(NASA-CR-162491) MODULAR THEORY OF INVERSE SYSTEMS Final Report, 1 Jun. - 31 Dec. 1979 (Notre Dame Univ.) 57 p HC A04/MF A01 CSCL 12A
N80-12782
Unclas
G3/64 46199

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# ELECTRICAL ENGINEERING

UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA

Final Technical Report

to the

National Aeronautics and Space Administration

on

NASA Grant NSG-2388*

# MODULAR THEORY OF INVERSE SYSTEMS

June 1, 1979 - December 31, 1979

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## Foreword

The following manuscript reports in detail upon researches carried out from June 1, 1979 to December 31, 1979 under support of NASA Grant NSG-2388, entitled "Modular Theory of Inverse Systems". A preliminary version of some of the results has been presented by Drs. Wyman and Sain at the 18th IEEE Conference on Decision and Control, December 1979. The results reported here have been submitted for journal publication.

# THE ZERO MODULE AND ESSENTIAL INVERSE SYSTEMS*

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Abstract: The theory of inverse dynamical systems has been and is continuing to be a keystone in the development of the theories of multivariable feedback control systems and of coding theory for reliable communication. Advances in understanding of the role of plant inverses in control system design have brought about additional insights, for example, in the general areas of decoupled design and of realistic possibilities for closed loop dynamical performance. Surprisingly enough, almost all the existing literature on inverse systems is cast in terms of matrices. Though the well known module theoretic approach to systems has been in place for a decade or more, this approach has not been fully exploited to bring out the foundations of a theory for inverse systems. This paper begins to lay such a foundation by developing a definition of zero module for a system. When inverse systems exist, their "pole modules" can be shown to contain the zero module in an appropriate algebraic sense. If the containment is tight, these inverses are called essential. Existence of and constructions for essential inverses are provided.

*This work was supported in part by the National Aeronautics and Space Administration under Grant NSG-2388.

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# 1. INTRODUCTION

Control engineers have been interested in the problem of inverting linear dynamical systems for more than a quarter century. A classic and pivotal illustration is the work of Bode and Shannon [1] in 1950. In that paper, the basic results of the famous Wiener-Kolmogorov theory for smoothing and prediction of stationary time series were reconsidered by methods based upon what was called at that time "electric circuit theory". Pointing out that the Wiener-Kolmogorov theory involved "formidable mathematics" and that Wiener's report had come to be known as "The Yellow Peril", Bode and Shannon saw the need for methods which had "the advantage of greater simplicity" and which had associated with them the insights of a "direct physical interpretation".

Of course, every student of the subject is now exposed to the resulting concept of a whitening filter, which was intimately related to the inversion of transfer functions. Moreover, these original steps have now extended to the more general modern interpretations of innovations processes.

Another basic step along the line of development of inverse systems tended to occur when engineers began their efforts to extend the classical frequency domain theories to the matrix case, in which the model was assumed to have more than one input and one output. Many matrix design equations were developed at that time. Basically, these matrix inversions were primarily to solve linear equations whose coefficients were from the field of rational functions in the Laplace variable $s$ and with coefficients from the real number field $R$. Some interesting examples of this literature may be found just prior to 1960 [2, 3, 4, 5].

These studies were to encounter some very nontrivial conceptual questions, however, inasmuch as there were cancellations of all types to be considered. Combined with the fact that the theory of the relationship between "interior" or state descriptions and "exterior" or transfer matrix descriptions was in its infancy, these cancellations left hard questions about internal stability unanswered. With the recent advent of the concept of multivariable zero, it is possible to see in retrospect just how difficult some of these questions were.

The formal study of the question of inverting linear dynamical systems arose again from several sources in the late 1960s. Occurring independently and within six months of each other, the papers by Youla and Dorato [6], Sain and Massey [7], and Silverman [8] initiated what is now becoming an increasingly important research area.

Though inverse system studies in the matrix sense have proceeded apace [9], and although inversion problems are demonstrably important in a wide variety of recent applications [10,11,12]including transport theory, meteorology and radar and optical imaging, almost no work has been carried out to extend the theory of inverse dynamical systems beyond the matrix viewpoint to the very fundamental underlying modular viewpoint. This is quite surprising, since a well known [13] module theoretic approach to dynamical systems has been in place for a decade or more.

In this paper, we explore further the relationship between multivariable zeros and inverse systems. Section 2 provides preliminaries and notation. Section 3 defines a zero module which is given in such a way that it is basis independent. Every system with a suitable abstract input/

output description has such a zero module, even if it does not have inverses. When a system input/output function is epic or monic, then there are right or left inverses. Section 4 shows that every such inverse has poles which "contain" (either as a quotient module or a submodule) the zeros of the system. If the inverse has no additional poles, it is said to be "essential". Section 5 establishes the existence of essential right and left inverses. The way in which the abstract zero module captures previous definitions of multivariable zeros is explained in Section 6; and examples are given in Section 7.

## 2. PRELIMINARIES AND NOTATION

Let $k$ be an arbitrary field. We denote by $k[z]$ the ring of polynomials in the indeterminate $z$, with coefficients from $k$, and by $k(z)$ the quotient field of $k[z]$.

If $U$ is an m-dimensional k-vector space, then

$$U[z] = \{ \sum_{i=0}^{n} u_i z^i : n \geq 0 \quad , \quad u_i \in U\}$$

is the free $k[z]$-module of polynomials with coefficients from $U$. As a free module, $U[z]$ has rank $m$, because it is free on any basis of $U$. Inasmuch as $U \approx k^m$, we can think of $U$ in terms of column vectors of height $m$ with entries from $k$. Then $U[z]$ may be visualized also as column vectors with entries from $k[z]$, which is a statement of isomorphism $U[z] \approx (k[z])^m$. In this spirit, we denote by $U(z)$ the $k(z)$-vector space of column vectors of height $m$ with entries from $k(z)$.

Now $U[z]$ is a $k[z]$-submodule of $U(z)$, and so it follows that the factor module

$$\Gamma U = U(z)/U[z]$$

is a $k[z]$-module, sometimes called a "Kalman output module". In this language, $U[z]$, also denoted by $\Omega U$, is called a "Kalman input module". Actually, $\Gamma U$ as here defined is the torsion submodule of the Laurent series module $U((z^{-1}))/U[z]$ which is more commonly used in the theory [14 , 15].

If $Y$ is a p-dimensional k-vector space, then the corresponding $k(z)$-vector space $Y(z)$ and $k[z]$-modules $Y[z]$ and $\Gamma Y$ can be formed in the same way.

A <u>transfer function</u> is a $k(z)$-linear map

$$G(z) : U(z) \to Y(z).$$

Alternatively, $G(z)$ can be considered as a $p \times m$ matrix with entries from $k(z)$. The matrix $G(z)$ will not be assumed proper. In general, then, we may write

$$G(z) = G_{poly}(z) + G^{\#}(z) ,$$

where $G_{poly}(z)$ has entries from $k[z]$ and $G^{\#}(z)$ is strictly proper in the sense that each of its entries has numerator degree strictly less than denominator degree. We will call $G^{\#}(z)$ the <u>strictly proper part</u> of $G(z)$. The map $G^{\#}(z)$ defines a $k[z]$-module map

$$G^{\#}(z) : \Omega U \to \Gamma Y$$

as shown in the upper part of the diagram in Figure 1.

A strictly proper $G^{\#}(z)$ can be <u>realized</u> by a linear dynamical system

$$\Sigma = (X, U, Y ; A, B, C) .$$

Here $X$, $U$, and $Y$ are finite-dimensional $k$-vector spaces; and

$$A : X \to X \quad , \quad B : U \to X \quad , \quad C : X \to Y$$

are $k$-linear maps. The system $\Sigma$ realizes $G^{\#}(z)$ if

$$G^{\#}(z) = C(z I - A)^{-1} B ,$$

or, equivalently, if there exists a commutative realization diagram of $k[z]$-modules as shown in the lower part of Figure 1. In this figure, $\overline{A^X}$ is a $k[z]$-module with underlying vector space given by $\overline{X}$ and module action given by

$$
\begin{tikzcd}
 & U(z) \arrow[rr, "G(z)"] & & Y(z) \arrow[dr, "p"] & \\
\Omega U \arrow[ur, "i"] \arrow[rrr, "G^{\#}(z)"] \arrow[drr] & & & & \Gamma Y \\
 & & A^X \arrow[urr] & &
\end{tikzcd}
$$

Figure 1.

$$z\,x = A\,x\,.$$

The maps $\tilde{B}$ and $\tilde{C}$ are $k[z]$-module maps defined by

$$\tilde{B}(uz^i) = A^i\,B\,u\,,$$

$$\tilde{C}(x) = C\,x\,z^{-1} + C\,A\,x\,z^{-2} + \ldots \pmod{Y[z]}.$$

The system $\sum$ is a <u>minimal</u> <u>realization</u> of $G^{\#}(z)$ if $\tilde{B}$ is epic (reachability) and $\tilde{C}$ is monic (observability). The module ${}_AX$ is then called the <u>minimal</u> <u>state</u> <u>module</u> of the realization $(A, B, C)$. Since all minimal realizations of $G^{\#}(z)$ are isomorphic as systems, it follows that all minimal state modules are isomorphic as $k[z]$-modules.

