N O T I C E

THIS DOCUMENT HAS BEEN REPRODUCED FROM MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED IN THE INTEREST OF MAKING AVAILABLE AS MUCH INFORMATION AS POSSIBLE

RECEIVED BY TIC JUN 4 1979

ORO 5126-57
U. of Md. TR #79-046
U. of Md. PP #79-103

# THE DOMINANT PARTITION METHOD*†

Robert M. Dixon
Department of Physics
Morgan State University
Baltimore, Maryland 21239

and

Edward F. Redish**
Laboratory for Astronomy and Solar Physics
NASA/Goddard Space Flight Center
Greenbelt, Maryland 20770
and
Department of Physics and Astronomy††
University of Maryland, College Park, Maryland 20742

March 1979

MASTER

394 7000

UNIVERSITY OF MARYLAND
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLLEGE PARK, MARYLAND

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

ORO 5126-57
U. of Md. TR #79-046
U. of Md. PP #79-103

# THE DOMINANT PARTITION METHOD*†

Robert M. Dixon
Department of Physics
Morgan State University
Baltimore, Maryland 21239

and

Edward F. Redish**
Laboratory for Astronomy and Solar Physics
NASA/Goddard Space Flight Center
Greenbelt, Maryland 20770
and
Department of Physics and Astronomy††
University of Maryland, College Park, Maryland 20742

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

March 1979

## ABSTRACT

Employing the L'Huillier, Redish and Tandy (LRT) wave function formalism we develop a partially connected method for obtaining few body reductions of the many body problem in the LRT and Bencze, Redish and Sloan (BRS) formalisms. This method for systematically constructing fewer body models from the N-body LRT and BRS equations is termed the Dominant Partition Method (DPM). The DPM maps the many body problem to a fewer body one using the criterion that the truncated formalism must be such that consistency with the full Schrödinger equation is preserved. The DPM is based on a class of new forms for the irreducible cluster potential, which is introduced in the LRT formalism. Connectivity is maintained with respect to all partitions containing a given

---

* Work based on material submitted in partial fulfillment of requirement for Ph.D., University of Maryland, 1977.

† Supported in part by U.S. Department of Energy.

** N.R.C.-N.A.S. Senior Resident Research Associate.

†† Permanent Address.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

partition which is referred to as the dominant partition. Degrees of freedom corresponding to the breakup of one or more of the clusters of the dominant partition are treated in a disconnected manner. This approach for simplifying the complicated BRS equations is appropriate for physical problems where a few body reaction mechanism prevails. We also show that the dominant partition truncated form of the BRS equations may be obtained by distributing the residual interaction in the exit channel in a manner consistent with the dominant partition truncations of the irreducible cluster potential.

# THE DOMINANT PARTITION METHOD

## I. INTRODUCTION

Connected Kernel Equations (CKE's) have enjoyed considerable prominence in the recent history of reaction theory. The equations due to Alt, Grassberger and Sandhas[1] (AGS), Bencze, Redish and Sloan[2] (BRS) and Kouri, Levin and Tobocman[3] (KLT) are representative examples. In the formalisms of AGS, BRS, and KLT the many body scattering problem is formulated in terms of a set of coupled integral equations for the transition operators. These CKE's are often viewed as extensions of the three-body formalism of Faddeev[4].

Although the CKE's provide mathematically correct formulations of the N-body scattering problem, these equations have not inspired extensive uasge in direct reaction analysis. The Distorted Wave Born Approximation continues to be the primary method employed in the analyses of direct reactions. Significantly, there exists in the community an understanding that many body effects should be included in reaction analysis;[5] however, the CKE's are not generally regarded as offering a viable approach for such inclusions. Even in the three-body case, the Faddeev Equations are often regarded as useful for mathematical proofs but not as feasible for calculations[6].

The complicated nature of CKE's as well as an uncertainty about how the dynamics is distributed in these equations have been important factors in limiting the role of CKE's in reaction analysis. Necessarily any use of CKE's must involve truncations. This compelling necessity for methods of truncating CKE's probably has contributed to their limited use.

The application of the CKE's to nuclear and atomic systems is rendered difficult because of the number of equations involved and the fact that many channels are theoretically treated on an equal footing. We therefore consider

what simplification can be achieved by reducing the number of equations and/or channels. In practice since certain channels may be ignored or treated phenomenologically it may not be necessary to preserve connectedness in them. We call the formalism derived from relaxation of full connectedness in a set of CKE's a partially connected formalism. A partially connected approach has been advocated as a means of circumventing the difficulty imposed by the coupling of all rearrangement channels by Hahn and Watson in the three-body problem[7].

The choice of criteria (by which one truncates a CKE) is an open question. A possible approach to simplifying these equations for some problems is to map the many body space into that of a fewer body problem. This approach will be useful in the case that the physics seems to be dominated by a few-body mechanism. One example is the deuteron-alpha scattering at energies below the threshold for breakup of the alpha. Notably, such a mapping does not destroy all of the many-body information which would be lost in the arbitrary imposition of a few-body model on a given many-body system.

Actually, when one considers such a truncation it becomes clear that many of the CKE's do not lend themselves to such a reduction method. The AGS and KLT equations are notable examples. The explicit dependence on the number of particles as exhibited by the AGS equations and the dependence on the number of channels as exhibited by the KLT equations tend to make the structures of these equations rather rigid. The AGS equations make explicit the number of particles through kernels which contain all subsystem transition operators. The KLT equations are written for a fixed number of channels which structurally excludes the possibility of later dropping one of the channels.

In this article we develop a method of truncating the many-body BRS equations to a fewer body problem. The method is developed in the LRT[8] connected kernel wave function formalism, and is similar in spirit to the Hahn-Watson reduction method[7]. This truncation is termed the Dominant Partition Method (DPM). It maps the given many-body problem to a fewer body problem whose solutions satisfy the full Schrödinger equation. The equations obtained constitute a partially connected set, the disconnectedness appearing in those channels which are not considered explicitly.

