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80-FM-3

JSC-16318

Simple Computations For Near-Optimum Ascent and Abort Maneuver Targets and Deorbit Ignition Time

(NASA-TM-80850) SIMPLE COMPUTATIONS FOR NEAR-OPTIMUM ASCENT AND ABORT MANEUVER TARGETS AND DEORBIT IGNITION TIME (NASA) 24 p HC A02/MF A01 CSCL 22A

N80-15185

Unclas

G3/12 45174

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Mission Planning and Analysis Division

January 1980

National Aeronautics and
Space Administration

Lyndon B. Johnson Space Center
Houston, Texas

80FM3

80-FM-3

JSC-16318

SHUTTLE PROGRAM

SIMPLE COMPUTATIONS FOR NEAR-OPTIMUM ASCENT AND ABORT MANEUVER TARGETS AND DEORBIT IGNITION TIME

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1.0 SUMMARY

This document presents closed-form solutions for a two-burn orbit insertion, AOA, and ATO maneuver targets, and for time of ignition of a one-burn deorbit. Based on the assumption that the orbits involved deviate only to the first order about a reference circular orbit, these solutions are nearly fuel-minimum. They are expressed in terms of linear terminal velocity constraint (LTVC) Shuttle guidance targets and, thus, are readily applicable to both ground and onboard software use. In addition to potential application to real-time targeting, the equations may be used as a mission design aid in preliminary definition of target loads and definition of abort mode boundaries.

2.0 INTRODUCTION

Guidance targets for the post-MECO orbit insertion, AOA, and ATO maneuvers are currently determined premission and loaded into the onboard and ground computers as constants.

A simple numerical scheme for solving the impulsive two-burn problem, assuming tangency of the second burn to the final orbit, is presented in reference 1. Using this scheme, which requires iterating until a necessary condition of optimality is satisfied, suitable guidance targets are obtained. Still lacking, however, is a scheme to determine optimum ignition time for a single deorbit burn to cover late AOA aborts.

Although the scheme presented in reference 1 satisfies the two-burn requirements, a closed-form solution requiring less computations and no iterations is presented in this document. A consequence of the derivation, which is obtained by assuming near-circular orbit transfers, implies that the second burn of the optimum two burns is a horizontal maneuver, which verifies the assumption used in reference 1. Additionally, using a similar near circular orbit transfer assumption, a closed-form solution for optimum ignition of a single deorbit burn is derived.

3.0 SYMBOLS AND ACRONYMS

AOA	abort-once-around
ATO	abort-to-orbit
μ	Earth gravitational constant
r_c	reference circular orbit radius
V_c	reference circular orbit velocity
θ	orbit transfer angle
W	cotangent of half-θ
M^*	sensitivity matrix of state deviations about a reference circular orbit
LTVC	linear terminal velocity constraint
$\delta\rho_0$	initial radial difference divided by r_c
$\delta\beta_0$	initial horizontal velocity difference divided by V_c
$\delta\alpha_0$	initial radial velocity difference divided by V_c
r_{i0}	initial orbit radius at initial location
V_{hi0}	initial orbit horizontal velocity at initial location
$\dot{r}_{i0}$	initial orbit radial velocity at initial location
r_{f0}	final orbit radius at initial location
V_{hf0}	final orbit horizontal velocity at initial location
$\dot{r}_{f0}$	final orbit radial velocity at initial location
$\delta\rho_2^+$	radial difference divided by r_c after second impulse
$\delta\beta_2^+$	horizontal velocity difference divided by V_c after second impulse
$\delta\alpha_2^+$	radial velocity difference divided by V_c after second impulse
$\delta\rho_2^-$	radial difference divided by r_c before second impulse
$\delta\beta_2^-$	horizontal velocity difference divided by V_c before second impulse
$\delta\alpha_2^-$	radial velocity difference divided by V_c before second impulse
$\Delta\beta_1$	horizontal component of first velocity impulse divided by V_c