The major technical tool used in this paper is the theory of modules over a principal ideal domain ring. A readable account of the whole theory can be found in [16 ]; and a quick summary (without proofs) is presented here in the special case of modules over the polynomial ring $k[z]$.

Let $M$ be a finitely generated $k[z]$-module. Then $M$ is isomorphic to a direct sum

$$M \cong F \oplus T\,,$$

where

$$F \cong (k[z])^n$$

is a free module of rank $n$ and $T$ is a torsion module. The rank of $F$, and the torsion module $T$, are uniquely determined by $M$. Concretely, the torsion module $T$ is given as a finite dimensional $k$-vector space, also called $T$, together with a $k$-linear endomorphism $A : T \to T$. The $k[z]$-module action on $T$ is given by

$$p(z)\ v = p(A)\ v$$

for all $v$ in $T$ and all polynomials $p(z)$. The module $T$ can also be described in the manner

$$T \cong k[z]/(\Psi_1(z)) \oplus \ldots \oplus k[z]/(\Psi_s(z)) ,$$

where $\{\Psi_i(z),\ i = 1,2,\ldots,s\}$ are nonzero polynomials with the property $\Psi_i | \Psi_{i+1}$, $i = 1,2,\ldots,s-1$. The $\Psi_i$ are called the <u>invariant factors</u> of $T$ and are uniquely determined by $T$. $(\Psi_i)$ is the ideal generated by $\Psi_i$ in $k[z]$.

Two torsion modules are isomorphic if and only if they have the same invariant factors.

The module $M$ is frequently given as the <u>cokernel</u> of a polynomial matrix. Suppose $N(z)$ is such a matrix, say of size $p \times m$. Then $N(z)$ represents a morphism

$$N(z) : (k[z])^m \to (k[z])^p$$

of free modules. The cokernel of $N(z)$ is defined by

$$\text{coker } N(z) = (k[z])^p / N(z)(k[z])^m ,$$

which is a finitely generated module. The invariant factors of coker $N(z)$ can be computed from the <u>Smith form</u> of $N(z)$.

The Smith form is given as follows. There exist a $p \times p$ polynomial matrix $E(z)$ and an $m \times m$ polynomial matrix $F(z)$, both with unit determinant in $k[z]$, such that

$$E(z)\ N(z)\ F(z) = S(z) ,$$

where $S(z)$ is as "diagonal as possible" with invariant factors on the

"diagonal". For instance, if $p \geq m$, then

$$S(z) = \begin{bmatrix} \Psi_1(z) & 0 & \cdots & 0 \\ 0 & \Psi_2(z) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_p(z) \end{bmatrix}$$

where, for some $s \leq p$,

$$\Psi_{s+1}(z) = \ldots = \Psi_p(z) = 0 ,$$

and $\Psi_i(z)$, $i = 1,2,\ldots,s$, are nonzero with $\Psi_i | \Psi_{i+1}$ for $i = 1,2,\ldots,s-1$. The Smith form $S(z)$ is uniquely determined by $N(z)$.

Furthermore, $S(z)$ and $N(z)$ have isomorphic cokernels; and

$$\text{coker } N(z) \cong k[z]/(\Psi_1(z)) \oplus \ldots \oplus k[z]/(\Psi_s(z)) \oplus (k[z])^{p-s} .$$

This means that the rank of the free part of coker $N(z)$ and the invariant factors of the torsion part are all determined by the Smith form of $N(z)$.

In particular, note that coker $N(z)$ is finite dimensional if and only if $s$ is equal to $p$, that is, if $N(z)$ has full rank.

## 3. THE ZERO MODULE OF A TRANSFER FUNCTION

Consider a given transfer function

$$G(z) : U(z) \to Y(z).$$

The purpose of this section is to capture, in an abstract, module theoretic sense, the multivariable generalization of the classical notion of zero. In so doing, it is intended not only to characterize the multivariable zeros of the transfer function $G(z)$, with multiplicity, but also to describe a finitely generated, torsion module structure which gives rise to the zeros.

This goal is accomplished in two steps. The first step is to give a definition of the module which depends only on the basic concepts of module theoretic system theory and not upon any of its particular matrix embodiments. The second step is to supply a module isomorphism which makes clear the finitely generated torsion structure.

It is helpful to give a brief intuitive prelude to the basic module definition. Consider the classical case in which

$$p = m = 1.$$

Then the matrix of the transfer function $G(z)$ could be visualized in the manner

$$\frac{a(z)}{b(z)},$$

where $a(z)$ and $b(z)$ are relatively prime elements of $k[z]$. The basic idea of a "zero" was understood as follows. If $u(z)$ in $U(z)$ had representation

$$u(z) = \frac{n(z)}{d(z)} ,$$

for $n(z)$ and $d(z)$ relatively prime in $k[z]$, and if $d(z)$ and $a(z)$ had non-unit factors in common, then the "modes" of $u(z)$ indicated by these common factors failed to appear in the corresponding output

$$y(z) = \frac{n(z)\ a(z)}{d(z)\ b(z)} ,$$

because of cancellation. Note that the identically zero output was of little interest in this regard, since it was the failure of certain exciting modes to appear in the response that was paramount. Though this is an academic point in the classical case, when there are no nonzero excitations which produce zero responses, it is an important observation in a multivariable generalization where $\ker G(z)$ is not necessarily zero. And its meaning is that $\ker G(z)$ can safely be neglected. Note also that, if

$$u(z) = p(z) \in k[z] ,$$

no zero effect could have been observed in the classical case, because $u(z)$ would have had no "modes" which could fail to appear in $y(z)$. Thus, $k[z]$ is of no interest insofar as producing test inputs to discover zeros; and thus $\Omega U$, which is its generalization, can safely be neglected in an abstract multivariable definition. In defining this abstract module, then, it is consistent with the classical case to ignore

$$\ker G(z) + \Omega U ,$$

which can be accomplished by forming an algebraic quotient .

It remains to describe abstractly what is meant by a "zero". Here again the classical case can be quite motivating. Simply focus on the

excitations which can produce no "modes" whatsoever in the response. As an example,

$$u(z) = \frac{b(z)}{a(z)} p(z) ,$$

for $p(z)$ in $k[z]$. Such excitations produce response

$$y(z) = p(z) \in k[z]$$

having no "modes". The "modes" of these excitations, which are expressed by $G^{-1}(k[z])$, capture the classical concept of "zero"; since $k[z]$ generalizes to $\Omega Y$, the class of excitations of interest can be extended easily to

$$G^{-1} (\Omega Y).$$

With these motivations, define the <u>zero module</u> $Z(G)$ of the transfer function $G(z)$ by

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\ker G(z) + \Omega U} ,$$

where the addend $\Omega U$ in the numerator of the quotient is provided so that the denominator is contained in the numerator. Because $\Omega Y$ and $\Omega U$ are $k[z]$-modules, it follows that $Z(G)$ is a $k[z]$-module.

Before establishing the finitely generated, torsion structure of $Z(G)$, it is a useful illustration to check the definition algebraically for the classical case. In that case, $\ker G(z)$ vanishes, and

$$G^{-1}(\Omega Y) = \{u(z) \in k(z) : \frac{a(z)}{b(z)} u(z) = p(z) \in k[z]\}$$

$$= \{\frac{b(z)p(z)}{a(z)} : p(z) \in k[z]\}.$$

Then

$$G^{-1}(\Omega Y) + \Omega U = \{\frac{b(z)p(z)}{a(z)} + q(z) : p(z), q(z) \in k[z]\}$$

$$= \{\frac{b(z)p(z) + a(z)q(z)}{a(z)} : p(z),\ q(z) \in k[z]\}$$

$$= \frac{1}{a(z)}\ \Omega U.$$

For the last step, use the fact that $a(z)$ and $b(z)$ are relatively prime so that any polynomial $r(z)$ in $\Omega U$ can be written

$$r(z) = b(z)\ p(z) + a(z)\ q(z)$$

for suitable $p(z)$ and $q(z)$ in $k[z]$. Accordingly,

$$Z(G) = \frac{1}{a(z)}\ \Omega U/\Omega U$$

for the classical case.

Notice that there exists a $k[z]$-module isomorphism

$$\frac{\frac{1}{a(z)}\ \Omega U}{\Omega U} \quad \rightarrow \quad \frac{\Omega U}{a(z)\ \Omega U} \quad = \quad \frac{k[z]}{a(z)k[z]}$$

defined by the action

$$\frac{r(z)}{a(z)} \bmod \Omega U \ \rightarrow\ r(z) \bmod a(z)\ \Omega U\ ,$$

so that the zero module gives the expected result in the classical case.

Next consider the second step, which establishes a $k[z]$-module isomorphism between $Z(G)$ and a finitely generated torsion module.

