\ \rho uv - \tau_{yx} \\ \rho v^2 - \sigma_{yy} \\ E v - (\theta - 1) \kappa^2 \left[ u \sigma_{yx} + v \sigma_{yy} \right] - \frac{\theta}{Pr \cdot R} \left[ \mu e_y \right] \end{bmatrix} = 0
\end{align*}
\]
\[\sigma_{xx} = -\frac{\sigma_{xx}}{(9-1)H^2} + \frac{1}{R} \left( \mu - \frac{2}{3} \mu \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\mu}{R} \left( \frac{2 \partial u}{\partial y} \right) \]  \hspace{1cm} (3.3a)

\[\sigma_{yy} = -\frac{\sigma_{yy}}{(9-1)H^2} + \frac{1}{R} \left( \mu - \frac{2}{3} \mu \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\mu}{R} \left( \frac{2 \partial v}{\partial y} \right) \]  \hspace{1cm} (3.3b)

\[\tau_{xy} = \tau_{yx} = \frac{\mu}{R} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]  \hspace{1cm} (3.3c)

\[p = (9-1)\rho e \]  \hspace{1cm} (3.4)

\[\mu = (h)^{3/2} \left[ (1 + h_1) / (h + h_1) \right] \]  \hspace{1cm} (3.5)

**Body Surface Boundary Conditions**

Velocity:

\[u(x_s, y_s, t) = 0 \]  \hspace{1cm} (3.6a)

\[v(x_s, y_s, t) = 0 \]  \hspace{1cm} (3.6b)

Thermal:

\[T(x_s, y_s, t) = T_c \]  \hspace{1cm} (3.7a)

or

\[\frac{\mu^6}{Pr \cdot R} \left[ e \frac{x}{x_s} + e \frac{y}{y_s} \right] \cdot \hat{n}_s = 0 \]  \hspace{1cm} (3.7b)

**Far Field Boundary Conditions**

Velocity:

\[u(x_\infty, y_\infty, t) = \cos \phi \]  \hspace{1cm} (3.8a)

\[v(x_\infty, y_\infty, t) = \sin \phi \]  \hspace{1cm} (3.8b)

Density:

\[\rho(x_\infty, y_\infty, t) = 1 \]  \hspace{1cm} (3.9)
Total Energy:

\[ E(x_\infty, y_\infty, t) = \frac{1}{\theta} + \frac{(\theta-1)M_\infty^2}{2} \]  \hspace{1cm} (3.10)

Initial Conditions:

\[ u(x,y,o) = u_0(x,y) \] \hspace{1cm} (3.11a)

\[ v(x,y,o) = v_0(x,y) \] \hspace{1cm} (3.11b)

\[ \rho(x,y,o) = \rho_0(x,y) \] \hspace{1cm} (3.11c)

\[ E(x,y,o) = E_0(x,y) \] \hspace{1cm} (3.11d)

Far from the body, boundary conditions were specified for all dependent variables. Velocity components, density and temperature were required to equal their reference values. Consequently, it was possible to determine a constant value for total energy from (3.1b). These boundary conditions are displayed as Equations (3.8) - (3.10); \((x_\infty, y_\infty)\) denote points far from the body, and \(\theta\) is the chord line angle of attack.

Initial values for the velocity, density, and total energy fields were required to define the transient flow problem. The steady state, potential flow solution for each of the variables was chosen as the initial conditions (3.11). It was felt that this initial condition would give flow time histories that were physically realistic soon after starting the solution.

Far-field boundary conditions and initial conditions have been given for the rigorous problem formulation. However, in actual implementation of the difference solutions, these conditions were modified somewhat. These modifications are discussed in subsequent parts of the report.
The continuum problem in the physical plane has been defined in this section. In the next section, the problem definition in the transformed plane is presented.

B. Problem Description in the Transformed Plane

The partial differential equations and boundary conditions are presented for a body-fitted coordinate system. These coordinate systems are characterized by having transformed coordinate lines coincide with the body contour and with the far-field boundaries of the physical field. It is also required that the transformed coordinates \( \xi = g(x,y) \) and \( \eta = h(x,y) \) have an inverse, i.e., \( x = \bar{g}(\xi,\eta) \) and \( y = \bar{h}(\xi,\eta) \). Actual generation of the transformations is discussed in the next section.

Derivatives of dependent variables with respect to \( x \) and \( y \), in terms of \( \xi \) and \( \eta \) derivatives have been presented in Reference [14]. These transformations used in the course of this research are presented in Appendix A. By means of the first derivative transformations, (3.12) - (3.14), and the Equation (3.15),

\[
\frac{\partial f}{\partial x} \bigg|_{y,t} = \frac{y (\frac{\partial f}{\partial \xi})_{\eta,t} - y (\frac{\partial f}{\partial \eta})_{\xi,t}}{x_{\xi}y - y_{\xi}x} \tag{3.12}
\]

\[
\frac{\partial f}{\partial y} \bigg|_{x,t} = \frac{x (\frac{\partial f}{\partial \eta})_{\xi,t} - x (\frac{\partial f}{\partial \xi})_{\eta,t}}{x_{\xi}y - y_{\xi}x} \tag{3.13}
\]

\[
\frac{\partial f}{\partial t} \bigg|_{x,y} = \frac{\partial f}{\partial t} \bigg|_{\xi,\eta} \tag{3.14}
\]
The flow equations (3.2) were transformed to Equations (3.17). The coefficient $D$ in those equations is the parameter $(\theta - 1)\kappa_2$, and $\hat{u}$ and $\hat{v}$ are defined as

\[
\hat{u} = u_y - v_x
\]
\[
\hat{v} = v_x - u_y
\]

Although the stresses, e.g., $\sigma_{xx}$, and the specific internal energy derivatives have the same notation as in Equations (3.2), they are understood to be functions of $\xi$ and $\eta$ derivatives of dependent variables in Equations (3.17). Terms in Equations (3.17) containing the stresses and $e$ derivatives transformed to the expressions given by (3.18) - (3.20). The subscripted $c$'s are continuous functions of $\xi$ and $\eta$, not constants. The notation for the $c$ array was chosen to be compatible with the finite difference computer code. For that reason, the list of definitions is complete, although there are missing subscripts.

Thus, the continuity, momentum and energy equations in the transformed plane are given by (3.16) - (3.20) along with the definitions for the $c$ array. The equation of state (3.4) and the viscosity law (3.5) are unchanged.

Equations (3.17) are in conservative form. The area flux (per unit length of $\xi$) normal to a line of constant $\eta$ is given by $\hat{v}$. Likewise, $\hat{u}$ is the flux normal to a line of constant $\xi$. The $x$ component
of force per unit area (in the transformed plane) with a normal to a line of constant \( \xi \) is given by \( \sigma_{xx} \eta - \tau_{xy} \eta \). Other terms can be similarly identified.

\[
\begin{bmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta} \\
\end{bmatrix}
\begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\end{bmatrix}
= \begin{bmatrix}
\rho \hat{u} \\
\rho u \hat{u} + \tau_{yx} \eta - \sigma_{xx} \eta \\
\rho u \hat{v} + \sigma_{yy} \eta - \tau_{xy} \eta \\
E \hat{u} + D[\tau_{yx} \eta - \sigma_{xx} \eta]v - \frac{\theta u}{Pr + R} [y_e \eta - x_e \eta] \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\rho \hat{v} \\
\rho u \hat{v} + \sigma_{xx} \xi - \tau_{yx} \xi \\
\rho u \hat{v} + \tau_{xy} \xi - \sigma_{yy} \xi \\
E \hat{v} + D[\sigma_{xx} \xi - \tau_{yy} \xi]u \\
\end{bmatrix}
= 0
\]

\[
\tau_{yx} \eta - \sigma_{xx} \eta = \frac{py \eta}{D} - \frac{\mu}{R} [c_{9 \xi} \eta - c_{13 \xi} \eta - c_{16 \xi} \eta + c_{18 \xi} \eta] \quad (3.18a)
\]

\[
\sigma_{xx} \xi - \tau_{yx} \xi = -\frac{py \xi}{D} + \frac{\mu}{R} [c_{13 \xi} \eta - c_{11 \xi} \eta - c_{17 \xi} \eta + c_{15 \xi} \eta] \quad (3.18b)
\]

\[
\sigma_{yy} \eta - \tau_{xy} \eta = -\frac{px \eta}{D} + \frac{\mu}{R} [c_{16 \xi} \eta - c_{17 \xi} \eta - c_{10 \xi} \eta + c_{14 \xi} \eta] \quad (3.18c)
\]

\[
\tau_{xy} \xi - \sigma_{yy} \xi = -\frac{px \xi}{D} + \frac{\mu}{R} [c_{18 \xi} \eta - c_{15 \xi} \eta - c_{14 \xi} \eta + c_{12 \xi} \eta] \quad (3.18d)
\]
\[
(\tau_{yx}x_\eta - \sigma_{xx}y_\eta)u + (\sigma_{yy}x_\eta - \tau_{xy}y_\eta)v = \frac{\rho u}{D} - \frac{\mu}{R} \left( c_{9}u - c_{16}v \right)u_\xi
\]
\[
+ (c_{9}v - c_{13}u)u_\eta + (c_{18}u - c_{14}v)v_\eta
\]
\[
+ (c_{10}v - c_{15}u)v_\xi \] (3.19a)

\[
(\sigma_{xx}x_\xi - \tau_{yx}x_\xi)u + (\tau_{xy}y_\xi - \sigma_{yy}x_\xi)v = \frac{\rho v}{D} + \frac{\mu}{R} \left( c_{13}u - c_{18}v \right)u_\xi
\]
\[
+ (c_{15}v - c_{11}u)u_\eta + (c_{15}u - c_{12}v)v_\eta
\]
\[
+ (c_{14}v - c_{17}u)v_\xi \] (3.19b)

\[
e_{x}^y_\eta - e_{y}^x_\eta = c_{2}e_\xi - c_{3}e_\eta
\]
\[
e_{y}^x_\xi - e_{x}^y_\xi = c_{4}e_\eta - c_{3}e_\xi
\] (3.20a)

Coefficient Array:
\[
c_{2} = \alpha/J \quad c_{14} = (3\beta + x_{\xi}x_{\eta})/3J
\]
\[
c_{3} = \beta/J \quad c_{15} = x_{\xi}y_{\xi}/3J
\]
\[
c_{4} = \gamma/J \quad c_{16} = x_{\eta}y_{\eta}/3J
\]
\[
c_{9} = (3\alpha + y_{\eta}^2)/3J \quad c_{17} = (x_{\xi}y_{\eta} + 2J)/3J
\]
\[
c_{10} = (3\alpha + x_{\eta}^2)/3J \quad c_{18} = (x_{\eta}y_{\xi} - 2J)/3J
\]
\[
c_{11} = (3\gamma + y_{\xi}^2)/3J \quad \alpha = x_{\eta}^2 + y_{\eta}^2
\]
\[
c_{12} = (3\beta + x_{\xi}^2)/3J \quad \beta = x_{\xi}x_{\eta} + y_{\xi}y_{\eta}
\]
\[
c_{13} = (3\beta + y_{\xi}y_{\eta})/3J \quad \gamma = x_{\xi}^2 + y_{\xi}^2
\]
\[
J = x_{\xi}y_{\eta} - y_{\xi}x_{\eta}
\] (3.21)
Transformation of the physical plane boundary conditions was dependent on the manner in which the outer boundary of the physical field was transformed. Two general types of transformed coordinate systems were used in this research. The physical field outer boundary mapped onto the line of minimum $\xi$, the line of maximum $\eta$, and the line of maximum $\xi$ for the wake transformation (Figure 1). In the wrap around transformation (Figure 2), the physical field outer boundary was transformed as the line of maximum $\eta$. Transformed physical plane boundary conditions for these two coordinate system types are discussed below.

In the discussion of boundary conditions, $I$ and $I$ denote the minimum and maximum values of $\xi$ in the transformed region; $I$ and $J$ have similar connotations for $\eta$. The minimum and maximum values of $\xi$ on the body are denoted by $\xi_{T1}$ and $\xi_{T2}$, respectively.

All of the transformed boundary conditions are shown on the following pages as Equations (3.22) - (3.29). Since the coordinate transformation technique described in the next section does not have the capability of transforming an infinite field into a finite one, the far-field boundary conditions are applied at finite distances from the airfoil.

The rectangular transformed field was obtained by cutting the doubly-connected physical field. This cut was made to coincide with a line of constant $\eta$ (wake coordinate system) or a line of constant $\xi$ (wrap-around coordinate system); the resulting boundaries of the transformed field were termed reentrant segments. Conditions along the reentrant segments of the wake coordinate system were that all dependent variables were even functions of $(\xi, \eta)$ about the midpoint of $\Gamma^w_1$. Along the reentrant segments of the wrap-around coordinate
system, all dependent variables were required to be periodic in \( \xi \) with period \((I-1)\).

Transformed initial conditions, given as Equations (3.30), complete the formulation of the continuous problem in both planes. In the following section, generation of coordinate transformations is discussed.