In section II the LRT wave function formalism and the related irreducible cluster potential are reviewed. In section III the Dominant Partition Theorem is presented and in section IV this reduction method is applied to the BRS equations. In section IV it is also shown that the reduced set of BRS equations may be obtained via a distribution method[4]. The summary and conclusion are presented in section V.

## II. THE L'HUILLIER, REDISH, TANDY WAVE FUNCTION FORMALISM

In this section a set of coupled connected kernel equations for the wave function describing the scattering between many-body ($N \geq 4$) clusters is obtained. These equations are derived by using the BRS equations and the Green function for the system. The system under consideration has N distinguishable particles which interact via two-body potentials. A division of the N particles into n clusters, is termed an $a_n$-partition. The Greek alphabet is used to label two cluster partitions and the N-cluster partition is labelled 0. The partition Hamiltonians $H_a$, residual interactions $V^a$ and associated Green functions are defined by

$$H_a = H_o + V_a, \tag{1}$$

$$V^a = H - H_a, \tag{2}$$

$$G_a = (Z - H_a)^{-1}, \tag{3}$$

where $V_a$ is the sum of two body interactions internal to the a-partition and Z is the complex energy parameter ($Z = E + i\varepsilon$). The full and free Green functions are given by

$$G = (Z - H)^{-1} \tag{4}$$

$$G_o = (Z - H_o)^{-1} \tag{5}$$

where H is the full N-particle Hamiltonian and $H_o$ is the total kinetic energy operator.

Consider the N-body scattering problem initiated by incoming bound states of the two clusters comprising the partition β. The full wave function is[9]

$$\Psi_\beta = \lim_{\varepsilon \to 0} i\varepsilon G \Phi_\beta, \tag{6}$$

where $\Phi_\beta$ describes a relative motion plane wave times the internal bound state wave functions for the two clusters. The Green function is expressible in terms of $G_\beta$,

$$G = G_\beta + GV^\beta G_\beta. \tag{7}$$

Using that

$$G_\beta \Phi_\beta = (i\varepsilon)^{-1} \Phi_\beta$$

we obtain

$$\Psi_\beta = (1 + GV^\beta)\Phi_\beta. \tag{8}$$

Using (7) we have

$$\Psi_\beta = G\, G_\beta^{-1} \Phi_\beta. \tag{9}$$

Noting that

$$V^a \Psi_\beta = T^{a\beta} \Phi_\beta, \tag{10}$$

where $T^{a\beta}$ is the transition operator[10] we write (9) as

$$\Psi_\beta = G_o G_\beta^{-1} \Phi_\beta + G_o T^{o\beta} \Phi_\beta . \tag{11}$$

Employing the BRS equation[2],

$$T^{a\beta} = V_\beta^a + \sum_\sigma K_\sigma^{ao} G_o T^{\sigma\beta} , \tag{12}$$

we obtain

$$\Psi_\beta = G_o (G_\beta^{-1} + V_\beta) \Phi_\beta + \sum_\sigma G_o K_\sigma^{oo} G_o T^{\sigma\beta} \Phi_\beta . \tag{13}$$

In (12) $V_\beta^a$ is the sum of two body interactions internal to $\beta$ and external to a. The kernel $K_\sigma^{ao}$ is the sum of all Weinberg graphs[11] of connectivity[2] $\sigma$ which begin with any interaction and do not end with an interaction in a. Defining $K_\sigma^{oo} \equiv K_\sigma$, using (3) and (10) we have

$$\Psi_\beta = \Phi_\beta + \sum_\sigma G_o K_\sigma G_o V^\sigma \Psi_\beta \ . \tag{14}$$

This integral equation has a completely connected kernel. The operator $K_\sigma$ is the sum of all $\sigma$-connected Weinberg graphs.

Decomposition of the wave function into parts associated with the two cluster partitions of the N-body problem is achieved by writing

$$\Psi_\beta = \sum_\gamma \Psi_\beta^{(\gamma)} , \tag{15}$$

where

$$\Psi_\beta^{(\gamma)} = \Phi_\beta \delta_{\gamma\beta} + G_o K_\gamma G_o V^\gamma \Psi_\beta . \tag{16}$$

The wave function $\Psi_\beta^{(\gamma)}$ only has outgoing waves of the $\gamma$-type, that is bound clusters of the $\gamma$-partition or direct breakup from the $\gamma$-partition. Equation (16) may be written in a convenient form by considering again (9) and writing G in terms of $G_\gamma$. We find

$$\Psi_\beta = G_\gamma G_\beta^{-1}\Phi_\beta + G_\gamma V^\gamma G\, G_\beta^{-1}\Phi_\beta. \tag{17}$$

Using (9) and (10) we have

$$\Psi_\beta = G_\gamma G_\beta^{-1}\Phi_\beta + G_\gamma T^{\gamma\beta}\Phi_\beta. \tag{18}$$

Multiplying from the left by $G_o G^{-1}$ we have

$$G_o G_\gamma^{-1}\Psi_\beta = G_o G_\beta^{-1}\Phi_\beta + G_o T^{\gamma\beta}\Phi_\beta. \tag{19}$$

Using (19) in (16) gives

$$\Psi_\beta^{(\gamma)} = (\delta_{\gamma\beta} - G_o K_\gamma G_o G_\beta^{-1})\Phi_\beta + G_o K_\gamma G_o G_\gamma^{-1}\Psi_\beta. \tag{20}$$