For this step, assume a left coprime factorization

$$G(z) = D^{-1}(z)\ N(z)$$

where the $k(z)$-vector space homomorphism $N(z)$ and automorphism $D(z)$ can also be understood as free $k[z]$-module homomorphisms

$$N(z)\ :\ \Omega U \rightarrow \Omega Y$$

and

$$D(z)\ :\ \Omega Y \rightarrow \Omega Y$$

which satisfy

$$D(z)\ A(z) + N(z)\ B(z) = 1_{\Omega Y}$$

for appropriate free k[z]-module homomorphisms

$$A(z) : \Omega Y \to \Omega Y$$

and

$$B(z) : \Omega Y \to \Omega U.$$

Notice that, as a k[z]-module homomorphism, $D(z)$ is only an endomorphism and not an automorphism.

The nature of the zero module $Z(G)$ is then established by the following result.

<u>Theorem 1</u>

Given any transfer function $G(z) : U(z) \to Y(z)$, with left coprime factorization $D^{-1}(z)\ N(z)$, then the zero module $Z(G)$ is isomorphic as a k[z]-module to the torsion submodule of $\Omega Y/N(z)\Omega U$.

Proof: Consider the k[z]-homomorphism

$$\alpha_1 : G^{-1}(\Omega Y) + \Omega U \to \Omega Y$$

whose action is given by

$$\alpha_1\ (u(z)) = N(z)\ u(z),$$

with the calculation in the right member following from $N(z)$ regarded as a k(z)-linear map. Notice that

$$\alpha_1(\Omega U) \subset \Omega Y$$

trivially. Moreover, if $u(z) \in G^{-1}(\Omega Y)$, then

$$D^{-1}(z)\ N(z)\ u(z) = y(z) \in \Omega Y,$$

so that

$$N(z)\ u(z) = D(z)\ y(z) \in \Omega Y$$

as required. Now let

$$p : \Omega Y \to \Omega Y/N(z)\Omega U$$

be the natural projection, and define

$$\alpha_2 = p\alpha_1$$

by composition. To establish the theorem, examine the diagram of Figure 2, where $q$ is the natural projection

$$G^{-1}(\Omega Y) + \Omega U \to \frac{G^{-1}(\Omega Y) + \Omega U}{\ker G(z) + \Omega U}\ .$$

To show the existence of the k[z]-module homomorphism $\bar{\alpha}_2$, together with the fact that it is monic, it suffices to verify that

$$\ker \alpha_2 = \ker G(z) + \Omega U.$$

Suppose that $u(z) \in \Omega U$. Then

$$\begin{aligned} \alpha_2\ u(z) &= p\ \alpha_1\ u(z) \\ &= p\ N(z)u(z) \\ &= 0. \end{aligned}$$

Moreover, if $u(z) \in \ker G(z)$, then

$$D^{-1}(z)\ N(z)\ u(z) = 0$$

from which

$$N(z)\ u(z) = 0$$

so that

$$\alpha_2\ u(z) = 0.$$

Accordingly,

$$\ker G(z) + \Omega U \subset \ker \alpha_2.$$

For the opposite inclusion, let

$$\begin{tikzcd}
G^{-1}(\Omega Y) + \Omega U \arrow[r, "q"] \arrow[dr, "\alpha_2"'] & \dfrac{G^{-1}(\Omega Y) + \Omega U}{\ker G(z) + \Omega U} \arrow[d, dashed, "\bar{\alpha}_2"] \\
& \Omega Y / N(z)\Omega U
\end{tikzcd}$$

Figure 2.

$$u(z) \in G^{-1}(\Omega Y) + \Omega U$$

be such that

$$\alpha_2 u(z) = 0.$$

Since $\alpha_2 = p\,\alpha_1$, it follows that

$$\alpha_1 u(z) \in \ker p = N(z)\Omega U$$

so that

$$\alpha_1 u(z) = N(z)\, u_1(z)$$

for some $u_1(z) \in \Omega U$. Therefore

$$u(z) - u_1(z) \in \ker N(z) = \ker G(z),$$

and it is a consequence that

$$u(z) = (u(z) - u_1(z)) + u_1(z)$$

with $u(z) - u_1(z)$ in $\ker G(z)$ and $u_1(z)$ in $\Omega U$. Thus every such $u(z)$ lies in $\ker G(z) + \Omega U$.

This establishes that the $k[z]$-module homomorphism $\bar{\alpha}_2$ exists, is unique, and is monic. It remains to show that the image of $\bar{\alpha}_2$ is the

torsion submodule of $\Omega Y/N(z)\Omega U$. For $u(z)$ in $U(z)$, there is a polynomial $t(z)$ in $k[z]$ such that

$$t(z)u(z) = u_1(z) \in \Omega U.$$

Then

$$\begin{aligned} t(z)N(z)u(z) &= N(z)t(z)u(z) \\ &= N(z)u_1(z) \in N(z)\Omega U, \end{aligned}$$

and

$$t(z)\ \alpha_2\ u(z) = 0,$$

which means that

$$\operatorname{im} \alpha_2 = \operatorname{im} \bar{\alpha}_2$$

is contained in the required torsion submodule. Next consider an arbitrary torsion element of $\Omega Y/N(z)\Omega U$, expressed as $p\ y(z)$ for $y(z)$ in $\Omega Y$. Using left-coprimeness, write

$$y(z) = D(z)\ A(z)\ y(z) + N(z)\ B(z)\ y(z).$$

Because

$$N(z)\ B(z)\ y(z) \in \ker p,$$

it follows that $p(D(z)\ A(z)\ y(z))$ is also a torsion element in $\Omega Y/N(z)\Omega U$. Thus there is a polynomial $t(z)$ in $k[z]$ such that, for some $u(z)$ in $\Omega U$,

$$t(z)\ D(z)\ A(z)\ y(z) = N(z)\ u(z).$$

Let

$$v(z) = \frac{1}{t(z)}\ u(z) + B(z)\ y(z).$$

Then

$$\begin{aligned} N(z)\ v(z) &= D(z)\ A(z)\ y(z) + N(z)\ B(z)\ y(z) \\ &= y(z). \end{aligned}$$

Furthermore, $B(z)\ y(z)$ is in $\Omega U$, and

$$\frac{1}{t(z)}\ u(z)\ \varepsilon\ G^{-1}(\Omega Y)\ ,$$

as may be verified by the calculation

$$\begin{aligned} G(\frac{1}{t(z)}\ u(z)) &= D^{-1}(z)\ N(z)\ (\frac{1}{t(z)}\ u(z)) \\ &= D^{-1}(z)\ D(z)\ A(z)\ y(z) \\ &= A(z)\ y(z)\ \varepsilon\ \Omega Y. \end{aligned}$$

This implies that

$$v(z)\ \varepsilon\ G^{-1}(\Omega Y) + \Omega U,$$

so that

$$y(z)\ \varepsilon\ \mathrm{im}\ \alpha_1,$$

so that

$$p\ y(z)\ \varepsilon\ \mathrm{im}\ \alpha_2 = \mathrm{im}\ \bar{\alpha}_2\ .$$

Thus $\bar{\alpha}_2$ maps onto the entire torsion submodule of $\Omega Y/N(z)\Omega U$, which completes the proof of the theorem, except for establishing the fact that $Z(G)$ is finitely generated. However, this conclusion follows from the discussion of coker $N(z)$ in Section 2, where it is shown that the torsion submodule is a direct sum of finitely many modules each of which has one generator.

It should be noted that no assumptions concerning the rank or nullity of $G(z)$ have been made in this section.

As an application of zero module concept, the next two sections relate $Z(G)$ to inverse systems associated with $G(z)$, for the cases in which such notions exist, namely when $G(z)$ is epic or monic.

## 4. ESSENTIAL POLE STRUCTURE OF INVERSE SYSTEMS

As pointed out in the introduction to this paper, the concept of inverse systems has played a very useful role in the various applications, including feedback control theory and coding for reliable communication. The notion of multivariable zero tends to arise quite naturally in such contexts along the lines of the intuitive statement "the zeros of $G(z)$ appear as poles of an arbitrary (left or right) inverse $\hat{G}(z)$ of $G(z)$". The purpose of this section is to give a precise algebraic version of this intuitive statement.

The discussion is divided into two parts, according to whether the transfer function $G(z)$ is epic or monic; and a theorem is given for each case.