Transformed Body Surface Boundary Conditions

Velocity:

\[
\begin{align*}
  u(\xi,1,t) &= 0 \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \\
  v(\xi,1,t) &= 0 \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2}
\end{align*}
\]

(3.22a) \hspace{1cm} (3.22b)

Thermal:

\[
\begin{align*}
  T(\xi,1,t) &= T_c \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \\
  \text{or} \\
  \frac{\mu_0}{Pr \cdot R} \frac{1}{\gamma} [c_4 e_\eta - c_3 e_\xi] &= 0 \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2}
\end{align*}
\]

(3.23a) \hspace{1cm} (3.23b)

Wake Type Transformed Far Field Boundary Conditions

Velocity:

\[
\begin{align*}
  u(\xi,\eta,t) &= \cos \phi \quad \text{for} \quad \eta = J, 1 \leq \xi \leq I \\
  v(\xi,\eta,t) &= \sin \phi \quad \text{for} \quad 1 \leq \eta \leq J, \xi = 1, I
\end{align*}
\]

(3.24a) \hspace{1cm} (3.24b)

Density:

\[
\begin{align*}
  \rho(\xi,\eta,t) &= 1.0 \quad \text{for} \quad \eta = J, 1 \leq \xi \leq I \\
  &\quad \text{or} \quad 1 \leq \eta \leq J, \xi = 1, I
\end{align*}
\]

(3.25)
Total Energy:
\[ E(\xi,\eta,t) = \frac{1}{6} + \frac{(\theta-1)}{2} \frac{M_\infty^2}{2} \quad \text{for} \quad \begin{cases} \eta = J, \quad 1 \leq \xi \leq J \\ 1 \leq \eta \leq \xi = 1, I \end{cases} \] (3.26)

Wrap Around Type Transformed Far Field Boundary Conditions

Velocity:
\[ u(\xi,J,t) = \cos \phi \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \] (3.27a)
\[ v(\xi,J,t) = \sin \phi \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \] (3.27b)

Density:
\[ \rho(\xi,J,t) = 1.0 \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \] (3.28)

Total Energy:
\[ E(\xi,J,t) = \frac{1}{6} + \frac{(\theta-1)}{2} \frac{M_\infty^2}{2} \quad \text{for} \quad \xi_{T1} \leq \xi \leq \xi_{T2} \] (3.29)

Initial Conditions:
\[ u(\xi,\eta,0) = u_\infty (\xi,\eta,t \to \infty) \]
\[ v(\xi,\eta,0) = v_\infty (\xi,\eta,t \to \infty) \]
\[ \rho(\xi,\eta,0) = \rho_\infty (\xi,\eta,t \to \infty) \]
\[ E(\xi,\eta,0) = E_\infty (\xi,\eta,t \to \infty) \] (3.30)

C. Coordinate System Generation

In the previous section the existence of a boundary-fitted coordinate transformation was assumed, and the governing flow equations were transformed to the new coordinate plane. This section discusses the procedure for generating the boundary-fitted coordinates; an extensive presentation is available in Reference [14].
The method is described as it was used to generate the transformed coordinate plane for the wake type region of Figure 1. One boundary of the physical Region D was the airfoil contour, curve \( \Gamma_1 \). Other physical boundaries of the Region D were the curves \( \Gamma_2 \) and \( \Gamma_3 \). To transform this doubly-connected region, a cut was made from the airfoil trailing edge to the downstream boundary. Consequently, two additional boundaries, \( \Gamma_4 \) and \( \Gamma_5 \), of D were introduced. This region was transformed to Region \( D^* \) in Figure 1; curve \( \Gamma_1 \) was mapped into \( \Gamma_1^* \), etc.

With appropriate boundary conditions and certain restrictions on \( P \) and \( Q \), the \( \xi \) and \( \eta \) fields described by the elliptic system

\[
\begin{align*}
\xi_{xx} + \xi_{yy} &= P(\xi, \eta) \quad (3.31a) \\
\eta_{xx} + \eta_{yy} &= Q(\xi, \eta) \quad (3.31b)
\end{align*}
\]

form a grid on Region D that is advantageous for finite difference computations. These fields have the essential features that \( \xi \) and \( \eta \) increase in a monotone manner over D and that there is a one-to-one correspondence between \((x, y)\) pairs and \((\xi, \eta)\) pairs. The desirable feature that one grid coordinate line coincides with each boundary curve of D is realized through the boundary conditions for \((3.31)\).

Since the transformed flow equations contain terms such as \( x_\xi \), \( x_\eta \), etc., and since analytical solutions of \((3.31)\) are difficult for arbitrary bodies, the Equations \((3.31)\) were transformed to

\[
\begin{align*}
ax_\xi + 2\beta x_\eta + \gamma x_\eta &= - J^2 [x_\xi P + x_\eta Q] \quad (3.32a) \\
ay_\xi + 2\beta y_\eta + \gamma y_\eta &= - J^2 [y_\xi P + y_\eta Q] \quad (3.32b)
\end{align*}
\]
and solved numerically. The boundary conditions used for the wake coordinate system were

\[ \Gamma_1: (x, y) = (x_s, y_s) \quad \text{for} \quad \eta = 1, \xi_{T1} \leq \xi \leq \xi_{T2} \]

\[ \Gamma_2: (x, y) = (x_o, y_o) \quad \text{for} \quad \eta = J, 1 \leq \xi \leq I \]

\[ \Gamma_3, \xi: x = x_d \quad \text{for} \quad 1 \leq \eta \leq J, \xi = 1 \]

\[ y_\xi = 0 \quad (3.33) \]

\[ \Gamma_3, \eta: x = x_d \quad \text{for} \quad 1 \leq \eta \leq J, \xi = I \]

\[ y_\xi = 0 \]

\[ \Gamma_4, \Gamma_5: x, y \text{ even functions about midpoint of } \Gamma_1^* \]

\((x_o, y_o)\) denote outer boundary coordinates; \(x_d\) denotes the location of the downstream boundary.

Previous experience showed that attraction was needed to bring coordinate lines into the boundary layer and to the trailing edge point. The attraction function used for the wake coordinate system was derived in Reference [2] and reviewed here.

The attraction \(Q(\eta)\) was obtained using a model for which \(y_\xi\) was zero. Equation (3.32b) reduced to

\[ \gamma y_{\eta \eta} + J^2 q_m y_{\eta} = 0 \quad (3.34) \]

For this model case \(y(\eta)\) was derived by approximating the non-dimensional boundary layer velocity \((u_m)\) profile by

\[ u_m = 1 - \exp(-By) \quad (3.35) \]

where \(B\) is determined from \(u_m = 0.99\) at \(y = \delta\). The \(\eta\) line at the boundary layer edge was denoted \(\eta_\delta\). By requiring the velocity to
change by the same percentage from one \( \eta \) line to the next, then in

the boundary layer

\[
y(\eta) = \frac{\delta \ln[1 - 0.99 \left( \frac{\eta-1}{\eta_\delta-1} \right)]}{\ln 0.01}
\]

(3.36)

and

\[
Q_m(\eta) = -\frac{\gamma_\eta}{\eta^2 (\eta_\delta-1) - 0.99 (\eta-1)}
\]

(3.37)

Outside the boundary layer, \( y_p(\eta) \) was approximated by a quartic

polynomial. For some value of \( \eta \leq \eta_\delta \), the first three derivatives of

the \( y_p(\eta) \) polynomial were required to equal those derivatives of

Equation (3.36). The fourth polynomial coefficient was chosen to

satisfy the outer boundary condition on \( y(\eta) \). By substituting \( y_p(\eta) \)

into (3.34), \( Q_p(\eta) \) outside the boundary layer was obtained.

For attraction of \( \xi \) lines to the trailing edge point, a \( P \)

attraction function of the form

\[
P = -\sum_{k=1}^{N} b_k \frac{(\xi - \xi_k)}{|\xi - \xi_k|} \exp \left\{ -d[(\xi - \xi_k)^2 + (\eta - \eta_k)^2]^{1/2} \right\}
\]

(3.38)

The magnitudes, \( b_k \), of the attraction were determined so that the \( \xi \)-

spacing was uniform at the trailing edge.

Thus, Equations (3.32) were solved with boundary conditions (3.33).

Attraction \( Q \) was given by \( Q_m \) and \( Q_p \), and \( P \) was given by (3.38).

Another \( Q \) attraction function was made available for the wrap-

around coordinate system. The same approach was used to develop it

except that (3.35) was replaced by the Blausius function for \( u_m \).

Following the previous development led to the wrap-around \( Q \) attraction

function. Its advantages are discussed later.
Boundary conditions for the wrap-around coordinate system were

\[ \Gamma_1: (x,y) = (x_s,y_s) \quad n = 1, 1 \leq \xi \leq I \]
\[ \Gamma_2: (x,y) = (x_o,y_o) \quad n = J, 1 \leq \xi \leq I \]  \hspace{1cm} (3.39)

\[ \Gamma_3, \Gamma_4: (x,y) \text{ periodic in } \xi \text{ with period } (I-1) \]

Coordinate systems described in this report were generated using the TOMCAT code described in Reference [14].
IV. NUMERICAL SOLUTION OF THE GOVERNING EQUATION

This chapter describes the numerical solution of transformed flow Equations (3.4, 3.5, 3.16-3.21) and their boundary conditions (3.22-3.30). Finite difference approximations of terms in the continuous equations and computation of body density are described in the first section. Following that, the scheme to solve for successive time levels of dependent variables is discussed. In the last two sections, the initial solution and various aspects of the numerical solution are presented.

A. Finite Difference Approximations

Difference approximations to derivatives at points in the field are discussed first. To demonstrate the approximations of the continuum derivatives, terms from the x momentum equation are used as examples. No special treatment was required for points along the cut as long as the periodicity of the transformation was taken into account. In the following the notation \( f^n_{i,j} \) denotes the value of any dependent variable \( f \) at time \( n\Delta t \) and at the spatial point \((i\Delta \xi, j\Delta \eta)\). For convenience, \( \Delta \xi \) and \( \Delta \eta \) were chosen as one; this value is included in the difference approximations of continuum derivatives.

Time derivatives at all points were approximated by backward differences. First order approximations were of the type

\[
\frac{\partial (\rho u)}{\partial t} \bigg|_{i,j}^n = \frac{(\rho u)^n_{i,j} - (\rho u)^{n-1}_{i,j}}{\Delta t} \quad (4.1a)
\]

For second order time accuracy

\[
\frac{\partial (\rho u)}{\partial t} \bigg|_{i,j}^n = \frac{3(\rho u)^n_{i,j} - 4(\rho u)^{n-1}_{i,j} + (\rho u)^{n-2}_{i,j}}{2\Delta t} \quad (4.1b)
\]
Convection derivatives and pressure work derivatives at field points were approximated by second order, central space differences. The two point, central difference approximation (4.2) is shown for the \( \eta \) derivative.

\[
\frac{\partial (\rho u \hat{v})}{\partial \eta} \bigg|^{\xi}_{i,j} = \frac{(\rho u \hat{v})_{i,j+1}^n - (\rho u \hat{v})_{i,j-1}^n}{2}
\] (4.2)

A six point approximation of these derivatives was also investigated. This approximation was obtained by averaging the two point, central difference approximation with a second order, four point approximation to the same derivative. In the resulting difference expression (4.3), the four point difference can be identified.

\[
\frac{\partial (\rho u \hat{v})}{\partial \eta} \bigg|^{\xi}_{i,j} = \frac{1}{4} \left[ (\rho u \hat{v})_{i,j+1}^n - (\rho u \hat{v})_{i,j-1}^n \right] + \frac{1}{8} \left[ (\rho u \hat{v})_{i+1,j+1}^n - (\rho u \hat{v})_{i+1,j-1}^n \right] + \frac{1}{8} \left[ (\rho u \hat{v})_{i-1,j+1}^n - (\rho u \hat{v})_{i-1,j-1}^n \right]
\] (4.3)

Pressure derivatives at field points were approximated by second order, central differences such as

\[
\frac{\partial (p y \hat{\eta})}{\partial \xi} \bigg|^{\xi}_{i,j} = \frac{(p y \hat{\eta})_{i+1,j}^n - (p y \hat{\eta})_{i-1,j}^n}{2}
\] (4.4)

Second derivatives with respect to \( \xi \) or with respect to \( \eta \) were approximated by successive central differences over \( \Delta \xi \) or \( \Delta \eta \) in the
field. For example, from the x momentum equation

\[
\frac{3}{\Delta \eta} \left( \frac{\partial \mu_{11}}{\partial \eta} \right)_{i,j} \approx \left( \mu_{11} \right)_{i,j+1/2} \left[ u_{i,j+1} - u_{i,j} \right] \\
- \left( \mu_{11} \right)_{i,j-1/2} \left[ u_{i,j} - u_{i,j-1} \right]
\] (4.5)

Any terms such as \( \left( \mu_{11} \right)_{i,j+1/2} \) were approximated by the average of the product, i.e.,

\[
\left( \mu_{11} \right)_{i,j+1/2} \approx 1/2 \left[ \left( \mu_{11} \right)_{i,j} + \left( \mu_{11} \right)_{i,j} \right]
\] (4.6)

Field point cross derivatives were approximated by successive central differences over \( 2\Delta \xi \) and \( 2\Delta \eta \) so that

\[
\frac{\partial}{\partial \xi} \left[ \frac{3}{\Delta \eta} \left( \frac{\partial \mu_{13}}{\partial \eta} \right) \right]_{i,j} \approx 1/4 \left( \mu_{13} \right)_{i+1,j} \left[ u_{i+1,j+1} - u_{i+1,j-1} \right] \\
- 1/4 \left( \mu_{13} \right)_{i-1,j} \left[ u_{i-1,j+1} - u_{i-1,j-1} \right]
\] (4.7)

The transformed velocities, \( \hat{u} \) and \( \hat{v} \), and the elements of the \( c \) array (3.21) were functions of the derivatives \( x_\xi, x_\eta, y_\xi, y_\eta \). At field points, the derivatives were approximated by second order, central differences, e.g., Equation (4.8a). On the body surface and on the maximum \( \eta \) boundary, it was necessary to approximate the \( \eta \) derivatives as second order, forward differences and second order, backward differences, respectively. Examples are given as Equations (4.8b) and (4.8c). For the wake coordinate system, \( \xi \) derivatives at the minimum \( \xi \) and maximum \( \xi \) boundaries were also represented by second order, forward and backward differences, respectively. Using
the periodicity property of the coordinate mapping permitted the treatment of points on the cut as field points.

\[(x_n)_{i,j} = (x_{i,j+1} - x_{i,j-1})/2 \quad (4.8a)\]

\[(x_n)_{i,1} = - (3x_{i,1} - 4x_{i,2} + x_{i,3})/2 \quad (4.8b)\]

\[(x_n)_{i,J} = (3x_{i,J} - 4x_{i,J-1} + x_{i,J-2})/2 \quad (4.8c)\]

Difference equations for the dependent variables at all points in the field were obtained by replacing the derivatives in the continuum equations by differences as described above. Values of the dependent variables were specified on the outer boundary through boundary conditions. However, on the body surface, only the velocity components were given for all cases. Body total energy was computed from the simple product \(\rho_{i,1} T_0\) for the fixed wall temperature condition; otherwise, the body total energy boundary condition was more complicated. There was no boundary condition at all on wall density. The remainder of this section discusses the difference equations for density and total energy on the body surface.

For those cases in which the thermal condition was an adiabatic wall, total energy (E) at the wall was obtained from the difference approximation to Equation (3.23b). The derivative \(e_\xi^n\) was replaced by second order, central differences, and \(e_\eta^n\) was replaced by second order, forward differences. Using the definition of E and the velocity boundary condition resulted in Equation (4.9) for total energy at the wall.

\[E_{i,1}^n = - \rho_{i,1} \left[ -4e_{i,2}^n + e_{i,3}^n \right] + \left[ \frac{E}{\rho} \right]_{i+1,1}^n - \left[ \frac{E}{\rho} \right]_{i-1,1}^n + \left[ \frac{c^3}{c_4} \right]_{i,1} \quad (4.9)\]
Since there was no boundary condition on body density, it was computed from the continuity equation. The time derivative was replaced by the appropriate backward difference. Since the body surface was a line of constant \( \eta \), only the \( \eta \) derivative was not identically zero. It was approximated by the second order forward difference expression

\[
\frac{\partial (\rho \hat{v})}{\partial \eta} \bigg|_{i,1} = -[-4(\rho \hat{v})^n_{i,2} + (\rho \hat{v})^n_{i,3}] / 2 \tag{4.10}
\]

which incorporated the velocity boundary condition.