We not introduce the operator $V_\gamma$ which is defined by

$$K_\gamma G_o \equiv V_\gamma G_\gamma. \tag{21}$$

The operator $V_\gamma$ is termed the <u>irreducible $\gamma$-connected potential</u>. It is the sum of $\gamma$-connected graphs which become less than $\gamma$-connected if the rightmost interaction is removed[12]. We have

$$\Psi_\beta^{(\gamma)} = (\delta_{\gamma\beta} - G_o V_\gamma G_\gamma G_\beta^{-1})\Phi_\beta + G_0 V_\gamma \Psi_\beta\ , \tag{22}$$

and $G_o V_\gamma G_\gamma G_\beta^{-1}$ on shell is the same as $G_o V_\beta \delta_{\beta\gamma}$. This yields

$$\Psi_\beta^{(\gamma)} = \delta_{\gamma\beta}(1 - G_o V_\beta)\Phi_\beta + G_o V_\gamma \Psi_\beta. \tag{23}$$

Noting that

$$(1 - G_o V_\beta)\Phi_\beta = 0, \tag{24}$$

which is most easily seen on examining the anti-cluster expansion for $V_\beta$ (section III) we obtain

$$\Psi_\beta^{(\gamma)} = G_o V_\gamma \Psi_\beta. \tag{25}$$

We write this in differential form as

$$(E - H_o - V_\gamma)\Psi_\beta^{(\gamma)} = V_\gamma \sum_{\sigma(\neq\gamma)} \Psi_\beta^{(\sigma)}, \tag{26}$$

which is reminiscent of the Faddeev three-body result. These are the LRT[7] equations. The relation of these equations to other N-body CKE's is discussed in detail in ref. 13.

## III. THE DOMINANT PARTITION THEOREM

The anti-cluster expansion for the kernel in the BRS equation has been previously given[14],

$$K_\sigma^{\beta o} G_o = \sum_{m=2}^{N-1} \sum_{(\sigma\supset)a_m} N(\sigma,a_m) V_{a_m}^\beta G_{a_m} \tag{27}$$

The N's in the above equation are termed counting coefficients[14]. They depend on both $\sigma$ and $a_m$. From (21) we have the anti-cluster expansion for the irreducible cluster potential.

$$V_\sigma = \sum_{m=2}^{N-1} \sum_{(\sigma\supset)a_m} N(\sigma,a_m) V_{a_m} G_{a_m} G_\sigma^{-1}. \tag{28}$$

Summing the components of equation (26) we note the interesting result that

$$(E - H_o - \sum_\sigma V_\sigma)\Psi_\beta = 0. \tag{29}$$

This property provides the underpinning for the truncation of the BRS equations that will be presented in this section. Insight into the method is afforded by the following theorem.

<u>Theorem I</u>. For arbitrary N, if $V_\sigma$ is given exactly then

$$\sum_\sigma V_\sigma \Psi_\beta = V\Psi_\beta$$

where V is the full potential and $\Psi_\beta$ solves the Schrödinger equation for the N-body system.

<u>Proof</u>: From (28) we have

$$\sum_\sigma V_\sigma \Psi_\beta = \sum_\sigma \sum_{m=2} \sum_{(\sigma\supset)a_m} N(\sigma,a_m) V_{a_m} G_{a_m} G_\sigma^{-1} \Psi_\beta. \tag{30}$$

Interchanging sums and using

$$G_\sigma^{-1} = G_{a_m}^{-1} - V_\sigma^{a_m}, \ \sigma \supset a_m \tag{31}$$

we have

$$\sum_\sigma V_\sigma \Psi_\beta = \sum_{m=2}^{N-1} \sum_{a_m} \sum_{\sigma(\supset a_m)} N(\sigma,a_m)(V_{a_m} - V_{a_m} G_{a_m} V_\sigma^{a_m})\Psi_\beta. \tag{32}$$

Noting that

$$V_\sigma^{a_m} = V_\sigma - V_{a_m} = V^{a_m} - V^\sigma, \ \sigma \supset a_m \tag{33}$$

we obtain

$$\sum_\sigma V_\sigma \Psi_\beta = \sum_{m=2}^{N-1} \sum_{a_m} \sum_{\sigma(\supset a_m)} N(\sigma,a_m) V_{a_m}(1-G_{a_m} V^{a_m} + G_{a_m} V^\sigma)\Psi_\beta. \tag{34}$$

We use the result[15],

$$\Psi_\beta = \delta_{\beta a_m} \Phi_\beta + G_{a_m} V^{a_m} \Psi_\beta \tag{35}$$

to obtain

$$\sum_\sigma V_\sigma \Psi_\beta = V_\beta \Phi_\beta + \sum_{m=2}^{n-1} \sum_{a_m} \sum_{\sigma(\supset a_m)} N(\sigma, a_m) V_{a_m} G_{a_m} V^\sigma \Psi_\beta . \tag{36}$$

In the appendix we show that

$$\sum_{\sigma(\supset a_m)} N(\sigma, a_m) V^\sigma = c_m V^{a_m}, \tag{37}$$

where $c_m = (-1)^m (m-1)!$ Using this in (36) and again employing (35) we obtain

$$\sum_\sigma V_\sigma \Psi_\beta = \sum_{m=2}^{n-1} c_m \sum_{a_m} V_{a_m} \Psi_\beta . \tag{38}$$

Now we employ the lemmas:

<u>Lemma 1</u>:[16]

$$\sum_{c_k} V_{c_k}^{a_m} = S_{N-1}^{(k)} V^{a_m} \tag{39}$$

and

Lemma 2: [17]

$$\sum_{n=2}^{N-1} C_n S_{N-1}^{(n)} = 1, \; N \geq 3. \tag{40}$$

The $S_{N-1}^{(k)}$ are Stirling numbers of the second kind. $S_{N-1}^{(k)}$ is the number of distinct ways of making k clusters out of (N-1) objects. From (38) we now obtain

$$\sum_\sigma V_\sigma \Psi_\beta = V \Psi_\beta . \quad \text{Q.E.D.} \tag{41}$$

This means that (29) with the $\sigma$ sum taken over the two cluster partitions is the Schrödinger equation. This result motivates the consideration of truncations of $\sum_\sigma$ and/or $V_\sigma$ such that the corresponding summed version of (29)

remains the full Schrödinger equation. We present a method which satisfies this condition through the following three results.

