The Mechanism of Fluid Resistance.[1]

By Th. v. Kármán and H. Rubach.

The resistance of a solid body moving with a uniform velocity in an unlimited fluid can be calculated theoretically only in the limiting cases of very slow motion of small bodies or of very high fluid viscosity. We are brought in such cases to a resistance proportional to the first power of the velocity, to the viscosity constant, and, for geometrically similar systems, to the linear dimensions of the body. To the domain of this "linear resistance"—which has aroused much interest, especially within recent years, on account of some important experimental applications—has to be opposed the limiting domain of comparatively large velocities, for which the so-called "velocity square law" holds with very good approximation. In this latter domain, which embraces nearly all the important technical applications, the resistance is nearly independent of fluid viscosity, and is proportional to the fluid density, the square of the velocity, and—again for geometrically similar systems—to a surface dimension of the body. In this domain of the "square law" is included the important case of air resistance, because it is easy to verify, by the calculation of the largest density variations which can occur for the speeds we meet in aeronautics and airscrews, that the air compression can be neglected without any sensible error. The influence of the compression first becomes important for velocities of the order of the velocity of sound. In fact, experiments show that the air resistance, in a broad range from the small speeds at which the viscosity plays a role up to the large speeds comparable to the velocity of sound, is proportional to the square of the velocity with very good approximation.[3] In general, fluid resistance depends upon the form and the orientation of the body in such a complicated way that it is extraordinarily difficult to predetermine the flow to a degree sufficient for the evaluation of the resistance of a body of given form, by a process of pure calculation, as can be done by aid of the Stokes formula in the case of very slow motions. We also will not succeed in this paper in reaching such a solution, but will still make the attempt to give a general view of the *mechanism of fluid resistance within the limit of the square law*.

We can state the problem of fluid resistance in the following somewhat more exact way.

Since the time of the fundamental considerations of *Osborne Reynolds* on the mechanical similitude of flow phenomena of incompressible viscous fluids of different density and viscosity and—under geometrical similitude—for different sizes of the system considered, it is known that the resistance phenomenon depends upon a single parameter which is a certain ratio of the above-mentioned quantities. Thus the fluid resistance of a body moving with the uniform velocity U in an incompressible unlimited fluid may be expressed by a formula of the form[4]

$$W=\mu l U f\left(\frac{Ul\rho}{\mu}\right) \tag{I}$$

where

μ is the viscosity constant

ρ the fluid density

l a definite but arbitrarily chosen linear dimension of the body, and $f\left(\frac{Ul\rho}{\mu}\right)$ a function of the single variable $R=\frac{U\rho l}{\mu}$. We will call "Reynolds' parameter" the quantity R which has a zero dimension.

Theory and experiment show that for very small values of R—that is, for low velocities, or small bodies, or great viscosity—the function $f(R)$ is very nearly constant; the resistance coefficient of the Stokes formula corresponds to the limiting case of $f(R)$ for $R=O$. The square law corresponds to the limiting case of $R=\infty$. We approach this latter case the more nearly the smaller the viscosity μ, so that in the limiting case of $R=\infty$, the fluid can be considered as frictionless. And we can ask ourselves, *to what limiting configuration does the flow of the viscous fluid around a solid body tend when we pass to the limiting case of a perfect fluid?* This is, according to our view, the fundamental point of the resistance problem.

The fact that we obtain in this case a resistance nearly independent of the viscosity constant—since according to formula (I) this corresponds to the square law—allows us to conjecture that in this limiting case the resistance is determined by flow types such as can occur in a perfect fluid.

[1] Translation of the paper of Th. v. Kármán and H. Rubach published in "Physikalische Zeitschrift," Jan. 15, 1912.

It is now certain that neither the so-called "continuous" potential flow, nor the "discontinuous" potential flow discovered by *Kirchhoff* and *v. Helmholtz,* can express properly this limiting case. Continuous potential flow does not cause any resistance in the case of uniform motion of a body, as may be shown directly by aid of the general momentum theorem; the theory of the discontinuous potential flow, which, in relation to the resistance problem has been discussed principally by Lord Rayleigh,[1] leads to a resistance which is proportional to the square of the velocity; the calculated values do not, however, agree with the observed ones. And, independent of the insufficient agreement between the numerical values, the hypothesis of the "dead water," which, according to this theory ought to move with the body, is in contradiction to nearly all observations. It is easy to see by aid of the simplest experiments that the flow, when referred to a system of coordinates moving with the body, is not stationary, as assumed in this theory. Furthermore, in the theory of discontinuous potential motion, the suction effect behind the body is totally missing, while in the dead water, which extends to infinity, we have everywhere the same pressure as in the undisturbed fluid at a great distance from the body. But according to recent measurements, in many cases the suction effect is of first importance for the resistance, and in any case contributes a sensible part of the last.