$\Delta\alpha_1$	radial component of first velocity impulse divided by V_c
$\Delta\beta_2$	horizontal component of second velocity impulse divided by V_c
$\Delta\alpha_2$	radial component of second velocity impulse divided by V_c
ΔV	total velocity impulse divided by V_c
ΔV_1	magnitude of first maneuver impulse
ΔV_2	magnitude of second maneuver impulse
$\Delta\beta$	horizontal component of velocity impulse
$\Delta\alpha$	radial component of velocity impulse
V_h	horizontal velocity
$\dot{r}$	radial velocity
c_1	linear terminal velocity constraint target line intercept
c_2	linear terminal velocity constraint target line slope
r_{iT}	initial orbit radius at target location
V_{hiT}	initial orbit horizontal velocity at target location
$\dot{r}_{iT}$	initial orbit radial velocity at target location
r_{fT}	final orbit radius at target location
V_{hfT}	final orbit horizontal velocity at target location
$\dot{r}_{fT}$	final orbit radial velocity at target location
$\delta\rho_T$	radial difference at target location divided by r_c
$\delta\beta_T$	horizontal velocity difference at target location divided by V_c
$\delta\alpha_T$	radial velocity difference at target location divided by V_c

4.0 NEAR-OPTIMUM TWO-BURN SOLUTION

To derive the closed-form solution for the near-optimum two-burn targets, the differences between the final and initial orbits are considered as first-order deviations about an arbitrary reference circular orbit. This requires that the initial states of both orbits be located at the first-burn ignition point, which is fixed in this application. Because the terminal orbits of nominal and ATO trajectories are constrained only by desired apsis radius, the final orbit state at the first-burn ignition may be determined directly. However, because the terminal constraints for an AOA are desired radius, velocity and flightpath angle (functions of velocity versus flightpath angle target relationship) at a desired entry angle, these elements must be first mapped from entry interface to the first-burn ignition point.

The approximate state sensitivity matrix that relates state perturbations of radius, horizontal and radial velocity after transfer through a central angle to initial state perturbations of the same elements about a reference circular orbit is given as follows (see appendix A):

$$\overset{*}{M} = \begin{bmatrix} \dfrac{W^2+3}{W^2+1} & \dfrac{4}{W^2+1} & \dfrac{2W}{W^2+1} \\[2ex] \dfrac{-2}{W^2+1} & \dfrac{W^2-3}{W^2+1} & \dfrac{-2W}{W^2+1} \\[2ex] \dfrac{2W}{W^2+1} & \dfrac{4W}{W^2+1} & \dfrac{W^2-1}{W^2+1} \end{bmatrix} \tag{1}$$

where W is the cotangent of half of the transfer angle

Hence,

$$(\delta\rho,\ \delta\beta,\ \delta\alpha)^T = \overset{*}{M}(\delta\rho_0,\ \delta\beta_0,\ \delta\alpha_0)^T \tag{2}$$

By defining $(\delta\rho_0,\ \delta\beta_0,\ \delta\alpha_0)^T$ as the difference in the state between the terminal and initial orbits at the first impulse, then

$$(\delta\rho_2^+,\ \delta\beta_2^+,\ \delta\alpha_2^+)^T = \overset{*}{M}(\delta\rho_0,\ \delta\beta_0,\ \delta\alpha_0)^T \tag{3}$$

where $\delta\rho_0 = (r_{f0} - r_{i0})/r_c$

$\delta\beta_0 = (V_{hf0} - V_{hi0})/V_c$

$\delta\alpha_0 = (\dot{r}_{f0} - \dot{r}_{i0})/V_c$

After the first impluse is applied, the state differences at the second impulse location as a result of the first impulse is

$$(\delta\rho_2^-, \delta\rho_2^-, \delta\alpha_2^-)^T = \overset{*}{M}(0, \Delta\beta_1, \Delta\alpha_1)^T \tag{4}$$

By definition, at the second impulse

$$\begin{aligned} \delta\rho_2^- &= \delta\rho_2^+ \\ \Delta\beta_2 &= \delta\beta_2^+ - \delta\beta_2^- \\ \Delta\alpha_2 &= \delta\alpha_2^+ - \delta\alpha_2^- \end{aligned} \tag{5}$$