<u>Remark</u>: For a given N, if $V_\sigma$ is given exactly the arbitrary sum $(\sum V_\sigma \Psi_\beta)$ over a subset of the two cluster partitions does not result in $(E - H_o - \sum_\sigma V_\sigma)\Psi_\beta = 0$ becoming the Schrödinger equation when the sum is over an arbitrary set of partitons $\sigma$.

<u>Proof</u>: Consider the case N = 3 for which $V_\sigma = V_\sigma$. We label the possible $\sigma$'s as $\sigma_1 = (1)(23)$, $\sigma_2 = (2)(13)$ and $\sigma_3 = (3)(12)$. Note that

$$\sum_{\sigma_1,\sigma_2} V_\sigma \Psi_\beta = \sum_{\sigma_1,\sigma_2} V_\sigma \Psi_\beta, \tag{42}$$

$$\sum_{\sigma_1,\sigma_2} V_\sigma \Psi_\beta \neq V \Psi_\beta. \quad \text{Q.E.D.} \tag{43}$$

<u>Remark</u>: For a given N and $V_\sigma$ truncated arbitrarily then the sum over all $\sigma$ does not result in $(E-H_o - \sum_{\text{all } \sigma} V_\sigma^T \Psi_\beta = 0$ becoming the Schrödinger equation, where $V_\sigma^T$ is a truncated version of $V_\sigma$.

<u>Proof</u>: Consider N = 4 and suppose we truncate $V_\sigma$ by taking $V_\sigma \sim V_\sigma^T = V_\sigma$.

Now

$$\sum_{\text{all } \sigma} V_\sigma^T \Psi_\beta = \sum_{\text{all } \sigma} V_\sigma \Psi_\beta. \tag{44}$$

From Lemma 1, $\sum_\sigma V_\sigma = S_{N-1}^{(2)} V$ and we have

$$\sum_{\text{all } \sigma} V_\sigma^T \Psi_\beta = S_3^{(2)} V \Psi_\beta. \quad \text{Q.E.D.} \tag{45}$$

We now introduce notation to represent a particular class of truncations of the irreducible cluster potential. The operator $V_\sigma^{a_m}$ is defined as a truncation of $V_\sigma$ that terminates with a single $a_m$, $3 \quad m \leq N - 1$. This operator involves the set of partitions $a_k$ such that $a_k \geq a_L$ This truncated irreducible cluster potential is defined in terms of the anti-clus · expansion for (28) as

$$V_\sigma^{a_m} = \sum_{n=2}^{m} \sum_{(\sigma\supset)a_n(\geq a_m)} N(\sigma,a_n)V_{a_n} G_{a_n} G_\sigma^{-1}. \tag{46}$$

This class of truncated operators allows us to introduce the Dominant Partition Theorem (DPT). The term dominant partition derives from the role played by a fixed partition $a_m$ in the truncation of $V_\sigma$ and in limiting the sum on two cluster partitions.

Theorem II. (Dominant Partition Theorem)

For arbitrary N, and an arbitrary fixed partition $a_m$

$$\sum_{\sigma(\supset a_m)} V_\sigma^{a_m} \Psi_\beta = V \Psi_\beta,$$

where $\Psi_\beta$ solves the Schrödinger equation for the N-body system $(3 \leq m \leq N - 1)$.

Proof:

Using the definition (46) gives

$$\sum_{\sigma(\supset a_m)} V_\sigma^{a_m} \Psi_\beta = \sum_{\sigma(\supset a_m)} \sum_{n=2}^{m} \sum_{(\sigma\supset)a_n(\geq a_m)} N(\sigma,a_n)V_{a_n} G_{a_n} G_\sigma^{-1} \Psi_\beta. \tag{47}$$

We write (46) in a more convenient form by picking off the $a_m$ term

$$\sum_{\sigma(\supset a_m)} V_\sigma^{a_m} \Psi_\beta = \sum_{\sigma(\supset a_m)} \sum_{n=2}^{m-1} \sum_{(\sigma\supset)a_n(\geq a_m)} N(\sigma,a_n)V_{a_n} G_{a_n} G_\sigma^{-1} \Psi_\beta + \sum_{\sigma(\supset a_m)} N(\sigma,a_m)V_{a_m} G_{a_m} G_\sigma^{-1} \Psi_\beta. \tag{48}$$

Again employing (31) we obtain

$$\sum_{\sigma(\supset a_m)} V_\sigma^{a_m} \Psi_\beta = \sum_{\sigma(\supset a_m)} [\sum_{n=2}^{m-1} \sum_{(\sigma\supset)a_n(\geq a_m)} N(\sigma,a_n)V_{a_n}(1-G_{a_n} V_\sigma^{a_n}) + N(\sigma,a_m)V_{a_m}(1-G_{a_m} V^{a_m})]\Psi_\beta. \tag{49}$$

Using (33) and (35) we have

$$\sum_{\sigma(\succ a_m)} V_\sigma^{a_m} \Psi_\beta = V_\beta \Phi_\beta + \sum_{\sigma(\succ a_m)} \sum_{n=2}^{m-1} \sum_{(\sigma\succ)a_n(\succeq a_m)} N(\sigma,a_n) V_{a_n} G_{a_n} V^\sigma \Psi_\beta$$
$$+ \sum_{\sigma(\succ a_m)} N(\sigma,a_m) V_{a_m} G_{a_m} V^\sigma \Psi_\beta. \tag{50}$$