The resulting difference equation was used at all points on the airfoil except at the trailing edge point. In the wake coordinate system, the trailing edge point was treated as a field point in obtaining the density difference equation. Because of the highly contracted coordinates at the airfoil trailing edge of the wrap-around coordinate system, the trailing edge density was computed by extrapolation. The trailing edge density, \( \rho_{1,1}^n \), was extrapolated from adjacent points on the body surface and in the field using the equation

\[
\rho_{1,1}^n = 1/3 \left[ (3\rho_{2,1}^n - 3\rho_{3,1}^n + \rho_{4,1}^n) + (3\rho_{I-1,1}^n - 3\rho_{I-2,1}^n + \rho_{I-3,1}^n) + (3\rho_{1,2}^n - 3\rho_{1,3}^n + \rho_{4,1}^n) \right] \tag{4.11}
\]

For the higher Reynolds number computations, first order time differences in the body continuity equation were not acceptable because of slow iterative convergence. If all field points had first
order time differences, the time derivative in the body continuity
equation was approximated by

$$\frac{\partial \rho}{\partial t} \bigg|_{i,1}^{n} \approx \rho_{i,1}^{n} - \rho_{i,1}^{n-1} + \Delta t \frac{\partial^{2} \rho}{\partial t^{2}} \bigg|_{i,1}^{n} \quad (4.12a)$$

Differentiating the continuity equation and using it again to eliminate
\(\partial \rho / \partial t\) yielded

$$\frac{\partial^{2} \rho}{\partial t^{2}} \bigg|_{i,1}^{n} = \left[ \rho (\nabla \cdot \mathbf{V})^{2} - \rho \frac{\partial (\mathbf{V} \cdot \mathbf{V})}{\partial t} \right]_{i,1}^{n} \quad (4.12b)$$

Substitution of Equation (4.12b) into (4.12a) and replacing the
divergence time derivative by a first order difference resulted in
an acceptable body density time difference. Although a time difference
of the type (4.1b) could be used, this alternative did not require
additional data storage capacity.

A common method of computing the body density utilizes the normal
component of momentum. The body pressure is computed from the normal
component of momentum at the body surface (Equation B.7); body density
is obtained from the equation of state. In evaluating this procedure
for computing body density, all first derivatives with respect to \(\eta\)
were replaced by second order, forward differences, e.g.,

$$\frac{\partial (p_{\xi})}{\partial \eta} \bigg|_{i,1}^{n} \approx \frac{1}{2} [3(p_{\xi})_{i,1}^{n} - 4(p_{\xi})_{i,2}^{n} + (p_{\xi})_{i,3}^{n}] / 2 \quad (4.13)$$

First derivatives with respect to \(\xi\) were approximated by second order,
central differences. Successive replacement of \(\xi\) derivatives by
central differences and \(\eta\) derivatives by forward differences yielded
the second order difference approximation of the cross derivatives.
Second derivatives with respect to \(\eta\) were expanded to two terms. The
resulting terms with first derivatives were represented by second order, forward differences and the second derivatives were replaced by the second order approximations

\[
\frac{\partial^2 u}{\partial \eta^2} |^{n}_{1,1} = 2u_{1,1} - 5u_{1,2} + 4u_{1,3} - u_{1,4} \quad (4.14)
\]

A truncation analysis was performed on selected difference expressions to develop guidelines for evaluating grid systems and solution accuracy. The principal truncation error for the convection terms was obtained because they contribute artificial viscosity to the difference solution. Because of the importance of second derivative terms near the body, a special case of a second derivative difference approximation was analyzed. Results of that truncation analysis were also included in determining grid system acceptability. These analyses are presented in Appendix C.

The right hand side of Equation (C.3) is the principal truncation error for the convection terms. Although the difference approximation is first order, the product of second derivatives was quite small at practically all points of the grids used. Thus, excluding the exception discussed below, the second order terms were of primary concern. There was a tendency at the leading edge stagnation point for body density to be less than at adjacent points on the body surface. This error was attributed to the term \(0.5 (x/y/\eta) x/\eta \partial^2 (\rho u)/\partial x^2\). In that region \(x/y/\eta\) was approximately unity. Since \(\partial^2 u/\partial x^2\) was of the order of the Reynolds number, \(|x/\eta|\) had to be much less than \(1/R\) to prevent \(\partial \rho/\partial t\) from being too large, which led to an error in \(\partial (\rho \ddot{v})/\partial \eta\). For Reynolds numbers of the order of \(10^4\) or \(10^5\), it was difficult to generate grid systems such that \(|x/\eta| < 1/R\).
By considering a difference approximation of \( f_{xx} \), the approximation was found to be first order (C.5) with error coefficient proportional to \( (x_{\eta\eta}/x_{\eta})^2 \). To reduce this error near the body, the Blasius attraction function was used in generating wrap-around coordinate systems. However, in the trailing edge region of this coordinate system, the derivative \( y_{\xi\eta} \) is sufficiently large to create truncation errors in the density computation. It was not determined whether these errors were due to the first order term \( \partial(\rho u)/\partial x \) or the second order term \( \partial^2(\rho u)/\partial y^2 \) in Equation (C.3). This problem was eliminated by using the central difference approximations of the non-conservative form of the convection terms. In Reference [14] this differencing was shown to have second order accuracy and no second order terms proportional to \( y_{\xi\eta} \).

Difference approximations and truncation analyses have been discussed in this section. From the truncation analyses, it was concluded that in regions where second derivatives of dependent variables are large, e.g., near the body surface, the finite difference grid should be such that second derivatives of \( x \) and \( y \) are minimized. In the next section, solution of the difference equations is discussed.

B. Solution of the Difference Equations

Backward differencing of the time derivatives resulted in a set of simultaneous difference equations. An iterative scheme was employed to solve these equations for the flow field at subsequent time steps after the initial time. The iterative scheme for each new time level of flow variables is discussed in this section; the starting solution is discussed in the next.
An accelerated Gauss-Seidel iterative scheme was used in this analysis. Each new iterate for a dependent variable, represented by \( f \), was computed with an equation of the type

\[
\left( f_{i,j}^n \right)^{(k)} = \left( f_{i,j}^n \right)^{(k-1)} + \omega_f \left[ \left( f_{i,j}^n \right)^\ast - \left( f_{i,j}^n \right)^{(k-1)} \right]
\]

Superscripts \( k \) and \( k-1 \) are used to denote the new and the previous iterates of \( f_{i,j}^n \), respectively. The term \( \left( f_{i,j}^n \right)^\ast \) is used to denote the value computed for \( f_{i,j}^n \) from its difference equation; in that computation, the most recent iterates for all quantities were used. \( \omega_f \) is referred to as the acceleration parameter; they were independent of \( i, j \) and \( n \). A new time level iteration was begun by choosing \( \left( f_{i,j}^n \right)^{(1)} \) as \( f_{i,j}^{n-1} \) and continued until the following convergence criteria was met.

\[
\max \left| \left( f_{i,j}^n \right)^{(k)} - \left( f_{i,j}^n \right)^{(k-1)} \right| < \varepsilon_f
\]

\( 1 \leq i \leq I \)

\( 1 \leq j \leq J \)

Five different acceleration parameters were used in the scheme; there was one each for the field density, body density, and field total energy computations. Both velocity components had the same acceleration parameter, and there was one for the body total energy when the adiabatic wall boundary condition was used. Values of the acceleration parameters were determined at the beginning of a solution by systematically varying each parameter and noting the number of iterations to converge the first time step. The set of parameters that required the minimum number of iterations to converge the first time step was
used in all subsequent time step computations. Typically, the solutions were underrelaxed; that is, the acceleration parameters were less than one.

Implicit difference formulations such as this one generally permit larger time steps than explicit formulations from the standpoint of computational stability. However, the number of computations required to solve the simultaneous difference equations of an implicit scheme can offset its larger time step advantage. Consequently, it is important to determine the functional dependence of the iterative solution convergence rate, particularly when it is time step dependent.

No theory was available to determine the convergence rate of an accelerated Gauss-Seidel iteration of the nonlinear, coupled difference equations of this research. To apply available theory, it was necessary to assume local linearization of the difference equations and consider each individually. Although this approach was not expected to give quantitative results, it was felt that it would indicate functional dependence. The continuity equation's convergence rate analysis is presented in Appendix D.

The continuity equation was considered for two reasons. First, only the assumption of local linearization was necessary; the momentum and energy equations have cross differences which are not encompassed by the linear theory. Moreover, the continuity equation was the slowest to converge; therefore, it was of primary importance.

A measure of convergence rate is the spectral radius; a smaller spectral radius indicates more rapid convergence. The functional
dependence of the continuity equation spectral radius is given by Equations (D.3) and (D.7) for the leading edge region where iterative convergence was slowest. From (D.7), it is seen that decreasing the ratio of convection area flux to cell area increases the convergence rate. This result was verified by running the same flow conditions on two grids with different $\xi_\eta$ in the leading edge region. The solution with the larger values of $\xi_\eta$ converged faster; however, the increased convergence rate was accomplished at the sacrifice of accuracy. Consequently, other methods for reducing the computer time per converged time step were investigated.

Upwind differencing of the convection terms was tried as a means of increasing the convergence rate. Convection derivatives were replaced by forward or backward differences exemplified by Equations (4.8b) and (4.8c), respectively. The difference form at each point was chosen on the basis of the direction of $\hat{u}$ and $\hat{v}$. The one sided differences were taken in the direction opposite to the direction of $\hat{u}$ and $\hat{v}$. When this form of differencing was used, the solution converged until the norms reached some value higher than the convergence criteria. After that, the norms never decreased. It was determined that, at selected field points, each new iterate was oscillating and not converging.

Finally, a variation of the basic iterative scheme was employed that reduced computation time. It accounted for the lower convergence rate near the body and avoided unnecessary computations in the basically potential region. In this iterative variation, the iteration proceeded from the previous time step data as described
above, except that iteration was limited to those grid points in the viscous region. When the maximum change norms dropped below specified values, the iteration continued, but over the entire field, until final convergence criteria were met. Consequently, the new time step values satisfied the simultaneous difference equations at all points in the field, but needless computations at approximately two thirds of the grid points were avoided.

In the following section, the initial solution from which the iterations were started is discussed.

C. Initial Solution

In the formulation of the continuous solution, the steady state, potential flow was proposed as a realistic initial condition for the viscous, transient flow problem. However, a compressible, potential flow solution for an arbitrary airfoil was not readily available at the time initial solutions were being formulated. Since the initial solution was a small aspect of the research, time was not devoted to obtaining a compressible, potential flow solution. Instead, a starting solution was obtained from the steady, incompressible velocity field about the body of interest.

The first step in obtaining the initial solution was to compute the incompressible, potential velocity field about the body using the computer code discussed in Reference [14]. From this velocity field, the non-dimensional temperature field was computed from the isentropic flow energy Equation (4.17). The stagnation temperature \( T_o \) was

\[
\frac{T}{T_o} = 1 - \frac{(\theta-1)}{2} \frac{N_m^2}{T_o} V^2
\]  

\( (4.17) \)
computed from (4.17) using the boundary conditions that $T = 1$ when $V = 1$. Density in the field and on the body was computed from

$$\frac{\rho}{\rho_o} = \left(\frac{T}{T_o}\right)^{1/(\theta - 1)} = \left[1 - \frac{(\theta - 1) M_o^2}{T_o} v^2\right]^{1/(\theta - 1)}$$  \hspace{1cm} (4.18)

where the non-dimensional stagnation density ($\rho_o$) was determined from the boundary conditions $\rho = 1$ when $V = 1$. Field and outer boundary total energy was computed from

$$E = \frac{\rho}{\theta} \left\{ T_o + \frac{[(\theta - 1) M_o^2]}{2} v^2 \right\}$$  \hspace{1cm} (4.19)

Total energy on the body was computed from (4.19) if the thermal boundary condition was an adiabatic wall; otherwise, it was computed as wall density times $T_c/\theta$.

As might be expected, this initial solution resulted in a compression wave formation at the leading edge which propagated upstream and an expansion wave formation at the trailing edge which propagated downstream.

An alternative initial solution was investigated to determine if the time dependent solution would reach steady state in less computation time. This alternative initial solution consisted of a uniform flow field, that is,

$$u = \cos \phi \hspace{1cm} (4.20a)$$
$$v = \sin \phi \hspace{1cm} (4.20b)$$
$$\rho = 1 \hspace{1cm} (4.20c)$$
$$E = \frac{1}{\theta} \left\{ T_o + [(\theta - 1) M_o^2]^2/2 \right\} \hspace{1cm} (4.20d)$$

at all points in the field and on the outer boundary. At the body, the velocity boundary conditions were imposed, and density had a value
of one. Body total energy was computed from (4.19) if the thermal boundary condition was an adiabatic wall. In the case of a constant wall temperature, body total energy was computed as described above.

This initial solution proved to be too abrupt, and no acceleration parameters could be found to converge the first time step.

A variation of this initial solution, called the penetration start, was set up for future research. The velocity, density, and total energy fields were computed from Equations (4.20). However, the velocity components on the body surface were prescribed as a function of time which decreased to zero. Density at the body surface was computed from the continuity equation with the $\xi$ derivative term retained. For a fixed body temperature, total energy on the body was computed from (3.1b) with $\varepsilon = T_c/\theta$; for the adiabatic wall condition, total energy on the body was computed from (3.23b) with the $\xi$ derivatives of velocity retained.

In the last section of this chapter, a characteristic of simultaneous difference equations is discussed. This characteristic, denoted as "wiggles" in Reference [10], is a spatial oscillation of the dependent variable.

D. Wiggles

The term "wiggles" is used to describe the cell-to-cell oscillation of a dependent variable. These oscillations occurred primarily in the $\eta$ direction. Wiggles appeared most often in the regions of large gradients. They were experienced most often by density and total energy, but all variables experienced them to some degree in the leading edge region of higher Reynolds number flows.
In Reference [10], it was demonstrated that the cellular oscillations described above were not necessarily a result of some type of instability. Terms in the differential equation

\[-\nu \frac{du}{dx} + \frac{v}{R} \frac{d^2u}{dx^2} = 0\]  \hspace{1cm} (4.21)

with boundary conditions \(u(0) = 0\) and \(u(-L) = 1\), were replaced by second order, central differences. The resulting difference equation

\[(1 - \frac{RV\Delta x}{2v})u_{j+1} - 2u_j + (1 + \frac{RV\Delta x}{2v})u_{j-1} = 0\]  \hspace{1cm} (4.22)

has the conditions \(u_N = 0\) and \(u_0 = 1\), where \(N = \frac{L}{\Delta x}\). The homogeneous equation (4.22) has the solution

\[u_j = K_1 \left[ \frac{1 + \frac{RV\Delta x}{2v}}{1 - \frac{RV\Delta x}{2v}} \right]^j + K_2\]  \hspace{1cm} (4.23)

Using the boundary conditions to evaluate the constants \(K_1\) and \(K_2\) gave the complete solution

\[u_j = \frac{\left[ \frac{1 + \frac{RV\Delta x}{2v}}{1 - \frac{RV\Delta x}{2v}} \right]^N - \left[ \frac{1 + \frac{RV\Delta x}{2v}}{1 - \frac{RV\Delta x}{2v}} \right]^j}{\left[ \frac{1 + \frac{RV\Delta x}{2v}}{1 - \frac{RV\Delta x}{2v}} \right]^N - 1}\]  \hspace{1cm} (4.24)

If \((1 - \frac{RV\Delta x}{2v})\) is negative, the solution (4.24) has a cell to cell oscillatory form even though there was no time derivative term in the differential equation. The oscillation is a result of the second term of the numerator.

