NASA Contractor Report 3209

System Theory as Applied Differential Geometry

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GRANT NSG-2252
NOVEMBER 1979

NASA

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Prepared for
Ames Research Center
under Grant NSG-2252

National Aeronautics
and Space Administration

**Scientific and Technical
Information Branch**

1979

TABLE OF CONTENTS

SYSTEM THEORY AS APPLIED DIFFERENTIAL GEOMETRY

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SUMMARY

The opening sections sketch the mathematical framework for System Theory. It has long been recognized that there are (at least) three levels at which it is necessary to study systems: Individual systems, systems depending on parameters, and "adaptive" or "self-tuning" systems. Systematic and thorough mathematical study at the second level is no doubt necessary for successful future attack on the third level; it is in this study that differential-geometric methodology becomes most useful, indeed, probably essential. It is also pointed out that some of the most basic and important problems in System Theory (even at the practical level) involve properties of orbits and orbit spaces of certain types of Lie groups acting on manifolds.

The core of the report is the description of a new technique for computing these orbits for the case of combined feedback and state equivalence groups acting on linear systems. Although it is equivalent to the well-known Brunovsky canonical form, which is an algebraic method, this technique involves differential geometry, particularly the classification of holomorphic vector bundles on the Riemannian sphere due to A. Grothendieck. It also generalizes to distributed parameter systems; certain examples are discussed in a tentative way here. Finally, a generalization to Riemannian geometry of the "pseudo-inverse" construction is presented as a possible useful tool in certain identification problems.

1. INTRODUCTION

Looking at history, we can recognize that many areas of mathematics have developed in symbiosis with science and technology. For example, let us cite the development of differential calculus and differential equations in the 18th and 19th centuries in conjunction with analytical mechanics; the development of the theory of linear partial differential equations in conjunction with continuum mechanics and electromagnetism; in the 20th century, functional analysis and differential geometry have been paired with quantum mechanics and relativity, although the relation between the mathematics and physics has not been as close, direct and stimulating as in earlier times.

A new feature in the science and technology of our day is the use of the hybrid scientific-technological "system", analyzed, designed and run by computers. One has only to think of the space program, and the analogy with such 19th century facts as the development of electricity and the automobile, to gain a historical perspective. Just as the hybrid scientific-technological projects of the 19th century stimulated development of the mathematical topics cited above, in our day there has arisen an amorphous subject called System and Control Theory. By its nature it sprawls over disciplinary boundaries, and is difficult to pin down; one cannot go to the library or to most major universities and systematically learn about it as one would learn, say, differential equations or quantum mechanics. Of course there are textbooks--the treatises by Anderson and Moore [25], Brockett [26], Rosenbrock [272 and Wonham [28] are the best for our purposes. At the research level the journals *IEEE Transactions in Automatic Control, SIAM Journal of Control and Optimization*, and *Automatica* are standard ones. A striking characteristic of this material is its strong reliance on mathematics, in contrast to other major contemporary areas of science and technology. This is in the nature of the subject--the basic idea is to translate broad areas of science and technology into mathematical models that can be handled on the computer in a way that is as independent as possible of the origin of the models. Most of the research work involves analysis and probability theory, and some relatively concrete aspects of algebra (e.g., matrix theory, combinatorics) that were developed in the 19th century. (In fact, Gantmacher's treatise [6] is the only complete reference for some of the algebraic ideas.) However, there has also been a contact with differential geometry which has played a role in guiding research in the discipline. This report will focus on certain aspects of this interface with geometry, with emphasis on work that I have done with Clyde Martin on the geometry of *linear* systems. I will try to concentrate on areas which I believe have the greatest potential utility *combined* with the possibility of providing new research insights into System Theory. My overall thesis is that modern differential geometry is, among all areas of contemporary mathematics, the best qualified to provide new thought in diverse areas of science and technology.

A short glance at the literature suggests that probability theory and matrix algebra are closely linked to system and control theory, but why differential geometry? Now, differential geometry is most commonly thought of as the study of curves and surfaces in "our" Euclidean three space. Although it has expanded enormously in recent times, allied at times with topology, group-theory, and algebraic geometry, this classical theme remains the core. The point I want to bring to the foreground in this report is that system and control theory also involve curves and surfaces--in more exotic spaces, it is true, but the methodology of "modern" differential geometry has been oriented precisely to the task of making this generalization.

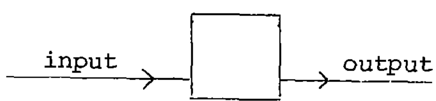
This methodology has involved the theory of what mathematicians call *differential manifolds*. Unfortunately this is a difficult subject to learn, since it involves fragments from all the major mathematical disciplines. At the point in this report that we get down to the real work, it will be necessary to assume that the reader has at least some acquaintance with its principles and notation--the treatises by Boothby [20], Dieudonné [21], and the author [22] may be cited. We shall also make use of material which interfaces differential and algebraic geometry--the standard reference is now the very recent book by Griffiths and Harris [23].

In geometry, a *manifold* is a space that looks locally like a piece of Euclidean space, but globally may not be one. This means that, given any point, one can coordinatize a neighborhood of that point with n real numbers (n is the *dimension* of the manifold), but that these number labels vary as the point moves over the space. However, as in tensor analysis, one wants to use these number assignments to perform the usual operations of differential and integral calculus. This requires that the number labels one assigns to different points (called *coordinate systems*) be related to each other in an *infinitely differentiable* ("C^∞") way. So, a space with a topological structure to assure over-all consistency, and a way of attaching local coordinates to points, not uniquely, but changing in a C^∞ way, is called a *differentiable manifold*. In modern mathematics, it has been found to be just about the most appropriately "general" geometric object to study, in the sense that it includes the class of Euclidean space and most of the other spaces (e.g., spheres, tori, surfaces) which appear most frequently, and yet its "differential" and "integral" structure can be studied in a unified way.

The basic idea in differential geometry is to study *geometric* objects with *analytic* techniques. (The idea of "curvature" of a curve or surface is a familiar example.) Often, it is most convenient to mediate between the geometry and analysis with *algebra*. (In fact, it was Descartes, in the 17th century, who taught us to do this.)

2. SOME GENERAL PRINCIPLES OF SYSTEM THEORY

System theory may be thought of as the study of machines, devices, automata, computers, physical objects, etc., from the point of view of *input-output behavior*. The famous "black box" is the key concept--a device which accepts certain inputs, subjects them to certain operations which depend on the "state" at which the box finds itself, and then puts out an output



The theory can be developed in this abstract context as a theory of mappings

$$\phi: (\text{space of inputs}) \times (\text{state}) \to (\text{space of outputs})$$

However, for the purposes of geometry it is most convenient to work in the context of differential equations and continuous time. An *input-output system* for us will be a system of ordinary first order differential equations of the form:

$$\frac{dx}{dt} = f(x,u)$$

$$y = g(x) \quad . \tag{2.1}$$

Here, $x = (x_1,\ldots,x_n)'$ is a real column vector, i.e., an element of R^n. It is called the *state vector*. $u = (u_1,\ldots,u_m)' \in R^m$ is the *input vector* $y = (y_1,\ldots,y_p)' \in R^p$ is the *output vector*. $x \to g(x)$ is *observation map*. The time parameter t runs over $0 \leq t < \infty$. We assume that all data and functions are sufficiently smooth (e.g., differentiability class C^∞) to perform all needed operations of differential and integral calculus, and to use freely the existence-uniqueness theorems for differential equations. The symbol ' stands for matrix transpose.

Choosing u as a function $\underline{u}: t \to u(t)$ of t, solving (2.1) for $t \to x(t)$ with a value x_0 of $x(t)$ at $t = 0$, then applying the observation map g, gives an *output curve* $\underline{y}: t \to y(t) = g(x(t))$. Thus, we obtain a mapping

$$(\text{curves in } R^m) \times R^n \to (\text{curves in } R^p)$$

$$\underline{u}, x_0 \to \underline{y}$$

which determines the system. The *input-output relations* are the set of pairs $(\underline{u},\underline{y})$ of curves in $R^m \times R^p$ which may appear in this way for some choice of x_0. In practical applications we are often given only these input-output relations, perhaps in partial, sampled-data, or approximate form. (We put to the side, for this work, the whole spectrum of stochastic problems. They also have an interesting and important geometric structure, but this is a much less well developed subject.)

This is the *open loop* description of a system. The alternative is the *closed loop*. A *feedback strategy* is a map $F: X \to U$. The solutions

$$t \to (u(t),\ x(t);\ y(t))$$

of (2.1) such that

$$u(t) = F(x(t))$$

are then defined. They are determined as solutions of the differential equations

$$\frac{dx}{dt} = f(x,F(x)) \quad . \tag{2.2}$$

Of particular importance are the feedback strategies which are *stable* in the sense that the system of differential equations (2.2) is stable, in one of the senses which are customary in differential equation theory.

A basic idea in system theory is to consider a collection, Σ, of systems and a corresponding collection IO of input-output relations. An important case is that where Σ depends on a finite number of parameters; call them

$$\lambda = (\lambda_1, \ldots, \lambda_r) \quad .$$

For example, Σ might be defined by differential equations of the following form:

$$\frac{dx}{dt} = f(x,u;\lambda)$$
$$y = g(x;\lambda) \quad . \tag{2.3}$$

Now, two different values λ, λ' of the parameters might give the same input-output relations. Thus, the corresponding input-output relations might depend on fewer parameters.

> The basic object of study in "Geometric" System Theory are parameterized families of systems and input-output relations.

It is in this study of "dependence on parameters" that modern differential geometry--based as it is on differential manifold theory--is so much more powerful than the traditional methodology. Every problem in System Theory has two aspects: First, study solutions for a single system, i.e., for a fixed value of the parameters, then see how this relation varies when the parameters are varied.

For our purposes, it is convenient to focus on three main problems:

A) *The Identification Problem.* Given a set of input-output relations, find a set of systems which realize these relations. Often it is not possible in problems with parameters; singularities are encountered. The goal is then to study how to avoid the singularities or to live with them. For nonlinear systems very little that is of great practical use is known about this problem.

B) *The Observer-Compensator Problem.* Given a system of form (2.1), design another system

$$\frac{d\hat{x}}{dt} = \hat{f}(\hat{x},u,y)$$

$$\hat{y} = \hat{g}(\hat{x}) \quad ,$$

where inputs are the inputs and/or outputs of (2.1), which perform in certain prescribed ways. For example, the "observer-tracker" problems require that $\hat{y}$ and x live on the same space, and that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0$$

C) *The Classic Regulator-Servomechanism-Optimal Control Problems.* I will not go into these in any detail--see the Treatises by Anderson-Moore [16] and Wonham [19]. Here, the traditional theory only is developed in the context of *linear systems*; about everything remains to be done for the nonlinear cases.

Each of these problems also has a "hierarchical" structure suggested by the wide spectrum of applications. I have mentioned the dependence-on-parameters as a "higher" problem than the fixed parameters one. Even "higher" than this are the *adaptive* and *decentralized-large scale* aspects, on which there has been much work (most of it inconclusive) in the last twenty years. The adaptive problem is probably the most interesting from the philosophical-social point of view (e.g., cybernetics, learning theory), as well as having wide practical ramifications (automata, self-tuning machines, robotics, artificial intelligence, etc.). Roughly, this involves a system with unknown, perhaps changing parameters, interaction with some outside system which can sample parts of the input and output. On the basis of these samples, it is desired--preferably by an iterative process--to estimate the parameters to a first approximation, perhaps then perform certain operations and adjustments, then measure and identify again, etc. hopefully converging on something useful. (There is some analogy with the theory of "measurement" in quantum mechanics; one must consider not only the system in isolation, but in interaction with certain measuring apparatus.)

Having as complete as possible a theory of systems depending on parameters is a necessary condition for a satisfactory adaptive control theory. There has recently been important progress in this area by Feuer and Morse [24], Goodwin, Ramadge and Caines [25], involving the analytical and stochastic, discrete-time side of system theory--it may be expected that analogous results may be obtained using geometric ideas for continuous time systems.

Although these concepts obviously have a wide range and validity, carrying them out in an appropriate mathematical context is a slow business which requires formidable mathematical experience and expertise. In other areas of science, it is possible to make use of *experiment* to get around mathematical difficulties. Thanks to computers, the analogue of "experimentation" is *possible* in System Theory, but is not as completely satisfactory as in the traditional physical sciences. After all, if one spends several million dollars simulating an aircraft control system, it is of no particular use in the study of the US economy, although in principle they might involve much the same sort of system-theoretic structure, whereas data one collects about most basic physical systems is of wide utility in many other contexts. The difference lies in the role that "physical laws" play in the two disciplines! Of course, the *ideal* would be to have a close interactive relation between advances in theory--which are eventually going to be highly mathematical!--and efforts to carry out useful modelling and calculation.

As in mathematical physics, the precise, but nonlinear, system equations must be linearized for any hope of tractability. Although in fundamental physics we are beginning to reach situations where the essential "nonlinearity" cannot be ignored, in System Theory--a much younger discipline--we are still in the linear stage. In fact, the theory of linear system, when approached from this general point of view, presents enough challenges to the mathematician. As I hope to show in this report, the rich geometric structure of the linear systems is still relatively unexplored! This is especially marked when one tries to expand known linear theory to cover distributed parameter and two-dimensional problems.

3. NOTIONS OF LINEAR SYSTEM THEORY

From the general point of view of Section 2, a *linear system* is one of form (2.1) which is linear in x and u, i.e., of the form:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \quad , \end{aligned} \tag{3.1}$$

where A,B,C are matrices of appropriate size. However, it is more in tune with modern differential geometry to use, instead of the classical and more "concrete" matrix theory, the theory of finite dimensional vector spaces. (Notice that Wonham's treatise [28] is written from this point of view. This formalism lends itself to a greater precision of thought and statement than classical matrix theory, although it is, of course, completely equivalent to it. Another motivation is that it is better suited to application of powerful group-theoretic ideas.)