Interchanging sums we have

$$\sum_{\sigma(\succ a_m)} V_\sigma^{a_m} \Psi_\beta = V_\beta \Phi_\beta + \sum_{n=2}^{m-1} \sum_{a_n(\succeq a_m)} \sum_{\sigma(\succ a_m)} N(\sigma,a_n) V_{a_n} G_{a_n} V^\sigma \Psi_\beta$$
$$+ \sum_{\sigma(\succ a_m)} N(\sigma,a_m) V_{a_m} G_{a_m} V^\sigma \Psi_\beta. \tag{51}$$

Use of (37) and (35) yields

$$\sum_{\sigma(\succ a_m)} V_\sigma^{a_m} \Psi_\beta = [\sum_{n=2}^{m-1} \sum_{a_n(\succeq a_m)} C_n V_{a_n} + C_m V_{a_m}] \Psi_\beta. \tag{52}$$

We now use

<u>Lemma 3</u>:

$$\sum_{a_n(\succ a_m)} V_{a_n}^{a_m} = S_{m-1}^{(n)} V^{a_m} \tag{53}$$

A proof is given in the appendix. This yields

$$\sum_{\sigma(\succ a_m)} V_\sigma^{a_m} \Psi_\beta = [\sum_{n=2}^{m-1} C_n (S_m^{(n)} V_{a_m} + S_{m-1}^{(n)} V^{a_m}) + C_m V_{a_m}] \Psi_\beta, \tag{54}$$

where we have used (33) (with $\sigma$ replaced by $a_n$) to express $V_{a_n}$ in (52) as $V_{a_m} + V_{a_n}^{a_m}$.

Noting the results

$$\sum_{n=2}^{m-1} C_n S_m^{(n)} = 1 - C_m \tag{55}$$

and

$$\sum_{n=2}^{m-1} C_n \, S_{m-1}^{(n)} = 1 \tag{56}$$

which follow from Lemma 2 we have

$$\sum_{\sigma(\supset a_m)} V_\sigma^{a_m} \Psi_\beta = V \Psi_\beta . \qquad \text{Q.E.D.} \tag{57}$$

Theorem II provides the basis for the DPM. It shows that we may truncate $V_\sigma$ through the anti-cluster expansion by retaining only those partitions that contain a given dominant partition. Note that the full partition Green functions $G_{a_n}$ are retained in (46). They are <u>not</u> projected on the Hilbert space corresponding to bound states of the dominant partition $a_m$. The description of the breakup of these clusters is contained in these Green functions.

The reduced problem is then solved by solutions to the original Schrödinger equation. This theorem provides a consistent means of reducing the many body problem in the LRT wave function formalism to a few body problem.

Nuclear reactions are commonly analyzed in terms of a few body picture. For a given N there are $S_N^{(2)} = 2^{N-1} - 1$ two cluster channels. Any realistic attempt to solve the many body problem cannot treat all of these channels on an equal footing. Moreover, it is reasonable to expect that in direct reactions the processes involved are not so extensive that all possible rearrangement and inelastic processes must be included. In many cases a realistic approach to many-body reaction theory will be afforded by sytematically building a few body models.

## IV. THE DOMINANT PARTITION AND THE BRS EQUATION

We now obtain dominant partition truncations of the BRS equations. This is accomplished by restricting the sum on two cluster partitions to the class defined by $\sigma(\supset a_m)$, where $a_m$ is taken to be the dominant partition. Correspondingly we introduce the appropriate truncation of the BRS kernel by using the anti-cluster expansion (27). Restricting the sum on two cluster partitions to those that contain a particular $a_m$ terminates the anti-cluster expansion with that term explicitly involving $a_m$. We write the truncated kernel as $\kappa^{a_m}_{\beta\sigma}$, that is

$$\kappa^{a_m}_{\beta\sigma} = \sum_{n=2}^{m} \sum_{(\sigma\supset)a_n(\supseteq a_m)} N(\sigma,a_n)V^{\beta}_{a_n} G_{a_n} . \tag{58}$$

With $\beta,\alpha(a_m)$ we write the truncated BRS equations as

$$T^{\beta\alpha} = V^{\beta}_{\alpha} + \sum_{\sigma(\supset am)} \kappa^{a_m}_{\beta\sigma} T^{\sigma\alpha} . \tag{59}$$

It will be recalled that in the derivation of the BRS equations that a crucial step was the democratic distribution of the residual interaction over all partitions.[16] If we restrict the distribution to those partitions containing $a_m$ and proceed with the derivation, are the truncated equations obtained the same as those we have termed the dominant partition truncated BRS equations? The answer is yes and provides the next theorem.

<u>Theorem III</u>. The dominant partition truncated BRS equations (59) are obtained by distributing the residual interaction $V^{\beta}$ over the subset of all possible partitions containing $a_m$ and proceeding as in the derivation of the BRS equations (ref. 16).

This theorem provides a satisfying degree of consistency in the reduction of the BRS equations. These considerations are displayed in Fig. 1.