The reason why in a perfect fluid the discontinuous potential flow, although hydrodynamically possible, is not realized is without any doubt the instability of the surfaces of discontinuity, as has already been recognized by v. Helmholtz and specially mentioned by Lord Kelvin.[2] A surface of discontinuity can be considered as a vortex sheet; and it can be shown in a quite general way that such a sheet is always unstable. This can also be observed directly; observation shows that vortex sheets have a tendency to roll themselves up; that is, we see the concentration around some points of the vortex intensity of the sheet originally between them. This observation leads to the question: Can there exist

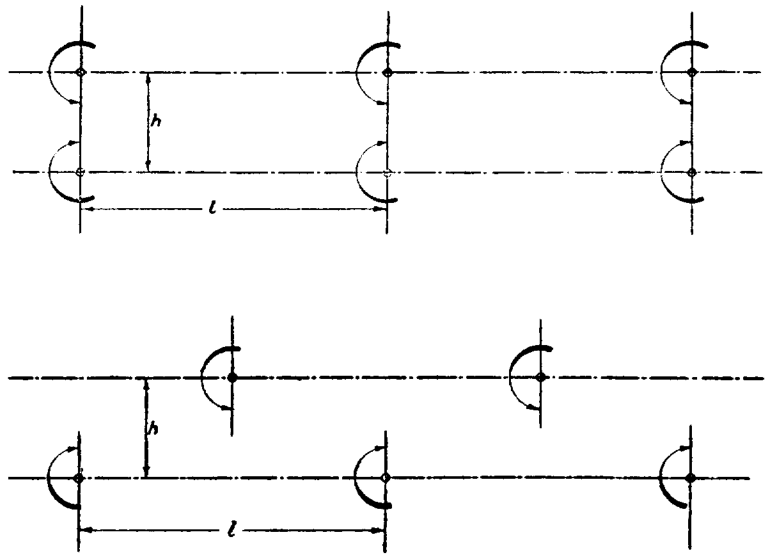

FIG. 1.

stable arrangements of isolated vortex filaments, which can be considered as the final product of decomposed vortex sheets? This question forms the starting point of the following investigations; it will, in fact, appear that at least for the simplest case of uniplanar flow, to which we will limit ourselves, we will be led to a "flow picture" which in all respects corresponds quite well to reality.

THE INVESTIGATION OF STABILITY.

We will investigate the question whether or not two parallel rows of rectilinear infinite vortices, of equal strength but of inverse senses, can be so arranged that the whole system, while maintaining an invariable configuration, will have a uniform translation and be stable at the same time. It is easy to see that there exist two kinds of arrangements for which two parallel vortex rows can move with a uniform and rectilinear velocity. The vortices may be placed one opposite the other (arrangement a, fig. 1), or the vortices of one row may be placed opposite the middle points of the spacing of the vortices of the other row (arrangement b). In the case of equality of spacing of the vortices in both rows, as a consequence of symmetry for the two arrangements a and b, it appears that each vortex has the same velocity in the sense of the X axis, and that the velocity in the sense of the Y axis is equal to zero. We have to answer the question, which of these two arrangements is stable?

To illustrate first by a simple example the method of the investigation of stability, we will start with the consideration of an infinite row of infinite vortices disposed at equal distances l and having the intensity ζ, and will study

[1] On the resistance of fluids, Mathematical and Physical Papers, Vol. I, p. 287.

[2] Mathematical and Physical Papers, Vol. IV, p. 215. This paper contains a detailed critique of the theory of discontinuous motion.

the stability of such a system. If we designate by x_p, y_p, the coordinates of the $p-th$ vortex, and by x_q, y_q the coordinates of the $q-th$ the velocity impressed on the latter vortex by the former is given by the formulæ

$$u_{pq}=\frac{\zeta}{2\pi}\cdot\frac{y_p-y_q}{(x_p-x_q)^2+(y_p-y_q)^2}$$

$$v_{pq}=-\frac{\zeta}{2\pi}\cdot\frac{x_p-x_q}{(x_p-x_q)^2+(y_p-y_q)^2}$$

These formulæ express the fact that each vortex communicates to the other a velocity which is normal to the line joining them and is inversely proportional to their distance apart. Therefore the resultant velocity of the $q-th$ vortex due to all the vortices is equal to

$$\frac{dx_q}{dt}=\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{y_p-y_q}{(x_p-x_q)^2+(y_p-y_q)^2}$$

$$\frac{dy_q}{dt}=-\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{x_p-x_q}{(x_p-x_q)^2+(y_p-y_q)^2}$$

where $p=q$ is excluded from the summation. If now the vortices are disturbed from their equilibrium position, the small displacements being ζ_p, η_p, the vortex velocities can be developed in terms of these quantities, and we will be brought to a system of differential equations for the disturbances ξ_p, η_p, i. e., for small oscillations of the system.