Thus, let x belong to a vector space X, called *state space*; $u \in U$, *control space*; $y \in Y$, *output space*. If V,W are vector spaces, $L(V,W)$ denote the vector space of linear maps $V \to W$. (Thus, if $V = R^n$, $W = R^m$, $L(V,W)$ is the space of $m \times n$ matrices.)

The system (3.1) is then determined by the tuple (A,B,C) with

$$A \in L(X,X); \quad B \in L(U,X); \quad C \in L(X,Y) \quad .$$

Let Σ be the space of all systems of form (3.1). As a space, it is just the Cartesian product $L(X,X) \times L(U,X) \times L(X,Y)$.

The structural properties of system (2.1), called *controllability* and *observability* are important; it will be assumed that the reader is familiar with them. (For linear systems, see any of the treatises cited above. For nonlinear systems, see [4].) The following results are well-known:

The system (3.1) is *controllable* iff

$$X = B(U) + AB(U) + A^2B(U) + \cdots \tag{3.2}$$

It is *observable* iff

$$0 = \text{kernel }(C) \cap \text{kernel }(CA) \cap \cdots \tag{3.3}$$

To the system (3.1) assign the *impluse response*

$$t \to Ce^{tA}B \tag{3.4}$$

or, its Laplace transform

$$T(s) = C(s-A)^{-1}B \quad , \tag{3.5}$$

which is called the *transfer function* or *frequency response.*

The importance of these mathematical objects is that two linear systems, with the same input and output spaces, have the same input-output relations if and only if their transfer functions are the same.

$T(s)$ admits a Laurent series in s:

$$\begin{aligned} T(s) &= \frac{1}{s} C \left(1 - \frac{A}{s}\right)^{-1} B \\ &= \frac{1}{s}\left(CB + \frac{CAB}{s} + \frac{CA^2B}{s^2} + \cdots\right) \end{aligned}$$

The *Hankel data* is:

$$H = (CB, CBA, \ldots, CA^nB, \ldots) \tag{3.6}$$

Here is a basic theorem about linear systems.

Theorem 3.1. Suppose given one linear system (3.1) which is controllable and observable. Let

$$\begin{aligned} \frac{dx}{dt} &= \hat{A}x + \hat{B}u \\ y &= \hat{C}x \end{aligned} \tag{3.7}$$

be another linear system which has the same input-output data as (3.1). Then, there is an invertible linear map $g: X \to X$ which takes the system (3.7) into the system (3.1). This implies that $(\hat{A},\hat{B},\hat{C})$ is related to (A,B,C) via the following formulas:

$$\boxed{\begin{aligned} A &= g\hat{A}g^{-1} \\ B &= g\hat{B} \\ C &= \hat{C}g^{-1} \end{aligned}} \tag{3.8}$$

These formulas can be interpreted in an interesting group-theoretic way. Let $GL(X)$ denote the group of all invertible linear maps: $X \to X$. (Thus if $X = R^n$, $GL(X)$ is the group of all $n \times n$ real matrices of nonzero determinant. In Lie group theory, this is denoted as $GL(n,R)$. Thus, $GL(X)$ is isomorphic to $GL(n,R)$, where n is the dimension of the vector space X.) Let Σ_{co} be the subset of all

$$(A,B,C) \in L(X,X) \times L(U,X) \times L(X,Y)$$

such that the system (3.1) is controllable and observable. $L(X,X) \times L(U,X) \times L(X,Y)$ is a vector space (of dimension $n^2 + nm + np$, where n = dimension of states, m = dimension of inputs, p = dimension of outputs) and Σ_{co} is an open subset of this vector space. Formulas (3.8) define a transformation group action of $GL(X)$ on Σ_{co}.

Remark. Here it is useful to keep in mind some general concepts concerning Lie groups acting as transformation groups acting on manifolds, which will be briefly reviewed. On page 3, the notion of "differentiable manifold" has been recalled.

Groups are encountered as algebraic structures. (By definition, a group is a set G, together with a map $G \times G \to G$ satisfying certain natural rules.) Groups with the "local Euclidean" structures of manifolds (so that they can be studied with the methods of calculus) are common. They are called *Lie groups* (after Sophus Lie, a 19th century mathematician who first isolated and studied them. He and Elie Cartan, a French mathematician who worked from 1890-1950, are the Mozart and Beethoven of the subject.) Precisely, they are sets G with a manifold structure and a group multiplication operation $G \times G \to G$ which is a differentiable map.

Let Z be a manifold, G a Lie group. An *action* of G on Z is a mapping $G \times Z \to Z$ denoted as $(g,z) \to gz$, such that

$$g_1(g_2z) = (g_1g_2)z$$

for $g_1, g_2 \in G$, $z \in Z$

$$1z = z$$

where "1" is the unit element of G.

The *orbits* are the subsets of Z of the form

$$Gz \quad ,$$

i.e., the set of points of Z which can be reached by applying all operations of G to one point of Z. If

$$G^z = \{g \in G: \quad gz = z\} \quad ,$$

G^z is called the *stability* or *isotropy* subgroup of G at z. The orbit Gz is then identified with the coset space G/G^z. The action of G is said to be *simple* if G^z = identity, for all $z \in Z$.

A space W together with a map $\pi: Z \to W$ is said to be an *orbit space* for the action of G if the following condition is satisfied:

> For each $w \in W$, the fiber $\pi^{-1}(w)$ is equal to an orbit of G.

We often denote this orbit space as $G \backslash Z$.

Here is the motivation for this terminology. Let us say that two points z_1, z_2 are *equivalent* (under the action of G) if there is a $g \in G$ such that $gz_1 = z_2$. This defines what is called an "equivalence relation". A subset S of Z is called an *equivalence class* if, whenever $z_1, z_2 \in S$, z_1 and z_2 are equivalent, and if $z \in S$, all z_1 which are equivalent to z also lie in S. One can now construct a *new* set, whose elements are precisely the equivalence classes; it is called the *quotient* by the equivalence relation. In this group-action case, the equivalence classes are the *orbits*, and G acts on the *left*, hence it is appropriate to denote it as "$G \backslash Z$", read "quotient of Z by G, acting on the left". The reader should also be aware of the notation G/H for the coset space of a group G by a subgroup H. This is an orbit space for H acting on the *right*: $(g,h) \to gh^{-1}$).

Return to system theory. The orbit space

$$\bar{\Sigma} \equiv GL(X) \backslash \Sigma_{co} \tag{3.9}$$

can be identified with either the transfer function $T(s)$ (which is a rational function of the complex variable s, with values in $L(U,Y)$) or the Hankel data H. Each of these objects is an element of an infinite dimensional space.

One can prove that the action of $GL(X)$ in Σ_{co} is *simple*, i.e., the isotropy subgroup of each point of Σ_{co} is the identity. Further, one can prove that this orbit space is a *manifold* and that the natural projection map $\Sigma_{co} \to \bar{\Sigma}$ is a *submersion*. One can also prove that this manifold does not have a single, global coordinate system, nor is any relatively simple and amenable description of it in terms of algebraic equations known. Further, the map $\Sigma_{co} \to \bar{\Sigma}$ is a *principal fiber bundle*, with base $\bar{\Sigma}$ and structure group $GL(X)$. (See [27].) It is known that this bundle has non-trivial topological invariants.

The "identification problem" can now be stated in the following way:

> Given a parameter space, Λ, and a map $\phi: \Lambda \to \Sigma_{co}$, let $\bar{\phi}: \Lambda \to \bar{\Sigma}$ be the map which assigns to each $\lambda \in \Lambda$ the input-output data of the system $\phi(\lambda)$. Knowing only $\bar{\phi}$, find an algorithm to lift $\bar{\phi}$ to a map $\alpha: \Lambda \to \Sigma_{co}$.

Thus, the first mathematical question is: Given a map $\bar{\phi}: \Lambda \to \bar{\Sigma}$, does a lifting map $\alpha: \Lambda \to \Sigma_{co}$ *exist*. More important, does it have appropriate stability, smoothness and robustness properties so that approximations constructed in conditions of practice (e.g., on a computer) can be used with some confidence. (Mathematically, this is a question of "structural stability"--do mappings $\beta: \Lambda \to \bar{\Sigma}$ which are in some sense close to $\bar{\phi}$ have a lifting which is close to the lifting of $\bar{\phi}$.)

In any case, the topological fiber bundle $(\Sigma_{co}, \bar{\Sigma}, GL(X))$ must inevitably play a role in identification theory. (Much the same sort of questions arise in *stochastic* identification theory, which is, in fact, the topic which is more often encountered in practice.)

As an example, here is a description of $\bar{\Sigma}$ in the simplest case, namely:

$$\dim X = 2 ; \qquad \dim U = \dim Y = 1 \quad .$$

The transfer function $T(s)$ is then a rational function of the form

$$T(s) = \frac{b_0 + b_1 s}{s^2 + a_1 s + a_0} \quad . \tag{3.10}$$

Note that Σ has dimension $4+2+2 = 8$. $GL(X)$ has dimension four. Hence, $\bar{\Sigma}$ has dimension $8-4 = 4$. Note that $T(s)$ also has four free parameters, a_0, a_1, b_0, b_1.

However, not all of these parameters in $T(s)$ correspond to controllable-observable systems with state space dimension two. We must throw away the

T(s) whose numerator and denominator have a common root. The roots of all denominators are

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_1}}{2} \quad .$$

Thus, $G \backslash \Sigma_{co}$ can be identified with the subset of (a_0, a_1, b_0, b_1)-space, i.e., R^4, such that

$$b_0 + b_1 \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_1}}{2} \right) \neq 0 \quad . \tag{3.11}$$

R. Brockett and V. Krishnaprasad [20,29] have studied diverse geometric and topological properties of the orbit space $\bar{\Sigma}$ in the case of single input-output systems.

4. STATE FEEDBACK AND LUENBERGER OBSERVERS OF LINEAR SYSTEMS

Continue to focus on the space of linear systems (3.1) with given input, state and output spaces. We have seen that one group acts on this space, GL(X), the *group of changes of basis in state space*. We have seen that the action of this group, particularly the structure of its orbit space, is a pertinent question in an important practical problem--the identification problem. We will now show that there is different group action associated with another important practical topic, *feedback*.

Consider an open-loop solution

$$t \to (x(t), u(t), y(t))$$

of the system

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \quad . \end{aligned} \tag{4.1}$$

Let $F: X \to U$ be a linear map. Use it to construct another curve

$$t \to (\hat{x}, \hat{u}, \hat{y})(t)$$

with

$$\hat{x} = x; \quad \hat{y} = y; \quad \hat{u}(t) = u(t) + Fx(t) \quad .$$

Then,

$$\frac{d\hat{x}}{dt} = A\hat{x} + B(\hat{u} - F\hat{x})$$

$$= (A - BF)\hat{x} + B\hat{u}$$

$$\hat{y} = C\hat{x} \quad .$$

The curve $(\hat{u},\hat{x},\hat{y})$ is then an open loop solution of the system defined by the triple $(\hat{A},\hat{B},\hat{C})$, with:

$$\begin{aligned} \hat{A} &= A - BF \\ \hat{B} &= B \\ \hat{C} &= C \end{aligned} \tag{4.2}$$

For each F, we obtain a linear transformation

$$\Sigma \to \Sigma$$

This map is called the (state) *feedback* transformation. As F varies over $L(X,U)$, it defines a group transformation on Σ called the (state) *feedback group*. It is an *abelian group*, since the result of applying two feedbacks F,F' successively is to obtain the feedback $F+F'$.

A key question is then to describe when two systems differ by feedback. We shall investigate this point in the next section by the introduction of a powerful algebraic theory, the *Kronecker pencil theory*.

A related question is to construct *Luenberger observers* (or *deterministic state estimates*) for the linear system (4.1). Such an object may be constructed using the following chain of reasoning. Let us consider systems with the state space as (4.1), but whose input space is the direct sum of the input and output spaces of (4.1). Such a system would have the following form:

$$\frac{d\hat{x}}{dt} = \hat{A}\hat{x} + \hat{B}\hat{u} + K\hat{y} \quad . \tag{4.3}$$

Since we are only interested in states, there is no need to put outputs into (4.3). Let us consider an open loop solution (x,u,t) of (4.1), and feed its inputs and outputs into the system (4.3). Set:

$$e(t) = x(t) - \hat{x}(t) \quad . \tag{4.4}$$

Consider e as an "error", which we would like to drive to zero. This will be done by arranging things so that e satisfies a *linear* differential equation which is asymptotically stable, i.e., such that all solutions go to zero.

There are two questions here; one algebraic, the other analytic. Suppose that $e(t)$ satisfies an equation of the following form:

$$\frac{de}{dt} = \alpha e \quad , \tag{4.5}$$

with $\alpha \in L(X,X)$. Let us work out the algebraic equations which must be satisfied:

$$\begin{aligned} \frac{de}{dt} &= \frac{dx}{dt} - \frac{d\hat{x}}{dt} \\ &= Ax + Bu - \hat{A}\hat{x} - \hat{B}u - Ky \\ &= Ax + Bu - \hat{A}(x-e) - \hat{B}u - KCx \quad . \end{aligned}$$

Thus (4.5) requires that the following conditions be satisfied:

$$\begin{aligned} B &= \hat{B} \\ \hat{A} &= A - KC \\ \alpha &= A - KC \end{aligned} \tag{4.6}$$

These formulas again define a transformation from one system to the other, depending on the linear map K. Notice that K plays a dual role to that of F used for "feedback", i.e., one transforms K into an F for another system by dual operation of linear maps (or matrix transpose).