This example was considered to demonstrate three points. First, any linear difference equation of the form (4.22), with the first and last coefficients of opposite sign, has an oscillatory homogeneous solution. Secondly, for a given problem, the oscillatory solution
appears for a particular step size and is present for all larger step sizes. And, from sample calculations with $RV/\nu = 22$ and $x = 0.1$, the oscillatory solution is not significantly in error outside regions of large curvature of the dependent variable.

A plausible explanation for observed leading edge density oscillation is presented. For one dimension, the continuity equation is

$$- \frac{\Delta t}{2J} \hat{v}_j v_{j+1} \rho^n_{j+1} - \rho^n_j + \frac{\Delta t}{2J} \hat{v}_j v_{j-1} \rho^n_{j-1} = - \rho^{n-1}_j$$  \hspace{1cm} (4.25)

Assuming local linearization yields the solution

$$\rho^n_j = K_1 (r_1)^j + K_2 (r_2)^j + \rho^n_p(j)$$  \hspace{1cm} (4.26)

where $\rho^n_p(j)$ is the particular solution of (4.25) and

$$r_1, r_2 = \frac{1 \pm [1 + \left( \frac{\Delta t}{2J} \right)^2 \hat{v}_j v_{j+1} \hat{v}_j v_{j-1} ]^{1/2}}{\left[ - \frac{\Delta t \hat{v}_j}{2J} \right]}$$  \hspace{1cm} (4.27)

Since $\hat{v}$ is negative in the leading edge region, one of the homogeneous solutions of the locally linearized Equation (4.25) will be oscillatory there.

Hence, assuming the complete solution density variation in the $\eta$ direction is described by that obtained from local linearization, it is seen from (4.26) and (4.27) that the wiggles appear due to the combination of $\Delta t$, spatial grid, and the flow problem. As demonstrated by (4.24), they are not a result of computational error amplification because they appear in the analytical solution of the difference equations.
Extensive effort was devoted to eliminating the wiggles from the difference solution. From the observation that wiggles were most pronounced where second derivatives with respect to \( \eta \) were large, attempts at eliminating the wiggles consisted of adding terms containing \( \rho \eta \) to the continuity difference equation. The actual form of the terms was obtained from a truncation analysis of the continuity difference equation. None of these explicit artificial viscosity terms was satisfactory in eliminating wiggles.

The artificial viscosity terms used in the explicit solution of Reference [5] were evaluated as a wiggle smoother. There was difficulty in obtaining satisfactory coefficients of the second order differences of dependent variables on the variable spacing grids of this research. Consequently, those viscosity terms were unacceptable.

A method was found to be effective in controlling wiggles that did not rely on artificial viscosity terms. This algorithm was based on the flux corrected transport algorithm of Reference [3]. It was called at the end of each time step iteration, and it had the effect of eliminating local extrema from the previously converged solution. This technique was used for the highest Reynolds number solutions.

Solutions obtained using the numerical techniques discussed in this chapter are presented in the next chapter.
V. COMPUTATIONAL RESULTS

Computational results for the laminar, transonic flow about two different bodies are presented in this chapter. These two bodies are a circular cylinder and a NACA-0018 airfoil section. The circular cylinder served as a test case for computer code verification and for evaluation of refinements to the numerical algorithm. Several considerations led to the choice of the NACA-0018 section. Previous incompressible flow research provided baseline data on mesh spacing requirements for the NACA-0018 section. In addition, a simple analytical expression for the body surface was available so that redistribution of body grid points could be easily accomplished. A final consideration was the section's thickness. It was hoped that a shock wave would form at a relatively low Reynolds number on this section as a result of its thickness; consequently, any numerical problems associated with the shock wave could be addressed without the additional complications of turbulence modeling. Thus, although it is unlikely that a NACA-0018 section would be flown transonically, it was a satisfactory choice for developing the implicit, laminar flow numerical method.

All of the transient solutions presented had the following in common. They were started from an approximate, potential solution as discussed in Chapter IV. The Prandtl number was 0.71 for all solutions, and the freestream temperature was 273° Kelvin. This temperature corresponds to the standard atmosphere temperature at approximately 7500 feet.
The results presented include details of the flow such as velocity fields, Mach contours, and heat transfer rates at the body surface as well as aerodynamic coefficients, which are derived in Appendix E. Circular cylinder results are discussed first followed by the airfoil solutions.

A. Circular Cylinder Results

Circular cylinder results are presented for a Reynolds number of $10^3$ and a freestream Mach number of 0.80; the body thermal condition was an adiabatic wall. The inner portion of the grid system is shown in Figure 3; this grid was of the wrap around type. Time derivatives were first order; convection and pressure work terms were approximated by two point differences. Since there was no sharp trailing edge, the continuity equation was used there to compute density.

Plots of density versus position along the horizontal axis of symmetry are presented for two purposes. They demonstrate the waves generated as a result of the initial solution, and the wiggles are quite evident.

In Figures 4 through 8, density along the rear axis of symmetry is displayed for various non-dimensional times. At a time of 0.1, an initial compression wave is shown forming at the trailing edge. This wave was attributed to the large velocities near the body in the initial solution. Wiggles are seen in the region between the body surface at $x = 0.50$ and $x = 0.65$ where the density profile curvature is largest. At a time of 0.50, this wave has propagated downstream and an expansion wave is beginning to form. By the time of 1.00, the initial compression
wave has attenuated, and an expansion wave has begun propagating downstream. Wiggles are still present near the body. For the subsequent non-dimensional times of 1.50 and 2.00, a solution with significant wiggles has replaced the continuous variation of density for $x$ greater than 1.50. These wiggles resulted when the solution could not describe the reflection of the initial compression wave from the downstream boundary, where the density value was fixed.

Density along the axis of symmetry forward of the cylinder is shown in Figures 9 through 13. Formation of a compression wave is evident from these density profiles; however, the propagation upstream is very slow as should be expected. Wiggles are again present in the regions of large density curvature.

In an attempt to eliminate the wiggles from the density solution, density on the body surface was computed from the continuity equation with explicit time step differencing. No other change was made to the computational procedure. In Figures 14 and 15 density is plotted along the rear axis of symmetry for these computations. By comparing with Figures 4 and 5, respectively, it is seen that there is little difference in the values of density at the body, but the wiggles are noticeably smaller for the explicit density computation. This result substantiated earlier statements that the wiggles are a solution of the difference equations as opposed to an instability. Any stability problems should be more pronounced when part of the computations were explicit.

Although this explicit body density computation had a favorable effect on wiggles, it was not employed further. It was felt that
instabilities would accompany higher Reynolds number computations and, thereby, negate any advantages.

Velocity fields in the vicinity of the cylinder are shown for times of 1.00 and 2.00 in Figures 16 and 17. In Figure 16, two vortices have formed behind the cylinder. At a time of 2.00, there are two small separation bubbles on the rear upstream of the vortices. There was no indication of wiggles in the velocity solution.

The heat transfer rate to the fluid at a time of 1.00 is shown in Figure 18. These results exemplify how well the boundary condition was satisfied.

Aerodynamic coefficients for times of 0.20, 1.00, and 2.00 are provided as Figures 19 through 21. Upper and lower surface pressure coefficients appear as single curves due to the symmetry of the flow. An incorrect body surface pressure distribution is indicated by the pressure coefficient of Figure 21. The pressure at the leading edge was less than at adjacent points on the body surface; it was the result of computing a density that was too low at the leading edge. When this error was first observed, it was attributed to wave reflections from the outer boundary. However, it also occurred on the airfoil solutions and is discussed with them.

A primary problem with the circular cylinder solution was the reflection of waves from the downstream boundary. One attempt to let the waves pass through the boundary consisted of using the boundary conditions \( \partial f / \partial \eta = 0 \) at the outer boundary points for which \( x \) was positive. On another attempt to let the waves pass through the outer boundary, \( \partial f / \partial t \) was extrapolated along lines of constant \( \eta \) to the
outer boundary points for which $x$ was positive. A new value of each dependent variable was computed from the previous time step data and the time derivative. Both of these boundary conditions on the dependent variables eventually led to large $v$ velocities on the outer boundary for slightly positive $x$ values, and subsequent time step iterations would not converge.

It was felt that any waves emanating from an airfoil would be much smaller than those from the circular cylinder. Hence, since primary interest was in airfoil solutions, subsequent computations were performed on the NACA-0018 airfoil. Those investigations are discussed in the next two sections.

B. NACA-0018 Airfoil Results for Wake Coordinate Systems

First computations for this body were made at a Reynolds number of $10^3$ and a freestream Mach number of 0.80. The airfoil was at zero angle of attack, and the body thermal condition was again an adiabatic wall. First order time derivatives were used, and two point central differences approximated the first derivative terms.

The grid system for these computations is shown in Figure 22. It was typical of the wake coordinate systems except that the $y$ coordinates were specified on the downstream boundary of the computational field. On all other wake coordinate systems the Neumann condition (3.33) was used, and lines of constant $\eta$ were parallel to the $x$ axis at the downstream boundary.

Contours of constant Mach number are displayed in Figures 23 through 26. The region of supersonic flow, enclosed by the contour line marked with a triangle, is seen to grow as time increases.
Development of the boundary layer can be seen from the velocity fields of Figures 27 through 30. At a time of 1.50, a small reverse flow region had formed near the trailing edge.

In Figure 31 the aerodynamic coefficients are presented for the non-dimensional time of 2.00. It is seen that the symmetrical pressure distribution in the leading edge region has the correct variation. The leading edge density was not lower than the value at adjacent body points. However, density wiggles in the $\eta$ direction were present on the axis of symmetry forward of the leading edge for three $\xi$ lines above and below the axis of symmetry.

These computations were discontinued at a time of 2.00. The small supersonic region in Figure 26 gave little indication that a shock wave would ever form for these flow conditions. Part of the reason for the small supersonic region was the thermal boundary condition. The adiabatic wall condition led to a thick boundary layer as indicated by the distance between the 0.8 Mach contour and the body surface in the airfoil's mid and aft region.

In an effort to identify any computational problems associated with a shock wave, a subsequent computation was started with a free-stream Mach number of 0.90 and a constant wall temperature of 0.80. Nothing else was changed from the previously described solution.

Constant Mach number contours for these conditions are given for non-dimensional times of 0.90 and 2.10 in Figures 32 and 33, respectively. At the time of 2.10 the supersonic region is quite large; it extended for one chord length above and below the airfoil. The
supersonic region and the Mach number greater than 1.10 region have downstream boundaries that are normal to the airfoil as would be the case with a shock wave. In addition, the lower, downstream segment of the supersonic contours have the shape of a lambda shock characteristic of laminar boundary layers in a supersonic flow. However, the deceleration to subsonic flow is spread over too broad a region to consider this a shock wave.

The lack of a shock formation for these computations was attributed to the relatively low Reynolds number. Since the objective of this research was to develop the laminar solution for high enough Reynolds number that a turbulence model became appropriate, it was decided to concentrate on computational problems associated with higher Reynolds numbers. This approach, as opposed to trying to artificially induce a shock wave by varying temperature conditions, was followed for the remainder of the research.

A coordinate system was generated that was appropriate for Reynolds number of $10^4$ computations, and a solution was started for that Reynolds number and a Mach number of 0.80. The wall temperature was specified as 1.00; time and spatial differences were the same as previously described. The angle of attack remained zero.

At the non-dimensional time of 1.00, densities in the leading edge region were clearly in error. Density at the forward stagnation point was less than the densities at adjacent points on the body surface. This error in the $\xi$ variation of density about the axis of symmetry occurred on the first nine $\eta$ lines forward of the body in the leading edge region. This problem was the same as observed earlier on the circular cylinder.
An error in the velocity at the trailing was also observed. At the first point in the wake on the rear axis of symmetry, the velocity had reversed, and its magnitude was approximately 0.50. The velocity error was analyzed as a truncation problem which could be easily corrected with stronger attraction during grid generation. Consequently, an attempt was made to continue the solution to identify any other computational problems.

For subsequent computations, a wall temperature of 0.90 was used. This change led to the correct density distribution in the leading edge region, but the trailing edge velocity errors became rapidly worse until the time step iterations would no longer converge. Mach number contours for the last time step are presented as Figure 34; the large velocities near the trailing edge are indicated by the small region with Mach number greater than the freestream value. Velocity profiles which are inconsistent with physical reasoning are evident in Figure 35. The large velocities on the trailing edge axis of symmetry are not plotted in the figure.

To demonstrate the heat transfer rates per unit area that accompany reduced wall temperatures, Figure 36 is provided from these $10^4$ Reynolds number computations. Heat transfer to the body is negative.

From the initial $R = 10^4$ results, it was felt that the leading edge region posed the most difficult computational problems. In addition to the errors in the density distribution, time step convergence problems developed in trying to start the $10^4$ Reynolds number computations. The time step of 0.01 used for $R = 10^3$ had to be reduced to 0.0025 for this case. Thus, increasing the Reynolds number an order
of magnitude required a 75 per cent reduction in time step to obtain iterative convergence. The number of iterations to converge a time step was approximately the same as the specified wall temperature, $10^3$ Reynolds number case.

A series of investigations was conducted to identify improvements to the iterative convergence and to eliminate the density distribution errors. In these investigations, it was found that a higher order time derivative approximation for body density was essential for iterative convergence with larger time steps. Equation (4.12) was satisfactory. With this change it was possible to start the solution with a time step of 0.0025 and, after four time steps, increase the time step to 0.005. However, the error in density distribution along the first seven grid lines intersecting the flow field axis of symmetry occurred at a non-dimensional time of 0.02 rather than at 1.00.

During an investigation of first derivative differencing techniques, it was determined that six point differences (4.3) of convection and pressure work terms improved the iterative convergence characteristics and decreased the density distribution errors. Six point pressure gradient approximations greatly reduced the body density distribution problem. However, this type of pressure gradient differencing caused velocity reversals on the flow field axis of symmetry for this grid system.