Let us accordingly put

$$x_p=pl+\xi_p$$
$$y_p=\eta_p$$

and, neglecting the small quantities of higher orders, we will get

$$\frac{d\xi_q}{dt}=\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{\eta_p-\eta_q}{(p-q)^2l^2}$$

$$\frac{d\eta_q}{dt}=\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{\xi_p-\xi_q}{(p-q)^2l^2}$$

The differential equations so obtained, which are infinite in number, are reduced to two equations by the substitution

$$\xi_p=\xi_o e^{ip\varphi};\ \eta_p=\eta_o e^{ip\varphi}$$

These two equations are

$$\frac{d\xi_o}{dt}=\eta_o\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{e^{ip\varphi}-1}{p^2l^2}$$

$$\frac{d\eta_o}{dt}=\xi_o\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{e^{ip\varphi}-1}{p^2l^2}$$

with $p\neq o$

The physical meaning of this substitution is easy to see: we consider a disturbance in which each vortex undergoes the same motion only with a different phase φ. Under such conditions we have to do with a wave disturbance and the system will be called stable, when for any value of φ, that is, for any phase difference between two consecutive vortices, the amplitude of the disturbance does not increase with the time.

Let us introduce the notation

$$\kappa(\varphi)=\frac{\zeta}{2\pi}\sum_{p=-\infty}^{\infty}\frac{e^{ip\varphi}-1}{p^2l^2}=\frac{\zeta}{\pi l^2}\sum_{p=1}^{\infty}\frac{\cos(p\varphi)-1}{p^2}$$

The foregoing equations then take the form

$$\frac{d\xi_o}{dt}=\kappa(\varphi)\eta_o$$

$$\frac{d\eta_o}{dt}=\kappa(\varphi)\xi_o$$

Let us put ξ_o and η_o proportional to $e^{\lambda t}$; we will then find for each value of φ two values for λ, that is

$$\lambda=\pm\kappa(\varphi)$$

It follows that the vortex system considered is unstable for any periodic disturbance, because there is always present a positive real value of λ, that is, the disturbance is of increasing amplitude.

Applying this method in the case of two vortex rows we will find that the arrangement a, that is, the symmetrical arrangement, is likewise unstable, but that for the arrangement b there exists a value of the ratio h/l (h is the distance between the two rows, l is the distance between the vortices in the row) for which the system is stable.

In both cases λ can be brought to the form

$$\frac{\pi}{\zeta}\lambda = \pm i(B \pm \sqrt{C^2 - A^2})$$

where A, B, C are functions of the phase difference φ. The system will be stable if $(C^2 - A^2)$ is positive for any value of φ.

For the symmetrical arrangement a, the functions A, B, C are expressed by the formulæ:

$$A(\varphi) = \frac{1}{2h^2} - \sum_{p=1}^{\infty} \frac{p^2l^2 - h^2}{(p^2l^2 + h^2)^2} + \sum_{p=1}^{\infty} \frac{1 - \cos p\varphi}{p^2l^2}$$

$$B(\varphi) = \sum_{p=1}^{\infty} \frac{p^2l^2 - h^2}{(p^2l^2 + h^2)^2} \sin(p\varphi)$$

$$C(\varphi) = \frac{1}{2h^2} - \sum_{p=\lambda}^{\infty} \frac{p^2l^2 - h^2}{(p^2l^2 + h^2)^2} \cos(p\varphi)$$

But for $\varphi = \pi$ we get

$$A(\pi) = \frac{\pi^2}{8l^2}\left[\operatorname{ctgh}^2\left(\frac{h\pi}{l}\right) - \operatorname{tgh}^2\left(\frac{h\pi}{l}\right)\right]$$

$$C(\pi) = \frac{\pi^2}{8l^2}\left[\operatorname{ctgh}^2\left(\frac{h\pi}{l}\right) - \operatorname{tgh}^2\left(\frac{h\pi}{l}\right)\right]$$

so that this arrangement is unstable for any values of h and l.