Combining equations (3), (4), and (5) leads to the following:

$$(0, \Delta\beta_2, \Delta\alpha_2)^T = \overset{*}{M}(\delta\rho_0, \delta\beta_0 - \Delta\beta_1, \Delta\alpha_0 - \Delta\alpha_1)^T \tag{6}$$

By defining $\tan\phi_2 = \Delta\alpha_2/\Delta\beta_2$, and thus $\Delta\alpha_2 = \Delta\beta_2 \tan\phi_2$, and solving equation (6) simultaneously for $\Delta\beta_2$, $\Delta\beta_1$, and $\Delta\alpha_1$, the following is obtained.

$$\begin{aligned} \Delta\beta_2 &= \delta\rho_0(W^2 + 1)/2(2 - W\tan\phi_2) \\ \Delta\beta_1 &= -\Delta\beta_2 + \delta\beta_0 + \delta\rho_0 \\ \Delta\alpha_1 &= \Delta\beta_2 \tan\phi_2 + W\delta\rho_0 + \delta\alpha_0 \end{aligned} \tag{7}$$

The minimum of the sum of the two impulses must be found. This is functionally expressed as

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{8}$$

Where $\Delta V_1 = (\Delta\beta_1^2 + \Delta\alpha_1^2)^{\frac{1}{2}}$ (9)

$$\Delta V_2 = (\Delta\beta_2^2 + \Delta\alpha_2^2)^{\frac{1}{2}} \tag{10}$$

Differentiating equation (8) with respect to $\tan\phi_2$ and W yields:

$$\frac{d\Delta V}{d\tan\phi_2} = \frac{\Delta\beta_2}{(2 - W\tan\phi_2)}\left[\left(\frac{\Delta\beta_2}{\Delta V_2} - \frac{\Delta\beta_1}{\Delta V_1}\right)W + 2\left(\frac{\Delta\alpha_1}{\Delta V_1} + \frac{\Delta\alpha_2}{\Delta V_2}\right)\right] \tag{11}$$

$$\frac{d\Delta V}{dW} = \frac{\Delta\alpha_2 + W\delta\rho_0}{(2 - W\tan\phi_2)}\left[\frac{\Delta\beta_2}{\Delta V_2} - \frac{\Delta\beta_1}{\Delta V_1} + \tan\phi_2\left(\frac{\Delta\alpha_1}{\Delta V_1} + \frac{\Delta\alpha_2}{\Delta V_2}\right)\right]$$

$$+ \delta\rho_0\frac{\Delta\alpha_1}{\Delta V_1} \tag{12}$$

Equation (8) is minimum when equations (11) and (12) are simultaneously zero. By performing the indicated operation, the following necessary condition of optimality is obtained.

$$\Delta\alpha_2\left(\frac{\Delta\alpha_1}{\Delta V_1} - \frac{\delta\alpha_0}{\Delta V}\right) = 0 \tag{13}$$

Except for when $\frac{\Delta\alpha_1}{\Delta V_1} = \frac{\delta\alpha_0}{\Delta V}$, the second maneuver is horizontal (i.e., $\Delta\alpha_2 = 0$). Results presented in reference 1 indicate that a tangent (near-horizontal) second maneuver is very nearly optimal.

Substituting this result $(\Delta\alpha_2 = 0)$ into equations (7) and (11) yields the following:

$$\begin{aligned}\Delta\beta_2 &= \delta\rho_0\,(W^2 + 1)/4\\ \Delta\beta_1 &= \delta\beta_0 + \delta\rho_0\,(3 - W^2)/4\\ \Delta\alpha_1 &= \delta\alpha_0 + W\,\delta\rho_0\end{aligned} \tag{14}$$

and

$$\Delta\beta_1\,W - 2\,\Delta\alpha_1 = \Delta V_1\,\mathrm{sgn}\,(\delta\rho_0)\,W \tag{15}$$

