Refraction Corrections for Surveying

Mission Planning and Analysis Division

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REFRACTION CORRECTIONS FOR SURVEYING

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1.0 INTRODUCTION

Optical measurements of range and elevation angle are distorted by the Earth's atmosphere as shown in figure 1.

![Figure 1: Refraction effects.](image)

For long ranges, the measured elevation angle $E_M$ may deviate from the true elevation angle $E$ by several milliradians, and measured range $P_M$ may differ from true range $P$ by several meters. Even short-range measurements are affected by refraction, and the effects are cumulative.

Intuition may lead one to think that refraction effects are minor when the elevation angle is near zero. Exactly the opposite is true. $E_M - E$ increases with a decreasing elevation angle, and the ray path becomes more curved. Also to be noted is that a laser ranging device cannot be compensated to give a true range measurement by using the speed of light in the atmosphere, at the instrument. This will not include the effects of the ray path bending and the passage through higher or lower layers of atmosphere, where the speed of light differs from that at the instrument. The equations in this report assume that measured range is obtained by using the speed of light in a vacuum (index of refraction $n=1$).

The work presented in this report is a spinoff of the Space Shuttle Program. The high precision refraction correction equations in this report are being used to evaluate approximate refraction correction equations developed at the NASA Johnson Space Center. It was determined that the equations in this report were ideally suited for surveying since their inputs are optically measured range and optically measured elevation angle. The outputs are true straight-line range and true geometric elevation angle. The "short" distances used in surveying allow the calculations of true range and true elevation angle to be quickly made using a programmable pocket calculator.
2.0 THE SPHERICAL FORM OF SNELL'S LAW

Let $n$ be the index of refraction. A light ray traveling from one media ($n_i$) to another ($n_{i+1}$) is bent according to Snell's law (fig. 2).

\[ n_{i+1} \sin \phi_{i+1} = n_i \sin \phi_i \]

Figure 2.- Snell's law.

Figure 3 shows how Snell's law can be applicable to concentric spherical surfaces.
Figure 3. - Ray path across spherical surfaces.

From the ABC triangle in figure 3 and from the law of sines,

\[ \frac{R_i}{\sin \psi} = \frac{R_{i+1}}{\sin(180 - \phi_i)} = \frac{R_{i+1}}{\sin \phi_i} \]

or

\[ R_{i+1} \sin \psi = R_i \sin \phi_i \]

But from Snell's law of refraction,
$$\sin \psi = \frac{n_{i+1}}{n_i} \sin \phi_{i+1}$$

Thus,

$$R_{i+1} \frac{n_{i+1}}{n_i} \sin \phi_{i+1} = R_i \sin \phi_i$$

or

$$n_{i+1} R_{i+1} \sin \phi_{i+1} = n_i R_i \sin \phi_i$$

In the next layer, it is easily seen that

$$n_{i+2} R_{i+2} \sin \phi_{i+2} = n_i R_i \sin \phi_i$$

or, in general,

$$nR \sin \phi = n_i R_i \sin \phi_i$$

Elevation angle $E_M$ is defined by

$$E_M = 90 - \phi$$

As a result, the spherical form of Snell's law is developed:

$$nR \cos E_M = n_i R_i \cos E_M$$  \hfill (1)

The elevation angle $E_M$ at radius $R$ is a function only of $n$ at $R$ and the initial conditions: $n_i$, $R_i$, and $E_{M1}$. That is, $E_M$ is independent of the shape of the ray path (any crazy curve) from $R_i$ to $R$. 
3.0 **THE RAY PATH EQUATIONS**

Figure 4 shows the ray path geometry from the initial altitude $h_i$ to the final altitude $h_f$. 
The following parameters are defined:

- \( R_0 \) = reference radius of Earth = 6,378,165 meters; value not critical
- \( R = R_0 + h \)
- \( h \) = altitude above \( R_0 \)
- \( h_i \) = initial value of \( h \)
- \( h_f \) = final value of \( h \)
- \( \theta \) = central angle
- \( \theta_i \) = zero
- \( \theta_f \) = value at target
- \( E_M \) = elevation angle of ray path
- \( E_Mf \) = value of \( E_M \) at target, final value
- \( E_Mi \) = initial value of \( E_M \) as measured by equipment
- \( ds \) = differential path length
- \( \rho \) = true, straight-line, geometric range
- \( E \) = true, geometric elevation angle
- \( c \) = speed of light in a vacuum
- \( n \) = index of refraction at \( h \), equal to \( c/(\text{speed of light at } h) \)
- \( n_0 \) = value of \( n \) at \( R_0 \)
- \( \rho_M \) = measured range

\[
\rho_M = \int_n ds
\] (2)

The value \( N = n^{-1} \) is the modulus of refraction. An exponential atmosphere is assumed. That is,

\[
N = n_0 \exp(-h/H_s)
\] (3)

The symbol \( H_s \) is defined as atmospheric scale height.

Values of \( n_0 \) and \( H_s \) are supplied by meteorologists. However, the following empirical relationship may be used for \( H_s \) (ref. 1):

\[
H_s = \frac{1000}{n} \ln \left( \frac{n_0}{n_0 - 7.32 \cdot 10^{-5} \exp(5577n_0)} \right) \text{ meters} \] (4)

6
This value of $H_0$ is used at Johnson Space Center for processing radar tracking data. However, it may not be the best for optical survey data.

Figure 5(a) shows the differential path length element $\delta s$ and its two components. Figure 5(b) shows the photon velocity along $\delta s$ and its two components.

![Diagram](image)

Figure 5. - Differential geometry.

Figure 5 shows that

$$\frac{ds}{dt} = \frac{c}{n}$$

(5)

$$\frac{dh}{dt} = -\frac{c}{n} \sin E_M$$

(6)

$$\frac{d\theta}{dt} = -\frac{1}{R} \frac{c}{n} \cos E_M$$

(7)

where, from equation (3), $n$ is given as a function of $h$ by
\[ n = 1 + N = 1 + N_0 \exp(-h/H_S) \]  

(8)

Note from equation (2) that

\[ \delta n = nds \]

And, using equation (5)

\[ \frac{d\delta n}{dt} = c \]  

(9)

An expression for \( E_M \) is now required. Equation (1), the spherical form of Snell's law of refraction, could be used; however, it is fraught with numerical problems. For example, if \( E_{M_1} = 0 \), then \( h \) (eq. (6)) will remain at its initial value, \( h_1 \). It has been determined that it is more accurate to develop a differential equation for \( E_M \) and integrate it. In equation (1),

\[ nR \cos E_M = \text{constant} \]

Differentiating with respect to time gives

\[ nR \cos \dot{F} \quad n \dot{h} \cos E_M - nR \dot{E}_M \sin E_M = 0 \]

But, from equations (8) and (3)

\[ \dot{h} = - \frac{N_0}{H_S} = \frac{N}{H_S} \]

Thus,

\[ -R \frac{N}{H_S} \dot{h} \cos E_M + nh \cos E_M - nR \dot{E}_M \sin E_M = 0 \]

Using equation (6) for \( \dot{h} \)
\[ N \frac{c}{H_S} \sin \theta_M \cos \theta_M + c \sin \theta_M \cos \theta_M - nR \dot{\theta}_M \sin \theta_M = 0 \]

And the equation for \( \dot{\theta}_M \) is

\[
\frac{d\theta_M}{dt} = \left( \frac{1 - \frac{N}{R}}{\frac{nH_S}{n}} \right) \frac{c}{\cos \theta_M} \]

(10)

Now, let

\[ a = ct \]

(11)

Then \( da = cd\dot{t} \) and the resulting equations are summarized as follows.

\[
\begin{align*}
\dot{\theta}_M &= a \\
\frac{dh}{da} &= \frac{1}{n} \sin \theta_M \\
\frac{d\theta}{da} &= \frac{1}{n} \cos \theta_M \\
\frac{d\theta_M}{da} &= \left( \frac{1 - \frac{N}{R}}{\frac{nH_S}{n}} \right) \frac{1}{\cos \theta_M} \\
\end{align*}
\]

(12) \( \quad \) (13) \( \quad \) (14) \( \quad \) (15)