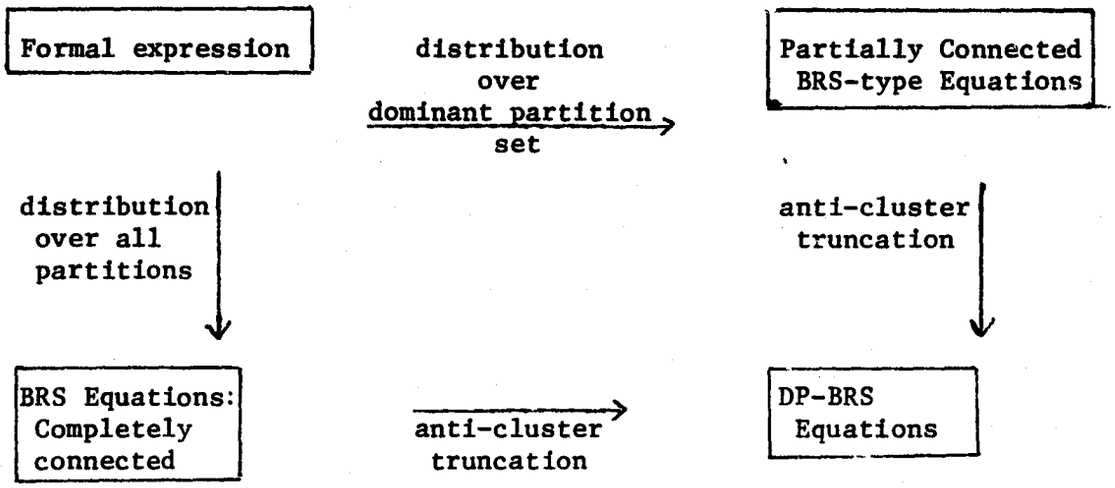


Fig. 1

Proof: From (39) and (53)

$$\sum_{d_j} V^{\beta}_{d_j} = S^{(j)}_{N-1} V^{\beta}$$

and

$$\sum_{d_j (\supset a_m)} V^{a_m}_{d_j} = S^{(j)}_{m-1} V^{a_m}.$$

Also note that

$$\sum_{d_j (\supset a_m)} V^{\beta}_{d_j} = S^{(j)}_{m-1} V^{\beta}, \quad \beta \supset a_m.$$

Using (56) we obtain

$$V^{\beta} = \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j V^{\beta}_{d_j}, \quad \beta \supset a_m.$$

We use this in the definition of the transition operator[10]

$$T^{\beta\alpha}_{+} = V^{\beta} G G^{-1}_{\alpha}. \tag{60}$$

We obtain

$$T_+^{\beta\alpha} = \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \, V_{d_j}^{\beta} \, G_{d_j} \, G_\alpha^{-1} + \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \, V_{d_j}^{\beta} \, G_{d_j} \, T_+^{d_j \alpha}, \quad \beta \supset a_m .$$

Using the Lippmann identity[16,18] to transform the Born term gives the following equation for a new set of operators $T^{\beta\alpha}$ which are equal to $T_+^{\beta\alpha}$ on the half-shell

$$T^{\beta\alpha} = \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \, V_{d_j}^{\beta} \, \delta_{d_j \alpha} + \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \, V_{d_j}^{\beta} \, G_{d_j} \, T^{d_j \alpha}, \quad \beta \supset a_m .$$

This yields

$$T^{\beta\alpha} = V_\alpha^{\beta} + \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \, V_{d_j}^{\beta} \, G_{d_j} \, T^{d_j \alpha}, \quad \alpha \supset a_m, \ \beta \supset a_m .$$

We use the Yakubovskii cluster expansion[19] to decompose the transition operator internal to partition $d_j$ into pieces of different connectivities. We write

$$T_{+d_j}^{\gamma\alpha} = \sum_{n=j}^{m-1} \sum_{(d_j \supset) d_n} K_{d_n}^{\gamma\alpha}$$

and

$$T_{+d_j}^{\beta o} \, G_o = V_{d_j}^{\beta} \, G_{d_j} = \sum_{n=j}^{m-1} \sum_{(d_j \supset) d_n} K_{d_n}^{\beta o} \, G_o .$$

The anti-cluster truncation procedure is to replace $K_{d_n}^{\beta o} \, G_o$ by $\kappa_{d_n}^{a_m}$ for $N > m$ and to drop all other terms.

We then get

$$T^{\beta\alpha} = V_\alpha^{\beta} + \sum_{j=2}^{m-1} \sum_{d_j (\supset a_m)} C_j \sum_{n=j}^{m-1} \sum_{(d_j \supset) d_n} \kappa_{d_n}^{a_m} \, T^{d_j \alpha} .$$

We interchange the n and j sums realizing that m is the largest number of clusters that we can have in the limited space. We obtain

$$T^{\beta\alpha} = V^{\beta}_{\alpha} + \sum_{n=2}^{m-1} \sum_{d_n (\supset a_m)} \kappa^{a_m}_{\beta d_n} \sum_{j=2}^{n} \sum_{d_j (\supset d_n)} C_j V^{d_j} G G^{-1}_{\alpha},$$

where (60) has been used. Employing

<u>Lemma 6</u>:[16]

$$\sum_{j=2}^{n} \sum_{d_j (\supset d_n)} C_j V^{d_j} = \delta_{2n} V^{d_n},$$

yields

$$T^{\beta\alpha} = V^{\beta}_{\alpha} + \sum_{n=2}^{m-1} \sum_{d_n (\supset a_m)} \kappa^{a_m}_{\beta d_n} \delta_{2n} T^{d_n \alpha}$$

and

$$T^{\beta\alpha} = V^{\beta}_{\alpha} + \sum_{\sigma (\supset a_m)} \kappa^{a_m}_{\beta\sigma} T^{\sigma\alpha}, \qquad \beta, \alpha \supset a_m. \qquad \text{Q.E.D.}$$

## V. SUMMARY AND CONCLUSIONS

We have developed a generalization of the Hahn and Watson's[7] 'partially-connected" strategy appropriate for constructing n-cluster models for N-body problems where it is intended that $n \ll N$. The cases treated are those in which the only channels treated explicitly are obtained by combining the clusters of an n-cluster "dominant" partition, $a_n$. We obtain n-body equations of the BRS type for transition operators and of the LRT type for wave functions. The resulting equations are connected in the degrees of freedom corresponding to the relative motion of the clusters of $a_n$, but not in those internal to a single cluster of $a_n$. Following Hahn and Watson we assume that these degrees of freedom are to be handled in some manner different from operator integral equations (e.g., by statistical or phenomenological methods).