To determine the influence of grid spacing on the leading edge problem, a coordinate system with reduced $\eta$ line attraction was generated. The distribution of $x(\eta)$ on the axis of symmetry forward of the body is shown as curve 3 in Figure 37. This reduced attraction
led to a larger y spacing between grid points in the leading edge region also. Shown as curve 2 in Figure 37 is the distribution of x(\eta) for the original $10^4$ Reynolds number computations. Using Equation (4.12) for body density and six point first derivatives permitted doubling the time step to 0.005 after a time of 0.01 with this new coordinate system. At a time of 0.02, the computed stagnation point density was less than that at adjacent points on the body surface. However, this pattern was not repeated on \eta lines in the field.

From the Reynolds number of $10^4$ solutions on two different grids, problems with density on the stagnation streamline were identified. As the stagnation point density increased, the velocity at the first point off the body rapidly decreased until $|\rho \hat{v}_{j=3}| > |4 \rho \hat{v}_{j=2}|$ on the flow axis of symmetry. Once this inequality was satisfied, the difference approximation of $\partial (\rho \hat{v}) / \partial \eta$ at the stagnation point was positive even though the computed values of $\rho \hat{v}$ monotonically decreased as \eta increased from the body. The errors in density along \xi lines where they crossed the axis of symmetry were attributed to wiggles in that direction.

A third grid was generated for Reynolds number of $10^4$ computations. However, before computing the grid a new distribution of body points was obtained; grid points in the leading edge region were more closely spaced. Time derivatives were approximated by second order differences in order to begin the evaluation of second order accuracy on the numerical solution. First order spatial differences were approximated by two point differences. This solution was continued long enough to
permit any leading edge density distribution errors to develop. When they did not, preparations were made for a Reynolds number of $10^5$ solution.

A $10^5$ Reynolds number solution was initiated by generating a suitable wake coordinate system. For the flow computations that followed, two point first derivatives were used and the time derivatives were second order. A wall temperature of 0.85 was used since this value facilitated the convergence of early time step iterations. The angle of attack was not changed from zero degrees. With the increase in Reynolds number there was the accompanying decrease in allowable time step to a value of 0.0025.

Wiggles developed in the leading edge region velocity component solutions as well as in the density and total energy solutions for this Reynolds number. The wiggles in the $\eta$ direction were present for five values of $\xi$ on either side of the flow field axis of symmetry, and the magnitude was such that some values of $u$ were negative. Starting on the third $\eta$ line forward of the body there were wiggles in the $\xi$ direction of the density solution also. When these computations were discontinued at a time of 0.23, the density at the forward stagnation point had decreased significantly compared to adjacent points on the body surface. As discussed previously, this decrease occurred as a result of the three point approximation of $\partial(\rho \hat{v})/\partial \eta$ in a region where $\partial^2 \hat{v}/\partial \eta^2$ was large.

This error in the body density computations prompted the truncation analysis presented in Appendix C. A truncation term containing the product $x_{\eta\eta} u_{xx}$ was identified in this analysis. Since $u_{xx}$ is of the
order of the Reynolds number, only an extremely small value of $x_{\eta}$ can prevent the computed value of $\partial \rho / \partial t$ from being too large. As the stagnation point density and, hence, pressure increased too rapidly, the velocity adjacent to the body was most strongly affected. This velocity decreased too rapidly and the difference approximation of $\partial (\rho \hat{v}) / \partial \eta$ finally yielded the incorrect sign. It is felt that the lack of influence of the body density on the velocity at the second line off the body was due to two problems. One problem was the first order accuracy of the viscous terms; another problem was that the $u$ velocity wiggle pattern was such that the computed velocity at the first point off the body was too low and at the second point off the body it was too large.

Attempts to solve the problems associated with higher Reynolds number solutions first consisted of trying to adjust the grid system. This approach was followed since the truncation terms were functions of second derivatives of $x$ and $y$ with respect to $\xi$ and $\eta$. In addition, it had been observed that wiggles first occurred in regions where these second derivatives were large such as the region $5 \leq \eta \leq 9$ on curve 2 of Figure 37.

In generating the second grid for Reynolds number of $10^5$, it was desired to have a small value of $x_{\eta}$ near the leading edge. To obtain such a wake type coordinate system, it was necessary to use an option that decreased grid attraction on the forward portion of the body relative to that on the rear. However, in order to prevent a large spatial step size normal to the body at the trailing edge, the overall attraction had to be increased. While this grid system was an
improvement over the previous one, $x_{\eta\eta}$ became significant much closer to the body than was desired.

New $R = 10^5$ computations were begun with a body temperature of 1.05 and a time step of 0.001. All other parameters remained the same. The forward stagnation point density decrease occurred at the very early time of 0.04 in this solution. This error in the computation of $\partial(p\hat{v})/\partial\eta$ was attributed to the starting solution and the higher body temperature value. The large $u_{xx}$ of the initial solution was represented much more accurately than on the previous grid system and the higher body temperature permitted a rapid compression near the body. As a result, $\partial(p\hat{v})/\partial\eta$ was computed incorrectly very early. Since an erroneous value of $\partial(p\hat{v})/\partial\eta$ tended to perpetuate itself, the computations were not continued.

For the same coordinate system and flow parameters the normal component of momentum at the body surface (B.7) was evaluated as a boundary condition. This condition first caused a reversal of $u$ velocity components at points on the first $\eta$ line in the field forward of the stagnation point. In subsequent computations, $u$ velocity components on the third line in the field reversed also. The extent in the $\xi$ direction of these reversed velocities also increased. This boundary condition did eliminate density oscillations near the body, but not in the region where $x_{\eta\eta}$ became significant. Similar results were obtained with the condition $\partial p/\partial\eta = 0$.

In subsequent investigations, the addition of explicit diffusion terms to the difference equations was tried to control the wiggles. Difference approximations of some truncation terms were not able to
reduce the wiggles. The smoother reported in Reference [5] was also evaluated. Addition of these terms to the difference equation resulted in a lack of iterative convergence after a few time steps.

The inability to reduce wiggles through coordinate spacing and the failure of explicit diffusion terms to eliminate them led to the evaluation of other attraction functions in coordinate system generation. Results obtained with an improved attraction function are discussed in the next section.

C. NACA-0018 Airfoil Results for Wrap Around Coordinate Systems

The Blasius attraction function for grid system generation was found to work best with the wrap around outer boundary. With the wake type outer boundary, it was not possible to obtain a converged solution and, at the same time, have \( \eta \) lines close enough to the body near the trailing edge.

A coordinate system with outer boundary radius of 8.00 was generated for a \( 2 \times 10^4 \) Reynolds number solution. The distribution of \( x(\eta) \) on the axis of symmetry forward of the body is shown as curve 1 in Figure 37. This distribution is nearly linear for \( x \) less than \(-0.505\). In the region where \( x_{\eta\eta} \) began to increase, it was less than the corresponding values for the wake coordinate systems with exponential type attraction functions.

Some problems with this type coordinate system arose in the region near the sharp trailing edge. As shown in Figure 38, \( \eta \) lines are highly contracted there. Some of the second derivatives of \( x \) and \( y \) with respect to \( \xi \) and \( \eta \) are large. The change in airfoil surface
slope at the trailing edge is due to an error in the analytical expression for the surface; \( y \) is not exactly zero for the trailing edge abscissa.

In an attempt to improve the coordinate system in this region, more than one \( \xi \) line was allowed to emanate from the trailing edge. This distribution of \( \xi \) lines did reduce some of the second derivatives of \( x \) and \( y \), but the closely contracted \( \eta \) lines did remain. Since there was no decisive advantage and since no approach in extrapolating values of the \( c \) array to the trailing edge had been developed, the concept of more than one \( \xi \) line intersecting the trailing edge was not pursued.

With this new coordinate system, a series of computations was begun with a Reynolds number of \( 2 \times 10^4 \). An adiabatic wall was the thermal boundary condition, the freestream Mach number was 0.90, and the angle of attack was three degrees. First order time differences were used since previous comparisons showed no less accuracy than second order time differences. The time step was 0.002, and the partial field followed by total field iteration scheme was used as described in Chapter IV.

Density at the body surface was computed with the continuity equation except at the trailing edge. Where the continuity equation was used the higher order correction of Equation (4.12) was included. Trailing edge density was computed by Equation (4.11).

Convection and pressure work derivatives were approximated by six point differences at all points in the field except on the \( \xi \) line passing through the trailing edge. As discussed in Chapter IV,
truncation errors yielded incorrect densities near the body for this value of \( \xi \). Central difference approximations of the non-conservative form of all first derivative terms were used to eliminate the errors.

For non-dimensional time less than 0.29, spatial oscillations appeared in the solution of all dependent variables. Density and total energy oscillations were in the \( \eta \) direction. A cell to cell density magnitude change of 0.17 was not uncommon near the body. Velocity component oscillations were present on the second through the fifth \( \eta \) lines forward of the leading edge region. The magnitude of these oscillations were such that, for a line of constant \( \eta \), the \( u \) velocity at alternate points in the \( \xi \) direction was negative. On those \( \xi \) values where \( u \) was negative the computed values of \( \rho \dot{v} \) increased from zero as \( \eta \) increased from the body value. For alternate \( \xi \) values, \( \rho \dot{v} \) decreased from zero as \( \eta \) increased from the body value. Consequently, the sign of \( \partial \rho / \partial t \) at the body alternated in the \( \xi \) direction and an up-down pattern was set up in the leading edge body density.

Upwind differencing was evaluated at this point as a means of reducing the wiggles. In a previous evaluation of second order upwind differences on the circular cylinder, the iteration norms decreased to a value larger than the convergence criteria and never decreased further. For this airfoil solution, first order upwind differences in the transform plane were considered. In a brief search for acceleration parameters, none were found that resulted in iterative convergence.

At this point, a different approach was taken to control the wiggles. The flux corrected transport algorithm reported in
Reference [3] was included in the solution technique. This algorithm was used on the converged time step data, and it successfully filtered the grid point to grid point oscillations in the dependent variables.

Mach number contours for this series of computations, which included the flux corrected transport algorithm are shown as Figures 39 and 40. Asymmetry of the flow field is evident in these figures. At the time of 1.013, the downstream segments of the supersonic Mach contour lines are nearly normal to the body chord line as would be expected. However, they are much too widely spaced to be a shock wave.

Portions of the velocity field are shown for the time of 1.013 in the next three figures. In Figure 41, the velocity field in the forward stagnation region is shown. The velocity profiles are characteristic of compressible flows near an adiabatic wall. The relatively high temperatures result in the velocity overshoots in the boundary layer. Velocity profiles in the boundary layer slightly forward of the maximum thickness are shown in Figure 42. The rear half of the airfoil is shown in Figure 43; there is extensive separation on the upper surfaces and some on the lower surface. It is noted that in all of these figures there are eight to ten lines in the boundary layer.

Aerodynamic coefficients are shown in Figure 44. Again the asymmetry of the flow field is apparent from the upper and lower surface $C_p$ plots.

The heat transfer rate to the fluid at the body surface and the body surface temperature are shown in Figures 45 and 46, respectively, for the non-dimensional time of 1.013. It can be seen that the heat
transfer rate is not exactly zero near the trailing edge. In addition, the body surface temperature is not a smooth curve. Both of these anomalies are attributed to the use of the flux corrected transport algorithm, but it is felt that they do not cause significant error in aerodynamic coefficient computations.

The transient flow computations were continued long enough to verify the success of the flux corrected transport algorithm in controlling wiggles and to confirm the benefits afforded by coordinate systems generated with the Blasius attraction function. The research of the laminar, transonic solution was completed with the inclusion of the eddy viscosity terms, characteristic of turbulent flow, in the solution. This particular solution was continued toward steady state with the turbulent viscosities incorporated.

In summarizing the computational results, the time dependent solution yielded transient solutions of the Navier-Stokes equations which agreed qualitatively with theory and experimental data. Quantitative comparisons were reserved for future research since the turbulent boundary layer is expected to have a significant impact on aerodynamic coefficients. The formation of a shock wave was not consistent with the laminar flow Reynolds numbers; hence, any computational problems associated with a shock wave formation were not identified. However, it is felt that the wiggles which occurred in the airfoil leading edge region are the source of the often reported spatial oscillations of the dependent variables near shocks. Thus, the extensive attention given to the oscillatory solution of the difference equations in this research should contribute to the analysis of shock waves occurring with the turbulent flow solution.
Wiggles were analyzed using the model Equations (4.22) and (4.25). In applying the results to two dimensional flow, a separation of variables solution is assumed.

Although Equation (4.22) is a steady state difference equation for $u$, the inclusion of time dependent terms does not alter the conclusion that oscillation can occur when the cell Reynolds number, $Ru\Delta x/\nu$, is greater than two. This result is altered somewhat when applied to the transformed difference equations containing diffusion terms. As seen in Equation (C.4), a term such as $f_{xx}$ contributes a first derivative term to the transformed difference equation which must be considered in predicting the onset of wiggles in the velocity solution. The criterion that the coefficient of $u_{j+1}$ and $u_{j-1}$ have opposite sign for an oscillatory solution is still appropriate for the transformed difference equations.

Density oscillations were analyzed with the transformed model Equation (4.25). If a uniform grid is assumed, the density model yields the result that wiggles are likely as the product $(u\Delta t/\Delta x)_{j+1} \cdot (u\Delta t/\Delta x)_{j-1}$ increases. This presents the interesting conclusion that, at a given point of a uniform grid, decreasing $\Delta x$ reduces the likelihood of wiggles in equations with diffusion terms. However, it increases the likelihood of wiggles in the density solution; only decreasing $\Delta t$ can reduce the density wiggles.

In concluding this chapter, several tables are presented that may be beneficial to further computational research. The $x$ coordinate of airfoil surface grid points is given in Table 1 for three Reynolds
numbers. Corresponding \( y \) coordinates were computed from Equation (5.1) for the leading edge and Equation (5.2) at other points.

\[
y = \pm 1.1019t^2 \tag{5.1}
\]

\[
y = \pm \frac{t}{0.20} \left[ 0.29690 \sqrt{x + 0.5} - 0.12600(x + 0.5) - 0.35160(x + 0.5)^2 \\
+ 0.28430(x + 0.5)^3 - 0.10150(x + 0.5)^4 \right] \tag{5.2}
\]

The percent thickness, \( t \), was 0.18. Field size and the spatial distance between the airfoil and the closest grid line is given in Table 2; \( \Delta y_{LE} \) is given for the first \( \eta \) line off the body. Acceleration parameters and convergence criteria for the various solutions are given in Table 3. The non-dimensional time step, the physical characteristic time \((c/V_\infty)\) assuming standard atmosphere conditions and the number of iterations to converge a time step are given in Table 4. In Table 4 it is seen that fewer iterations per time step were required when the thermal condition was a specified body temperature rather than an adiabatic wall.