Consider first the case in which $G(z)$ is epic. Then there exist right inverses

$$\hat{G}(z) : Y(z) \to U(z)$$

such that

$$G(z)\ \hat{G}(z) = 1_{Y(z)}.$$

Let $\hat{G}(z)$ be any such right inverse. It follows from Section 2 that $\hat{G}\#$ has a uniquely determined minimal state module $X(\hat{G})$, as indicated in Figure 3. Because $X(\hat{G})$ is isomorphic to a submodule of $\Gamma U$, it is possible to write

$$X(\hat{G}) \underset{\sim}{\sim} \hat{G}^{\#}\ (\Omega Y)$$
$$= \frac{\hat{G}(\Omega Y) + \Omega U}{\Omega U}.$$

The first of the two theorems can now be stated.

$$\begin{tikzcd}
\Omega Y \arrow[rr, "\hat{G}^{\#}"] \arrow[dr] & & \Omega U \\
& X(\hat{G}) \arrow[ur] &
\end{tikzcd}$$

Figure 3.

$$\begin{tikzcd}
\hat{G}(\Omega Y) + \Omega U \arrow[r, "\text{inclusion}"] \arrow[d, "q"'] \arrow[dr, "\pi"] & G^{-1}(\Omega Y) + \Omega U \arrow[d, "p"] \\
\dfrac{\hat{G}(\Omega Y) + \Omega U}{\Omega U} \arrow[r, dashed, "\bar{\pi}"'] & Z(G)
\end{tikzcd}$$

Figure 4.

Theorem 2A (Right Inverse Case)

Suppose given an epic transfer function $G(z) : U(z) \to Y(z)$. Let $G(z) : Y(z) \to U(z)$ be an arbitrary right inverse, and let $X(\hat{G})$ be the minimal state module of $\hat{G}$. Then there is an epic $k[z]$-module homomorphism

$$\bar{\pi} : X(\hat{G}) \to Z(G).$$

Proof: Let $u(z) \in \hat{G}(z)\Omega Y$, so that

$$u(z) = \hat{G}(z)\ y(z)$$

for a suitable $y(z)$ in $\Omega Y$. Then

$$\begin{aligned} G(z)\ u(z) &= G(z)\ \hat{G}(z)\ y(z) \\ &= y(z), \end{aligned}$$

which implies that $u(z) \in G^{-1}(\Omega Y)$. Thus

$$\hat{G}(\Omega Y) \subset G^{-1}(\Omega Y),$$

and the diagram of Figure 4 can be constructed. The natural projection $p$ has kernel $\ker G(z) + \Omega U$; $\pi$ is defined by composition; and the natural projection $q$ has kernel $\Omega U$. To complete the proof, it is to be shown that the $k[z]$-module homomorphism $\bar{\pi}$ exists, is unique, and is epic. Existence and uniqueness follows from the fact that

$$\Omega U \subset \ker p \cap (\hat{G}(\Omega Y) + \Omega U) = \ker \pi.$$

To show that $\bar{\pi}$ is epic, suppose given some element $\zeta$ in $Z(G)$. Write

$$\zeta = p\ u(z),$$

and assume without loss of generality that $u(z)$ is in $G^{-1}(\Omega Y)$. Then for some $y(z)$ in $\Omega Y$,

$$G(z)\ u(z) = y(z).$$

Now calculate

$$\begin{aligned} \pi\ \hat{G}(z)\ y(z) - \zeta &= p\ \hat{G}(z)\ y(z) - p\ u(z) \\ &= p[\hat{G}(z)\ y(z) - u(z)]. \end{aligned}$$

But

$$G(z)\ [\hat{G}(z)\ y(z) - u(z)] = G(z)\ \hat{G}(z)\ y(z) - G(z)\ u(z)$$
$$= y(z) - G(z)\ u(z)$$
$$= 0,$$

so that

$$\hat{G}(z)\ y(z) - u(z)\ \varepsilon\ \mathrm{Ker}\ G(z) \subset \mathrm{Ker}\ p,$$

and thus

$$\pi\ \hat{G}(z)\ y(z) - \zeta = 0,$$

which means that $\pi$ is epic. Finally,

$$\mathrm{im}\ \pi = \mathrm{im}\ \bar{\pi}\ ,$$

so that $\bar{\pi}$ is epic; and the theorem is proved.

In general, the homomorphism $\bar{\pi}$ is not monic. If $\bar{\pi}$ is monic, then the right inverse $\hat{G}(z)$ is called an <u>essential right inverse</u>. Notice that the minimal state module of an essential right inverse is isomorphic as a $k[z]$-module to the zero module $Z(G)$ of $G(z)$. Further discussion of essential right inverses, including a proof of their existence, is provided in Section 5.

Next consider the case in which $G(z)$ is monic. Then there exist left inverses

$$\hat{G}(z)\ :\ Y(z) \to U(z)$$

such that

$$\hat{G}(z)\ G(z) = 1_{U(z)}.$$

If $\hat{G}(z)$ is any such left inverse, it follows once again from Section 2 that $\hat{G}^{\#}$ has a uniquely determined minimal state module $X(\hat{G})$ which satisfies, as before,

$$X(\hat{G}) \approx \frac{\hat{G}(\Omega Y) + \Omega U}{\Omega U} .$$

<u>Theorem 2B</u> (Left Inverse Case)

Suppose given a monic transfer function $G(z) : U(z) \to Y(z)$. Let $\hat{G}(z) : Y(z) \to U(z)$ be an arbitrary left inverse, and let $X(\hat{G})$ be the minimal state module of $\hat{G}$. Then there is a monic $k[z]$-module homomorphism

$$\bar{i} : Z(G) \to X(\hat{G}) .$$

Proof: Because $G(z)$ is monic, $\ker G(z)$ vanishes; and

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} .$$

Now let $u(z) \in G^{-1}(\Omega Y)$, so that

$$G(z)\ u(z) = y(z)$$

with $y(z) \in \Omega Y$. Then

$$u(z) = \hat{G}(z)\ G(z)\ u(z) = \hat{G}(z)\ y(z),$$

so that $u(z) \in \hat{G}(\Omega Y)$ also. Thus

$$G^{-1}(\Omega Y) \subset \hat{G}(\Omega Y),$$

and the diagram of Figure 5 applies. In the figure, $p$ and $q$ are natural projections with kernels $\Omega U$, and $i$ is defined by composition. Observe that

$$\begin{aligned} \ker i &= \ker p \bigcap (G^{-1}(\Omega Y) + \Omega U) \\ &= \Omega U \bigcap (G^{-1}(\Omega Y) + \Omega U) \\ &= \Omega U = \ker q, \end{aligned}$$

so that $\bar{i}$ exists and is unique. Moreover, because of the equality

$$\ker i = \ker q,$$

$\bar{i}$ is monic as well, which establishes the theorem.

Again, $\bar{i}$ need not be epic. Should it happen, however, that $\bar{i}$

$$\begin{tikzcd}
G^{-1}(\Omega Y) + \Omega U \arrow[r, "\text{inclusion}"] \arrow[d, "q"'] \arrow[dr, "i"] & \hat{G}(\Omega Y) + \Omega U \arrow[d, "p"] \\
Z(G) \arrow[r, dashed, "\bar{i}"'] & \dfrac{\hat{G}(\Omega Y) + \Omega U}{\Omega U}
\end{tikzcd}$$

Figure 5.

is epic as well, then the left inverse $\hat{G}(z)$ is called an <u>essential left inverse</u>. Just as in the previous case, the minimal state module of an essential left inverse is isomorphic as a $k[z]$-module to the zero module $Z(G)$ of $G(z)$.

In the following section, the existence of essential inverses is demonstrated by construction for both cases.

## 5. CONSTRUCTION OF ESSENTIAL INVERSES

The preceding section examined transfer functions $G(z) : U(z) \to Y(z)$ which were either epic or monic. If $\hat{G}(z) : Y(z) \to U(z)$ is a (right or left) inverse, then it was established that the minimal realization of $\hat{G}(z)$ has poles which "contain" (either as a factor module or as a submodule) the zeros of $G(z)$. In this context, the zero module of $G(z)$ is called the <u>essential pole module</u> of $\hat{G}(z)$. If $\hat{G}(z)$ has no additional "inessential" poles, then $\hat{G}(z)$ is called an <u>essential inverse</u>.

More formally, suppose $G(z) : U(z) \to Y(z)$ is an (epic or monic) transfer function, and suppose $\hat{G}(z)$ is a (right or left) inverse of $G(z)$. Let $X(\hat{G})$ be the minimal state module of $\hat{G}(z)$, and let $Z(G)$ be the zero module of $G(z)$. Then $\hat{G}(z)$ is an <u>essential inverse</u> of $G(z)$ if

$$X(\hat{G}) \approx Z(G)$$

as $k[z]$-modules.

The purpose of this section is to show that essential inverses exist by giving an explicit construction for them. Right inverses and left inverses will be treated separately.