In order that the system (4.3) serve as an "observer" or "state estimator" for the system (4.1), it is necessary that all solutions (4.5) go to zero. This requires that:

All eigenvalues of $\alpha = A - KC$ have negative real parts.

Since eigenvalues are invariant under duality operation, this is the same thing as requiring that $\alpha^* = A^* - C^*K^*$ be stable: hence, K^* serves as a *stabilizing feedback* for the dual system.

Clyde Martin and I have suggested [26] a method for finding such a K that fits in well with the point of view of this paper. Given $A,B,C,$ we construct the following map

$$\phi_{(A,B,C)}(g,K) = g(A-KC)g^{-1}$$

of $GL(X) \times L(Y,X) \to L(X,X)$

We proved that if the system (4.1) is observable, then the map $\phi_{(A,B,C)}$ is a *submersion map* on some non-empty open subset of $GL(X) \times L(Y,X)$.

Remark. Here is more differential-geometric background. Let $\pi\colon Z \to W$ be a map between manifolds with the dimension of Z greater than or equal to the dimension of W. For $z \in Z$, let Z_z be the tangent space to Z at z. Let $\pi_*\colon Z_z \to W_{\pi(z)}$ be the linear map on tangent vectors induced by π. z is said to be a *regular point* if $\pi_*(Z_z) = W_{\pi(z)}$. π is said to be a *submersion* if each point is a regular point. (Think of "submersion" as "projection".)

This suggests the "robustness" property that is needed for handling dependence on parameters and related adaptive control questions. For example, if (A,B,C) depend on parameters λ, one might want to find a $K(\lambda)$ depending on λ, so that the error equations (4.5) had a desired rate of convergence to zero The implicit function theorem offers a useful tool for deciding when this is possible. Suppose again that $\pi\colon Z \to W$ is a map between manifolds and that Λ is another manifold (which serves as the "parameters"). Let $\alpha\colon \lambda \to w(\lambda)$ be a map $\Lambda \to W$. Our basic problem is to find a map $\beta\colon \lambda \to z(\lambda)$ such that $\pi z(\lambda) = w(\lambda)$. This means that the following diagram of maps is "commutative"

$$\begin{tikzcd} \Lambda \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & Z \arrow[dl, "\pi"] \\ & W & \end{tikzcd}$$

β is said to be a "lifting" of α. The "local" existence of such a lifting map can be decided using the implicit function theorem if π is a submersion. The question of its global existence, and the situation where π is not everywhere a submersion, i.e., has "singularities", is more complicated and must be attacked with more powerful tools from topology and the theory of singularities of mappings.

Another interesting direction of research is the generalization of this argument to cover more complicated situations or to take advantage of different experimental data. Here is one possibility and an illustration.

Suppose that $t \to \hat{\hat{x}}(t)$ is a curve in X with

$$e = x - \hat{\hat{x}} ,$$

such that e satisfies an equation of the following form:

$$\frac{d^2e}{dt^2} = \alpha e \quad . \tag{4.7}$$

Now, from (4.1),

$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + B\frac{du}{dt}$$

$$= A(Ax+Bu) + B\frac{du}{dt}$$

$$\frac{d^2e}{dt^2} = A^2x + ABu + B\frac{du}{dt} - \frac{d^2\hat{x}}{dt^2}$$

$$= \alpha x - \alpha\hat{x} \quad .$$

We can satisfy this relation if $\hat{x}$ satisfies an equation of the following type:

$$\frac{d^2\hat{x}}{dt^2} = \alpha\hat{x} - B\frac{du}{dt} - ABu - Ky \quad , \tag{4.8}$$

with

$$\alpha = A^2 - KC \quad . \tag{4.9}$$

Again, we have the question of choosing K so that equation (4.7) is asymptotically stable. There are clearly many schemes like this one to find asymptotic state estimators for a linear system. It is a major open problem to usefully generalize these schemes to cover nonlinear systems.

5. THE KRONECKER THEORY OF PENCILS OF LINEAR MAPS

Although the Kronecker theory is one of the highlights of 19th century algebra and invariant theory, it is very difficult to find complete expositions of it. The only one in recent times is in Gantmacher's magnificent *Theory of Matrices* [6], but even that becomes obscure at a key place. In [7] I translated Gantmacher's ideas into coordinate-free language, and (I believe) made it clearer, but did not fill in that expository gap. Here I want to present it in a different way, which is much more closely attuned to System Theory.

Before plunging into the abstract vector-space treatment of Kronecker's ideas, which we will find most useful for system theory, it is useful to recall the more classical way that the theory developed in terms of matrices whose entries are polynomials. (See Gantmacher [6] and Rosenbrock [18] for more details.) Let $M(s)$ and $M_1(s)$ be $m\times n$ matrices, whose coefficients are polynomials in a variable s. Let us say that $M_1(s)$ is *equivalent* to $M(s)$ (in Weierstrass's sense) if there are $m \times m$ and $n \times n$ matrices $A(s)$, $B(s)$ with polynomial entries, such that

$$M_1(s) = A(s)M(s)B(s) \tag{5.1}$$

$$\det A(s) = 1 = \det B(s) \quad .$$

The "invariants" of matrices under this equivalence relation are the so-called *elementary divisors*, i.e., the diagonal entries after the matrix is put into a "canonical form" after elementary row and column operations. Let us say that $M_1(s)$ and $M(s)$ are equivalent *in the sense of Kronecker* if a relation (5.1) holds, with A and B *independent of* s. $M(s)$ is called a *pencil of matrices* if each polynomial occurring in its entries is at most first degree. A remarkable theorem, proved in fact by Kronecker himself, is that two *pencil* matrices are Kronecker-equivalent if and only if they are Weierstrass equivalent. The "Kronecker pencil theory", which we describe briefly below, then provides a constructive *algorithm* for describing these equivalence classes. Such "pencils" of matrices are often encountered in system theory, and the Kronecker theory is *the* basic structural fact in many applications. The term "pencil" comes from geometry--the coefficients of matrices, when set equal to constants, define straight lines, since they are linear in s. Think of a "pencil" as a geometric collection of straight lines.

Now return to abstract vector spaces. Let $V, W, \ldots$ be finite dimensional vector spaces over the real or complex numbers as field of scalars. (In fact, everything carries over to arbitrary scalar fields.) $L(V,W)$ denotes the vector space of linear maps

$$\alpha: V \to W$$

$L(V)$ denotes $L(V,V)$. $GL(V)$ denotes the group of $A \in L(V)$ such that A^{-1} exists. The product group $GL(W) \times GL(V)$ acts on $L(V,W)$:

$$(g_1, g_2)(\alpha) = g_1 \alpha g_2^{-1}$$

$$\text{for } g_1 \in GL(W), \quad g_2 \in GL(V) \quad .$$

The *orbits* of this group are readily described: They are the elements of $L(V,W)$ of a given rank and nullity. The quantities $\dim(\text{kernel } \alpha)$, $\dim(\alpha(V))$ are the *invariants* of this action and label the orbits. Thus, we are dealing with a transformation group with only a finite number of orbits and with an algorithm to construct the invariants of the group action. It is a very difficult problem to do this for *general* transformation group actions--it is unknown for all but a few cases. (For example, it was a major labor of 19th century invariant theory to do this in *part* for algebraic actions of $SL(2,C)$, the group of 2×2 complex matrices of determinant one. Thus, invariant theory "died" before it could be extended to other groups, and to this day it has barely begun to revive.)

The Kronecker pencil theory fits in very well with this point of view. Consider $GL(W) \times GL(V)$ acting on $L(V,W) \times L(V,W)$ as follows:

$$(g_1,g_2)(\alpha_0,\alpha_1) = (g_1\alpha_0 g_2^{-1}, g_1\alpha_1 g_2^{-1})$$

$$\text{for } g_1 \in GL(W), \quad g_2 \in GL(V), \quad \alpha_0,\alpha_1 \in L(V,W) \quad .$$

The Kronecker theory is an algorithm for computing orbits, invariants, and "canonical forms" for orbits for this particular transformation group action. This is one of the very rare "invariant theoretic" situations which can be described in this way. The theory of "quivers" due to Gabriel and Gelfand [8] offers one general insight into this.

Suppose $\alpha_0,\alpha_1 \in L(V,W)$. A pair $(V' \subset V, W' \subset W)$ of linear subspaces is said to reduce (α_0,α_1) if

$$\alpha_0,\alpha_1(V') \subset W' \quad .$$

Such a pair is said to be a *Kronecker reduction* if there is a map

$$\beta: V' \to V'$$

such that the following conditions are satisfied:

$$\alpha_1 = -\alpha_0\beta \tag{5.2}$$

$$\beta \text{ is nilpotent} \quad , \tag{5.3}$$

i.e., $\beta^n = 0$ for n sufficiently large

$$V' = \text{kernel}(\alpha_0) + \beta(V') \quad .$$

Such a triple $(V',W';\beta)$ is said to be an *elementary* Kronecker reduction if the following additional conditions are satisfied:

$$\dim(\text{kernel } \alpha_0) = 1 \quad . \tag{5.4}$$

Here is the motivation for this definition in terms of the classical material. Consider the *pencil* of linear maps

$$\alpha(s) = \alpha_0 + s\alpha_1 \quad . \tag{5.5}$$

(Thus, the term "pencil" refers to the fact that $\alpha(s)$ is a *linear* polynomial in s.) Consider vector-valued polynomials

$$s \to v_0 + v_1 s + \cdots + s^n v_n$$

in

$$v_0, \ldots, v_n \in V$$

such that

$$\alpha(s)(v(s)) = 0 \tag{5.6}$$

Theorem 5.1. (Kronecker). If $\alpha(s)$ is a singular pencil in the sense that

$$\text{kernel } \alpha(s) \neq 0$$

$$\text{for all } s \quad ,$$

but

$$\text{kernel } (\alpha_0) \cap \text{kernel } (\alpha_1) = (0) \quad ,$$

and if $v(s)$ is chosen to be the *minimal* degree polynomial such that (5.6) is satisfied, then the elements $v_0, \ldots, v_n$ are linearly independent. Further, if V' is a linear subspace of V whose basis is $v_0, \ldots, v_n$, then

$$\alpha_0(V') = \alpha_1(V') = W' \quad .$$

There are linear subspaces $V'' \subset V$, $W'' \subset W$ such that

$$\begin{aligned} V &= V' + V'' \\ W &= W' + W'' \\ \alpha(s)(V'') &\subset W'' \quad , \qquad \text{for all } s \quad . \end{aligned} \tag{5.7}$$

Thus, the process can be applied again to $\alpha(s)$ acting on V''. □

In order to see where the nilpotent map β comes in, let us make (5.6) explicit:

$$0 = (\alpha_0 + s\alpha_1)(v_0 + sv_1 + \cdots + s^n v_n)$$

$$= \alpha_0 v_0 + s\alpha_0 v_1 + \cdots + s^n \alpha_0 v_n + s\alpha_1 v_0 + s^2 \alpha_1 v_1 + \cdots \quad .$$

Equating the coefficients of powers of s to zero gives the following relations:

$$\alpha_0(v_0) = 0 \tag{5.8}$$

$$\alpha_1 v_n = 0 \tag{5.9}$$

$$\begin{aligned} \alpha_0 v_1 + \alpha_1 v_0 &= 0 \\ \alpha_0 v_2 + \alpha_1 v_1 &= 0 \\ &\vdots \\ \alpha_0 v_n + \alpha_1 v_{n-1} &= 0 \quad . \end{aligned} \tag{5.10}$$

Let V' be the linear subspace of V spanned by the vectors $v_0,\ldots,v_n$. Define

$$\beta\colon V' \to V'$$

as follows:

$$\begin{aligned} \beta(v_0) &= v_1 \\ \beta(v_1) &= v_2 \\ &\vdots \\ \beta(v_n) &= 0 \end{aligned} \tag{5.11}$$

Remark. β is the map which operator-theorists call the *shift*. It plays an important underlying role in many systems-theoretic areas.

We can now rewrite relations (5.10) as:

$$\begin{aligned} \alpha_1 v_0 &= -\alpha_0 \beta v_0 \\ \alpha_1 v_1 &= -\alpha_0 \beta v_1 \\ &\vdots \\ \alpha_1 v_{n-1} &= -\alpha_0 \beta v_{n-1} \end{aligned} \tag{5.12}$$

(5.9) means that

$$\alpha_1 v_n = -\alpha_0 \beta v_n = 0$$

Thus, relations (5.9)-(5.10) can be summarized in the following relation:

$$\alpha_1 v' = -\alpha_0 \beta v' \tag{5.13}$$

for all $v' \in V'$.

Only relation (5.8), i.e., $v_0 \in \text{kernel } \alpha_0$ is not implied by this relation.

Notice that we can write the polynomial map $s \to v(s)$, which satisfies (5.6) in a more convenient basis-independent form:

$$v(s) = (1 - s\beta)^{-1}(v_0) \quad . \tag{5.14}$$

(This formula might be useful for generalization to infinite dimensional situations.)

We can now continue to split up V'' in this way until there are no more polynomials with coefficients in V which annihilate $\alpha(s)$. Then we work on the dual situation. Finally, we end up with a non-singular pencil. This process can be described in an overall way as follows.