The decrease in allowable time step for iterative convergence with increasing Reynolds number is evident in Table 4. The \( 10^5 \) Reynolds number time step is not inconsistent with the trend; those computations were performed with second order time differences. Consequently, the effective time step for the iterative scheme was actually two thirds of 0.0025.
VI. CONCLUSIONS

An implicit finite difference solution of the Navier-Stokes equations was investigated. Time histories of the transonic, laminar flow development about two-dimensional bodies were obtained. Reynolds numbers ranged from those corresponding to purely laminar flow to Reynolds numbers corresponding to a significant region of turbulence in the boundary layer. Body thermal conditions of an adiabatic wall and a specified body temperature were considered.

Versatility in treating arbitrary bodies was incorporated in the solution approach by using the numerically generated, body-fitted coordinate transformations. Difference equations were obtained from the transformed differential equations.

Solution of the simultaneous difference equations for the dependent variables at each time step was obtained using an accelerated Gauss-Sidel iterative scheme. The solution was started from physically realistic initial conditions. Acceleration parameters for the iteration were determined by numerical experimentation and, for simplicity, were maintained constant over the field and for all time steps.

The accuracy of the flow solutions was judged on a qualitative basis and was found to be good. Comparisons with experimental data were reserved for future research which incorporates turbulent modeling.

It was demonstrated that the fully implicit method is a viable alternative to the explicit method of Reference [8] for transonic, airfoil flow solutions. No published results were available to permit a detailed comparison of the computational efficiency of this implicit
method with the explicit method. Data was provided for such a comparison in the form of time step sizes and the number of iterations for time step convergence.

The allowable time steps in this research were limited by the iterative solution scheme. It was felt that the time step restrictions imposed by the iterative scheme were more stringent than necessary from an accuracy standpoint. Using the Reynolds number of $10^3$ as a baseline, the rule of thumb was formulated that the time step had to be reduced approximately 75% for each order of magnitude increase in Reynolds number when first order time differences were used. However, for a given Reynolds number solution, the time step for second order time differences can be 50% higher than the corresponding time step for first order time differences. Consequently, the detrimental effect of increasing Reynolds number on allowable time step can be offset with second order time differences. In addition, the computational time increases associated with higher Reynolds numbers can be reduced using the partial field iteration near the body and then total field iteration.

The flow field in the leading edge region of the body provided the most taxing conditions for iterative convergence. Strictly first order time differences for the continuity equation at the body surface were inadequate in the leading edge region for Reynolds numbers above $10^3$. Higher order time derivative approximations for the body continuity equation were used to preclude even larger time step reductions than described for Reynolds numbers above $10^3$. Assuming local linearization permitted the application of linear theory to
estimate the density solution convergence rate in the leading edge region. Since the difference equations were obtained in the transformed plane, this convergence rate functional dependence was in terms of transformed field properties. A method of relating this dependence to physical field properties was described. The analytical expression predicted the correct trends in density convergence rate for the limited number of coordinate systems, time steps and flow conditions considered in this research.

Difference representation accuracy was investigated by performing truncation analyses of the convection terms and a representative second derivative term. From these analyses it was determined that gradients of the grid spacing, especially normal to the body in the boundary layer, should be kept small. It was determined that the Blasius attraction function was superior to previous coordinate generation attraction functions in this regard. Body points should be distributed so that gradients in the step size parallel to the body surface are small, particularly near stagnation points.

Grid point-to-grid point oscillations of the dependent variables were observed near the leading edge in the airfoil solutions. By using locally linear analyses, these oscillations were concluded to be solutions of the difference equations and not instabilities. Six point differences for the convection and pressure work, but not pressure gradient, terms reduced these oscillations. However, the flux corrected transport algorithm was found to be the most effective in controlling these dependent variable oscillations.
Based on these research results, recommendations for future efforts were formulated. It is recommended that alternative initial solutions be considered to reduce the computational effort associated with reaching a steady state. One such possibility is a compressible, potential flow starting solution. It is also recommended that a simple stagnation point flow solution be obtained. With a simple model of flow impinging on a flat surface, the effects of time step, spatial grid, spatial grid gradients, and Reynolds number on wiggles and iterative convergence could be quantified without the complications of surface curvature or indirect control of the grid spacing. Such a model would permit a more detailed investigation of upwind differencing to see if it could eventually be used to control wiggles.
Table 1. X Coordinates of Body Grid Points

<table>
<thead>
<tr>
<th>$R = 10^3$ and $R = 10^4$</th>
<th>$R = 10^5$</th>
<th>$R = 2 \times 10^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 0.5000</td>
<td>- 0.50000</td>
<td>- 0.50000</td>
</tr>
<tr>
<td>- 0.4990</td>
<td>- 0.49997</td>
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<td>- 0.49880</td>
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<td>0.01024</td>
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<tr>
<td>Solution Conditions</td>
<td>Field Size ($\xi$ Lines x $\eta$ Lines)</td>
<td>Leading Edge Region</td>
</tr>
<tr>
<td>---------------------</td>
<td>----------------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>$R = 10^3$, $M_\infty = 0.80$</td>
<td>111 x 27</td>
<td>$\Delta x = 0.106(10^{-2})$</td>
</tr>
<tr>
<td>$q_w = 0., \phi = 0.$ AND $R = 10^3$, $M_\infty = 0.90$ $T_c = 0.80, \phi = 0.$</td>
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<td>$\Delta y = 0.782(10^{-2})$</td>
</tr>
<tr>
<td>$R = 10^4$, $M_\infty = 0.80$</td>
<td>111 x 27</td>
<td>$\Delta x = 0.350(10^{-3})$</td>
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<td>$\Delta y = 0.761(10^{-2})$</td>
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<tr>
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<td>113 x 26</td>
<td>$\Delta x = 0.910(10^{-3})$</td>
</tr>
<tr>
<td>$T_c = 0.85, \phi = 0.$</td>
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<td>$\Delta y = 0.179(10^{-2})$</td>
</tr>
<tr>
<td>$R = 2 \times 10^4$, $M_\infty = 0.90$</td>
<td>68 x 45</td>
<td>$\Delta x = 0.520(10^{-3})$</td>
</tr>
<tr>
<td>$q_w = 0., \phi = 3.0^\circ$</td>
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<td>$\Delta y = 0.456(10^{-2})$</td>
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Table 3. Acceleration Parameters and Convergence Tolerances

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<tr>
<th>Solution Conditions</th>
<th>$\omega_{u,v}$</th>
<th>$\omega_\rho$</th>
<th>$\omega_E$</th>
<th>$\left(\omega_\rho\right)_{\text{Body}}$</th>
<th>$\left(\omega_E\right)_{\text{Body}}$</th>
<th>$\varepsilon_f$</th>
</tr>
</thead>
<tbody>
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<td>$R = 10^3$, $M_\infty = 0.80$</td>
<td>0.95</td>
<td>0.70</td>
<td>0.50</td>
<td>0.20</td>
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<td>$10^{-5}$</td>
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<tr>
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</tr>
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<td>$R = 10^3$, $M_\infty = 0.90$</td>
<td>0.95</td>
<td>0.60</td>
<td>0.40</td>
<td>0.20</td>
<td>--</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>$T_c = 0.80$, $\phi = 0^\circ$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R = 10^4$, $M_\infty = 0.80$</td>
<td>0.45</td>
<td>0.40</td>
<td>0.50</td>
<td>0.20</td>
<td>--</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>$T_c = 1.00$, $\phi = 0^\circ$</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$R = 10^5$, $M_\infty = 0.80$</td>
<td>0.45</td>
<td>0.45</td>
<td>0.40</td>
<td>0.30</td>
<td>--</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>$T_c = 0.85$, $\phi = 0^\circ$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R = 2 \times 10^4$, $M_\infty = 0.90$</td>
<td>0.50</td>
<td>0.30</td>
<td>0.50</td>
<td>0.30</td>
<td>0.60</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>$q_w = 0.0$, $\phi = 3.0^\circ$</td>
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Table 4. Time Steps and Number of Iterations for Time Step Convergence

<table>
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<tr>
<th>Solution Conditions</th>
<th>Time Step</th>
<th>Number of Iterations For Convergence</th>
<th>Characteristic Time, c/V∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = 10^3, M_∞ = 0.80 )</td>
<td>0.01</td>
<td>38 for ( T = 0.25 )</td>
<td>0.25 ( (10^{-6}) \text{sec} )</td>
</tr>
<tr>
<td>( q_w = 0., \phi = 0. )</td>
<td></td>
<td>31 for ( T = 0.50 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>34 for ( T = 1.00 )</td>
<td></td>
</tr>
<tr>
<td>( R = 10^3, M_∞ = 0.90 )</td>
<td>0.01</td>
<td>20 for ( T = 0.25 )</td>
<td>0.20 ( (10^{-6}) \text{sec} )</td>
</tr>
<tr>
<td>( T_c = 0.80, \phi = 0^\circ )</td>
<td></td>
<td>19 for ( T = 0.50 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>18 for ( T = 1.00 )</td>
<td></td>
</tr>
<tr>
<td>( R = 10^4, M_∞ = 0.80 )</td>
<td>0.0025</td>
<td>16 for ( T = 0.25 )</td>
<td>2.49 ( (10^{-6}) \text{sec} )</td>
</tr>
<tr>
<td>( T_c = 1.00, \phi = 0^\circ )</td>
<td></td>
<td>17 for ( T = 0.50 )</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>15 for ( T = 1.00 )</td>
<td></td>
</tr>
<tr>
<td>( R = 10^5, M_∞ = 0.80 )</td>
<td>0.0025</td>
<td>21 for ( T = 0.23 )</td>
<td>24.86 ( (10^{-6}) \text{sec} )</td>
</tr>
<tr>
<td>( T_c = 0.85, \phi = 0^\circ )</td>
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<tr>
<td>( R = 2 \times 10^4, M_∞ = 0.90 )</td>
<td>0.002</td>
<td>43 for ( T = 0.25 )</td>
<td>3.93 ( (10^{-6}) \text{sec} )</td>
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<tr>
<td>( q_w = 0., \phi = 3.0^\circ )</td>
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<td>79 for ( T = 0.50 )</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>49 for ( T = 1.00 )</td>
<td></td>
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</tbody>
</table>
a. Physical Field

Figure 1. Wake Type Coordinate System
Figure 2. Wrap Around Coordinate System

a. Physical Field
Figure 3. Circular Cylinder Coordinate System
Outer Boundary Radius = 6.0
Figure 4. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry 
$M_\infty = 0.8, R = 10^3$, Time = 0.10
Figure 5. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry
$M = 0.8, R = 10^3, \text{Time} = 0.5$
Figure 6. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry
\( U_\infty = 0.8, R = 10^3, \text{ Time } = 1.0 \)
Figure 7. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry

\( M_\infty = 0.8, R = 10^3, \text{Time} = 1.5 \)
Figure 8. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry
$M_\infty = 0.8$, $R = 10^3$, Time = 2.0
Figure 9. Density Versus Position Along Circular Cylinder's Upstream Axis of Symmetry
$M_\infty = 0.8$, $R = 10^3$, Time = 0.1
Figure 10. Density Versus Position Along Circular Cylinder's Upstream Axis of Symmetry
$M_\infty = 0.8$, $R = 10^3$, Time = 0.5
Figure 11. Density Versus Position Along Circular Cylinder's Upstream Axis of Symmetry

\( \infty = 0.8, R = 10^3, \text{Time} = 1.0 \)
Figure 12. Density Versus Position Along Circular Cylinder's Upstream Axis of Symmetry
$M_\infty = 0.8$, $R = 10^3$, Time = 1.5
Figure 13. Density Versus Position Along Circular Cylinder's Upstream Axis of Symmetry
\( M_\infty = 0.8, R = 10^3, \text{ Time } = 2.0 \)
Figure 14. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry, Explicit Continuity Equation for Wall Density
\[ M_\infty = 0.8, \quad R = 10^3, \quad \text{Time} = 0.1 \]
Figure 15. Density Versus Position Along Circular Cylinder's Downstream Axis of Symmetry, Explicit Continuity Equation for Wall Density $M_\infty = 0.8$, $R = 10^3$, $T = 0.5$
Figure 16. Velocity Field for Circular Cylinder

$M_\infty = 0.8, R = 10^3, T = 1.0$
Figure 17. Velocity Field for Circular Cylinder
\( M_\infty = 0.8, R = 10^3, \text{Time} = 2.0 \)
Figure 18. Heat Transfer Rate to Fluid at Circular Cylinder Surface
\( M_\infty = 0.8, R = 10^3, \text{Time} = 1.0 \)
LIFT COEFFICIENT = -0.0024
DRAG COEFFICIENT = 1.1879
MOMENT COEFFICIENT = 0.0000

Figure 19. Aerodynamic Coefficients for Circular Cylinder
$M_\infty = 0.8, R = 10^3, \text{Time} = 0.2$
LIFT COEFFICIENT = -0.0008
DRAG COEFFICIENT = 1.7952
MOMENT COEFFICIENT = -0.0000

Figure 20. Aerodynamic Coefficients for Circular Cylinder
$M_\infty = 0.8, R = 10^3$, Time = 1.0
LIFT COEFFICIENT = 0.0014
DRAG COEFFICIENT = 1.6851
MOMENT COEFFICIENT = -0.0000

Figure 21. Aerodynamic Coefficients for Circular Cylinder
$M_\infty = 0.8$, $R = 10^3$, Time = 2.0
Figure 22. Wake Coordinate System for $R = 10^3$
Figure 23. Mach Number Contours for NACA-0018 Section
$M_\infty = 0.8$, $R = 10^3$, Time = 0.10
Figure 24. Mach Number Contours for NACA-0018 Section
\( M_\infty = 0.8, R = 10^3, \text{Time} = 0.5 \)
Figure 25. Mach Number Contours for NACA-0018 Section
$M_\infty = 0.8$, $R = 10^3$, Time = 1.0
Figure 26. Mach Number Contours for NACA-0018 Section
$M_\infty = 0.8$, $R = 10^3$, Time = 2.0
Figure 27. Viscosity Field for NACA-0018 Section

$M_\infty = 0.8$, $R = 10^3$, Time = 0.1
Figure 28. Velocity Field for NACA-0018 Section
\( M_\infty = 0.8, \; R = 10^3, \; \text{Time} = 1.0 \)
LIFT COEFFICIENT = 0.0001
DRAG COEFFICIENT = 0.2372
MOMENT COEFFICIENT = 0.0000