Begin with the case in which the transfer function $G(z) : U(z) \to Y(z)$ is epic. Suppose that

$$G(z) = D^{-1}(z)\, N(z)$$

is a left-coprime factorization as discussed in Section 3. As a $k(z)$-linear map, $N(z)$ must be epic, inasmuch as $G(z)$ is assumed to be epic. On the $k[z]$-linear level, then, $N(z)\Omega U$ is a free module of rank $p$; and it then follows from Section 2 that $\Omega Y/N(z)\Omega U$ is a finitely generated torsion module. Moreover, by Theorem 1,

$$Z(G) \overset{\sim}{\sim} \Omega Y / N(z)\Omega U.$$

To construct an essential right inverse $\hat{G}(z) : Y(z) \to U(z)$ with minimal state module $Z(G)$, start by choosing a basis

$$\{y_1, y_2, \ldots, y_p\}$$

for the free module $N(z)\Omega U$. Next choose

$$\{u_1, u_2, \ldots, u_p\}$$

in $\Omega U$ such that

$$N(z)u_i = y_i \quad , \quad i = 1,2,\ldots,p.$$

Because the $N(z)\Omega U$-basis is also a $k(z)$-vector space basis of $Y(z)$, there exists a $k(z)$-linear map

$$\hat{N}(z) : Y(z) \to U(z)$$

with action satisfying

$$\hat{N}(z)y_i = u_i \quad , \quad i = 1,2,\ldots,p.$$

Now regard $D(z)$ as a $k(z)$-linear map on $Y(z)$ to itself, and define

$$\hat{G}(z) : Y(z) \to U(z)$$

by

$$\hat{G}(z) = \hat{N}(z)\ D(z).$$

The claim is that $\hat{G}(z)$ is an essential right inverse of $G(z)$.

It is straightforward to verify that $\hat{G}(z)$, so defined, is a right inverse. Indeed, the calculation

$$\begin{aligned} G(z)\ \hat{G}(z) &= D^{-1}(z)\ N(z)\ \hat{N}(z)\ D(z) \\ &= D^{-1}(z)\ D(z) \\ &= 1_{Y(z)} \end{aligned}$$

is sufficient, provided that the fact

$$N(z)\ \hat{N}(z) = 1_{Y(z)},$$

which follows by the construction, is recognized.

The next goal is to show that $\hat{G}(z)$ is essential; and this is accomplished by constructing a minimal realization diagram with state module isomorphic to $Z(G)$. Regard $D(z)$ as a $k[z]$-linear map $\Omega Y \to \Omega Y$ of free modules. Let

$$p : \Omega Y \to \Omega Y/N(z)\Omega U \overset{\sim}{\sim} Z(G)$$

be the natural projection, and set

$$D' = pD(z)$$

by composition. Then

$$D' : \Omega Y \to \Omega Y/N(z)\Omega U \overset{\sim}{\sim} Z(G)$$

is a $k[z]$-module homomorphism. Now regard $\hat{N}(z) : Y(z) \to U(z)$ as a transfer function, which gives rise to

$$\hat{N}^{\#} : \Omega Y \to \Gamma U$$

as in Section 2. Because $\{y_i, i = 1,2,\ldots,p\}$ is a basis for $N(z)\Omega U$, and

$$\hat{N}(z)\ y_i = u_i \in \Omega U$$

for $i = 1,2,\ldots,p$, it follows that

$$N(z)\Omega U \subset \ker \hat{N}^{\#}.$$

But this means that $\hat{N}^{\#}$ induces a unique $k[z]$-module homomorphism

$$\hat{N}' : \Omega Y/N(z)\Omega U \to \Gamma U$$

which makes the diagram of Figure 6 commute. Piecing these ideas together yields a candidate for a realization diagram, as in Figure 7. To complete the proof, it is necessary to show that the diagram commutes, which means

$$\hat{G}^{\#} = \hat{N}'\ D',$$

that $D'$ is epic (reachability), and that $\hat{N}'$ is monic (observability).

$$\begin{tikzcd}
\Omega Y \arrow[r, "p"] \arrow[dr, "\hat{N}^{\#}"'] & \Omega Y/N(z)\Omega U \arrow[d, dashed, "\hat{N}'"] \\
 & \Gamma U
\end{tikzcd}$$

Figure 6.

$$\begin{tikzcd}
 & Y(z) \arrow[r, "\hat{G}(z)"] & U(z) \arrow[dr] & \\
\Omega Y \arrow[ur] \arrow[rrr, "\hat{G}^{\#}(z)"] \arrow[dr, "D'"'] & & & \Gamma U \\
 & \Omega Y/_{N(z)\Omega U} \arrow[urr, "\hat{N}'"'] & &
\end{tikzcd}$$

Figure 7.

Consider commutativity of the diagram. Select $y(z)$ in $\Omega Y$. Then

$$\begin{aligned}\hat{G}^{\#} y(z) &= \hat{N}(z)\ D(z)\ y(z) \mod \Omega U,\\ &= \hat{N}^{\#} D(z)\ y(z)\\ &= \hat{N}'\ p\ D(z)\ y(z)\\ &= \hat{N}'\ D'\ y(z),\end{aligned}$$

as required. Next consider whether $\hat{N}'$ is monic. This property is a consequence of the equation

$$\ker \hat{N}^{\#} = N(z)\,\Omega U.$$

The inclusion

$$N(z)\Omega U \subset \ker \hat{N}^{\#}$$

follows from the fact that $\hat{N}(z)$ takes a basis for $N(z)\Omega U$ into $\Omega U$. To establish the reverse inclusion

$$\ker \hat{N}^{\#} \subset N(z)\Omega U,$$

suppose that

$$\hat{N}^{\#} y(z) = 0,$$

for some $y(z)$ in $\Omega Y$. Then

$$\hat{N}(z)\ y(z) = u(z) \in \Omega U.$$

Apply $N(z)$ to obtain

$$\begin{aligned}N(z)\ u(z) &= N(z)\ \hat{N}(z)\ y(z)\\ &= y(z),\end{aligned}$$

which means that $y(z) \in N(z)\Omega U$ as needed. Finally, consider whether $D'$ is epic. From Section 3, recall the existence of $A(z) : \Omega Y \to \Omega Y$ and $B(z) : \Omega Y \to \Omega U$ such that

$$D(z)\ A(z) + N(z)\ B(z) = 1_{\Omega Y}.$$

Then

$$\begin{aligned}p\ y(z) &= p\ D(z)\ A(z)\ y(z) + p\ N(z)\ B(z)\ y(z)\\ &= D'\ A(z)\ y(z),\end{aligned}$$

so that for every $y(z) \in \Omega Y$, $p\, y(z)$ is in

$$\operatorname{im} D' A(z) \subset \operatorname{im} D' .$$

This discussion shows that essential right inverses exist and gives a prescription for constructing them. There is an element of freedom of choice in the procedure, particularly in the choice of $u_i$ such that $N(z)u_i = y_i$ (where $y_1, \ldots, y_p$ is a fixed basis for $N(z)\Omega U$.) Although details will not be included here, this procedure does <u>not</u> give all possible essential inverses.

The next step is to consider the construction of essential left inverses. To provide an alternate construction procedure, this part of the development proceeds in a manner slightly different from the right inverse case.

Suppose that $G(z) : U(z) \to Y(z)$ is monic. In this case, the zero module is given by

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} .$$

The goal of the present discussion is to construct an essential left inverse $G(z) : Y(z) \to U(z)$. This means that the minimal state module $X(\hat{G})$ must satisfy

$$X(\hat{G}) \approx Z(G).$$

Consider the set

$$M = G(z)U(z) \cap \Omega Y.$$

$M$ is a free $k[z]$-module of rank $m$, because it is a submodule of $\Omega Y$ and because $G(z)$ is monic. Choose a $k[z]$-module basis

$$\{y_1, y_2, \ldots, y_m\}$$

for $M$. Because $G(z)$ is monic, there exist uniquely determined vectors

$$\{u_1, u_2, \ldots, u_m\}$$

in $U(z)$ such that

$$G(z)\ u_i = y_i \quad , \quad i = 1, 2, \ldots, m.$$

Notice also that $\{u_i, i = 1, 2, \ldots, m\}$ is a basis for the free module $G^{-1}(\Omega Y)$. To see this, suppose $u(z) \in U(z)$ and

$$G(z)\ u(z) \in \Omega Y.$$

Then

$$G(z)\ u(z) \in M$$

and there exist unique $a_i(z) \in k[z]$, $i = 1, 2, \ldots, m$ such that

$$\begin{aligned} G(z)\ u(z) &= \sum_{i=1}^{m} a_i(z)\ y_i \\ &= \sum_{i=1}^{m} a_i(z)\ G(z)\ u_i \\ &= G(z) \sum_{i=1}^{m} a_i(z)\ u_i. \end{aligned}$$

Inasmuch as $G(z)$ is monic, this means that

$$u(z) = \sum_{i=1}^{m} a_i(z)\ u_i$$

uniquely.