For the unsymmetrical arrangement b we find

$$A(\varphi) = -\sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2l^2 - h^2}{[(p+\frac{1}{2})^2l^2 + h^2]^2} + \sum_{p=1}^{\infty} \frac{1 - \cos(p\varphi)}{p^2l^2}$$

$$B(\varphi) = \sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2l^2 - h^2}{[(p+\frac{1}{2})^2l^2 + h^2]^2} \sin(p+\tfrac{1}{2})\varphi$$

$$C(\varphi) = \sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2l^2 - h^2}{[(p+\frac{1}{2})^2l^2 + h^2]^2} \cos(p+\tfrac{1}{2})\varphi$$

We see now that $C(\pi) = o$, so that in the place where $\varphi = \pi$, A must also be equal to zero, because, on account of the double sign, λ takes a positive real value. This brings us to the condition

$$\sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2l^2 - h^2}{[(p+\frac{1}{2})^2l^2 + h^2]^2} = \sum_{p=0}^{\infty} \frac{2}{(2p+1)^2l^2}$$

But

$$\sum_{p=0}^{\infty} \frac{[(p+\frac{1}{2})^2l^2 - h^2}{[(p+\frac{1}{2})^2l^2 + h^2]^2} = \frac{\pi^2}{2l^2 \cosh^2\frac{\pi h}{l}}$$

and

$$\sum_{p=0}^{\infty} \frac{2}{(2p+1)^2l^2} = \frac{\pi^2}{4l^2}$$

so that, as the necessary condition of stability we find the relation

$$\cosh\frac{h\pi}{l} = \sqrt{2}$$

and for the ratio h/l we find the value

$$h/l = 0{,}283\ldots.$$

For a certain value of the wave length of the disturbance, corresponding to $\varphi=\pi$, we get $\lambda=o$, that is, the system is in a neutral state. But it can be shown by calculation that our system is stable for all other disturbances. This unique disturbance has to be tested by further investigations. It can, however, be seen that a zero value for λ must appear, because only one stable configuration exists. If this were not so, we would find for l/h a finite domain of stability.[1]

THE "FLOW PICTURE."

The consideration of the question of stability has brought us to the result that there exists a particular configuration of two vortex rows which is stable. The vortices of both rows have then such an arrangement that the vortices of one row are placed opposite the middle of the interval between the vortices of the other row, and the ratio of the distance h between the two rows to the distance l between the vortices of the same row has the value

$$\frac{h}{l}=\frac{1}{\pi}\text{ arc cosh }\sqrt{2}=0{,}283$$

The whole system has the velocity

$$u=\frac{\zeta}{\pi}\sum_{p=o}^{\infty}\frac{h}{(p+\frac{1}{2})^2l^2+h^2}$$

which can also be written

$$u=\frac{\zeta}{2l}\text{tg h}\frac{\pi h}{l}$$

or, introducing the value of h/l found by the stability investigation, we get

$$u=\frac{\zeta}{l\sqrt{8}}$$

The flow is given by the complex potential (φ potential, ψ flow function)

$$\chi=\varphi+i\psi=\frac{i\zeta}{2\pi}\,lg\,\frac{\sin\,(z_o-z)\frac{\pi}{l}}{\sin\,(z_o+z)\frac{\pi}{l}}$$

where

$$z_o=\frac{l}{4}+\frac{hi}{2}$$

By aid of this formula we have calculated the corresponding streamlines and have represented them in Fig. 2. We see that some of the streamlines are closed curves around the vortices, while the others run between the vortices.

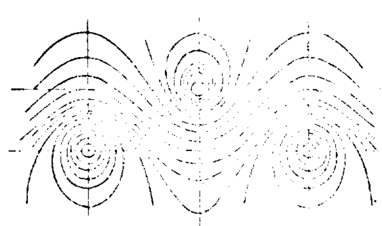

FIG. 2.

On the other hand, we have tried to make visible the flow picture behind a body, e. g., a flat plate or circular cylinder, moved through immobile water, by aid of lycopodium powder sifted on the surface of the water, and to fix these pictures photographically (exposure one-tenth of a second).

The regularly alternated arrangement of the vortices can not be doubted. In most cases the vortex centers can also be well determined; sometimes the picture is disturbed by small "accidental vortices" produced in all probability by small vibrations of the body, which in our provisional experiments could not be avoided. We had a narrow tank whose floor was formed by a band running on two rolls, and the bodies tested were simply put on the moving band and carried by it. It is to be expected that by aid of an arrangement especially made for the purpose much more regular flow pictures could be obtained, while in the actual experiments the flow was disturbed on the one hand by the vibrations of the body and on the other by the water flow produced by the moving band itself.

The alternated arrangement of the vortices rotating to the right and to the left can only be obtained when the vortices periodically run off first from one side of the body, then from the other, and so on, so that behind the body there appears a periodic motion, oscillating from one side to the other, but with such a regularity, however, that the frequency of this oscillation can be estimated with sufficient exactness. The periodic character of the motion in the so-called "vortex wake" has often been observed. Thus, Bernard [2] has remarked that the flow picture behind a narrow obstacle can be decomposed into vortex fields with alternated rotations. Also for the flow of water around balloon models the oscillation of the vortex field has been observed.[3] Finally, v. d. Borne [4] has observed and photographed recently the alternated formation of vortices in the case of air flowing around different obstacles. The

[1] From a mathematical standpoint our stability investigation may be considered as a direct application of the theorems of Mr. O. Toplitz on Cyclanten with an infinite number of elements, which he has in part published in two papers (Gottingen Nachrichten, 1907, p. 110; Math. Annalen 1911. p. 351), and in part been so kind as to communicate personally to us.