Figure 31. Aerodynamic Coefficients for NACA-0018 Section
$M_\infty = 0.8$, $R = 10^3$, Time = 2.0
Figure 32. Mach Number Contours for NACA-0018 Section

\( M_\infty = 0.9, \quad R = 10^3 \), Wall Temperature Specified,
Time = 0.9
Figure 33. Mach Number Contours for NACA-0018 Section
$M = 0.9$, $R = 10^3$, Wall Temperature Specified,
Time = 2.1
Figure 34. Mach Number Contours for NACA-0018 Section
$M = 0.8, R = 10^4$, Wall Temperature Specified,
Time = 1.15
Figure 35. Velocity Field for NACA-0018 Section
$M_o = 0.8$, $R = 10^3$, Wall Temperature Specified,
Time $= 1.15$
Figure 36. Heat Transfer to Fluid at NACA-0018 Surface
\( M_\infty = 0.8, \quad R = 10^4, \) Wall Temperature Specified,
Time = 1.15
Figure 37. X Coordinate Variation Along NACA-0018's Leading Edge Axis of Symmetry
Figure 38. Trailing Edge Region of NACA-0018 Wrap Around Coordinate System
Figure 39. Mach Number Contours for NACA-0018 Section

\[ M_\infty = 0.9, \ R = 2 \times 10^4, \ \phi = 3^\circ, \ \text{Time} = 0.6 \]
Figure 40. Mach Number Contours for NACA-0018 Section
$M_\infty = 0.9$, $R = 2 \times 10^4$, $\phi = 3^\circ$, Time = 1.013
Figure 41. Velocity Field in the Leading Edge Region of NACA-0018 Section

\( M_\infty = 0.9, R = 2 \times 10^4, \phi = 3^\circ, \text{ Time } = 1.013 \)
Figure 42. Velocity Field in the Maximum Thickness Region of NACA-0018 Section

$M_\infty = 0.9$, $R = 2 \times 10^4$, $\phi = 3^\circ$, Time = 1.013

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Figure 43. Velocity Field on Rear Half of NACA-0018 Section
\( M_\infty = 0.9, R = 2 \times 10^4, \phi = 3^\circ, \text{Time} = 1.013 \)
LIFT COEFFICIENT = 0.4014
DRAG COEFFICIENT = 0.1355
MOMENT COEFFICIENT = -0.0418

Figure 44. Aerodynamic Coefficients for NACA-0018 Section
$M_\infty = 0.9$, $R = 2 \times 10^4$, $\phi = 3^\circ$, Time = 1.013
Figure 45. Heat Transfer to Fluid at NACA-0018 Surface
$M_\infty = 0.9, R = 2 \times 10^4, \phi = 3^\circ, \text{Time} = 1.013$
Figure 46. Temperature of NACA-0018 Surface
\( M_\infty = 0.9 \), \( R = 2 \times 10^4 \), \( \phi = 3^\circ \), Time = 1.013
APPENDIX A

Various Expressions in the Transformed Plane

Various transformations from the (x,y) plane to the (ξ,η) plane are summarized in this appendix. Most of these definitions have appeared in earlier works, Reference [13,14]; however, some, in the form presented, were unique to this research. Definitions used in this appendix are:

\[ f(x,y,t), g(x,y,t) \] - twice continuously differential scalar functions of x, y, and t.

\[ F(x,y) = f_1(x,y) + j F_2(x,y) \] - a continuously differentiable vector valued function; \( f \) and \( j \) are the conventional cartesian coordinate unit vectors.

Definitions of the Transformation

\[ \alpha \equiv x^2 + y^2 \eta \] (A.1)

\[ \beta \equiv x \xi + y \eta \eta \] (A.2)

\[ \gamma \equiv x^2 \xi + y^2 \xi \] (A.3)

\[ J \equiv x \xi \eta - x \eta \xi \] (A.4)

where J is the determinant of the Jacobian matrix.

Derivative Transformations

\[ f_x \equiv \left( \frac{\partial f}{\partial x} \right)_{y,t} = (y \eta f_\xi - y \xi f_\eta) / J \] (A.5)

\[ f_y \equiv \left( \frac{\partial f}{\partial y} \right)_{x,t} = (x \xi f_\eta - x \eta f_\xi) / J \] (A.6)
\[ \frac{\partial}{\partial x} \left[ g \frac{\partial f}{\partial x} \right]_{x,t} = \frac{y_n}{J} \frac{\partial}{\partial \xi} \left[ g \frac{y_n}{J} f_{\xi} - y_{\xi n} f_{n} \right] \]
\[ - \frac{y_{\xi n}}{J} \frac{\partial}{\partial \eta} \left[ g \frac{y_n}{J} f_{\xi} - y_{\xi n} f_{n} \right] \tag{A.7} \]

\[ \frac{\partial}{\partial y} \left[ g \frac{\partial f}{\partial y} \right]_{x,t} = \frac{x_{\xi n}}{J} \frac{\partial}{\partial \eta} \left[ g \frac{x_{\xi n}}{J} f_{n} - \frac{x_{\xi n}}{J} f_{\xi} \right] \]
\[ - \frac{x_{\xi n}}{J} \frac{\partial}{\partial \xi} \left[ g \frac{x_{\xi n}}{J} f_{n} - \frac{x_{\xi n}}{J} f_{\xi} \right] \tag{A.8} \]

\[ \frac{\partial}{\partial x} \left[ g \frac{\partial f}{\partial y} \right]_{x,t} = \frac{y_n}{J} \frac{\partial}{\partial \eta} \left[ g \frac{x_{\xi n}}{J} f_{n} - \frac{x_{\xi n}}{J} f_{\xi} \right] \]
\[ - \frac{y_{\xi n}}{J} \frac{\partial}{\partial \xi} \left[ g \frac{x_{\xi n}}{J} f_{n} - \frac{x_{\xi n}}{J} f_{\xi} \right] \tag{A.9} \]

**Vector Derivative Transformations**

**Gradient:**

\[ \nabla f = [(y \frac{\partial f}{\partial \xi} - y_{\xi n} f_{n}) \mathbf{i} + (x \frac{\partial f}{\partial \eta} - x_{\xi n} f_{\xi}) \mathbf{j}] / J \tag{A.10} \]

**Divergence:**

\[ \nabla \cdot \mathbf{F} = [y_{\xi n} (F_1)_{\xi} - y_{\xi n} (F_1)_{n} + x_{\xi n} (F_2)_{\xi} - x_{\xi n} (F_2)_{n}] / J \]
\[ = \frac{\partial}{\partial \xi} [y_{\xi n} F_1 - x_{\xi n} F_2] + \frac{\partial}{\partial \eta} [x_{\xi n} F_2 - y_{\xi n} F_1] \tag{A.11} \]

**Unit Tangent and Normal Vectors**

**Normal to \( n \)-line:**

\[ \mathbf{n}^{(n)} \equiv \nabla n / |\nabla n| = (-y_{\xi n} \mathbf{i} + x_{\xi n} \mathbf{j}) / \sqrt{\gamma} \tag{A.12} \]
Normal to $\xi$-line:
\[ \mathbf{n}(\xi) \equiv \frac{\nabla \xi}{|\nabla \xi|} = \frac{(y_{\xi} \mathbf{i} - x_{\xi} \mathbf{j})}{\sqrt{\alpha}} \] (A.13)

Tangent to $\eta$-line:
\[ \mathbf{t}(\eta) \equiv \mathbf{n}(\eta) \times \mathbf{k} = \frac{(x_{\xi} \mathbf{j} + y_{\xi} \mathbf{i})}{\sqrt{\gamma}} \] (A.14)

Tangent to $\xi$-line:
\[ \mathbf{t}(\xi) \equiv \mathbf{n}(\xi) \times \mathbf{k} = \frac{-(x_{\eta} \mathbf{i} + y_{\eta} \mathbf{j})}{\sqrt{\alpha}} \] (A.15)

Vector Components Tangent and Normal to $\xi$- and $\eta$-Lines
\[ \mathbf{F}(\eta) = \mathbf{n}(\eta) \cdot \mathbf{F} = \frac{(-y_{\xi} F_1 + x_{\xi} F_2)}{\sqrt{\gamma}} \] (A.16)
\[ \mathbf{F}(\xi) = \mathbf{t}(\xi) \cdot \mathbf{F} = \frac{(x_{\eta} F_1 + y_{\eta} F_2)}{\sqrt{\gamma}} \] (A.17)
\[ \mathbf{F}(\xi) = \mathbf{n}(\xi) \cdot \mathbf{F} = \frac{(y_{\eta} F_1 - x_{\eta} F_2)}{\sqrt{\alpha}} \] (A.18)
\[ \mathbf{F}(\xi) = \mathbf{t}(\xi) \cdot \mathbf{F} = \frac{-(x_{\eta} F_1 + y_{\eta} F_2)}{\sqrt{\alpha}} \] (A.19)

Directional Derivatives
\[ \frac{\partial f}{\partial \eta}(\eta) = \mathbf{n}(\eta) \cdot \mathbf{F} = \frac{(\gamma f_{\eta} - \beta f_{\xi})}{(J \sqrt{\gamma})} \] (A.20)
\[ \frac{\partial f}{\partial \eta}(\xi) = \mathbf{n}(\xi) \cdot \mathbf{F} = \frac{(\alpha f_{\xi} - \beta f_{\eta})}{(J \sqrt{\alpha})} \] (A.21)

Integral Transforms
Scalar Function:
\[ \int_{\mathbb{R}} f(x,y) \, dx \, dy = \int_{\mathbb{R}^*} f(x(\xi,\eta), y(\xi,\eta)) \, |J| \, d\xi \, d\eta \] (A.22)
Vector Integral:

\[ I \equiv \int_C f(x,y)n(\eta) \, ds = \int_{\xi_{\min}}^{\xi_{\max}} f(\xi,\eta) (x_\xi \cdot n_x - y_\xi \cdot n_y) \, d\xi \quad (A.23) \]

or

\[ I \equiv \int_C F(x,y) \, ds = \int_{\xi_{\min}}^{\xi_{\max}} F(\xi,\eta) \sqrt{n} \, d\xi \quad (A.24) \]

where \( C \) is any line of constant \( \eta \), \( s \) is the arc length along the contour \( C \), and \( \eta_1 \) is the value of the line of constant \( \eta \).

Similarly for a line of constant \( \xi \),

\[ I \equiv \int_C f(x,y)n(\xi) \, ds = \int_{\eta_{\min}}^{\eta_{\max}} f(\xi,\eta) (y_\eta \cdot n_y - x_\eta \cdot n_x) \, d\eta \quad (A.25) \]

or

\[ I \equiv \int_C F(x,y) \, ds = \int_{\eta_{\min}}^{\eta_{\max}} F(\xi,\eta) \sqrt{n} \, d\eta \quad (A.26) \]

where \( C \) is any line of constant \( \xi \), \( s \) is the arc length along the contour \( C \), and \( \xi_1 \) is the value of the line of constant \( \xi \).

Scalar Product:

\[ \int_C F(x,y) \cdot n \, ds = \int_{\xi_{\min}}^{\xi_{\max}} (x_\xi F_2 - y_\xi F_1) \, d\xi \quad (A.27) \]

\[ \int_C F(x,y) \cdot n \, ds = \int_{\eta_{\min}}^{\eta_{\max}} (y_\eta F_1 - x_\eta F_2) \, d\eta \quad (A.28) \]
APPENDIX B

Normal Component of Momentum at the Body Surface

In this appendix, the normal component of the momentum equation at the body surface is derived. This expression was evaluated as a means of computing body density.

To obtain the normal component of momentum, the momentum equation (3.2b, 3.2c) was written as

\[
\rho \frac{D\mathbf{v}}{Dt} + \nabla \cdot \rho \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot \mathbf{\sigma} \tag{B.1}
\]

where

\[
\mathbf{v} = u_1 \mathbf{i} + v_1 \mathbf{j} \tag{B.2}
\]

\[
\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \tag{B.3}
\]

\[
\mathbf{v} = \frac{1}{\eta} \frac{\partial \mathbf{i}}{\partial x} + \frac{1}{\eta} \frac{\partial \mathbf{j}}{\partial y} \tag{B.4}
\]

\[
\mathbf{\sigma} = \mathbf{i} \sigma_{xx} + \mathbf{j} \sigma_{xy} + \mathbf{i} \mathbf{j} \tau_{yx} + \mathbf{i} \mathbf{j} \tau_{yy} + \mathbf{j} \mathbf{i} \sigma_{yx} + \mathbf{j} \mathbf{j} \sigma_{yy} \tag{B.5}
\]

Since the body surface was a line of constant \( \eta \), a unit normal vector at the body surface is given by Equation (A.12). Taking the dot product of the momentum equation (B.1) with the unit normal (A.12) yielded the normal component of momentum (B.6) in the physical plane.

\[
\left[ - \frac{y_1}{\sqrt{Y}} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} \right) + \frac{x_1}{\sqrt{Y}} \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \right) \right]_B = 0 \tag{B.6}
\]

The velocity condition at the body surface was used to obtain (B.6).
Equation (B.6) was transformed to the \((\xi, \eta)\) plane. Since the first and second derivatives of \(u\) and \(v\) with respect to \(\xi\) were zero on the body surface, the transformed normal component of momentum at the body surface was written as

\[
\{y_\xi \frac{\partial (py_\xi)}{\partial \eta} + x_\xi \frac{\partial (px_\xi)}{\partial \eta}\}_{i,1} = \{y_\xi \frac{\partial (py_\eta)}{\partial \xi} + x_\xi \frac{\partial (px_\eta)}{\partial \eta}\}_{i,1} \tag{B.7}
\]

Before the derivatives were replaced by differences in (B.7), those terms involving second derivatives with respect to \(\eta\) were expanded.

This equation (B.7) and the equation of state (3.4) were used to compute a body pressure and then a body density at all points on the body surface except the trailing edge point of a wake type coordinate system. At that point \(u_{\xi\xi}\) and \(v_{\xi\xi}\) are not zero; however, it was convenient to use the continuity equation for that point.

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APPENDIX C

Difference Approximation Truncation Analysis

In Chapter IV, the difference approximations were referred to as second order accurate approximations of derivatives in the \((\xi, \eta)\) plane. However, the accuracy of ultimate interest was that of approximating the derivatives in the physical plane equations (3.2), (3.3), and (3.7). This accuracy is evaluated in this appendix.