The initial conditions are

\[
\begin{align*}
a &= 0 \\
h &= h_1 \\
\theta &= \theta_i = 0 \\
\theta_M &= \theta_{M1} \\
\end{align*}
\]

And where
\[ R = R_0 + h \quad (16) \]

\[ N = N_o \exp(-h/H_0) \quad (17) \]

\[ n = N + 1 \quad (18) \]

The equations are integrated from \( a = 0 \) to \( a = \) measured range. All that remains is to obtain the expressions for \( \rho \) and \( E \). It can be seen in figure 4 that

\[ R_1 = R_0 + h_1 \]
\[ R_f = R_0 + h_f \]
\[ T_1 = R_f \cos \theta_f - R_i \]
\[ T_2 = R_f \sin \theta_f \]

\[ \rho = \sqrt{T_1^2 + T_2^2} \quad (19) \]
\[ E = \arctan \left( \frac{T_1}{T_2} \right) \quad (20) \]

4.0 INTEGRATING THE EQUATIONS

A fourth order Runge-Kutta-Gill integrator has been found to be very suitable for most purposes. Other integrators may be found in reference 2. The Runge-Kutta-Gill integrator allows a maximum integration step size of about 10 000 meters for even the most precise surveying work.

Let the state vector \( \mathbf{x} \) be defined by

\[
\mathbf{x} = \begin{bmatrix}
    h \\
    \theta \\
    E_M
\end{bmatrix}
\]
Let

\[ f(x) = \begin{bmatrix} dh/da \\ d\theta/da \\ dE_H/da \end{bmatrix} \]  

Then the fourth order Runge-Kutta integrator equations (ref. 2) are:

\[
\begin{align*}
    x_n &= x \\
    \rho_H &= \rho_H + \Delta \theta \\
    k_1 &= \Delta a f(x) \\
    x &= x_n + a_1 k_1 \\
    k_2 &= \Delta a f(x) \\
    x &= x_n + b_1 k_1 + b_2 k_2 \\
    k_3 &= \Delta a f(x) \\
    x &= x_n + c_1 k_1 + c_2 k_2 + c_3 k_3 \\
    k_4 &= \Delta a f(x) \\
    x &= x_n + d_1 k_1 + d_2 k_2 + d_3 k_3 + d_4 k_4 
\end{align*}
\]

The Runge-Kutta-Gill constants (ref. 2) are:

\[
\begin{align*}
    a_1 &= 1/2 \\
    b_1 &= (\sqrt{2}-1)/2 \\
    b_2 &= (2-\sqrt{2})/2 \\
    c_1 &= 0 \\
    c_2 &= -\sqrt{2}/2 \\
    c_3 &= (2+\sqrt{2})/2 \\
    d_1 &= 1/6 \\
    d_2 &= (2-\sqrt{2})/6 \\
    d_3 &= (2+\sqrt{2})/6 \\
    d_4 &= 1/6 
\end{align*}
\]

5.0 **EXAMPLES**

Tables I and II show examples of refraction corrections for

\[
\begin{align*}
    N_0 &= 0.000395 \\
    H_0 &= 5446 \text{ meters} 
\end{align*}
\]
\[ h_i = 0 \text{ meters} \]
\[ E_{M_1} = -0.239 \text{ degrees} \]

In tables I and II, \( \Delta \rho = \rho_M - \rho \) meters and \( \Delta \theta = E_{M_1} - E \) milliradians.
### TABLE I. SHORT- AND MEDIUM-RANGE REFRACTION CORRECTIONS

<table>
<thead>
<tr>
<th>$\rho_M$, m</th>
<th>$\Delta \rho$, m</th>
<th>$\Delta \Theta$, mrad</th>
<th></th>
<th>$\rho_M$, m</th>
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<th>$\Delta \Theta$, mrad</th>
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<td>100</td>
<td>0.0395</td>
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<td></td>
<td>1 000</td>
<td>0.3950</td>
<td>0.03 624</td>
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<td>1.1859</td>
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### TABLE II.- LONG-RANGE REFRACTION CORRECTIONS

<table>
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<tr>
<th>$\rho_M$, km</th>
<th>$h_M$, m</th>
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<th>$\Delta E$, mrad</th>
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6.0 REFERENCES