Now consider the factor module

$$N = \Omega Y/M\ ,$$

which is <u>torsion-free</u>. In fact if $y(z)$ in $\Omega Y$ represents a torsion element in $N$, then for some $t(z)$ in $k[z]$ and $u(z)$ in $U(z)$

$$t(z)\ y(z) = G(z)\ u(z)\ .$$

But then

$$y(z) = G(z)\ \left(\frac{1}{t(z)}\ u(z)\right)$$

is itself in $M$, and represents $0$ in $N$. But $N$ is finitely generated over $k[z]$, hence free; and since $M$ has rank $m$, the module $N$ has rank $p-m$. Choose $y_{m+1}, \ldots, y_p$ in $\Omega Y$ which represent a basis of $N$. Then $\{y_1, \ldots, y_m, y_{m+1}, \ldots, y_p\}$ is a module basis for $\Omega Y$. To see this, suppose $y(z)$ lies in $\Omega Y$. Then there exist $b_{m+1}, \ldots, b_p$ in $k[z]$ such that

$$y \equiv b_{m+1}\, y_{m+1} + \ldots + b_p\, y_p \bmod M,$$

that is,

$$y - b_{m+1}\, y_{m+1} - \ldots - b_p\, y_p \in M,$$

so there exist $b_1, b_2, \ldots, b_m$ in $k[z]$ such that

$$y - b_{m+1}\, y_{m+1} - \ldots - b_p\, y_p = b_1\, y_1 + \ldots + b_m\, y_m .$$

Thus $\{y_1, \ldots, y_p\}$ span $\Omega Y$. It is easy to see that these $y_i$ are independent. If

$$a_1(z)\, y_1 + \ldots + a_p(z)\, y_p = 0,$$

then

$$a_{m+1}\, y_{m+1} + \ldots + a_p\, y_p = 0 \bmod M ;$$

and so $a_{m+1}, \ldots, a_p$ are all zero. But then $a_1, \ldots, a_m$ are zero too because $y_1, \ldots, y_m$ are independent. It follows immediately that $\{y_1, y_2, \ldots, y_p\}$ is a $k(z)$-vector space basis for $Y(z)$. Notice that $\{y_{m+1}, \ldots, y_p\}$ is highly non-unique; different choices of these vectors will give different essential left inverses. Define

$$\hat{G}(z) : Y(z) \to U(z)$$

by the action

$$\hat{G}(z)\, y_i = u_i \quad , \quad i = 1, 2, \ldots, m$$
$$\hat{G}(z)\, y_i = 0 \quad , \quad i = m+1, m+2, \ldots, p.$$

Clearly,

$$\hat{G}(z)\ G(z)\ u_i = u_i \quad , \quad i = 1, 2, \ldots, m$$

so that $\hat{G}(z)$ is a left inverse for $G(z)$, with the help of the observation that $\{u_i, i = 1,2,\ldots,m\}$ are a basis for $U(z)$. The claim is that $\hat{G}(z)$ is an essential left inverse.

Proof involves the explicit construction of a minimal realization diagram. First note that since $\{y_1, \ldots, y_p\}$ is a module basis for $\Omega Y$, the map $\hat{G}(z)$ can be considered as a $k[z]$-module homomorphism

$$\hat{G}(z) : \Omega Y \to G^{-1}(\Omega Y) + \Omega U$$

defined by $\hat{G}\ y_i = u_i$ in $G^{-1}(\Omega Y)$, $i = 1, \ldots, m$, and $\hat{G}\ y_j = 0$, $j = m+1,\ldots,p$. Now let

$$p : G^{-1}(\Omega Y) + \Omega U \to \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U}$$

be the natural projection. Then define

$$\hat{G}_1 : \Omega Y \to \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U}$$

by

$$\hat{G}_1 = p\ \hat{G}|\Omega Y.$$

$\hat{G}_1$ is a $k[z]$-module homomorphism because $p$ and $\hat{G}(z)|\Omega Y$ are $k[z]$-module homomorphisms. Moreover, $\hat{G}_1$ is epic. To see this, note that

$$\hat{G}_1\ y_i = p\ \hat{G}(z)\ y_i$$

$$= p\ u_i \quad , \quad i = 1, 2, \ldots, m\ ;$$

and $\{p\ u_i, i = 1, 2, \ldots, m\}$ span

$$\frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U}$$

because $\{u_i, i = 1, 2, \ldots, m\}$ span $G^{-1}(\Omega Y)$. Next, consider the inclusion

$$i : G^{-1}(\Omega Y) + \Omega U \to U(z).$$

Let $\pi : U(z) \to \Gamma U$ be the natural projection. Then

$$\ker \pi i = \Omega U \subset G^{-1}(\Omega Y) + \Omega U,$$

and therefore $\pi i$ induces a monic $k[z]$-module map

$$i_1 : \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} \to \Gamma U$$

as indicated in the diagram of Figure 8. This means that a candidate for realization diagram is given by Figure 9.

Inasmuch as $\hat{G}_1$ and $i_1$ have already been shown to be epic and monic, respectively, it only remains to establish that the diagram commutes. Select an arbitrary $y(z)$ in $\Omega Y$; then

$$\begin{aligned} i_1 \hat{G}_1 y(z) &= i_1 p \hat{G}|\Omega Y \, y(z) \\ &= \pi i \hat{G}|\Omega Y \, y(z) \\ &= \pi \hat{G}(z) y(z) \\ &= \hat{G}^{\#} y(z) , \end{aligned}$$

as desired.

In this case, then,

$$X(\hat{G}) = Z(G) .$$

Of course, the minimal state module is only unique up to isomorphism.

Section 5 then has established the existence of essential right inverses and essential left inverses for the cases in which $G(z)$ has right and left inverses, respectively---namely when $G(z)$ is epic or monic. Thus a fundamental connection has been established between the zero module and the prolific area of inverse systems. In the next section, it is shown that the zero module concept captures the various matrix notions of multivariable zero which have been discussed in the literature.

$$\begin{tikzcd}
G^{-1}(\Omega Y) + \Omega U \arrow[r, "P"] \arrow[dr, "\pi i"'] & \dfrac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} \arrow[d, dashed, "i_1"] \\
 & \Gamma U
\end{tikzcd}$$

Figure 8.

$$\begin{tikzcd}
 & Y(z) \arrow[r, "\hat{G}(z)"] & U(z) \arrow[dr] & \\
\Omega Y \arrow[ur] \arrow[rrr, "\hat{G}^{\#}"] \arrow[dr, "\hat{G}_1"'] & & & \Gamma Y \\
 & Z(G) \arrow[urr, "i_1"'] & &
\end{tikzcd}$$

Figure 9.

## 6. THE ZERO MODULE AND MULTIVARIABLE ZEROS

One of the most interesting developments over the past few years in system theory has been the definition and application of the concept of "multivariable" zeros. The purpose of this section is to show that the zero module concept captures the various notions of multivariable zero which have been discussed, for the system considered in this paper.

Begin by showing that the zero module $Z(G)$ as defined in this paper is a natural sharpening of the popular definition of multivariable zeros in terms of the Smith-Macmillan form of the matrix of $G(z)$.

Suppose $G(z) : U(z) \to Y(z)$ is a transfer function and choose bases

$$\{u_1, u_2, \ldots, u_m\}$$
$$\{y_1, y_2, \ldots, y_p\}$$

for $U$ and $Y$, respectively, over $k$. Notice that these bases serve also for $U(z)$ and $Y(z)$ over $k(z)$, so that there arises a $p \times m$ matrix

$$[G(z)]$$

for the transfer function $G(z)$. Then from Section 2 it follows that

$$M(z) = R(z)\ [G(z)]\ L(z)$$

for $L(z)$ and $R(z)$ unimodular matrices over $k[z]$ is

$$M(z) = \left[\begin{array}{c|c} G^* & 0 \\ \hline 0 & 0 \end{array}\right],$$

where the $0$'s are zero matrices of appropriate sizes and $G^*$ is a square matrix

$$G^* = \operatorname{diag}\{\varepsilon_i(z)/\Psi_i(z)\}$$

for relatively prime elements $\varepsilon_i(z)$ and $\Psi_i(z)$ in $k[z]$ satisfying

$$\varepsilon_i(z) \mid \varepsilon_{i+1}(z) \quad , \quad i = 1,2,\ldots, r-1,$$

$$\Psi_i(z) \mid \Psi_{i-1}(z) \quad , \quad i = r,r-1,\ldots, 2.$$

The product

$$Z(z) = \prod_{i=1}^{r} \varepsilon_i(z)$$

is called the <u>zero polynomial</u> of $G(z)$. Now $Z(z)$ is the characteristic polynomial of the finitely generated torsion module $Z(G)$ defined in the present paper, and in fact a much stronger result is true.

<u>Theorem 3</u>. The polynomials $\varepsilon_1(z),\ldots,\varepsilon_r(z)$ defined above are the invariant factors of the zero module $Z(G)$.