Squaring both sides of equation (15) and factoring yields the following:

$$\Delta\alpha_1 \left(\Delta\alpha_1 W^2 + 4\Delta\beta_1 W - 4\Delta\alpha_1\right) = 0 \tag{16}$$

Either $\Delta\alpha_1 = 0$, or $\left(\Delta\alpha_1 W^2 + 4\Delta\beta_1 W - 4\Delta\alpha_1\right) = 0$

If $\Delta\alpha_1 = 0$, then the two-impulse transfer trajectory is near tangent at both impulses. This, of course, occurs for the special case in which a gravity turn maneuver is optimal. For this case,

$$W = -\delta\alpha_0/\delta\rho_0 \tag{17}$$

(Note: Equation (17) is an approximate form of the exact equation derived in appendix B for a gravity turn maneuver to achieve a desired apsis magnitude).

If $\Delta\alpha_1 W^2 + 4\Delta\beta_1 W - 4\Delta\alpha_1 = 0$, then the solution for W is

$$W = 8\delta\alpha_0 / \left((4\delta\beta_0 - \delta\rho_0) - \mathrm{sgn}(\delta\rho_0) \left((4\delta\beta_0 - \delta\rho_0)^2 + 16\delta\alpha_0^2\right)^{\frac{1}{2}}\right) \tag{18}$$

that was originally derived in reference 2.

Hence, the location of the second impulse from the first impulse is given as

$$\theta = \pi + \tan^{-1}\left((2W/(W^2 - 1)\right)$$

To obtain the near-optimum LTVC guidance targets for the first maneuver, the state of the final orbit at the first ignition location is extrapolated through the angle θ using Kepler equations of elliptic motion (e.g., equations (A-1), (A-2), and (A-3) of appendix A). The first-burn target radius is defined as the final orbit radius at the second-burn location. The target velocity line, defined as

$$\dot{r} = c_1 + c_2 V_h$$

is chosen to ensure optimality of the resulting solution by defining c_1 (target line intercept) as the final orbit radial velocity at the second-impulse location and c_2 (target line slope) as zero.

The second-burn ignition time is obtained by first using the LTVC powered flight guidance (ref. 3) to determine the finite burn effects of the first maneuver. A resulting transfer angle from the predicted cutoff state to the target is then determined. Using an appropriate set of Kepler equations for time of flight such as presented in reference 4, the second-burn ignition time can be defined.

5.0 NEAR-OPTIMAL IGNITION FOR A SINGLE-BURN DEORBIT

For the analysis of the minimum-impulse solution to the single-impulse deorbit problem, first consider the difference between the terminal orbit and the initial orbit at the entry interface location.

$$\delta\rho_T = (r_{fT} - r_{iT})/r_c \tag{19}$$

$$\delta\beta_T = (V_{hfT} - V_{hiT})/V_c \tag{20}$$

$$\delta\alpha_T = (\dot{r}_{fT} - \dot{r}_{iT})/V_c \tag{21}$$

For the deorbit problem, the linear terminal velocity constraint is applied to express $\dot{r}_T$ as a linear function of V_{hT} as follows:

$$\dot{r}_{fT} = c_1 + c_2 V_{hfT} \tag{22}$$

Thus, equation (21) becomes a linear function of equation (20) as follows:

$$\delta\alpha_T = K + c_2\delta\beta_T \tag{23}$$

where $K = (c_1 + c_2 V_{hiT} - \dot{r}_{iT})/V_c$

The minimum impulse on the initial orbit that satisfies the terminal orbit constraints of $\delta\rho_T$, $\delta\beta_T$, $\delta\alpha_T$, must be located. They are considered as state perturbations about a reference circular orbit. Using the approximate state sensitivity matrix that relates final state perturbations of radius, horizontal, and radial velocity to initial state perturbations of the same elements at a central angle θ about a reference circular orbit, the following is obtained:

$$(\delta\rho_T,\ \delta\beta_T,\ K + c_2\delta\beta_T)^T = \overset{*}{M}(0,\ \Delta\beta,\ \Delta\alpha)^T \tag{24}$$

where $\Delta\beta$, $\Delta\alpha$ are the impulsive velocity components.