Because of the implicit viscosity they can introduce, the convection terms were considered first. For each dependent variable, these terms were approximated by

\[
\frac{1}{2J_{i,j}} [(f_\xi)_{i+1,j}^n - (f_\xi)_{i-1,j}^n + (f_\nu)_{i,j+1}^n - (f_\nu)_{i,j-1}^n]
\]

The truncation analysis was conducted by replacing each flow variable in this expression by its Taylor series expansion about the point \([x(\xi_i, \eta_j), y(\xi_i, \eta_j)]\), denoted by \((x', y')\). For example,

\[
f[x_{i+1,j}, y_{i+1,j}, t_n] = f(x', y')^n + [x_{i+1,j} - x'] \frac{\partial f}{\partial x} |_{x', y'}^n
\]

\[
+ [y_{i+1,j} - y'] \frac{\partial f}{\partial y} |_{x', y'}^n
\]

\[
+ \frac{[x_{i+1} - x']^2}{2} \frac{\partial^2 f}{\partial x^2} |_{x', y'}^n
\]

\[
+ [x_{i+1,j} - x'][y_{i+1,j} - y'] \frac{\partial^2 f}{\partial x \partial y} |_{x', y'}^n
\]
After simplifying terms, subscripted x's and y's were replaced by series expansions such as

\[
y_{i+1,j+1} = y_{i,j} + y_{\xi} \big|_{i,j} + y_{\eta} \big|_{i,j} + \frac{1}{2} y_{\xi\xi} \big|_{i,j} \\
+ y_{\xi\eta} \big|_{i,j} + \frac{1}{2} y_{\eta\eta} \big|_{i,j} + \ldots\tag{C.2}
\]

The truncation error is defined as the difference approximation less the differential expression, and the principal truncation error consists of the lower order derivatives of this difference. For the convection terms, the principal truncation error, \(T_{i,j}^n\), was determined to be

\[
T_{i,j}^n = \frac{y_{\xi\eta}}{2J} (x_{\xi} - x_{\eta}) \frac{\partial (fu)}{\partial x} + \frac{x_{\xi\eta}}{2J} (y_{\eta} - y_{\xi}) \frac{\partial (fv)}{\partial y} \\
+ \frac{x_{\xi\eta}}{2J} (x_{\eta} - x_{\xi}) \frac{\partial (fv)}{\partial x} + \frac{y_{\xi\eta}}{2J} (y_{\xi} - y_{\eta}) \frac{\partial (fu)}{\partial y} \\
+ \frac{1}{2J} [x_{\xi} y_{\eta} x_{\xi} - x_{\eta} y_{\xi} x_{\eta} + y_{\xi} (x_{\xi}^2 - x_{\eta}^2)] \frac{\partial^2 (fu)}{\partial x^2} \\
+ \frac{1}{2J} [y_{\eta} y_{\xi} (y_{\xi}^2 - y_{\eta}^2) + y_{\xi\eta} (y_{\xi}^2 - y_{\eta}^2)] \frac{\partial^2 (fu)}{\partial y^2} \\
+ \frac{1}{2J} [x_{\xi} y_{\eta} y_{\xi\xi} + y_{\xi} y_{\eta} (x_{\xi}^2 - x_{\eta}^2) - x_{\eta} y_{\xi} y_{\eta}] \\
+ 2y_{\xi\eta} (x_{\xi} y_{\xi} - x_{\eta} y_{\eta}) \frac{\partial^2 (fu)}{\partial x \partial y}
\]
As mentioned earlier, the convection terms were considered because they can introduce artificial viscosity into the difference solution. An example of this viscosity is the term \(0.5(x_\eta y_\eta/J) x_\eta^n \frac{\partial^2 (fv)}{\partial x^2}\). However, since the difference approximation was not second order accurate due to the presence of terms such as \(\partial (fu) / \partial x\) in the principal truncation errors, second differences were investigated for accuracy also.

A simplified model was chosen for the second derivative truncation analysis because of their complication. The term \(f_{xx}\) was considered for a grid such that \(x_\xi\) and \(y_\eta\) were zero and \(y_\xi\) was constant. With those restrictions,

\[
f_{xx} = \frac{1}{x_\eta^2} [f_{\eta \eta} - \frac{x_{\eta \eta}}{x_\eta} f_\eta]
\]

using second order, central differences gave the approximation

\[
[f_{j+1} - 2f_j + f_{j-1} - \frac{(x_{j+1} - 2x_j + x_{j-1})}{(x_{j+1} - x_{j-1})} (f_{j+1} - f_{j-1})]/ \frac{(x_{j+1} - x_{j-1})^2}{4}
\]

Replacing \(f_{j+1}\) and \(f_{j-1}\) by series expansions and retaining the first order terms yielded the truncation error

\[
\tau_{1,j}^n = -\frac{1}{4} \left(\frac{x_{\eta \eta}}{x_\eta}\right)^2 f_{xx}
\]
Truncation error is a measure of the error in the analytical solution of the difference equations as compared to the analytical solution of the differential equations. From a practical standpoint, it does provide a bound on the error in the computed numerical solution. Thus, in light of Equations (C.3) and (C.5), care must be exercised in generating grid systems with rapidly expanding or contracting grid lines. In particular, based on Equation (C.5), grid expansion normal to a body surface must be kept small. Otherwise, the second derivatives of velocity normal to the wall will be underestimated in the numerical solution.
APPENDIX D

Convergence of the Iterative Scheme

An analytical determination of the factors affecting iterative convergence is not possible for the difference equations of this research. No theoretical results are available concerning the convergence rate for an accelerated Gauss-Sidel iteration of a coupled system of nonlinear equations. In this appendix, local linearization is assumed and those theoretical results for a single linear equation are employed to identify trends affecting the iterative convergence.

If a linear system of the form

\[ a_1(f_{i+1,j} + f_{i-1,j}) + a_2(f_{i,j+1} + f_{i,j-1}) + b_1(f_{i+1,j} - f_{i-1,j}) \]

\[ + b_2(f_{i,j+1} - f_{i,j-1}) + d f_{i,j} = e_{i,j} \]  

(D.1)

is solved using accelerated Gauss-Sidel iteration for \( f_{i,j} \) at \( (i-2, j-2) \) number of field points, the optimum acceleration parameter and a measure of the convergence rate can be determined analytically. The convergence rate is proportional to the logarithm of the reciprocal of the spectral radius. For the system of Equations (D.1), there are two cases.

\[ \rho^*_\text{AGS} = \frac{1 - \sqrt{1 - \rho_j^2}}{1 + \sqrt{1 - \rho_j^2}} \text{ if } a_1^2 \geq b_1^2, a_2^2 \geq b_2^2 \]  

(D.2)

\[ \rho^*_\text{AGS} = \frac{\sqrt{1 + \rho_j^2} - 1}{\sqrt{1 + \rho_j^2} + 1} \text{ if } a_1^2 \leq b_1^2, a_2^2 \leq b_2^2 \]  

(D.3)
where, if \( a_i^2 - b_i^2 \) and \( a_1^2 - b_2^2 \) have the same sign,

\[
\rho_J = 2 \sqrt{\left| \frac{a_1^2}{d} - \frac{b_1^2}{d} \right| \cos \left( \frac{\pi}{J-1} \right) + 2 \sqrt{\left| \frac{a_2^2}{d} - \frac{b_2^2}{d} \right| \cos \left( \frac{\pi}{J-1} \right)} \quad (D.4)
\]

\( \rho_J \) is the spectral radius for Jacobi iteration, and \( \rho_{AGS}^* \) is the spectral radius for accelerated Gauss-Sidel iteration using the optimum acceleration parameter. No analytical results are available for \( \rho_{AGS}^* \) if the conditions on the coefficients are not met. It should be noted that if \( \rho_J \) is greater than one, and the conditions on (D.3) are met, \( \rho_{AGS}^* \) is still less than one. Hence, accelerated Gauss-Sidel iteration will converge, but as \( \rho_J \) increases above one the rate of convergence will decrease.

The system of equations describing the density at field points is

\[
\left( \frac{\Delta t \hat{u}}{2J} \right)_{i,j}^{n+1} = \rho_{i+1,j}^n - \left( \frac{\Delta t \hat{u}}{2J} \right)_{i-1,j}^n \rho_{i-1,j}^n + \left( \frac{\Delta t \hat{v}}{2J} \right)_{i,j+1}^n \rho_{i,j+1}^n
\]

\[
- \left( \frac{\Delta t \hat{v}}{2J} \right)_{i,j-1}^n \rho_{i,j-1}^n + \rho_{i,j}^n = \rho_{i,j}^{n-1}
\]

(D.5)

In the notation of the standard form (D.1),

\[
a_1 = - \frac{\Delta t}{2J} [\hat{u}_{i+1,j}^n - \hat{u}_{i-1,j}^n], \quad a_2 = - \frac{\Delta t}{2J} [\hat{v}_{i,j+1}^n - \hat{v}_{i,j-1}^n] \quad (D.6a,b)
\]

\[
b_1 = - \frac{\Delta t}{2J} [\hat{u}_{i+1,j}^n + \hat{u}_{i-1,j}^n], \quad b_1 = - \frac{\Delta t}{2J} [\hat{v}_{i,j+1}^n + \hat{v}_{i,j-1}^n] \quad (D.6c,d)
\]

and

\[
a_1^2 - b_1^2 = - \left( \frac{\Delta t}{2J} \right)^2 \hat{u}_{i+1,j}^n \hat{u}_{i-1,j}^n \quad (D.6e)
\]
In the leading edge region where the iteration most often diverged, \( \dot{v} \) is negative. Except on the dividing streamline, \( \dot{u}^\text{n}_{i+1,j} \) and \( \dot{u}^\text{R}_{i-1,j} \) had the same sign. Thus, the Jacobi spectral radius is approximately

\[
\rho_j \approx \frac{\Delta t}{2J_{i,j}} \left[ \sqrt{\left| \frac{\partial u}{\partial x} \right|_{i+1,j} \frac{\partial u}{\partial x} \left|_{i-1,j} \right| + \sqrt{\left| \frac{\partial v}{\partial y} \right|_{i,j+1} \frac{\partial v}{\partial y} \left|_{i,j-1} \right|} } \right]
\]

The cosine factors are approximately one for the field sizes used in this research.

The Jacobi spectral radius is given in terms of transformed velocities in Equation (D.7). It can be estimated as a function of physical plane and grid system quantities by using the Taylor series expansion approach of Appendix C. However, that was not accomplished as part of this research.
Appendix E

Aerodynamic Coefficients

Expressions for the aerodynamic coefficients of two-dimensional bodies are obtained, and the procedure for computing them is described.

The pressure coefficient at any point is defined in terms of dimensional variables as

\[ c_p = \frac{\bar{p} - \bar{p}_\infty}{\frac{1}{2}(\rho V)_\infty^2} \]  

(E.1)

Using the definition of the non-dimensional variables yielded the equation

\[ c_p = \frac{p - p_\infty}{\frac{1}{2}(\theta-1)M_\infty^2} \]  

(E.2)

Included in Equation (E.2) are the freestream conditions on the non-dimensional density and velocity. This equation, with \( p \) as the surface pressure, was used to compute the pressure coefficient at the body surface.

Force coefficients were obtained from Equation (E.3) which describes the force on any closed fluid surface \( S_F \).

\[ \mathbf{F}_F = \rho_\infty V_\infty^2 \oint_{S_F} \hat{n} \cdot \mathbf{a} \, dS \]  

(E.3)

The non-dimensional stress tensor, \( a \), is defined by Equation (B.5), and \( \hat{n} \) is an outward pointing, unit normal vector. At the body surface, the force on the body, \( \mathbf{F}_B \), equals \( -\mathbf{F}_F \), and the outward normal to the body surface has opposite sign to the fluid surface outward normal.
From Equation (E.3), the force on body is given by

$$F_B = \rho_\infty V^2 \hat{n} \cdot \sigma(cds)$$  \hspace{1cm} (E.4)

The surface integral of Equation (E.3) reduced to a line integral around the body contour for two-dimensional bodies. In Equation (E.4), the differential arc length, ds, is non-dimensional with respect to the chord. In this research, the body contours were lines of constant $\eta$. Thus, transforming the integration to the $\xi, \eta$ plane yielded

$$F_B^\xi = (\rho V)^2 \omega \int_{\xi_{T1}}^{\xi_{T2}} (x_\xi \tau_{yx} - y_\xi \tau_{xx})d\xi$$  \hspace{1cm} (E.5a)

$$F_B^\eta = (\rho V)^2 \omega \int_{\eta_{T1}}^{\eta_{T2}} (x_\eta \tau_{yy} - y_\eta \tau_{xy})d\xi$$  \hspace{1cm} (E.5b)

Derivatives with respect to $x$ and $y$ in the stress tensor components (3.3) were transformed to derivatives with respect to $\xi$ and $\eta$ using Equations (3.12) and (3.13).

The lift coefficient, $C_L$, is defined as the component of force normal to the freestream direction divided by $1/2(\rho V^2)\omega c$; the drag coefficient, $C_D$, is the component of force parallel to the freestream direction and normalized by the same quantity. Thus,

$$C_L = \frac{-F_B^x \sin \phi + F_B^y \cos \phi}{1/2(\rho V^2)\omega c}$$ \hspace{1cm} (E.6)

$$C_D = \frac{F_B^x \cos \phi + F_B^y \sin \phi}{1/2(\rho V^2)\omega c}$$ \hspace{1cm} (E.7)
Computation of $C_L$ and $C_D$ was accomplished by trapezoidal rule integration of Equations (E.5) and substitution into Equations (E.6) and (E.7). Values of the stress tensor at body points in the transformed plane were obtained from forward differences of $\eta$ derivatives and central differences of $\xi$ derivatives.

The moment on a closed fluid surface about the coordinate system origin is described by the equation

$$M_F = (\rho V)^2 \oint_{S_F} \left[ \mathbf{r} \times \hat{n} \cdot \mathbf{g} \right] dS$$ \hspace{1cm} (E.8)

Position of a point on the surface is given by the vector $\mathbf{r}$ which is non-dimensionalized by the chord length. At the fluid-body interface, the moment on the body, $M_B$, is equal to $-M_F$, and the body surface outward normal has direction opposite to the fluid surface outward normal. Consequently, for a two-dimensional body

$$M_B = (\rho V)^2 \oint_{C_B} \left[ \mathbf{r} \times \mathbf{n} \cdot \mathbf{g} \right] (cdS)$$ \hspace{1cm} (E.9)

In the transformed plane, this integral became

$$M_B = (\rho V)^2 \oint_{C_B} \frac{\xi^2 T_2}{T_1} \left[ x(\sigma_{yy} x - \tau_{yy} y) - y(\tau_{yy} x - \sigma_{yy} y) \right] d\xi$$ \hspace{1cm} (E.10)

For two-dimensional bodies, the moment vector has only one component, and the moment coefficient, $C_M$, is defined by

$$C_M = \frac{|N_B|}{\frac{1}{2}(\rho V)^2 c^2}$$ \hspace{1cm} (E.11)

In computing the moment coefficient (E.11), the integration indicated in Equation (E.10) had to be performed. It was accomplished in the same manner as were those integrations for $(F_B)_x$ and $(F_B)_y$. 

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