[2] Comptes Rendus, Paris, 148, 839, 1908.

[3] Technical report of the Advisory Committee for Aeronautics (British), 1910–11.

[4] Undertaken on the initiative of the representatives of aeronautical science in Gottingen, November, 1911.

phenomenon could not be explained until now; according to our stability investigation the periodic variations appear as a natural consequence of the instability of the symmetrical flow.[1]

It is also very interesting to observe how the stable configuration is established. When, for example, a body is set in motion from rest (or conversely, the stream is directed onto the body) some kind of "separation layer" is first formed, which gradually rolls itself up, at first symmetrically on both sides of the body, till some small disturbance destroys the symmetry, after which the periodic motion starts. The oscillatory motion is then maintained corresponding to the regular formation of left hand and right hand vortices.

We have also made a second series of photographs for the case of a body placed at rest in a uniform stream of water. For this case the flow picture can be obtained from Fig. 2 by the superposition of a uniform horizontal velocity. We will then see on the lines drawn through the vortex centers perpendicular to the stream direction, some ebbing point where the stream lines intersect and the velocity is equal to zero. However, in the same way as the motion is affected by the vibrations of the experimental body in the case of the motion of a body in the fluid, so in this case the turbulence of the water stream gives rise to disturbances.

As to the quantitative agreement attained by the theory, it must be noted that our stability conditions refer to infinite vortex rows, so that an agreement of the ratio h/l with the measured values is to be expected only at a certain distance from the body. The measurements on the photographs show that the distance l between vortices in a row is very regular, so that l may be measured satisfactorily, but per contra the distance h is much more variable, because the disturbance of the vortices takes place principally in the direction normal to the rows, that is, the latter undergo in the main transverse oscillations. The best way to determine the mean positions of the centers of the vortices would be by aid of cinematography, but we can also, without any special difficulty, find by comparison the mean direction of each vortex row directly from photographs. So in the case of the photograph of a circular cylinder 1.5 cm. in diameter, when making measurements beyond the first two or three vortex pairs we have found the following mean values for h and l

$$h=1.8 \text{ cm.}; \; l=6.4 \text{ cm.}$$

So that for the ratio h/l we obtain the value

$$h/l=0.28.$$

For the flow around a plate of 1.75 cm. breadth we found

$$h=3 \text{ cm.}; \; l=9.8 \text{ cm.}$$

Accordingly

$$h/l=0.305.$$

The agreement with the theoretical value 0.283 is entirely satisfactory.

For the first vortex pair behind the body, h/l comes out sensibly larger, somewhere near $h/l=0.35$. But in the first investigation of Kármán, mentioned at the beginning of this paper, the stability of the vortex system was investigated in such a way that all the vortices with the exception of one pair were maintained at rest and the free vortex pair considered oscillating in the velocity field of the others. Under such assumptions it was found that $h/l=1/\pi \operatorname{arc} \cosh \sqrt{3}=0.36$. We therefore think that the conclusion can be drawn, that in the neighborhood of the body, where the vortices are even more limited in their displacements, the ratio h/l is greater than 0.283 and approaches rather the value of 0.36.

APPLICATION OF THE MOMENTUM THEOREM TO THE CALCULATION OF FLUID RESISTANCE.

Let us assume that at a certain distance behind the body there exists a flow differing but slightly from the one of stable configuration which we have established theoretically in the foregoing, but that at a distance in front of the body, which is great in comparison with the size of the body, the fluid is at rest—as it is quite natural to assume. We will then be brought by the application of the momentum theorem to a quite definite expression for the resistance which a body moving with a uniform velocity in a fluid must experience. Practically, by such a calculation for the uniplanar problem, we will obtain the resistance of a unit of length of an infinite body placed normally to the plane of the flow.

We will use a system of coordinates moving with the same speed u as the vortex system behind the body. In this coordinate system, according to our assumptions, at a sufficient distance from the body the vortex motion behind the body as well as the fluid state in front of the body will be steady, and we will have, when referred to this system of coordinates, a uniform flow of speed $-u$ in front of the body, but behind the body the velocity components will be expressed by

$$-u+\frac{\partial \psi}{\partial y} \text{ and } -\frac{\partial \psi}{\partial x}$$

where ψ is the real part of the complex potential

$$\chi=\varphi+i\psi=\frac{i\zeta}{2\pi} lg \frac{\sin (z_0+z)\frac{\pi}{l}}{\sin (z_0-z)\frac{\pi}{l}}$$

[1] The tone that is emitted by a stick rapidly displaced in air is fixed by this periodicity, to which Prof. C. Runge has already drawn our attention.