Our main result is that the BRS and LRT equations for the small number of clusters is in fact exact if the subsystem Green functions are put in from some other source. This means that no incoming waves associated with

channels breaking the cluster of $a_n$ are to be admitted. Furthermore, it does not matter whether one obtains the partially-connected equations by truncating the anti-cluster expansion for the kernel or by only distributing the residual potential over the appropriate limited set of partitions. The same equations are obtained by both procedures.

A specific example where a procedure such as described here may be relevant is in the six-nucleon problem where the initial channel is a low energy ($E < 20$ MeV) deuteron incident on a $^4$He nucleus. As is well known[20] this is well described as a three-body problem. Here, our dominant partition would be $a_3 = (n)(p)(nnpp)$ where the effects of exchange are ignored. The three-body equation would fall out immediately upon approximating the Green function $G_{a_3}$ by its part having the $^4$He pole. This approximation would yield real effective nucleon-$^4$He interactions.

More general results, including the appearance of more complete effective interactions, can be obtained in a number of ways, the simplest of which is the introduction of projection operators at the Green function $G_{a_3}$. The part corresponding to everything but the $^4$He pole are then solved formally á la Feshbach[21]. This leads to the appearance of generalized optical potentials as effective interactions plus the well-known[22] effective three-body force.

## APPENDIX

We now prove results which were employed in section III on the DPT. These results are Eq. (31) and Lemma 3. According to (37)

$$\sum_{\sigma(\supset a_m)} N(\sigma,a_m)V^{\sigma} = C_m V^{a_m}, \tag{A.1}$$

where $C_m = (-1)^m(m-1)!$. An operator such as the T matrix for a given partition can be written in terms of a cluster decomposition

$$T_{a_n} = \sum_{m=n}^{N-1} \sum_{b_m} \delta_{a_n \supseteq b_m} [T]_{b_m}, \tag{A.2}$$

where $[T]_{a_m}$ is the part of $T_{a_n}$ with connectivity $a_m$ and $\delta_{a_n \supseteq b_m}$ is one provieded $a_n$ contains $b_m$ otherwise it is zero. Inverting the above expansion we obtain the $[T]_{b_m}$ in terms of the $T_{a_n}$. The inverted expansion is termed an anti-cluster expansion. In this expansion the numerical coefficients, N (the counting coefficients) occur naturally. From the cluster expansion for $T_{a_n}$ we have

$$[T]_{b_n} = \sum_{m=n}^{N-1} \sum_{(b_n \supset) a_m} N(b_n,a_m)\, T_{a_m} \tag{A.3}$$

The existence of the $N(a_n,a_m)$ follows from the fact that the matrix $\delta_{a_n \supseteq b_m}$ is invertible[23]. Moreover using (A.2) and (A.3) we easily note that

$$\sum_{m=j}^{k} \sum_{a_m(\subset c_j)} N(c_j,a_m)\, \delta_{a_m \supseteq b_k} = \delta_{c_j b_k} \tag{A.4}$$

We develop a recursion relation among the counting coefficients by following Ref. 24. We write (A.4) as

$$\sum_{m=j}^{k} \sum_{a_m} N(c_j,a_m)(\delta_{c_j \supseteq a_m} + \bar{\delta}_{c_j \supseteq a_m})\delta_{a_m \supseteq b_k} = \delta_{c_j b_k}, \tag{A.5}$$

where we have used $\delta_{c_j \supseteq a_m} + \bar{\delta}_{c_j \supseteq a_m} = 1$. We obtain

$$\sum_{m=j}^{k} \sum_{a_m} N(c_j,a_m)\delta_{c_j \supseteq a_m}\, \delta_{a_m \supseteq b_k} = \delta_{c_j b_k} \tag{A.6}$$

since $N(c_j,a_m) = 0$ for $c_j \not\geq a_m$.[24]

Employing (A.6) we easily obtain a useful relationship among the counting coefficients which allows us to determine $N(c_j,a_m)$ in terms of the $N(c_j,a_j)$, $N(c_j,a_{j+1}),\ldots,N(c_j,a_{m-1})$. From (A.6) we have

$$\sum_{a_k} N(c_j,a_k)\ \delta_{c_j \geq a_k}\ \delta_{a_k \geq b_k} + \sum_{m=j}^{k-1} \sum_{a_m} N(c_j,a_m)\ \delta_{c_j \geq a_m}\ \delta_{a_m \geq b_k} = 0, \tag{A.7}$$

which yields

$$N(c_j,b_k) = -\sum_{m=j}^{k-1} \sum_{a_m} N(c_j,a_m)\ \delta_{c_j \geq a_m}\ \delta_{a_m \geq b_k}. \tag{A.8}$$

We return to (A.1) and employ an inductive argument. We note that for $m = 2,3$ that

$$\sum_{\sigma} N(\sigma,a_2)V^{\sigma} = C_2\ V^{a_2}, \tag{A.9}$$

$$\sum_{\sigma} N(\sigma,a_3)V^{\sigma} = C_3\ V^{a_3}. \tag{A.10}$$

Assuming that

$$\sum_{\sigma} N(\alpha,a_k)V^{\sigma} = C_k\ V^{a_k} \tag{A.11}$$

we show that

$$\sum_{\sigma} N(\sigma,a_{k+1})V^{\sigma} = C_{k+1}\ V^{a_{k+1}}. \tag{A.12}$$

Consider

$$\sum_{\sigma} N(\sigma,a_{k+1})V^{\sigma} = -\sum_{\sigma} \sum_{m=2}^{k} \sum_{b_m} N(\sigma,b_m)\ \delta_{\sigma \geq b_m}\ \delta_{b_m \geq a_{k+1}}\ V^{\sigma}, \tag{A.13}$$

where we have used (A.8). Using the assumption (A.11) we have that

$$\sum_{\sigma} N(\sigma,a_{k+1})V^{\sigma} = -\sum_{m=2}^{k} \sum_{b_m} C_m\ V^{b_m}\ \delta_{b_m \geq a_{k+1}}. \tag{A.14}$$