Theorem 5.2 (Kronecker). Let $\alpha_0, \alpha_1 \colon V \to W$ be a pair of linear maps. V and W can be split up into direct sums

$$V = V' \oplus V'' \oplus V'''$$

$$W = W' \oplus W'' \oplus W'''$$

such that:

$$\alpha_0, \alpha_1 (V', V'', V''') \subset W', W'', W''' \quad ,$$

i.e., each of the pairs of linear spaces reduces the pairs (α_0, α_1) of linear maps. Further, there are *nilpotent* linear maps

$$\beta' \colon V' \to V'$$

$$\beta'' \colon W'' \to W''$$

such that the following conditions are satisfied:

$$\alpha_1 = \alpha_0\beta' \quad \text{on} \quad V' \tag{5.15}$$

$$\alpha_1 = \beta''\alpha_0 \quad \text{on} \quad V'' \tag{5.16}$$

$$V' = (\text{kernel } \alpha_0) \oplus \beta(V') \tag{5.17}$$

$$\alpha_0(V') = W' \tag{5.18}$$

$$W'' = \alpha_0(V'') + \text{kernel } \beta' \tag{5.19}$$

For all but a finite number of s's, $\alpha_0 + s\alpha_1$ is an isomorphism $V''' \to W'''$, i.e., this component of the pencil is "regular". □ (5.20)

This marvelous theorem gives a complete description of the algebraic structure of pairs of linear maps. As I mentioned above, it is a very rare event in invariant theory when such a complete story is available. Luckily, many of the situations encountered in the theory of linear, time-independent one-dimensional input-output systems involve pairs of maps. We will now look at another, more geometric, way of analyzing the structure of pairs of linear maps.

6. THE VECTOR BUNDLE ASSOCIATED WITH PAIRS OF LINEAR MAPS

Continue with V,W as finite dimensional vector spaces, and with α_0,α_1 as a pair of linear maps $V \to W$. Suppose also that the complex numbers $\mathbb{C}$ are the field of scalars. $\mathbb{C}^2$ denotes the set of pairs (s_0,s_1) of complex numbers. $\mathbb{C}^\#$ denotes the multiplicative group of nonzero complex numbers. $\mathbb{C}^\#$ acts on $\mathbb{C}^{2,\#}$ as follows: ($\mathbb{C}^{2,\#}$ denotes the pairs $(s_0,s_1) \in \mathbb{C}^2$ with either s_0 or $s_1 \neq 0$.)

$$\lambda(s_0,s_1) = (\lambda s_0,\lambda s_1) \quad .$$

The orbit space $\mathbb{C}^\#\backslash\mathbb{C}^{2,\#}$ is $P_1(\mathbb{C})$, the one (complex) *dimensional projective space.* (Alternate names are the *projective line* and the *Riemann sphere.*)

For each $s = (s_0,s_1) \in \mathbb{C}^{2,\#}$, let

$$V(s) = \{v \in V\colon (s_0\alpha_0 + s_1\alpha_1)(v) = 0\} \; .$$

Note that

$$V(\lambda s) = V(s)$$

$$\text{for } \lambda \in \mathbb{C}^{\#} \quad .$$

Thus, the assignment

$$s \rightarrow V(s)$$

of a complex vector space to each $s \in \mathbb{C}^{2,\#}$ is constant on the orbits of $\mathbb{C}^{\#}$, hence, to each orbit $\pi(s) \in P_1(\mathbb{C})$ one can assign the vector space $V(s)$. This assignment defines a *complex vector bundle* E, [10,11,32] namely, the set of pairs $(\pi(s),V(s))$, $s \in \mathbb{C}^{2,\#}$. It is called the *kernel bundle* of the pair (α_0,α_1). It is *holomorphic*, i.e., defined by holomorphic functions. Of course, it may be *singular*, i.e., the dimension of the fibers may vary. If it is non-singluar, a theorem of Grothendieck [12] gives the precise structure, a *direct sum of complex line bundles*. (A complex line bundle is a complex vector bundle with one-dimensional complex vector space fibers.)

Since the "vector bundle" concept is a central one in modern differential geometry, a few words about the general setting are appropriate. A *vector space* is a set which has two compatible algebraic structures: an addition and a scalar multiplication by elements of a field of scalars (the real or complex numbers, in the typical situations). Let S be a space. Suppose that for each $s \in S$, is given a vector space $V(s)$. This forms a space E, which consists of the pairs (s,v), with $s \in S$, $v \in V(s)$. Assign to $(s,v) \in E$ the point $s \in S$; this defines a map $\pi\colon E \rightarrow S$. The *fibers* of π, i.e., the sets $\pi^{-1}(s)$, are then the vector spaces $V(s)$. E is a "bundle" of vectors, in the sense that to each point $s \in S$ one sees attached the collection of vectors in $V(s)$.

For certain applications, particularly in algebraic geometry [23], it is useful to put additional structures on such vector bundles. Suppose that S is a *complex analytic manifold*, in the sense that local coordinates for S can be chosen as complex numbers, with the maps which interrelate two such coordinate systems given by holomorphic functions. (Recall that, in complex function theory, a "holomorphic" ($\equiv$ "complex analytic") function is one defined on open subsets of complex Euclidean space whose derivatives exist at each point.) In addition, suppose that the field of scalars for the vector spaces $V(s)$ are the complex numbers. Then, we say that (E,π,S) is a *complex analytic* or *holomorphic vector bundle* if E can be made into a complex analytic manifold such that π is a complex analytic map.

For this special way of defining the bundle using the pair (α_0,α_1) of linear maps, another, purely algebraic, analysis of the bundle is available using the Kronecker theory outlined in previous sections. (Grothendieck's theorem uses "transcendental", non-algebraic tools.) For example, suppose the first part of the Kronecker reduction process succeeds in filling up V completely:

$$V'' = V''' = (0) \quad .$$

Then, the following conditions are satisfied:

$$\alpha_1 = -\alpha_0\beta$$

$$\text{kernel } \alpha_0 \oplus \beta(V) = V$$

with $\beta\colon V \to V$ a *nilpotent* map. For $s = (s_0, s_1) \in \mathbb{C}^{2,\#}$,

$$V(s) = \text{set of all } v \in V \text{ such that}$$

$$\begin{aligned} 0 &= (s_0\alpha_0 + s_1\alpha_0\beta)v \\ &= \alpha_0(s_0 + s_1\beta)(v) \\ &= 0 \quad , \end{aligned}$$

or

$$(s_0 + s_1\beta)(v) \in \text{kernel } \alpha_0 \quad . \tag{6.1}$$

Now, if $s_0 \neq 0$,

$$\begin{aligned} V(s) &= \left\{ v \in V\colon \left(1 + \frac{s_1}{s_0}\beta\right)(v) \in \text{kernel } \alpha_0 \right\} \\ &= \left(1 + \frac{s_1}{s_0}\beta\right)^{-1} (\text{kernel } \alpha_0) \quad . \end{aligned}$$

For $s_0 = 0$,

$$V(s) = v \in V\colon \beta(v) \in \text{kernel } \alpha_0$$

and in view of (6.1),

$$\begin{aligned} V(s) &= \{v \in V\colon \beta(v) = 0\} \\ &= \text{kernel } \beta \quad . \end{aligned}$$

Using (6.1) again to compute the dimension of kernel β:

$$\dim V = \dim \text{kernel } (\alpha_0) + \dim \beta(V)$$

$$= \dim \text{kernel } (\beta_0) + \dim V - \dim (\text{kernel } \beta)$$

or

$$\dim \text{kernel } (\beta) = \dim \text{kernel } (\alpha_0) \quad . \tag{6.2}$$

This proves one of my main results.

<u>Theorem 6.1</u>. If $\alpha_0, \alpha_1 \colon V \to W$ are linear maps with only the "singular" Kronecker components present (i.e., the "cosingular" and "regular" components vanish), then the kernel bundle is non-singular. $\square$

We can use the nilpotent map β to exhibit the Grothendieck decomposition of the kernel bundle of (α_0, α_1) into line bundles. For $v_0 \in$ kernel α_0, $(s_0, s_1) \in \mathbb{C}^{2,\#}$, with $s_0 \neq 0$, set:

$$v_0(s_0, s_1) = \left(1 + \frac{s_1}{s_0}\beta\right)^{-1} (v_0) \quad . \tag{6.3}$$

This defines a cross-section of E over the open subset

$$P_1(\mathbb{C}) - (\text{the "point at infinity"}) \quad .$$

The one-dimensional linear singular space of $V(s_0, s_1)$ spanned by the value $v_0(s_0, s_1)$ of the cross-section can be continued to the point at infinity of $P_1(\mathbb{C})$, since

$$s_0^{n-1}\, v_0(s_0, s_1) = s_0^{n-1}\left(v_0 - \frac{s_1}{s_0} v_0 + \cdots + (-1)^{n-1}\left(\frac{s_1}{s_0}\right)^{n-1} \beta^{n-1}\, v_0\right)$$

$$\to (-1)^{n-1}\, s_1^{n-1}\, \beta^{n-1}(v_0)$$

as $s_0 \to 0$. (n = least integer such that $\beta^n v_0 = 0$.) We obtain in this way a one dimensional sub-bundle of E. As v_0 varies over a basis of kernel α_0, we obtain in this way a "concrete" version of the Grothendieck decomposition.

7. THE HOLOMORPHIC VECTOR BUNDLE INVARIANTS OF LINEAR, TIME-INVARIANT INPUT SYSTEMS

Next we can turn to material more familiar to system theorists. Consider an input system

$$\frac{dx}{dt} = Ax + Bu \quad .$$

Here (changing notation from that of previous sections so that it conforms with the more-or-less standard system-theory notation),

$$x \in X \equiv \text{state space}$$

$$u \in U \equiv \text{input space}$$

X,U are finite dimensional vector spaces; $A: X \to X$, $B: U \to X$ are linear maps. Set:

$$V = X \times U$$

$$W = X \quad .$$

Write an element v of V as a partitioned column vector:

$$v = \begin{pmatrix} x \\ u \end{pmatrix} \quad .$$

Define linear maps

$$\alpha_0, \alpha_1 : V \to W$$

as follows:

$$\alpha_1 \begin{pmatrix} x \\ u \end{pmatrix} = x \quad , \qquad \alpha_0 \begin{pmatrix} x \\ u \end{pmatrix} = -(Ax + Bu) \quad .$$

Thus Equations (7.1) take the following form:

$$\alpha_1 \frac{d}{dt} \begin{pmatrix} x \\ u \end{pmatrix} + \alpha_0 \begin{pmatrix} x \\ u \end{pmatrix} = 0 \quad . \qquad (7.2)$$

The pair (α_0, α_1) of linear maps then define a vector bundle E over $P_1(\mathbb{C})$, as described in previous sections.

For $s \in \mathbb{C}$,

$$E(s) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} : \alpha_0 \begin{pmatrix} x \\ u \end{pmatrix} + s\alpha_1 \begin{pmatrix} x \\ u \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} : sx - (Ax + Bu) = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} : x = (s - A)^{-1} Bu \right\} .$$

Thus, if $1/s$ is not an eigenvalue of A,

$$\dim E(s) = \dim U$$

$$E(\infty) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} : Ax + Bu = 0 \right\} .$$

Theorem 7.1. If the input system (7.1) is controllable, then $s \to E(s)$ defines a non-singular vector bundle over $P_1(\mathbb{C})$, whose fibers are equal to the dimension of U. □

This is proved in [9].

We can immediately use this to infer some important qualitative facts about controllable systems (7.1). The complex vector bundle is, following Grothendieck [12], a direct sum of line bundles. The degrees of these line bundles (i.e., their Chern classes evaluated on the generator of two-dimensional integral homology of $P_1(\mathbb{C})$) are then significant numbers for system theory. (For example, they would remain invariant under both *complex analytic* deformations and *feedback transformations* of the systems (7.1).) Martin and I show [9] that they are, in fact, the numbers in the Brunovsky canonical form. Their sum is then the first Chern class of the vector bundle E evaluated on the generator of $H_2(P_2(\mathbb{C}), Z)$. It is the *Macmillan degree*, i.e., the dimension of X. In particular, this shows that the Macmillan degree attached to the transfer function $T = C(s - A)^{-1} B$ of a controllable, observable system is a *topological* invariant.

What purpose does this serve? Notice that we have taken the standard algebraic structure of linear systems and interpreted it *geometrically* in a way that is much more amenable to dealing with more complicated systems. In [14] Martin and I have presented certain preliminary calculations which lead us to believe that the material does carry over to certain infinite-dimensional and time-varying systems. One obstacle to such applications is that the information we would need about the classification of holomorphic vector bundles on more complicated complex manifolds than $P_1(\mathbb{C})$ is not yet available from the work of the pure mathematicians, although it is a topic of intense development among algebraic geometers. There are also probably many more significant applied problems whose solution is dependent on information about the structure of holomorphic vector bundles. For example, the classification of holomorphic vector bundles on $P_3(\mathbb{C})$ has played a key role in the Atiyah-Singer-Ward theory of Yang-Mills "instantons", objects which appear naturally in *elementary particle physics*. C. Byrnes has constructed analogous bundles on $S \times P_1(\mathbb{C})$, with S a topological space, as useful gadgets for the study of the *delay systems*. In later sections I will show how certain N-dimensional systems (e.g., the Maxwell and Helmholtz equation) lead in a natural way to such vector bundles on higher dimensional complex manifolds.

There is also a possibility of an intermediate classification of systems by classifying the *structure groups* of their associated vector bundles. The structure group starts off as $GL(m,\mathbb{C})$, where m = dimension of input space. The Grothendieck theorem [12] says it can be reduced to the subgroup of diagonal matrices. However, it might be already given "by Nature" in such a way that the structure group is another subgroup of $GL(m,\mathbb{C})$. It appears that the systems occurring in circuit theory, analytical mechanics, etc., all have their typical and characteristic structure groups; in turn, this structure group can be related to properties of the Hankel matrix of the system. For all of these reasons, I believe that people interested in applications should, in this case, relax their natural skepticism (which I share in principle) about fancy mathematics and consider seriously the possibility that the theory of vector bundles gives a valuable way of describing unified properties of systems.