The body itself has, relative to this system of coordinates, the velocity $U-u$, where U is the absolute velocity of the body. If we designate by l the distance between the vortices of one row, there must take place, as a consequence of the displacement of the body, in the time $T=l/(U-u)$, the formation of a vortex on each side of the body. We will calculate the increment of the momentum, along the X axis, in this time interval T (that is, between two instants of time of identical flow state) and for a part of the flow plane, which we define in the following way (see fig. 3). On the sides the plane portion considered is limited by the two parallel straight lines $y=\pm\eta$; in front and behind, by two straight lines $x=$Const disposed at distances from the body which are great in comparison with the size of the body, the line behind the body being drawn so as to pass through the point half way between two vortices having inverse rotation. When the boundary lines are sufficiently far from the body we can consider the fluid velocities at those lines as having the values indicated in the foregoing.

For a space with the boundaries indicated above the relation must exist that the momentum imparted to the body $\int_0 W dt$ (where W is the resultant fluid resistance) is equal to the difference between the momentum contained in the space considered at the times $t=\tau$ and $t=\tau+T$ and the sum of the inflow momentum and the time integral of the pressure along the boundary lines. If we thus consider as exterior forces the force $-W$ and the pressure, which act on the whole system of fluid and solid, they must then correspond to the increment of the momentum—that is, to the excess of momentum after the time T less the inflow momentum.

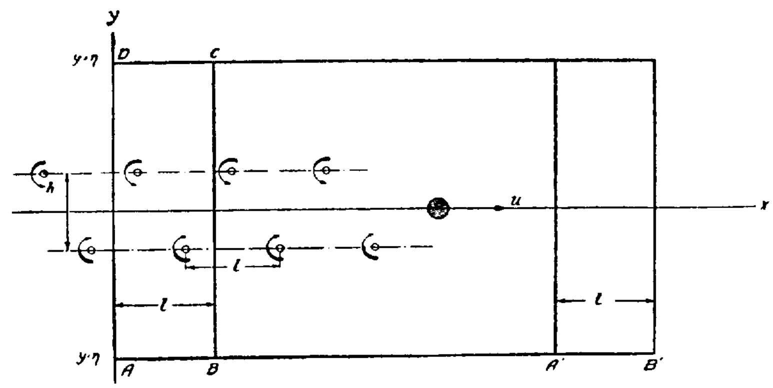

FIG. 3.

We will calculate these momentum parts separately. The excess of momentum after the time T is equal to the difference of the values that the double integral $\rho\int\int u\,(x,\,y)\,dx\,dy$ takes at the times $t=\tau$ and $t=\tau+T$. But the time interval has been chosen in such a way that the state of flow is identical, with the difference that the body has been displaced through the distance $l=(U-u)\,T$. The double integral reduces thus to the difference of the integrals taken over the strips $ABCD$ and $A'B'C'D'$ both of breadth l. For the strip $A'B'C'D'$ the fluid speed can be taken equal to $-u$ for the strip $ABCD$ equal to $-u+\frac{\partial\psi}{\partial y}$ so that we get

$$I_1=\rho\int_0^l\int_{-\eta}^{\eta}\frac{\partial\psi}{\partial y}dx\,dy$$

If we pass to side boundaries having $\eta=\infty$, we obtain for I_1 the very simple expression

$$I_1=\rho\zeta h$$

which can also be obtained directly by the application of the general momentum theorem to vortex systems.

We will unite in one single term the inflow momentum and the time integral of the pressure, because in such a way we will be led to more simple results. If we consider a uniplanar steady fluid motion with the velocity components $u\,(x,\,y)$ and $v\,(x,\,y)$ and consider a fixed contour in the plane, the inflow momentum in a unit of time in the direction of X is expressed by the closed integral $\rho\int(\bar{u}^2dy-\bar{u}\bar{v}dx)$ where $\bar{u}$, $\bar{v}$ are the velocities on the contour. The pressure gives the resultant $\int\bar{p}\,dy$ along the X axis, but since for a steady flow the relation

$$p=\text{Const}-\rho\frac{\bar{u}^2+\bar{v}^2}{2}$$

must hold, we thus obtain for the sum of both integrals, multiplied by T

$$I_2=T\int\rho(\bar{u}^2dy-\bar{u}\bar{v}dx)+T\int\boldsymbol{p}\;dy$$
$$=T\rho\int\left(\frac{\bar{u}^2-\bar{v}^2}{2}\,dy-\bar{u}\bar{v}dx\right)$$

Or, introducing the complex quantity,

$$\bar{w}=\bar{u}-i\bar{v}=\frac{\partial(\varphi+i\psi)}{\partial(x+iy)}=\frac{\partial\chi}{\partial z}$$

we get

$$I_2=\rho\ Im\int\left(\bar{w}^2dz\right)$$

where Im is to be understood as the complex part of the integral.