Proof: Abstractly, the zero module of $G$ has been defined as

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\ker G(z) + \Omega U} .$$

A choice of bases for $U$ and $Y$ as made above gives

$$G^{-1}(\Omega Y) = \{u(z) \in k(z)^m : [G(z)]\, u(z) \in k[z]^p\}.$$

On the other hand,

$$Z(M) = \frac{M^{-1}(k[z]^p) + k[z]^m}{\ker M(z) + k[z]^m} .$$

It is now shown that $Z(G) \cong Z(M)$ as $k[z]$-modules. In fact

$$\begin{aligned} M^{-1}(k[z]^p) &= \{u \text{ in } k(z)^m : M(z)\, u(z) \in k[z]^p\} \\ &= \{u \text{ in } k(z)^m : R[G]\, L\, u(z) \in k[z]^p\} \\ &= \{u \text{ in } k(z)^m : [G]\, L\, u(z) \in k[z]^p\} \\ &= \{u \text{ in } k(z)^m : L\, u(z) \in G^{-1}(\Omega Y)\}. \end{aligned}$$

Thus

$$L : M^{-1}(k[z]^p) \to G^{-1}(\Omega Y)$$

is an isomorphism of free $k[z]$-modules. It

is also easy to see that

$$L(z)\ (\ker M(z)) = \ker G(z)\ ,$$

by the relationships

$$M(z)\ u(z) = 0 \iff R(z)[G(z)]\ L(z)\ u(z) = 0$$

$$\iff [G(z)]\ L(z)\ u(z) = 0.$$

Furthermore,

$$L(z)\ (k[z]^m) = k[z]^m$$

because $L(z)$ is unimodular and over $k[z]$. Then by the diagram of Figure 10, there exists an epimorphism of $k[z]$-modules

$$\tilde{L} : M^{-1}(k[z]^p) + k[z]^m \to Z(G).$$

Now calculate

$$\ker \tilde{L} = \{u(z) : L(z)\ u(z) \in \ker G(z) + k[z]^m\}$$

$$= \{u(z) : u(z) \in \ker M(z) + k[z]^m\},$$

which then permits the construction of Figure 11, where $L_1$ is a $k[z]$-module isomorphism. This establishes that

$$Z(M) \cong Z(G)$$

as $k[z]$-modules. The proof is concluded by an explicit calculation of $Z(M)$. Write

$$M(z) = D^{-1}(z)\ N(z)$$

for

$$D(z) = \operatorname{diag}\{\Psi_1(z), \ldots, \Psi_r(z), 1, 1, \ldots, 1\}$$

and

$$N(z) = \left[\begin{array}{c|c} N^* & 0 \\ \hline 0 & 0 \end{array}\right]$$

with

$$N^* = \operatorname{diag}\{\varepsilon_1(z), \ldots, \varepsilon_r(z)\}.$$

$$\begin{tikzcd}
M^{-1}(k[z]^p) + k[z]^m \arrow[r, "L(z)"] \arrow[dr, "\tilde{L}"'] & G^{-1}(k[z]^p) + k[z]^m \arrow[d, "\text{projection}"] \\
& Z(G)
\end{tikzcd}$$

Figure 10.

$$\begin{tikzcd}
M^{-1}(k[z]^p) + k[z]^m \arrow[r] \arrow[dr, "\tilde{L}"'] & \dfrac{M^{-1}(k[z]^p) + k[z]^m}{\ker M(z) + k[z]^m} = Z(M) \arrow[d, dashed, "L_1"] \\
& Z(G)
\end{tikzcd}$$

Figure 11.

The reader may verify that $D(z)$ and $N(z)$ are left coprime. By Theorem 1, $Z(M)$ is isomorphic to the torsion submodule of

$$\Omega Y/N(z)\ \Omega U.$$

Moreover, in this case, Section 2 establishes that

$$\Omega Y/N(z)\Omega U \overset{\sim}{\sim} k[z]/(\varepsilon_1(z)) \oplus \ldots \oplus k[z]/(\varepsilon_r(z)) \oplus k[z]^{p-r} .$$

Then

$$Z(M) \overset{\sim}{\sim} k[z]/(\varepsilon_1(z)) \oplus \ldots \oplus k[z]/(\varepsilon_r(z)) ,$$

which means that the $\varepsilon_i(z)$, $i = 1, 2, \ldots, r$ are the invariant factors of $Z(G)$, as required.

Rosenbrock [17] defined the zeros of the transfer function matrix $[G(z)]$ in terms of the zero polynomial $Z(z)$. Clearly, the zero module $Z(G)$ agrees with that definition and extends it to an invariant factor structure. In a paper on the role of transmission zeros in linear multivariable regulators, Francis and Wonham [18] have established essential equivalences between the Rosenbrock definition and definitions given by other authors.

## 7. EXAMPLES

In this section some examples of epic and monic transfer functions are considered. In each case the zero module is computed, and several distinct essential inverses are derived. Various techniques are used which reflect in an algorithmic way the different proof techniques in the previous section.

Consider the epic transfer function $G(z) : U(z) \to Y(z)$ with matrix given by

$$\begin{bmatrix} \frac{z}{z+1} & \frac{z}{z+2} \end{bmatrix},$$

where $U$ and $Y$ are real vector spaces of dimension two and one, respectively. As an $R(z)$-linear transformation, $G(z)$ has rank one and nullity one. A left coprime factorization

$$D^{-1}(z)\, N(z)$$

for $G(z)$ can be given by

$$D(z) = (z+1)(z+2) ,$$

$$N(z) = [z(z+2) \;\; z(z+1)] .$$

The essential state module for right inverses is given by

$$\begin{aligned} Z(G) &= Y[z]/N(z)\, U[z] \\ &\cong R[z]/(z) . \end{aligned}$$

To compute an essential inverse for $G(z)$ explicitly, consider

$$N(z) : U[z] \to Y[z].$$

Let

$$u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} z+1 \\ -(z+2) \end{bmatrix},$$

so that

$$N(z)\ u_1 = z$$

is a basis for $N(z)\ U[z]$, $u_2$ is a basis for $\ker N(z)$, and $\{u_1, u_2\}$ is a basis for $U[z]$. Next define

$$\hat{N}_0 : N(z)\ U[z] \to U[z]$$

by

$$\hat{N}_0(z) = u_1 .$$

The $R[z]$-linear map $\hat{N}_0$ can also be considered as an $R(z)$-linear map $\hat{N}_0 : Y(z) \to U(z)$ given in the standard basis by

$$\hat{N}_0(1) = \begin{bmatrix} \frac{1}{z} \\ -\frac{1}{z} \end{bmatrix} .$$

Consequently, we obtain the essential right inverse

$$\hat{G}_0(z) = \hat{N}_0(z)\ D(z)$$

$$= \begin{bmatrix} \frac{(z+1)(z+2)}{z} \\ -\frac{(z+1)(z+2)}{z} \end{bmatrix} .$$

This particular construction, by a careful choice of $u_1$, ensures also that $G(z)$ is an essential <u>left</u> inverse for $\hat{G}_0(z)$. Such a special occurrence need not be the case. For example,

$$\hat{G}_1(z) = \begin{bmatrix} \frac{z+1}{z} \\ 0 \end{bmatrix}, \qquad \hat{G}_2(z) = \begin{bmatrix} 0 \\ \frac{z+2}{z} \end{bmatrix}$$

are both essential right inverses for $G(z)$; but $G(z)$ is <u>not</u> an essential left inverse for either $\hat{G}_1(z)$ or $\hat{G}_2(z)$. Incidentally, $\hat{G}_0(z)$, $\hat{G}_1(z)$, and $\hat{G}_2(z)$ have distinct strictly proper parts which come from three non-isomorphic canonical systems each having state module $Z(G)$.

Next consider the case $p = 2$, $m = 3$, with the matrix of $G(z)$ given by

$$\begin{bmatrix} \frac{z+3}{(z+1)(z+2)} & \frac{-2(z+5)^2}{(z+2)} & \frac{z(z+3)}{(z+1)(z+2)} \\ \frac{1}{(z+1)(z+2)} & \frac{(z+5)^2 z}{z+2} & \frac{z}{(z+1)(z+2)} \end{bmatrix}.$$

A suitable left coprime factorization $D^{-1}(z)\,N(z)$ for $G(z)$ is

$$D(z) = \begin{bmatrix} z & 2 \\ -1 & z+3 \end{bmatrix},$$

$$N(z) = \begin{bmatrix} 1 & 0 & z \\ 0 & (z+1)(z+5)^2 & 0 \end{bmatrix}.$$

The map $N(z)$ is regarded as an $R[z]$-module map $U[z] \to Y[z]$ with $U \approx R^3$ and $Y \approx R^2$. The essential pole module is given by

$$\begin{aligned} Z(G) &= Y[z]/N(z)\,U[z] \\ &\approx R[z]/(z+1) \oplus R[z]/(z+5)^2 . \end{aligned}$$

To compute an explicit essential inverse, choose a basis

$$y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0 \\ \Psi(z) \end{bmatrix},$$

for $N(z)\ U[z]$, where

$$\Psi(z) = (z+1)(z+5)^2 .$$

Furthermore, choose a basis $\{u_1, u_2, u_3\}$ for $U[z]$ where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \quad u_3 = \begin{bmatrix} -z \\ 0 \\ 1 \end{bmatrix} .$$

Then

$$N(z)\ u_1 = y_1 ,$$
$$N(z)\ u_2 = y_2 ,$$

and $u_3$ is a basis for $\ker N(z)$. Next, define

$$\hat{N}_0 : Y(z) \to U(z)$$

by

$$\hat{N}_0(y_i) = u_i \quad , \quad i = 1,2 .$$

With respect to standard bases, $\hat{N}_0(z)$ has the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\Psi(z)} \\ 0 & 0 \end{bmatrix} .$$

Finally, define

$$\hat{G}_0(z) = \hat{N}_0(z)\ D(z)$$
$$= \begin{bmatrix} z & 2 \\ -\frac{1}{\Psi(z)} & \frac{z+3}{\Psi(z)} \\ 0 & 0 \end{bmatrix} ,$$

which is an essential right inverse for $G(z)$. Again, essential inverses

are not unique. For illustration,

$$\hat{N}(z) = \begin{bmatrix} 1 & -\beta z \\ 0 & \frac{1}{\Psi(z)} \\ 0 & \beta \end{bmatrix}$$

gives essential inverses

$$\hat{G}(z) = \hat{N}(z)\ D(z)$$

when

$$\beta = \frac{1}{\phi(z)}\ ,$$

with $\phi(z)$ a nontrivial divisor of $\Psi(z)$; and these inverses have distinct strictly proper parts.