8. FEEDBACK EQUIVALENCE AND THE KRONECKER THEORY OF PAIRS OF LINEAR MAPS

The work that Martin and I have done can be usefully interpreted in the language of category-functor theory. We attach to the "category" of (linear, time-invariant) input systems essentially two "functors". One is *geometric*, with the category of holomorphic vector bundles, the other is *algebraic*, with the category of pairs of linear maps. The reader who is familiar with such things will recognize that this is a typical situation in such avante-garde branches of mathematics as algebraic topology and geometry. In these areas it has been found that it is precisely in reconciling the "geometric" and "algebraic" world view that some of the most significant and useful relations appear. Now, from the engineer's point of view, the natural "isomorphisms" of systems are the *feedback transformations*. The natural isomorphisms of pairs of linear maps

$$(\alpha_0,\alpha_1) \in L(V,W) \times L(V,W)$$

are the action of the group $GL(V) \times GL(W)$. The fact that they are the *same thing* is very worthwhile proving explicitly, because it plays such a fundamental role. (If it had been recognized earlier, say in 1970, it would have simplified the task of the authors of many papers.)

Now, it is obvious that feedback equivalence of systems implies equivalence under $GL(V) \times GL(W)$ of the corresponding pairs of linear maps. We must deal with the converse: Suppose then that

$$\frac{dx}{dt} = Ax + Bu \tag{8.1}$$

$$\frac{d'x}{dt} = A'x + B'u \tag{8.2}$$

are two linear systems. Set:

$$V = X \times U = \text{set of pairs } \binom{x}{u}$$

$$W = X \quad .$$

$$\begin{aligned} \alpha_1\binom{x}{u} &= x \\ \alpha_0\binom{x}{u} &= -Ax - Bu \\ \alpha_1'\binom{x}{u} &= x \\ \alpha_0'\binom{x}{u} &= -A'x - B'u \quad . \end{aligned} \tag{8.3}$$

Let us then suppose that (α_0, α_1) lies in the same $GL(V) \times GL(W)$ orbit as (α_0', α_1'), i.e.,

$$\begin{aligned} \alpha_0' &= g_1 \alpha_0 g_2^{-1} \\ \alpha_1' &= g_1 \alpha_1 g_2^{-1} \end{aligned} \tag{8.4}$$

$$g_1 \in GL(X)$$

$$g_2 \in GL(X \oplus U) \quad .$$

Then,

$$\alpha_1 g_2\binom{x}{u} = g_1 \alpha_1\binom{x}{u} = g_1 x \quad .$$

Suppose g_2 is written in partitioned form:

$$g_2\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} g_{2,11} & g_{2,12} \\ g_{2,21} & g_{2,22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} g_{2,11}x + g_{2,12}u \\ g_{2,21}x + g_{2,22}u \end{pmatrix} \tag{8.6}$$

$$g_{2,11} \in L(X,X); \qquad g_{2,12} \in L(U,X);$$

$$g_{2,21} \in L(X,U); \qquad g_{2,22} \in L(U,U) \quad .$$

Compare (8.5) and (8.6):

$$g_{2,11}x + g_{2,12}u = g_1 x \quad ,$$

hence

$$\boxed{g_1 = g_{2,11}} \tag{8.7}$$

$$\boxed{g_{2,12} = 0} \tag{8.8}$$

Plug (8.7) and (8.8) back into (8.4):

$$\alpha_0' g_2 \begin{pmatrix} x \\ u \end{pmatrix} = g_1 \alpha_\Omega \begin{pmatrix} x \\ u \end{pmatrix}$$

or

$$\alpha_0' \begin{pmatrix} g_1 x \\ g_{2,21}x + g_{2,22}u \end{pmatrix} = - g_1 (Ax + Bu)$$

or

$$A' g_1 x + B'(g_{2,21} x + g_{2,22} u) = g_1 (Ax + Bu)$$

or

$$\boxed{g_1^{-1} A' g_1 = - g_1^{-1} B g_{2,21} + A} \tag{8.9}$$

$$\boxed{B' g_{2,22} = g_1 B} \tag{8.10}$$

These formulas can be summarized in the following way.

Theorem 8.1. The two input systems (8.1), (8.2) lead to $GL(X) \times GL(X \oplus U)$-equivalent pairs of maps if and only if the systems differ by *state feedback* and by *change of basis in input and state space.* □

This result has two sorts of ramifications in system theory. First, it enables the *state feedback classes* to be classified by the *Kronecker pencil theory*. In fact, it was Brunovsky [5] who did this in a way independent of the general Kronecker theory. Second, it indicates a major algebraic difference between *state feedback* on the one hand and output feedback and various sorts of "compensators" on the other hand. If the latter sort define a group (which is not always clear), usually the orbits of these groups on systems are *not* equivalent to the problem Kronecker solved, i.e., equivalence of pairs of linear maps under isomorphisms of domain and range spaces. Problems of enumerating orbits are extremely difficult in our present state of knowledge of invariant theory if they do not reduce in one form or another to the Kronecker problem or are closely related to it. It is then a lucky accident (?) that *state feedback* is governed by the Kronecker invariant theory.

9. TRANSFER FUNCTIONS AND HOLOMORPHIC VECTOR BUNDLES FOR 1-D AND n-D SYSTEMS

The geometric methods introduced by Martin and me into system theory depend on associating with a finite dimensional, time-invariant input-output system a *vector bundle* on the Riemann sphere. Note that the Riemann sphere (the complex numbers with its one-point compactification) is the only *Riemann surface of genus zero*. This point of view is not really so avante-garde as it might sound, since it is very much in the spirit of the one-complex variable methods used by electrical engineers in the 1920's to '40s. (This is the material that was supplanted by "state space methods" in the late 1950's and 1960's.) We believe that there is considerable potential for applying these methods (enriched by the modern mathematical research) to more general systems. I will now develop material in this direction for systems described by ordinary *and* partial differential equations.

There is a vast field of engineering *practice* called the "theory of two-dimensional filters", which as yet has no systematization and mathematization (as the "state space" theory systematized and mathematized one-dimensional filters). I believe the ideas presented here might put us on the road to such a theory. Another interesting possibility is an application of complex manifold methods to the systems of *partial* differential equations which occur in physics.

My plan is to try to construct holomorphic vector bundles with such systems. We shall see that this is a modern algebro-geometric version of the traditional "transfer function" methods used in system theory and electrical engineering.

Let us begin with well-known material. Consider a finite dimensional, linear, time-invariant input-output system

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \quad . \end{aligned} \tag{9.1}$$

Convert this into an algebraic equation using the Laplace transform

$$\hat{x}(s) = \int_0^\infty x(t)e^{-st}\,dt \quad ,$$

with zero initial conditions.

$$\begin{aligned} s\hat{x}(s) &= A\hat{x} + B\hat{u} \\ \hat{y} &= C\hat{x} \end{aligned}$$

or

$$\begin{aligned} \hat{y} &= (C(s-A)^{-1}B)\,\hat{u} \\ T(s) &= C(s-A)^{-1}B \end{aligned} \tag{9.2}$$

is the *transfer function* or *frequency response*. If U and Y are the linear input and output spaces, T is the natural mapping

$$\mathbb{C} \to L(U,Y) \quad .$$

A more appropriate gadget from the Riemann surface is the sequence

$$\theta_n = C(s-A)^{-1} B s^n \, ds \tag{9.3}$$

of differential forms. Let us see how they behave at $s = \infty$. Set:

$$u = \frac{1}{s}$$

$$du = -u^2 \, ds$$

$$\theta_n = -C\left(\frac{1}{u} - A\right)^{-1} B u^{-n} u^{-2} \, du$$

$$= -C(1-Au)^{-1} B u^{-(n+1)} \, du$$

$$= -C(1 + Au + A^2u^2 + \cdots) B u^{-(n+1)} \, du \quad . \tag{9.4}$$

This shows that the θ_n are *meromorphic differential forms* on the Riemann sphere. Their residue at $u = 0$, i.e., $s = \infty$, is especially important; it is the "Hankel data".

$$\text{residue } \theta_n = -CA^n B \quad . \tag{9.5}$$

We are especially interested in the inverse Laplace transform

$$\int_{a-i\infty}^{a+i\infty} T(s) e^{st} \, ds = \sum_{n=0}^{\infty} \int \frac{\theta_n}{n!} \, t^n \quad . \tag{9.6}$$

This formula describes the input-output relations in terms of objects which have an a priori meaning in terms of Riemann surfaces, the "Abelian Integrals"

$$\int \theta_n \quad .$$

It is a well-known and classical idea (in mathematics) that the appropriate way to generalize these ideas is to replace the Riemann sphere with other compact complex manifolds.

In previous sections I have described the relation between system theory and Kronecker pencil theory. Here is a more general version. Let us rewrite (9.1) as

$$\frac{dx}{dt} - Ax + Bu = 0 \tag{9.7}$$

$$y - Cx = 0 \quad .$$

These equations have the form

$$\alpha_1 \frac{d}{dt}\begin{pmatrix} x \\ u \\ y \end{pmatrix} + \alpha_0 \begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0 \quad , \tag{9.8}$$

where α_0, α_1 are linear maps: $X \oplus U \oplus Y \to X \oplus Y$.

Thus, the *general* problem of the theory of linear, time-invariant, finite dimensional systems is covered by the following equations

$$\alpha_1 \frac{dv}{dt} + \alpha_0 v = 0 \quad , \tag{9.9}$$

where V, W are vector spaces, α_0, α_1 are linear maps $V \to W$. Associated with this is the *pencil*

$$\alpha(s) = \alpha_1 s + \alpha_0 \tag{9.10}$$

of linear maps.

We can assign to the pencil (9.10) a *vector bundle*

$$\pi\colon E \to P_1(\mathbb{C})$$

$$E = \{(s,v)\colon s \in \mathbb{C}, \quad v \in V\colon \alpha(s)v = 0\} \quad .$$

We can now generalize these constructions to cover "distributed parameter" systems.

Let $t_1, \dots, t_n$ be independent variables; V, W are finite dimensional vector spaces.

$$\alpha_0, \alpha_1, \dots, \alpha_n\colon V \to W$$

are linear maps. Consider the differential equations:

$$\alpha_1 \frac{\partial v}{\partial t_1} + \cdots + \alpha_n \frac{\partial v}{\partial t_n} + \alpha_0 v = 0 \quad . \tag{9.11}$$

Associated with this we have:

$$s = (s_1,\ldots,s_n) \in \mathbb{C}^n \tag{9.12}$$

$$\alpha(s) = \alpha_1 s_1 + \cdots + \alpha_n s_n \tag{9.13}$$

$$E = \{(s,v): \alpha(s)(v) = 0\} \tag{9.14}$$

$$\pi: E \to \mathbb{C}^n$$

$$\pi(s,v) = s \quad .$$

E should be *completed* to be a *vector bundle* over a compact complex analytic manifold X on which $\mathbb{C}^n$ is embedded as an open subset. How this is to be done *in general* will be left open for the moment. Consider some simple examples motivated by physics.

Suppose given two independent variables t_1,t_2. (In the physical applications, they may be two space variables or one space, one time variable.) Denote partial derivatives of functions of these variables by subscripts. Consider the equation of "Helmholtz" type:

$$y_{t_1 t_1} + y_{t_2 t_2} + \lambda y = u \quad . \tag{9.15}$$

λ is a constant, u a scalar function of t_1,t_2, u is the "intput", y the "output". This can, of course, be converted to an equation of type (9.11) by introducing more dependent variables, but it will be (for the moment) more convenient to work directly with Equation (9.15). The "transfer function" is obviously the following rational function of two complex variables s_1,s_2:

$$T(s_1,s_2) = \frac{1}{s_1^2 + s_2^2 + \lambda} \quad . \tag{9.16}$$

Thus,

$$y(t_1,t_2) = \iint_Y T(s_1,s_2)\, e^{(s_1t_1+s_2t_2)}\, \hat{u}(s_1,s_2)\, ds_1\, ds_2 \tag{9.17}$$

taken over an appropriately chosen two-dimensional submanifold of $\mathbb{C}^2$, might be a solution of (9.15).

Note that (9.15) is invariant under the rotation group SO(2,R) acting in R^2. This translates into T depending only on $s_1^2 + s_2^2$, as we know explicitly, as shown from formula (9.17).

We now want to consider (9.17) as an integral over a manifold. In order to do this, we shall utilize the connection between differential forms and integration on manifolds, which is explained in many references (e.g., [21-23]). This requires that we introduce the *exterior product* operator "∧" and the exterior derivative "d". Thus, the "element of integration" in (9.17) is

$$\text{"}ds_1 ds_2\text{"} \equiv ds_1 \wedge ds_2 \quad .$$

(Recall the algebraic rules

$$ds_1 \wedge ds_2 = - ds_2 \wedge ds_1$$

$$ds_1 \wedge ds_1 = 0 \quad .$$

These rules of "Grassmann algebra" provide a convenient algebraization of the familiar rules from advanced calculus for manipulating and changing variables in multiple integrals.)