We write this as

$$\sum_{\sigma} N(\sigma,a_{k+1})V = -\sum_{m=2}^{k} C_m \sum_{b_m (\geq a_{k+1})} V^{b_m}. \tag{A.15}$$

Note that

$$\sum_{a_n(\supset a_m)} V^{a_n} = S_m^{(n)} V^{a_m} - S_{m-1}^{(n)} V^{a_m} \tag{A.16}$$

where the $S_m^{(n)}$ are Stirling numbers of the second kind. This result follows almost immediately from Lemma 3 which will be proven. Employing (A.16) we find

$$\sum_{\sigma} N(\sigma, a_{k+1}) V^{\sigma} = - \sum_{m=2}^{k} C_m (S_{k+1}^{(m)} V^{a_{k+1}} - S_k^{(m)} V^{a_{k+1}}). \tag{A.17}$$

Finally we write

$$\sum_{\sigma} N(\sigma, a_{k+1}) V^{\sigma} = -[\sum_{m=2}^{k+1} C_m S_{k+1}^{(m)} - \sum_{m=2}^{k} C_m S_k^{(m)} - C_{k+1} S_{k+1}^{(k+1)}] V^{a_{k+1}} \tag{A.18}$$

and use the fact that[16]

$$\sum_{m=2}^{N-1} C_m S_{N-1}^{(m)} = 1 \tag{A.19}$$

to obtain

$$\sum_{\sigma} N(\sigma, a_{k+1}) V^{\sigma} = C_{k+1} V^{a_{k+1}}. \qquad \text{Q.E.D.}$$

We complete the discussion by proving Lemma 3:

$$\sum_{a_n(\supset a_m)} V_{a_n}^{a_m} = S_{m-1}^{(n)} V^{a_m}$$

<u>Proof</u>: Consider a particular pairwise interaction $V_j$ with j external to $a_m$. This interaction will be internal to some number of the $a_n$ clusters that include $a_m$. We determine how many by considering an m-cluster system. In all of the n clusters the particles of the pair j will have to be together so we construct n clusters out of the m clusters by initially joining the two particles of the pair j into a single 'particle'. Now the remaining (m-1) clusters may be joined into n clusters in any way. This number is $S_{m-1}^{(n)}$. This is independent of the pair index so every pair external to $a_m$ will appear this number of times. This yields Lemma 3.

## REFERENCES


1. E. O. Alt, P. Grassberger and W. Sandhas, Nucl. Phys. B2, 167 (1967); JINR Report No. 4-6688 (1972); W. Sandhas, in Few Body Dynamics, A. Mitra, I Slaus, V. Bhasin and V. Gupta, eds. (North-Holland, New York, 1976), pp. 540-557.
2. G. Bencze, Nucl. Phys. A210, 568 (1973); E. F. Redish, Nucl. Phys. A225, 16 (1974); I. H. Sloan, Phys. Rev. C6, 1945 (1972).
3. D. J. Kouri and F. S. Levin, Nucl. Phys. A253, 395 (1975); W. Tobocman, Phys. Rev. C9, 2466 (1974).
4. L. D. Faddeev, Sov. Phys. JETP 12, 1014 (1961).
5. E. F. Redish, in Modern Three Hadron Physics, A. W. Thamos, ed. (Springer-Verlag, New York, 1977), pp. 181-239.
6. S. R. Cotanch and C. M. Vincent, Phys. Rev. C14, 1739 (1976).
7. Y. Hahn and K. M. Watson, Phys. Rev. A5, 1718 (1972).
8. M. L'Huillier, E. F. Redish and P. C. Tandy, Bull. Am. Phys. Soc. 21, 49 (1976).
9. M. Goldberger and K. M. Watson, Collision Theory (John Wiley, New York, 1964).
10. S. Weinberg, Phys. Rev. 133, B232 (1964).
12. M. L'Huillier, E. F. Redish and P. C. Tandy, "The Structure of the Scattering Wave Function in the 1 body Problem," Int. Symp. on Nuclear Reaction Models, Balatonfüred, Hungary, June 27-July 1, 1977; J. Math. Phys. 19, 1276 (1978).
13. V. Vanzani, in Few-Body Nuclear Physics (I.A.E.A., Vienna, 1978), p. 57.
14. E. F. Redish, Nucl. Phys. A235, 82 (1974).
15. W. Glöckle, Nucl. Phys. 141, 620 (1970).

16. P. Benoist-Gueutal, M. L'Huillier, E. F. Redish and P. C. Tandy, Phys. Rev. C17, 1924 (1978).

17. Handbook of Mathematical Functions, M. Abramowitz and I. A. Stegun, eds. (National Bureau of Standards, 1964), p. 825.

18. B. A. Lippmann, Phys. Rev. 102, 264 (1956).

19. O. A. Yakubovskii, Sov. Journ. of Nucl. Phys. (U.S.A.) 5, 937 (1967).

20. G. Plattner, in Few-Body Systems and Nuclear Forces II, Proc. VIII Int. Conf., Graz, Austria, Aug. 24-30, 1978, H. Zingle, M. Haftel, and H. Zankel, eds. (Springer, Berlin, 1978), p. 376.

21. H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 287 (1962); Gy. Bencze, "Effective Three-Body Problems in Nuclear Reactions," U. of Helsinki preprint no. 2-73 (1973), unpublished.

22. N. Austern and K. C. Richards, Ann. Phys. (N.Y.) 49, 309 (1968).

23. W. N. Polyzou and E. F. Redish, Ann. Phys. (N.Y.), to be published.

24. K. L. Kowalski, Ann. Phys. (N.Y.), to be published.