Solving equation (24) for $\delta\beta_T$, $\Delta\beta$, $\Delta\alpha$ yields the following equations:

$$\delta\beta_T = \left[-2WK + (W^2 + 3)\delta\rho_T\right] /2(c_2W - 2) \tag{25}$$

$$\Delta\beta = \delta\beta_T + \delta\rho_T \tag{26}$$

$$\Delta\alpha = - K - c_2\delta\beta_T + W\delta\rho_T \tag{27}$$

The impulse is defi ,d as follows:

$$\Delta V = (\Delta\beta^2 + \Delta\alpha^2)^{\frac{1}{2}} \tag{28}$$

The derivative of ΔV with respect to W requires the derivative of $\delta\beta_T$ with respect to W, which is obtained by differentiating equation (25) as follows:

$$\frac{d\delta\beta_T}{dW} = \frac{\Delta\alpha}{c_2W - 2} \tag{29}$$

Then,

$$\frac{d\Delta V}{dW} = \frac{\Delta a}{C_2W - 2} \left[\delta\beta_T(1 + C_2^2) + C_2K - \delta\rho_T\right] /\Delta V \tag{30}$$

Therefore, for ΔV to be minimum, $\Delta\alpha$ must be zero.

After substituting this result into equation (24), the following equations are obtained.

$$\delta\rho_T = \frac{4}{W^2 + 1} \Delta\beta \tag{31}$$

$$\delta\beta_T = \frac{W^2 - 3}{W^2 + 1} \Delta\beta \tag{32}$$

$$K + c_2\delta\beta_T = \frac{4W}{W^2 + 1} \Delta\beta \tag{33}$$

After combining equations (31), (32), and (33), a quadratic equation in W is obtained as follows:

$$\frac{1}{2} c_2 W^2 - 2W + 2K/\delta\rho_T - \frac{3}{2}c_2 = 0 \tag{34}$$

Solving this quadratic equation yields the following:

$$W = 2 \left\{ 1 - \left[1 - c_2 \left(K/\delta\rho_T - \frac{3}{4}c_2 \right) \right]^{\frac{1}{2}} \right\} / c_2 \tag{35}$$

Hence, the approximate location of the minimum impulse from entry interface is given by

$$\theta = \pi + \tan^{-1} \left[2W / (W^2 - 1) \right] \tag{36}$$

and the approximate ΔV is given by

$$\Delta V = V_c \left| \delta\rho_T (W^2 + 1)/4 \right| \tag{37}$$

The deorbit maneuver ignition time is obtained by first determining the range angle from the current vehicle state to the maneuver location. It is given as

$$\theta_{tig} = \theta_T - \theta$$

where θ_T is the range angle from the current state to the deorbit maneuver target location. Then, using the current state and the resulting angle as inputs to the Kepler equations for time of flight, the time of ignition may be determined.

6.0 CONCLUDING REMARKS

By expressing the initial conditions of the initial and final orbits at a common location and assuming that their state differences are deviations about a reference circular orbit, simple near-optimum solutions for the guidance targets for ascent and abort maneuvers are obtained. This simplicity, in addition to the current availability of analytic J_2 compensation in the on-board program, make these solutions attractive for targeting and may aid in mission design. The way that the analytic J_2 compensation scheme can be used to bias these targets so that the vehicle can fly through the actual targets will be the subject of a forthcoming memorandum.

7.0 REFERENCES

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APPENDIX A

SENSITIVITY MATRIX FOR STATE PERTURBATIONS ABOUT A CIRCULAR ORBIT

The sensitivity matrix derived herein for extrapolating state deviations about a circular orbit has been widely used in orbit perturbation analyses. It is presented here as an easy reference for the applications of this document.

The approach taken is to first express the basic Kepler equations for elliptical orbits in terms of radius, horizontal, and radial velocities, and cotangent of half of the transfer angle; second, differentiate them with respect to the initial conditions; and finally, apply circular orbit conditions to the derivatives to obtain the desired result.