Introduce polar coordinates in the integral (9.17):

$$z^2 = s_1^2 + s_2^2$$

$$s_1 = z \cos\theta \quad , \qquad s_2 = z \sin\theta$$

$$ds_1 \wedge ds_2 = (dz \cos\theta - z \sin\theta \, d\theta) \wedge (dz \sin\theta + z \cos\theta \, d\theta)$$

$$= z \, dz \wedge d\theta \quad .$$

Set:

$$w = \cos\theta$$

$$dw = - \sin\theta \, d\theta \quad ,$$

$$d\theta = - \frac{1}{\sqrt{1 - w^2}} dw$$

$$\boxed{ds_1 \wedge ds_2 = \frac{z}{\sqrt{1-w^2}} \, dw \wedge dz} \tag{9.18}$$

Let us now suppose that u *is a function of* z *alone.* Then (9.17) can be rewritten as:

$$y(t_1,t_2) = \iint e^{z(t_1 w + t_2\sqrt{1-w^2})} \frac{\hat{u}(z)}{z^2+E} \frac{z}{\sqrt{1-w^2}} \, dw \wedge dz$$

$$= \iint \left(\int e^{z(t_1 w + t_2\sqrt{1-w^2})} \frac{dw}{\sqrt{1-w^2}} \right) \frac{z\hat{u}(z)}{z^2+\lambda} \, dz \tag{9.19}$$

This formula exhibits the potential "algebro-geometric" nature of the situation. We have the *algebraic correspondence*

$$(s_1,s_2) \to z = \sqrt{s_1^2 + s_2^2} \tag{9.20}$$

of $\mathbb{C}^2 \to \mathbb{C}^1$. (Note that is is *not* a "rational map".) The fibers are the orbit of the orthogonal group $SO(2,\mathbb{C})$. They are the *circles*

$$s_1^2 + s_2^2 = z_1^2 \quad .$$

In order to get a better idea of the algebro-geometric nature of the integrals (9.19), let us use a power series expansion for the exponential function. Set:

$$y_{n,m} = \iint (zw)^n \left(z\sqrt{1-w^2}\right)^m \frac{dw}{\sqrt{1-w^2}} \frac{z}{z^2+\lambda} \wedge \hat{u}(z) \, dz \tag{9.21}$$

i.e.,

$$y(t_1,t_2) = \sum_{n,m=0}^{\infty} y_{n_1 n_2} \frac{t_1^n t_2^m}{n!m!} \quad . \tag{9.22}$$

Thus, it might be appropriate to consider the differential forms

$$\theta_{n,m} = (zw)^n \left(z\sqrt{1-w^2}\right)^m \frac{dw}{\sqrt{1-w^2}} \wedge \frac{z}{z^2+\lambda} \quad dz \quad , \tag{9.23}$$

$$\theta_{n,m} = \frac{s_1^n s_2^m}{s_1^2+s_2^2+\lambda} \quad ds_1 \wedge ds_2 \tag{9.24}$$

as the characteristic "geometric objects" attached to the input-output system (9.15).

This suggests that we consider separately the differential forms

$$\alpha_{n,m} = (zw)^n \frac{\left(z\sqrt{1-w^2}\right)^m dw}{\sqrt{1-w^2}} \quad . \tag{9.25}$$

with z considered as a *parameter*. They are essentially differential forms on the Riemann surface whose local variable is w, i.e., the algebraic curve

$$w_1^2 + w_2^2 = 1 \quad .$$

(A *Riemann surface* is a complex analytic manifold that can be parameterized by one complex variable. Of course, another point of view is to regard

$$e^{z(t_1 w + t_2\sqrt{1-w^2})} \quad \frac{dw}{1-w^2}$$

as a differential form on this Riemann surface. (Its indefinite integral can be written down explicitly in terms of Bessel functions.)

This simple example clearly gives us much new material to consider for a *general* theory of systems from a complex manifold-algebraic geometric point of view.

10. VECTOR BUNDLES ON ORBIT SPACES

We can immediately see a general pattern to the example treated in the previous section.

Let X,Y be spaces,

$$\phi: X \to Y$$

a map. Suppose *given* a vector bundle

$$\pi: E \to Y \quad .$$

For $y \in Y$, the fiber $E(y) = \pi^{-1}(y)$ is a vector space. It defines a vector bundle

$$\phi^{-1}(E)$$

on X, called the *pull-back bundle*

$$\phi^{-1}(E) \;=\; \{(x,v): v \in E(\phi(x))\} \quad .$$

There is a commutative diagram

$$\begin{tikzcd} \phi^{-1}(E) \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

Consider a given bundle E' on X and a map $\phi: X \to Y$. We can ask *whether there is a bundle* E *on* Y *such that*

$$E' \;=\; \phi^{-1}(E) \quad .$$

One can also ask whether there is an *equivalent vector bundle* E'' on a bundle E such that

$$E'' \;=\; \phi^{-1}(E) \quad .$$

These questions are especially interesting for systems theorists (and physicists!) if the following additional structure is put on:

G is a transformation group on X. $Y = G\backslash X$ is the orbit space, $\phi: X \to G\backslash X = Y$ the map which sends $x \in X$ into the orbit Gx on which it lies. G acts as a group of automorphisms of the vector bundle E'.

Let us return to the context of Section 9.

11. VECTOR BUNDLES DEFINED BY SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS AND LINEAR GROUPS OF SYMMETRIES

Let V,W be complex vector spaces, n an integer, and let $\alpha_0,\alpha_1,\ldots,\alpha_n$: $V \to W$ be linear maps. We can then construct the system of *linear constant coefficient* partial equations:

$$\alpha_1 \frac{\partial v}{\partial t_1} + \cdots + \alpha_n \frac{\partial v}{\partial t_n} - \alpha_0 v = 0 \tag{11.1}$$

to be solved for a function

$$t = (t_1,\ldots,t_n) \to v(t) \quad ,$$

i.e., a map $R^n \to V$.

We can then associate with this system the n-complex variable "pencil"

$$\alpha(s) = \alpha_1 s_1 + \cdots + \alpha_n s_n - \alpha_0 \tag{11.2}$$

of linear maps: $V \to W$, and the *holomorphic vector bundle*

$$E = \{(s,v): s = (s_1,\ldots,s_n) \in \mathbb{C}^n; \quad \alpha(s)v = 0\} \tag{11.3}$$

Now let G be a group. Suppose given *three* linear actions on $\mathbb{C}^n$, V and W.

Definition. G acts as a *symmetry group* of the pencil $\alpha(s)$ if the following condition is satisfied:

$$g\alpha(s)g^{-1} = \alpha(g(s)) \tag{11.4}$$

for all $g \in G$.

Such an action determines an action of G on the vector bundle E:

$$g(s,v) = (gs,gv) \quad . \tag{11.5}$$

(If $\alpha(s)v = 0$, note that using (11.4)

$$\alpha(g(s))(gv) = g\alpha(s)(v) = 0 \quad ,$$

so that the action (11.5) really does map E onto itself.)

Thus we can form the orbit space $G\backslash\mathbb{C}^n$ and ask whether the bundle E comes from a bundle on this orbit space. This is clearly an interesting system-theoretic way of defining vector bundles!

12. SO(3,$\mathbb{C}$)-BUNDLES

If G has a known Lie-theoretic structure, we can use Lie group representation theory to analyze the bundles constructed in Section 11. One of the simplest cases (and the most important for physicists) is that where G is the three-dimensional rotation group. For algebraic reasons we complexify everything.

We are supposing that $n = 3$. Refer to [30] for notation and ideas used here. Thus, $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ are linear maps: $V \to W$. If

$$s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$$g(s) = gs \equiv \text{matrix multiplication} \quad .$$

It is convenient to write

$$\alpha \equiv (\alpha_1\alpha_2\alpha_3)$$

$$(\alpha)(s) \equiv (\alpha_1\alpha_2\alpha_3)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \alpha_0 \quad .$$

Suppose g is an orthogonal 3×3 complex matrix, i.e.,

$$g' = g^{-1} \quad .$$

(' denotes transpose of a matrix.) Let

$$\sigma_1 : G \to L(V)$$

$$\sigma_2 : G \to L(W)$$

be the given representations of G by linear transformations on V and W.

$$\sigma_2(g)(\alpha)(s)\sigma_1(g^{-1}) = (\alpha)(g(s))$$

or

$$\sigma_2(g)(\alpha_1\alpha_2\alpha_3)\sigma_1(g^{-1}) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \sigma_2(g)\alpha_0\sigma_1(g^{-1}) = (\alpha_1\alpha_2\alpha_3)g \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \alpha_0$$

or

$$\sigma_2(g)\alpha_0\sigma_1(g^{-1}) = \alpha_0 \tag{12.1}$$

$$\sigma_2(g)(\alpha_1\alpha_2\alpha_3)\sigma_1(g^{-1}) = (\alpha_1\alpha_2\alpha_3)g \quad . \tag{12.2}$$

Now,

$$L(V,W) \equiv V \otimes W^d \quad .$$

(W^d = dual space to W. In fact, all finite dimensional representations of G are self-dual.) The "Clebsch-Gordan" rules for decomposition of tensor product representations of $SO(3)$ tell how many such independent α's there are.

The *irreducible* (finite dimensional) representations of G are parameterized by an integer $j \geq 0$ ($\equiv$ *spin*). The vector space for the spin j-representation has dimension $(2j+1)$. Thus, V and W can be split up into a direct sum of vector spaces V_j, W_j in each of which G acts via a direct sum of spin j-representations. Using "Clebsch-Gordan" we see that

$$\alpha(s)(V_j) \subset W_{j+1} + W_j + W_{j-1} \tag{12.3}$$

$$\alpha_0(V_j) \subset W_j \quad , \qquad j = 0,1,2,\ldots \quad .$$

Example. Consider the Helmholtz operator with input u, output y;

$$s^2 y + \lambda y - u = 0 \quad .$$

Set:

$$x_i = s_i y \quad , \qquad i = 1,2,3 \quad .$$

Thus,

$$\sum_{i=1}^{3} s_i x_i = -\lambda y + u$$

or

$$\boxed{\begin{aligned} s_i y - x_i &= 0 \quad , \qquad i = 1,2,3 \\ \sum_i s_i x_i + \lambda y - u &= 0 \end{aligned}} \tag{12.4}$$

$$V = R^5$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y \\ u \end{pmatrix} \right\}$$

$$W = R^4 \quad .$$

$$(s)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y \\ u \end{pmatrix} = \begin{pmatrix} s_1 y - x_1 \\ s_2 y - x_2 \\ s_3 y - x_3 \\ \sum_i s_i x_i + \lambda y - u \end{pmatrix} \qquad (12.5)$$

Thus, we see that V is a direct sum of a spin one subspace S and two spin zero subspaces. W is the direct sum of an S_1 and an S_0. Now, Clebsch-Gordan gives:

$$S_1 \otimes S_1 = S_2 \oplus S_1 \oplus S_0$$

$$S_0 \otimes S_1 = S_1 \quad ,$$

i.e., there are intertwining maps

$$S_1 \otimes S_1 \to S_0$$

$$S_0 \otimes S_1 \to S_1 \quad .$$

It is these that are obviously present in (12.4).

13. THE COMPACTIFIED VECTOR BUNDLES DETERMINED BY THE HELMHOLTZ EQUATION

Continue with the situation of Sections 9-12, the linear differential equations which are invariant under the action of $SO(3,\mathbb{C})$ on $\mathbb{C}^3$. These equations cover most of those of interest in classical mathematical physics! For $s \in \mathbb{C}^3$ let

$$E(s) = \{v \in V: \alpha(s)(v) = 0\} \quad . \qquad (13.1)$$

As s varies over $\mathbb{C}^3$, $E(s)$ defines a *vector bundle* whose basis is $\mathbb{C}^3$. $G \equiv SO(3,\mathbb{C})$ acts linearly on E.

We want to examine the algebro-geometric structure of this bundle, and any possible bundles with orbit spaces

$$G \backslash \mathbb{C}^3 \quad .$$

The first step is to "homogenize" Equations (11.3), (12.4). Introduce homogeneous coordinates

$$\tau_0, \tau_1, \tau_2, \tau_3 \quad ,$$

with

$$s_i = \frac{\tau_i}{\tau_0} \quad , \qquad i = 1,2,3 \quad .$$

Equations (12.4) take the form

$$\boxed{\begin{aligned} \tau_i y - x_i \tau_0 &= 0 \\ \sum_{i=1}^{3} \tau_i x_i + (\lambda y - u)\tau_0 &= 0 \end{aligned}} \tag{13.2}$$

These equations *determine a vector bundle* E *whose base is* $P_3(\mathbb{C})$, *which restricts to the given bundle when* $\mathbb{C}^3$ *is embedded as the "affine" subspace of* $P_3(\mathbb{C})$.

Let us work out the fibers of this bundle. If $\tau_0 \neq 0$, these equations are equivalent to (11.3), and the solutions $E(s)$ obviously form a one-dimensional vector space. The "input-output" relations are

$$\boxed{y = \frac{1}{s^2 + \lambda} u}$$

This gives a "geometric", *systems-theoretic* meaning to $1/(s^2 + \lambda)$ as the *"transfer-function-fundamental solution-Green's function"* for the Helmholtz equation!

For

$$\tau_0 = 0 \quad ,$$

Equations (13.2) take the following form:

$$\begin{aligned} \tau_i x &= 0 \\ \sum_{i=1}^{3} \tau_i x_i &= 0 \quad . \end{aligned} \tag{13.3}$$

Since one of the τ_i is nonzero, (13.3) forces $y = 0$. Then, (13.3) reduces to

$$\boxed{\begin{aligned} y &= 0 \\ \sum_{i=1}^{3} \tau_i x_i &= 0 \end{aligned}} \tag{13.4}$$

$E(\tau)$ consists of the (y,u,x_i) such that *only* (13.4) is satisfied. Thus,

$$\boxed{\dim E(\tau) = 3}$$

The vector bundle E does *not* have constant dimension as τ ranges over $P_3(\mathbb{C})$. We see that there is here a distinct difference from the one-dimensional situation!