If we put for the contour

$$\bar{u}=-u+u'$$
$$\bar{v}=v'$$

then the terms in u^2 will at once be eliminated, and also the terms in u on account of the equality of the inflow and outflow; and there will remain only the terms in u'^2 and $u'v'$. The latter will give a finite value only for the boundary line passing through the vortex system (AD in fig. 3). Passing to $\eta=\infty$, we get

$$I_2=T\rho\, Im\left[\int_{-i\infty}^{i\infty}\left(\frac{d\chi}{dz}\right)^2dz\right]$$

and integrating along AD we get

$$I_2=T\rho Im\left[\int_{\chi(-i\infty)}^{\chi(i\infty)}\frac{d\chi}{dz}d\chi\right]$$

But

$$w=\frac{d\chi}{dz}=-\frac{\zeta}{l}\operatorname{tgh}\frac{h\pi}{l}-\frac{i\zeta\cos\frac{2\pi\chi}{\zeta}}{l\cosh\frac{h\pi}{l}}$$

so that, integrating and introducing the values

$$\chi(iw)=\frac{\zeta}{4}-i\frac{h\zeta}{2l}$$
$$\chi(-iw)=-\frac{\zeta}{4}+i\frac{h\zeta}{2l}$$
$$I_2=T\rho\left[\frac{\zeta uh}{l}-\frac{\zeta^2}{2\pi l}\right]$$

where u again has been written for $\frac{\zeta}{2l}\operatorname{tgh}\frac{\pi h}{l}$.

Thus the total momentum imparted to the body is

$$\int_0^T Wdt=\rho\zeta h-T\rho\left(\frac{\zeta uh}{l}-\frac{\zeta^2}{2\pi l}\right)$$

If for the mean value of $^1/_T\int_0^T Wdt$ we write W (as the time mean value of the resistance) we will obtain with $T=l/(U-u)$ the final formula

(II) $$W=\rho\zeta\frac{h}{l}(U-2u)+\rho\frac{\zeta^2}{2\pi l}$$

The fluid resistance appears here expressed by the three characteristic constants ζ, h, l of the vortex configuration (as u is expressed by the last). In the deduction of this last formula we did not take account of the stability conditions, so that this formula applies to any value of the ratio h/l. If we assume the vortices in the row to be brought all close together so that they are uniformly distributed along the row, but in such a way that the vortex intensity per unit of length remains finite, we thus pass to the case of continuous vortex sheets. In this case $\zeta/l=U$, but $\zeta^2/l=O$ and $u=\frac{U}{2}$, so that the fluid resistance disappears. The discontinuous potential flow of v. Helmholtz thus does not give any resistance when the depth of the dead water remains finite, as can also be shown from general theorems.

THE FORMULAE FOR FLUID RESISTANCE.

Let us now apply to our special case the general formula we have just found, introducing the relations between ζ and u, and h and l according to the stability conditions. For the speed u we have

$$u=\frac{\zeta}{l\sqrt{8}}$$

further,

$$h/l=0,283$$

so that we get

$$W=\rho l\left[0,283\sqrt{8}.u\ (U-2u)+\frac{4}{\pi}u^2\right]$$

If we introduce, as is ordinarily done, the resistance coefficient according to the formula

$$W=\psi_w\ \rho\ d\ U^2$$

where d is a chosen characteristic dimension of the body, to which we refer the resistance, we will obtain ψ_w expressed by the two ratios u/U and l/d in the following way

$$\text{(III)} \qquad \Psi_w=\left[0,799\frac{u}{U}-0,323\left(\frac{u}{U}\right)^2\right]\frac{l}{d}$$

We have thus obtained the resistance coefficient—which before could be observed only by resistance measurements—expressed by two quantities which can be taken directly from the flow phenomenon, viz, the ratio

$$\frac{u}{U}=\frac{\text{Velocity of the vortex system}}{\text{Velocity of the body}}$$

and

$$\frac{l}{d}=\frac{\text{Distances apart of the vortices in one row}}{\text{Reference dimension of the body}}$$

Both quantities, corresponding to the similitude of the phenomenon, within the limits of validity of the square law can depend only upon the dimension of the body.