Next, consider the monic transfer function $G(z) : U(z) \to Y(z)$ (with one-dimensional $U$ and two-dimensional $Y$) given by

$$G(z) = \begin{bmatrix} \frac{z+1}{z+2} \\ \frac{(z+1)^2}{z+3} \end{bmatrix}.$$

The zero module $Z(G)$ can be computed either from a matrix factorization or directly from the definition. For example, write

$$G(z) = \begin{bmatrix} z+2 & 0 \\ 0 & z+3 \end{bmatrix}^{-1} \begin{bmatrix} z+1 \\ (z+1)^2 \end{bmatrix},$$

so that

$$N(z) = \begin{bmatrix} z+1 \\ (z+1)^2 \end{bmatrix}.$$

Here

$$\Omega Y/N(z)\Omega U = k[z]^2/M$$

where

$$M = \{\begin{bmatrix} z+1 \\ (z+1)^2 \end{bmatrix} p(z) \ : \ p(z) \ \varepsilon \ k[z]\}$$

is a free rank-one submodule of $k[z]^2$. Thus

$$\Omega Y/N(z)\Omega U \approx k[z]/(z+1)k[z] \oplus k[z],$$

and Theorem 1 gives

$$Z(G) \approx k[z]/(z+1)k[z]$$

To compute $Z(G)$ directly from the definition

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} ,$$

write

$$G^{-1}(\Omega Y) = \{\alpha(z) \ \varepsilon \ k(z) \ : \ G(z) \ \alpha(z) \ \varepsilon \ k[z]^2\}$$

$$= \{\frac{(z+2)(z+3)}{(z+1)} \ p(z) \ : \ p(z) \ \varepsilon \ k[z]\},$$

and

$$G^{-1}(\Omega Y) + \Omega U = \{\frac{(z+2)(z+3)}{(z+1)} \ p(z) + q(z) \ : \ p(z), \ q(z) \ \varepsilon \ k[z]\}$$

$$= \{\frac{\tilde{p}(z)}{z+1} \ : \ \tilde{p}(z) \ \varepsilon \ k[z]\}.$$

So

$$\frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} = \frac{\frac{1}{z+1} \ k[z]}{k[z]} \approx k[z]/(z+1)k[z] \ .$$

To compute an essential inverse $\hat{G} : Y(z) \to U(z)$ using the method of proof used in Section 5, first compute a basis element $y_1$ of the rank-one free module

$$G(z) \ U(z) \cap \Omega Y \ .$$

Now

$$\begin{bmatrix} \frac{z+1}{z+2} \\ \frac{(z+1)^2}{z+3} \end{bmatrix} \alpha(z) \ \varepsilon \ \Omega Y$$

if and only if

$$\alpha(z) = (z+2)(z+3)q(z)/z+1$$

for some $q(z)$ in $k[z]$. In fact, if

$$u(z) = \frac{(z+2)(z+3)}{(z+1)} ,$$

then

$$G(z)\ u(z) = \begin{bmatrix} (z+3) \\ (z+1)(z+2) \end{bmatrix} = y_1$$

can be taken as a basis for $G(z)\ U(z) \cap \Omega Y$. Next, let $y_2$ in $\Omega Y$ be any vector such that $\{y_1, y_2\}$ is a $k[z]$-module basis for $\Omega Y$. According to Section 5, the map $\hat{G}(z) : Y(z) \to U(z)$ defined by

$$\hat{G}(z)(y_1) = u(z)$$
$$\hat{G}(z)\ y_2 = 0$$

is an essential left inverse for $G(z)$.

Because $\{y_1, y_2\}$ is a basis for $\Omega Y$ if, and only if, the partitioned $2 \times 2$ matrix $[y_1 \mid y_2]$ has a non-zero scalar determinant, it follows that

$$y_2 = \begin{bmatrix} 1 \\ z \end{bmatrix}$$

is a suitable choice. To compute a standard basis representation for the resulting $\hat{G}(z)$, note that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-1}{2}\ (z\ y_1 - (z^2 + 3z + 2)\ y_2)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\ (-y_1 + (z + 3)\ y_2)$$

so that

$$G(z) = -\frac{1}{2}\left[\frac{z(z+2)(z+3)}{z+1} \quad \frac{-(z+2)(z+3)}{z+1}\right] .$$

A different essential inverse, available in this case by inspection, is

$$\hat{G}_1(z) = \left[\frac{z+2}{z+1} \quad 0\right] .$$

Note, however, that the attempt

$$\hat{G}_2(z) = \left[0 \quad \frac{z+3}{(z+1)^2}\right]$$

leads to an inverse which is <u>not</u> essential.

Consider next a "decoupled" 2 x 3 example

$$G(z) = \begin{bmatrix} \frac{z}{z+1} & 0_2 \\ 0 & \frac{z}{z+2} \\ 0 & 0 \end{bmatrix} .$$

Then

$$G^{-1}(\Omega Y) = \left\{\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \in k(z)^2 : G(z)\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \in k[z]^3\right\}.$$

But

$$G(z)\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} = \begin{bmatrix} \frac{z\alpha_1}{z+1} \\ \frac{z^2\alpha_2}{z+2} \\ 0 \end{bmatrix} ,$$

so that

$$\alpha_1 = \frac{z+1}{z} p_1(z)$$

$$\alpha_2 = \frac{z+2}{z^2} p_2(z)$$

for some polynomials $p_1(z)$, $p_2(z)$ in $k[z]$. Thus, $G^{-1}(\Omega Y)$ is a rank-two free module with basis

$$u_1 = \begin{bmatrix} \frac{z+1}{z} \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ \frac{z+2}{z^2} \end{bmatrix}.$$

Because

$$u_1 = \begin{bmatrix} \frac{1}{z} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$u_2 = 2\begin{bmatrix} 0 \\ \frac{1}{z^2} \end{bmatrix} + z\begin{bmatrix} 0 \\ \frac{1}{z^2} \end{bmatrix},$$

it follows that $G^{-1}(\Omega Y) + \Omega U$ is a free module with basis

$$\begin{bmatrix} \frac{1}{z} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{z^2} \end{bmatrix}.$$

Therefore

$$\frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U} = \frac{k[z]\begin{bmatrix} \frac{1}{z} \\ 0 \end{bmatrix} \oplus k[z]\begin{bmatrix} 0 \\ \frac{1}{z^2} \end{bmatrix}}{k[z]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus k[z]\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\approx \frac{k[z] \cdot \frac{1}{z}}{k[z]} \oplus \frac{k[z] \cdot \frac{1}{z^2}}{k[z]}$$

$$\approx k[z]/(z)k[z] \oplus k[z]/(z^2)k[z] .$$

This establishes the expected result that

$$Z(G) = k[z]/(z)k[z] \oplus k[z]/(z^2)k[z] .$$

To compute an essential inverse, consider the vectors

$$y_1 = Gu_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$y_2 = Gu_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then the calculation of $G^{-1}(\Omega Y)$ above shows also that $\{y_1, y_2\}$ forms a basis for

$$G(z)\ U(z) \cap \Omega Y.$$

Now choose a vector $y_3$ such that $\{y_1, y_2, y_3\}$ is a basis for all of $\Omega Y$. In this case

$$y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

works. An essential inverse $\hat{G}(z)$ can then be defined by

$$\hat{G}(y_1) = u_1$$
$$\hat{G}(y_2) = u_2$$
$$\hat{G}(y_3) = 0$$

giving the expected matrix

$$\hat{G} = \begin{bmatrix} \frac{z+1}{z} & 0 & 0 \\ 0 & \frac{z+2}{z^2} & 0 \end{bmatrix}.$$

## 8. CONCLUSIONS

In this paper, we have given an abstract, module theoretic definition of zero module which captures the features of existing matrix definitions without being dependent upon any particular representation of the transfer function. Every transfer function has a zero module, whether or not it has right or left inverses. If such inverses exist, however, their minimal state modules must "contain" the zero module either as a quotient module or as a submodule. When containment is exact, inverses are called essential. Existence of essential inverses has been established by construction.

The existence of the zero module, together with its many useful features, suggests that a given system might be regarded as having a zero module as defined herein and a <u>pole module</u> which is a renaming for the previous usage minimal state module.

The pole module of an essential inverse system is then the zero module of the system itself.

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