However, we can construct an *equivalent* bundle E' on $\mathbb{C}^3$ which does have a non-singular extension to $P_3(\mathbb{C})$. Namely,

$$E' = \left\{ (y,u,s): \ y = \frac{1}{s^2 + \lambda} u \right\} \quad .$$

Again, put

$$s_i = \frac{\tau_i}{\tau_0} \quad .$$

Thus,

$$E'(\tau) = \left\{ (x,u) : \quad x = \frac{1}{\sum \frac{\tau_i \tau_i}{\tau_0^2} + \lambda} \, u \right\}$$

$$= \left\{ (x,u) : \quad x = \frac{\tau_0^2}{\sum \tau_i \tau_i + \tau_0^2 \lambda} \, u \right\}$$

Thus, if $\tau_0 = 0$,

$$E'(\tau) = \{ (x,u) : x = 0 \} \quad .$$

We see that for *all* $\tau \in P_3(\mathbb{C})$,

$$\boxed{\dim E'(\tau) = 1}$$

We can also examine how E' projects down to the orbit space $SO(3,\mathbb{C}) \setminus \mathbb{C}^3$. Map

$$\phi : \mathbb{C}^3 \to \mathbb{C}$$

as follows

$$\phi(s) = s^2 = z \quad .$$

The fibers are the orbits of $SO(3,\mathbb{C})$. We would now like to "projectify" this situation

$$s_i = \frac{\tau_i}{\tau_0}$$

$$z = \frac{z_1}{z_0}$$

$$\phi(\tau) = \left(\tau_0^2, \sum_{i=1}^{3} \tau_i\tau_i\right) \tag{13.5}$$

Note that this map sends one-dimensional linear subspaces of $\mathbb{C}^4$ into one-dimensional linear subspaces of $\mathbb{C}^2$, hence it passes to the equivalent to define a map

$$\phi: P_3(\mathbb{C}) \to P_1(\mathbb{C}) \quad . \tag{13.6}$$

The orbits of $SO(3,\mathbb{C})$ *lie in the fibers of* ϕ.

Thus ϕ and the vector bundle E' are very compatible.

$$E(\tau) = \left\{ (x,u): \quad x = \frac{\tau_0^2}{\sum_i \tau_i\tau_i + \lambda\tau_0^2} \, u \right\}$$

Set:

$$E''((z_0,z_1)) = \left\{ (x,u): \quad x = \frac{z_0}{z_1 + \lambda z_0} \, u \right\} \quad .$$

This formula defines a *non-singular* vector bundle on $P_1(\mathbb{C})$.

$$E' = \phi^{-1}(E'') \quad ,$$

i.e., E' is the *pull-back* of the bundle on $P_1(\mathbb{C})$.

14. LINEAR FILTERS AND GROUPS

So far, we have been considering the class of input-output systems defined in terms of differential equation models. Of course, "system theory" is much more extensive, and not necessarily tied to all such models. In this section I will briefly describe another foundational approach. It too has a basic geometric aspect that has barely been developed.

Consider a "system" as a mapping from certain input data to certain output data. It is important to specify some sort of mathematical structure on the input-output data and to consider systems which interrelate this structure in some way. For example, let us analyze the linear, finite dimensional, time-invariant state space models from this point of view. Consider one, say of the form

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx \quad . \tag{14.1}$$

x,u,y are vectors in finite dimensional real vector space, and A,B,C are matrices of the appropriate size. t is a continuous time parameter. Let $\underline{u}$: $t \to u(t)$ be an input function. We can then solve (14.1) *with zero initial conditions* to define the output $t \to \underline{y}(t)$:

$$\underline{y}(t) = \int_0^t Ce^{A(t-s)} B\underline{u}(s)\, ds \quad . \tag{14.2}$$

We can then consider the system as a mapping $\underline{u} \to \underline{y}$ which is linear and of the form:

$$\underline{y}(t) = \int_{-\infty}^{\infty} K(t,s)\, \underline{u}(s)\, ds \quad , \tag{14.3}$$

where $(t,s) \to K(t,s)$ is a mapping $R \times R \to$ (matrices) such that the following conditions are satisfied:

a) $K(t,s) = 0$ if $s < 0$ (14.4)

b) $K(t,s) = 0$ if $s > t$ (14.5)

c) $K(t+t',s) = K(t,s-t')$, for $t,t',s \in R$ (14.6)

Condition (14.4) is just a normalization of the output at $t = 0$, and is not particularly important. Condition (b) is crucial, since it expresses *causality*; if the incoming signal $\underline{u}$ is zero for time $t < 0$, it guarantees that the output $\underline{y}$ is also zero for $t < 0$. Condition (14.6), which expresses translation invariance, is a *group property*, and suggests that group theory might be the appropriate general setting for some of these ideas.

For example, one might replace the role of the real numbers R, i.e., set t by the integers, Z. An *input* is then a map $\underline{u}$: $n \to \underline{u}(n)$ of $Z \to$ (input vector space U), an output is a map $\underline{y}$: $Z \to$ (output vector space Y), and the "system" is a map

$$\underline{u} \to \underline{y}$$

of the form

$$\underline{y}(n) = \sum_{m=-\infty}^{\infty} K(n,m)\, \underline{u}(m) \tag{14.7}$$

We can immediately write down the analog of conditions (14.4)-(14.6). Such input-output relations might be generated by the difference equation analog of the differential equations (14.1):

$$\begin{aligned} x_{n+1} &= x_n + Ax_n + Bu_n \\ y_{n+1} &= Cx_{n+1} \quad . \end{aligned} \tag{14.8}$$

However, we can generalize relation (14.7) to other groups. For example, replace the integers Z with the direct product group $Z \times Z$. A *system* then might be a map (input) $\rightarrow$ (output) of the form:

$$\underline{y}(n_1,n_2) = \sum_{(m_1,m_2)} K((n_1,n_2),(m_1,m_2)\underline{u}(m_1,m_2)) \tag{14.9}$$

Relation (14.6) generalizes readily to this case. However, the right generalization of (14.5)--causality--is less existent. In fact, there is now a large body of current engineering literature-lore concerning such systems, called, say, *2-D digital filters*, and the lack of a natural generalization of "causality" is one complicating feature that has slowed down the development of a systematic theory paralleling the non-traditional 1-D theory. Of course, these non-traditional hybrid system-filter problems are extremely important in the state of today's technology, particularly in terms of realization of the ultimate potential of digital computers. Since I believe that geometric-group theoretic ideas are potentially important here also, I will now sketch a general setting for this approach. I will also indicate how some of the *stochastic* filter-system problems can be considered in the same way.

Suppose given the following data:

a) A real vector space U called the *input space*,

b) A real vector space Y called the *output space*,

c) A space T called the parameter\time-space.

U and Y may be infinite dimensional and T may be continuous or discrete. However, the functional analysis complications are reduced if we work with the case where T is discrete

d) A σ-field of subsets of T, and a measure $d\tau$ on this sigma-field.

(In this section it will be assumed that the reader is familiar with the rudiments of measure theory and functional analysis, say at the level of Ref. [33].)

Let $\mathcal{M}(T,U)$, $\mathcal{M}(T,Y)$ denote the space of mappings with domain τ and range U and Y. A *linear system*, in this framework, now might be defined as a triple $(\mathcal{I}, \mathcal{O}, \mathcal{K})$ with the following conditions satisfied:

a) $\mathcal{I} \subset \mathcal{M}(T,U)$

b) $\mathcal{O} \subset \mathcal{M}(T,Y)$

c) $\mathcal{I}$ and $\mathcal{O}$ are linear subspaces of the vector spaces in which they are embedded, by (a) and (b)

d) $\mathscr{K}$ is a linear map: $\mathscr{I} \to \mathscr{O}$, of the following form:

$$\mathscr{K}(\underline{u})(\tau) = \int_T K(\tau,\tau')u(\tau')\, d\tau' \quad , \tag{14.10}$$

where $(\tau,\tau') \to K(\tau,\tau')$ is a mapping of

$$T \times T \to (\text{space of linear maps } U \to Y)$$

(Of course, one must also postulate sufficient functional analysis detail in order that the terms in (14.10) even make sense. In certain situations it is desirable to relax the framwork to allow the "kernel" K to be a distribution-generalized function in the sense of L. Schwartz and F. Gelfand.)

In order to make contact between this general framework and contemporary engineering (and I want to emphasize how widely these ideas are dispersed throughout mathematics, statistics, and the signal-information-communication-filtering-prediction parts of technology), it is of course necessary to specialize further. There are two aspects I want to discuss briefly here because of their considerable geometric significance:

a) A group G acts on T as a transformation group, and certain relations of compatibility are presented between G and the system $(\mathscr{I}, \mathscr{O}, \mathscr{K})$. (The prototype is that where T = real numbers or integers, G = additive group, and the "compatibility" relations are those which describe "stationarity" of the processes.) The group action is denoted as

$$(g,\tau) \to g\tau \quad .$$

b) G acts on the products $T \times U$, $T \times Y$, in the following way:

$$\begin{aligned} g(\tau,u) &= (g\tau,\alpha(\tau,g)(u)) \\ g(\tau,y) &= (g\tau,\beta(\tau,g)(y)) \end{aligned} \tag{14.11}$$

for $\tau \in T$, $g \in G$, $u \in U$, $y \in Y$.

We require that for each (τ,g), $\alpha(\tau,g)$ are linear maps on U and Y.

(Note that the conditions that (14.12) define a transformation group action on $T \times U$ and $T \times Y$ requires that certain conditions be satisfied by α and β. We shall not write them down explicitly.)

This action (14.11) then defines an action of G on $\mathscr{M}(T,U)$ and $\mathscr{M}(T,Y)$, as follows:

$$\begin{aligned} (g\underline{u})(\tau) &= \alpha(\tau,g)(\underline{u}(g^{-1}\tau)) \\ (g\underline{y})(\tau) &= \beta(\tau,g)(\underline{y}(g^{-1}\tau)) \end{aligned} \tag{14.12}$$

One can now require that systems, defined as linear maps

$$\mathscr{K}\colon \mathscr{I} \to \mathscr{O}$$

commute with the action of G on $\mathscr{M}(T,U)$ and $\mathscr{M}(T,Y)$. This is a sort of generalized "stationarity". Here are some examples that appear with great frequency in the engineering literature.

<u>Example</u>. *2-D translation-invariant digital filters with scalar signals.*

Let $U = Y = \mathbb{C}$, the complex numbers, and let $T = Z \times Z$, $G = Z \times Z$ acting on itself. Let G act on $\mathscr{M}(T,U)$ and $\mathscr{M}(T,Y)$ via simple translation:

$$\begin{aligned} (g\underline{u})(n_1,n_2) &= \underline{u}((n_1,n_2) - g) \\ (g\underline{y})(n_1,n_2) &= \underline{y}((n_1,n_2) - g) \end{aligned} \tag{14.13}$$

$$\text{for } g \in G \equiv Z \times Z, \quad (n_1,n_2) \in Z \times Z \quad .$$

Let $K\colon T \to \mathbb{C}$ be a map and let

$$\mathscr{K}\colon \mathscr{M}(T,U) \to \mathscr{M}(T,Y)$$

be the map defined for the following formula:

$$\mathscr{K}(\underline{u})(n_1,n_2) = \sum_{(m_1,m_2) \in Z\times Z} K(n_1 - m_1, n_2 - m_2) \tag{14.14}$$

This linear map $\mathscr{K}$ is translation invariant, i.e.,

$$\mathscr{K}(g\underline{u}) = g\mathscr{K}(\underline{u}) \quad . \tag{14.15}$$

Let us now exploit the invariance under G. Decompose $\mathscr{M}(T,U)$ into linear subspaces which transform irreducibly under G. This can be done in the following way: Let G^* denote the set of all group-homomorphisms

$$\lambda\colon G \to \mathbb{C}$$

Two such homomorphisms can be multiplied; they form a group called the *dual group* to G. In this simple case (i.e., $G = Z \times Z$), G^* can be identified with $\mathbb{C} \times \mathbb{C}$: For $z = (z\ ,z\ ) \in \mathbb{C} \times \mathbb{C}$,

$$\lambda_z(n_1,n_2) = e^{n_1 z_1 + n_2 z_2} \quad . \tag{14.16}$$

For $\lambda = z \in G^*$, let $\underline{u}_z \in \mathscr{M}(T,U)$ be the map defined as follows:

$$\underline{u}_{\underline{z}}(n_1,n_2) = e^{-n_1 z_1 - n_2 z_2} \tag{14.17}$$

for $(n_1,n_2) \in T$.

Thus, for $g \in G$, $\lambda \in G^*$,

$$g(\underline{u}_\lambda) = \lambda(g) u_\lambda \tag{14.18}$$

This equation means that u_λ "transforms via the irreducible representation λ".

The "digital filter" $\mathscr{K}$ now commutes with G, hence maps $\underline{u}_\lambda$ into a linear subspace of $\mathscr{M}(T,Y)$ which transforms via λ. Set

$$\underline{y}_z((n_1,n_2)) = e^{-n_1 z_1 - n_2 z_2}$$

Thus, for each $\lambda \equiv z \equiv (z_1,z_2) \in \mathbb{C} \times \mathbb{C} \equiv G^*$,

$$\mathscr{K}(\underline{u}_z) = \tilde{T}(z) u_{-z} \tag{14.19}$$

with $\tilde{T}(z) \in \mathbb{C}$.

This function

$$z \to \tilde{T}(z) \tag{14.20}$$

is called the *transfer function* of the filter/system $\mathscr{K}$.

It can, of course, be computed in the more traditional way in terms of the kernel K.

$$\begin{aligned}
\mathscr{K}(\underline{u}_z)(n_1,n_2) &= \sum_{(m_1,m_2)} K(n_1 - m_1,\ n_2 - m_2)\ \underline{u}_z(m_1,m_2) \\
&= \sum K(n_1 - m_1,\ n_2 - m_2)\ e^{-m_1 z_1 - m_2 z_2} \\
&= \sum K(m_1,m_2)\ e^{-(m_1+n_1) z_1 + (m_2+n_2) z_2} \\
&= \underline{y}_z(n_1,n_2) \sum K(m_1,m_2)\ e^{-m_1 z_1 - n_2 z_2}
\end{aligned}$$

hence:

$$\tilde{T}(z) = \sum K(m_1,m_2)\ e^{-m_1 z_1 - m_2 z_2} \tag{14.21}$$

The right hand side of (14.21) is called (at least in the engineering literature) the z-*transform* of K.