These two quantities can be observed very easily experimentally. The ratio l/d can be taken directly from photographs, while the ratio u/U can be found easily by counting the number of vortices formed. If we designate by T the time between two identical flow states we can then introduce the quantity $l_0=UT$, which is the distance the body moves in the period T. This quantity must be independent of velocity for the same body, and the ratio l/l_0 for similar bodies must also be independent of the dimensions of the body but determined by the shape of the body. Remembering that $T=l/(U-u)$, we then find between u/U and l/l_0 the simple relation

$$\frac{u}{U}=1-\frac{l}{l_0}$$

By some provisional measurements we have proved the similitude rule and afterwards calculated the resistance coefficient for a flat plate and a cylinder disposed normal to the stream, for the purpose of seeing if the calculated values agreed with the air resistance measurements, at least in order of magnitude.

Our measurements were made first on two plates of width 1.75 and 2.70 cm. and 25 cm. length, and we have measured the period T and calculated the quantity $l_0=UT$ for two different velocities. We have used a chronograph for time measurements and the period was observed for each vortex row independently. Thus was found for the narrower plate

U=10.0 *cm/sec*	15.1 *cm/sec*
T=1.26 *sec*	0.805 *sec*
UT=12.6 *cm*	12.1 *cm*

for the wider plate

U=9.6 *cm/sec*	15.5 *cm/sec*
T=1.99 *sec.*	1.20 *sec.*
UT=19.1 *sec.*	18.6 *sec.*

Mean value UT=18.8 *cm*

The ratio of the plate width is equal to

$$\frac{2.\ 70}{1.\ 75}=1.54$$

and the ratio of the quantities $l_0=UT$ is equal to

$$\frac{18.\ 8}{12.\ 3}=1.52$$

So that the similitude rule is in any case confirmed.

A circular cylinder of 1.5 cm. diameter was also teasted at two speeds. We found the values

U=11.0 *cm/sec*	15.8 *cm/sec*
T=0.66 *sec.*	0.48 *sec.*
UT=7.3 *cm*	7.5 *cm*

Mean value UT=7.4 *cm*

Knowing the values of $l_0 = UT$ we can calculate for the plate and the cylinder the speed ratio u/U. Thus,

for the plate $u/U=0.20$.
for the cylinder $u/U=0.14$

and with the values of l indicated before we have

for the plate $l/d=5.5$
for the cylinder $l/d=4.3$

where d is the plate width or cylinder diameter. We thus find the resistance coefficients

for the plate $\psi_w=0.80$
for the cylinder $\psi_w=0.46$

The resistance measurements of Foppl[1] have given for a plate with an aspect ratio of 10:1 the resistance coefficient $\psi_w=0.72$ and the Eiffel[2] measurements, for an aspect ratio of 50:1; that is, for a nearly plane flow, the value $\psi_w=0.78$. Further, Foppl has found for a long circular cylinder $\psi_w=0.45$, so that the agreement between the calculated and measured resistance coefficients must be considered as fully satisfactory.

The theoretical investigations here developed ought to be extended and completed in two directions. First, we have limited ourselves to the uniplanar problem; that is, to the limiting case of a body of great length in the direction normal to the flow. It is to be expected that by the investigation of stable vortex configurations in space we will also be brought to a better understanding of the mechanism of fluid resistance. However, the problem is rendered difficult by the fact that the translation velocity of curved vortex filaments is not any longer independent of the size of the vortex section, because to an infinitely thin filament would correspond an infinitely great velocity. Nevertheless, it must not be considered that the extension of the theory to the case of space would bring unsurmountable difficulties.

Much more difficult appears the extension of the theory in another direction, which really would first lead to a complete understanding of the theory of fluid resistance, namely, the evaluation by pure calculation of the ratios l/d and u/U, which we have found from flow observations, and which determine the fluid resistance. This problem can not be solved without investigation of the process of vortex formation. An apparent contradiction is brought out by the fact that we have used only the theorems established for perfect fluids, which in such a fluid (frictionless fluid) no vortices can be formed. This contradiction is explained by the fact that we can everywhere neglect friction except at the surface of the body. It can be shown that the friction forces tend to zero when the friction coefficient decreases, but the vortex intensity remains finite. If we thus consider the perfect fluid as the limiting case of a viscous fluid, then the law of vortex formation must be limited by the condition that only those fluid particles can receive rotation which have been in contact with the surface of the body.

This idea appear first, in a perfectly clear way, in the Prandtl theory of fluids having small friction. The Prandtl theory investigates those phenomena which take place in a layer at the surface of the body, and the way in which the separation of the flow from the surface of the body occurs. It we could succeed in bringing into relation these investigations on the method of separation of the stream from the wall with the calculation of stable configuration of vortex films formed in any way whatever, as has been explained in the foregoing pages, then this would evidently mean great progress. Whether or not this would meet with great difficulties can not at the present time be stated.

[1] See the work of O. Foppl already mentioned.
[2] G. Eiffel, "La Resistance de l'Air et l'Aviation," p. 47, Paris, 1910.

O