The mathematical problems that are most important for applications involve conditions for *causality* and *stability* in terms of $\tilde{T}$.

Example 2. *Stationary stochastic process as inputs and outputs.*

This example is a traditional one in the engineering literature (and goes back to the original work of Kolmogoroff and Wiener). First, let us review some probability theory.

Let Ω be a space with a fixed σ-field of subsets and a probability measure $d\omega$ defined on this family of subsets. A *random variable* is a measurable mapping $f: \Omega \to R$. The integral of f with respect to the probability measure $d\omega$ is called the *expectation* of f, and denoted as follows:

$$E(f) \equiv \int_{\Omega} f \, d\omega \tag{14.22}$$

Let H denote the vector space of measurable functions $f: \Omega \to R$ such that

$$E(f^2) \equiv \int_{\Omega} f^2 \, d\omega < \infty \tag{14.23}$$

H then forms a *Hilbert space*--"the" L^2 of $d\omega$. The inner product is

$$(f_1,f_2) \to \langle f_1,f_2 \rangle \equiv E(f_1 f_2) \quad .$$

The *norm* is

$$||f|| \equiv \langle f,f \rangle^{1/2}$$

Let T again denote a parameter space. (It will suffice to suppose T discrete. This avoids certain analytic complications of the theory of stochastic processes.) For our purposes, it suffices to restrict attention to random variables that are square-integrable, i.e., lie in H. A *stochastic process* (parameterized by T) is a mapping

$$S: T \to H \quad .$$

Suppose such a process S is given, and in addition, G is a transformation group on T. Let $O(H)$ be the group of isomorphisms of the Hilbert space H, i.e., the space of linear maps $A: H \to H$ such that:

The inverse map A^{-1} exists

$$\langle Ah_1, Ah_2 \rangle = \langle h_1, h_2 \rangle$$

i.e., A is isometric .

Definition. The stochastic process $S: T \to H$ is *stationary* (relative to group G which acts on T) if there is a group-homomorphism $\alpha: G \to O(H)$ such that

$$\alpha(g)(S(\tau)) = S(g^{-1}\tau) \tag{14.24}$$

for $g \in G$, $\tau \in T$.

Remark. Condition (14.24) means that the "means" and "correlations"

$$T \to E(S(\tau))$$

$$(\tau_1, \tau_2) \to E(S(\tau_1)S(\tau_2))$$

are *invariant under translation by* G. In the case where $T = R$ or Z, G = additive group of translations, this is what is usually called "weak stationarity" in the theory of stochastic processes.

Having defined "stationary processes" in this way, we may now define *filters* as maps whose domain and range are stationary stochastic processes. For example. if $T = G = Z$, it is readily seen that such *linear* filters are of the form

$$f \to k*f,$$

where $k: T \to R$ is a real-valued function and $*$ is the usual *convolution*. Again, as in Example 1, this leads us back to "harmonic analysis" of the group G.

In summary, in this section we have briefly seen that there are possibilities for application of general Lie group-geometric methods in the theory of deterministic and stochastic filters.

15. CONCLUDING REMARKS

Certain general principles for the application of modern differential geometry to system theory have been presented. These principles have been applied (in joint work with Clyde Martin) to the feedback structure of linear time-invariant systems. The key technique is the theory of *holomorphic vector bundles*. Such bundles can also be defined for distributed parameter systems, although they now have singularities and their "poles" become submanifolds.

This approach constructs a bridge between the classical work in electrical engineering and control theory concerned with the description of systems via "transfer functions" and modern differential and algebraic geometry. Where the engineer speaks of "poles", the geometer considers "intersections". One might expect that mathematics of this sort will become essential for a systematic treatment of the "non-classical" control problems that are on the technological horizon.

APPENDIX

GENERALIZATION OF THE PSEUDO-INVERSE CONSTRUCTION TO RIEMANNIAN MANIFOLDS; SLICES AND CANONICAL FORMS FOR GROUP ACTIONS

One of the fundamental mathematical problems of system theory is to find liftings of mappings in the following sense:

Let X, Y, Λ be spaces and let

$$\pi: X \to Y$$

$$\phi: \Lambda \to Y$$

be mappings. We want to find mappings

$$\phi': \Lambda \to X \quad ,$$

such that

$$\pi\phi'(\lambda) = \phi(\lambda)$$

for all $\lambda \in \Lambda$.

Further, one wants to let ϕ' depend on ϕ in a reasonably "robust" way.

For example, the estimation problem takes this form:

X = a space of systems

Y = a space of input-output relations

$\pi: X \to Y$ the map which assigns to each system the input-output relations it determines .

In case X is the space of controllable and observable linear systems with given input, state, and output spaces, Y is the orbit space of the action of the group of change of bases on state space.

In case X and Y are finite dimensional vector spaces, the construction of ϕ' is usually handled via what is called (in statistics and numerical analysis) the *pseudo-inverse*. We shall first recall its definition, then sketch a possible generalization to Riemannian manifolds. Then we go on to the case (which has evident implications for system theory) where Y is the orbit space of the action of a group G on X. The simplest general theory occurs when X has a Riemannian metric, and G is a group of *isometries* of this Riemannian metric. In case G is *discrete*, Poincaré suggested (in classical work) a method of constructing a "fundamental domain", i.e., a slice of the orbit space. In [31], the author has suggested a method of generalization of Poincaré's construction. This construction will be reviewed in this appendix.

The Pseudo-Inverse

Suppose that X and Y are finite dimensional real vector spaces with a given positive-definite symmetric dot product:

$$(x_1,x_2) \to x_1 \cdot x_2 \in R$$

$$(y_1,y_2) \to y_1 \cdot y_2 \in R \quad .$$

The *norms* of x and y are then

$$\|x\| = (x \cdot x)^{1/2}$$

$$\|y\| = (y \cdot y)^{1/2}$$

$$d(x_1,x_2) = \|x_1 - x_2\|$$

$$d(y_1,y_2) = \|y_1 - y_2\|$$

are then *metrics* on X and Y.

Suppose that $\pi: X \to Y$ is a linear map. Let $\pi^*: Y \to X$ be the *dual map*, defined by the following relation

$$y \cdot \pi x = \pi^* y \cdot x$$

$$\text{for} \quad x \in X, \quad y \in Y \quad .$$

Let us now minimize the function $x \to \|x\|^2$, subject to the *constraint* $y - \pi x = 0$. Using the Lagrange multiplier rule, introduce the Lagrange multiplier vector $\lambda \in Y$ and define the function

$$f(x,y,\lambda) = \|x\|^2 + \lambda \cdot (y - \pi x)$$

We now look for the unconstrained critical points of the function $x \to f(x,y,\lambda)$ with (y,λ) held fixed:

$$\frac{d}{dt} f(x+t\delta x, y, \lambda)\bigg|_{t=0} = 2x \cdot \delta x - 2\lambda \cdot \pi(\delta x)$$

$$= 2(x - \pi^*(\lambda)) \cdot \delta x \quad .$$

x is a critical point of this function if and only if

$$x = \pi^*(\lambda) \quad . \tag{A.1}$$

The constraint condition gives:

$$y = \pi\pi^*(\lambda) \quad . \tag{A.2}$$

Definition. The *pseudo-inverses* of y under the map π are the points $x \in X$ of the form (A.1), where $\lambda \in Y$ satisfies (A.2).

Remark. Notice that conditions (A.1) and (A.2) *together* imply that $\pi x = y$. There is a pseudo-inverse if and only if $\pi\pi^*$ is an isomorphism $Y \to Y$. In this case,

$$y \to \pi^*(\pi\pi^*)^{-1}(y) \equiv \gamma(y)$$

is a map $Y \to X$ which is a cross-section of the map $\pi: X \to Y$, i.e., $\pi\gamma =$ identity.

Maps Between Riemannian Manifolds

Let X and Y be Riemannian manifolds. Denote the inner product on tangent vectors as

$$(v_1, v_2) \to \langle v_1, v_2 \rangle \quad .$$

(For notation or background of Riemannian geometry, see [22].) If $t \to x(t)$, $a \leq t \leq b$, is a curve in X, then

$$\int_a^b \left\langle \frac{dx}{dt}, \frac{dx}{dt} \right\rangle^{1/2} dt$$

is the *length* of the curve. The *distance* $d(x_1, x_2)$ between two points $x_1, x_2 \in X$ is the greatest lower bound of the length of all continuous, precise C^∞ curves joining x_1 to x_2. This satisfies all the usual axioms needed to define a *metric space*.

Suppose that $\pi: X \to Y$ is a map. Pick a point $x_0 \in X$ which is held fixed throughout the discussion. If Z is a subset of X, let

$$d(x_0,Z) = \inf_{z \in Z} d(x_0,z) \quad .$$

For $y \in Y$, set

$$\gamma(y) = \{x \in X: \; x \in \pi^{-1}(y), \text{ and } d(x_0,x) \equiv d(x_0, \pi^{-1}(y))\}$$

The points in $\gamma(y)$ are called the *pseudo-inverses* of x under the map π (relative to the fixed Riemannian metric).

We are especially interested in three questions:

a) When do there exist points in $\gamma(y)$?

b) When does there exist a unique point in $\gamma(y)$?

c) If there is a unique point in $\gamma(y)$ for each $y \in Y$, is the map $y \to \gamma(y)$ of $Y \to X$ (which is a cross-section for the map $\pi: X \to Y$) sufficiently smooth to provide the sort of "robust" liftings of mappings $\Lambda \to Y$ needed for applications?

Now, a reasonable sufficient condition for (a) is the following one:

a') The subset $\pi^{-1}(y)$ is a closed, regularly embedded submanifold of X and the Riemannian metric on X is complete.

In case (a') is satisfied, and x_0 lies sufficiently near $\pi^{-1}(x)$ (e.g., in a tubular neighborhood), then (b) will be satisfied. However, there is no known reasonable condition which will imply (b). (c) represents even more unknown territory. However, the powerful techniques available for handling questions like these ("global Riemannian geometry") give *some* hope that this is a construction which might prove useful in certain circumstances. Certainly, it is the natural generalization of the "pseudo-inverse", which has proved to be widely useful and deserves further study. To my knowledge, there is no work in the literature in this direction.

SLICES, FUNDAMENTAL DOMAINS, CANONICAL FORMS, ETC. FOR TRANSFORMATION GROUP ACTIONS

Let X be a manifold, G a Lie group, and $G \times X \to X$: $(g,x) \to gx$, be a transformation group action on X. The *orbits* of G are the subsets of X of the form Gx, with $x \in X$. An *orbit space* for G acting on X is another space Y (possibly having a more general structure than a manifold) and a map $\pi: X \to Y$ such that the fibers $\pi^{-1}(y)$, for $y \in Y$, are the orbits of G. Suppose such an orbit space is given.

A subset $S \subset X$ is a *slice* (or *fundamental domain*) for the action of G if S meets each orbit of G in precisely one point, i.e., if the following condition is satisfied:

π restricted to S is one-one and onto.

Thus, if $\gamma = (\pi$ restricted to $S)^{-1}$, γ is a cross-section map: $Y \to X$. (Conversely, if $\gamma: Y \to X$ is a cross-section map for π, then $S = \gamma(Y)$ is a slice.)

One usually wants S to have further properties. For example, it would be nice if it were a submanifold. (This is usually too much to hope for.)

The concept of "canonical form" is rather vague and indeterminate in the literature. Sometimes it means a slice, sometimes a more general subset which touches the "generic" orbit in precisely one point, but may have a larger intersection with certain "singular" orbits. In system theory (where it is the key concept in attempts to solve the "identification problem") it is often a *submanifold* of X, which touches a "generic" set of orbits precisely once, but does not necessarily touch all orbits.

In one case, generalizing the classical construction due to Poincaré for discrete groups [31], there is an elegant and useful definition of a "slice". Namely, suppose that X is a complete connected Riemannian manifold, and that G is a group of *isometries* of X which is a *closed* subgroup of the Lie group of all isometries of X. It follows [22] that all orbits of G are regularly embedded, closed submanifolds of X, and that each has a tubular neighborhood which is invariant under G. Pick a point $x_0 \in X$. Set:

$$P = \{x \in X: \ d(x,x_0) \leq d(x_0,Gx)\} \quad .$$

The proof that P is a slice, and has other nice properties, is given in [22], which also contains further useful general information about the orbits and the structure of the orbit space in this case. Finally, notice that P is precisely the "generalized inverse" set, as defined above, where Y = orbit space $G \backslash X$, and $\pi: X \to Y$ is the natural quotient map which assigns the orbit Gx to each $x \in X$.

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1. Report No. NASA CR-3209	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle "System Theory As Applied Differential Geometry"		5. Report Date November 1979
		6. Performing Organization Code
7. Author(s) Robert Hermann		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address American Mathematical Society Providence, Rhode Island		11. Contract or Grant No. NSG-2252
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The report examines in depth the invariants of input-output systems under the action of the feedback group. The approach uses the theory of Lie groups and concepts of modern differential geometry, thereby illustrating how the latter provides a basis for the discussion of the analytic structure of systems. The first half of the report considers finite dimensional linear systems in a single independent variable. This part has been written so as to be readily accessible to a reader with only a modest background in modern algebra and advanced calculus. The second half of the report extends the lessons of the first half to the more general situations (e.g., distributed parameter and multi-dimensional systems) which are increasingly encountered as technology advances.

17. Key Words (Suggested by Author(s)) Linear Systems, Feedback Systems, Control Theory, Differential Geometry, Lie Groups, Helmholtz Equation, Digital Filters		18. Distribution Statement Unclassified - Unlimited STAR Category - 63	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 65	22. Price* $5.25

*For sale by the National Technical Information Service, Springfield, Virginia 22161