The basic Kepler equations for elliptical orbits may be expressed as follows:

$$V_h = \left[(W^2 - 1)\, V_{ho} - 2W\dot{r}_o + 2\mu/r_o\, V_{ho}\right] /(W^2 + 1) \tag{A-1}$$

$$\dot{r} = \left[2W(V_{ho} - \mu/r_o\, V_{ho}) + (W^2 - 1)\, \dot{r}_o\right] /(W^2 + 1) \tag{A-2}$$

$$r = r_o V_{ho}/V_h \tag{A-3}$$

where r_o, V_{ho}, $\dot{r}_o$ are the initial radius and horizontal and radial velocities, respectively, and W is the cotangent of one-half of the transfer angle.

Let r_c be an arbitrary reference circular orbit radius. Then $V_c = (\mu/r_c)^{\frac{1}{2}}$ is the reference circular orbit velocity. Define as follows:

$$\begin{aligned} \beta &= V_h/V_c \\ \alpha &= \dot{r}/V_c \\ \rho &= r/r_c \end{aligned} \tag{A-4}$$

Equations (A-1), (A-2), and (A-3) then become

$$\beta = \left[(W^2 - 1)\beta_o - 2W\alpha_o + 2/\rho_o\beta_o\right] /(W^2 + 1) \tag{A-5}$$

$$\alpha = \left[2W\,(\beta_o - 1/\rho_o\beta_o) + (W^2 - 1)\, \alpha_o\right]/(W^2 + 1) \tag{A-6}$$

$$\rho = \rho_o\beta_o\, /\beta \tag{A-7}$$

In order to determine the sensitivity of β, α, ρ to β_o, α_o, ρ_o, the derivatives of equations (A-5), (A-6), and (A-7) are required. Then, deviations in β, α, ρ can be functionally related to deviations in β_o, α_o, ρ_o as follows:

$$\begin{pmatrix} \delta\rho \\ \delta\beta \\ \delta\alpha \end{pmatrix} = \begin{bmatrix} \frac{d\rho}{d\rho_o} & \frac{d\rho}{d\beta_o} & \frac{d\rho}{d\alpha_o} \\ \frac{d\beta}{d\rho_o} & \frac{d\beta}{d\beta_o} & \frac{d\beta}{d\alpha_o} \\ \frac{d\alpha}{d\rho_o} & \frac{d\alpha}{d\beta_o} & \frac{d\alpha}{d\alpha_o} \end{bmatrix} \begin{pmatrix} \delta\rho_o \\ \delta\beta_o \\ \delta\alpha_o \end{pmatrix} \tag{A-8}$$

The required derivatives are expressed as follows:

$$\frac{d\beta}{d\rho_o} = \frac{-2/\left(\rho_o^2\beta_o\right)}{W^2 + 1} \tag{A-9}$$

$$\frac{d\beta}{d\beta_o} = \frac{W^2 - 1 - 2/\left(\rho_o\beta_o^2\right)}{W^2 + 1} \tag{A-10}$$

$$\frac{d\beta}{d\alpha_o} = \frac{2W}{W^2 + 1} \tag{A-11}$$

$$\frac{d\alpha}{d\rho_o} = \frac{2W/\left(\rho_o^2\beta_o\right)}{W^2 + 1} \tag{A-12}$$

$$\frac{d\alpha}{d\beta_o} = \frac{2W\left(1 + 1/\left(\rho_o\beta_o^2\right)\right)}{W^2 + 1} \tag{A-13}$$

$$\frac{d\alpha}{d\alpha_o} = \frac{W^2 - 1}{W^2 + 1} \tag{A-14}$$

$$\frac{d\rho}{d\rho_o} = \left(1 - \frac{d\beta}{d\rho_o}\,\rho_o/\beta\right)\beta_o/\beta \tag{A-15}$$

$$\frac{d\rho}{d\beta_o} = \left(1 - \frac{d\beta}{d\beta_o}\,\beta_o/\beta\right)\rho_o/\beta \tag{A-16}$$

$$\frac{d\rho}{d\alpha_o} = \frac{2W\,\rho_o\,\beta_o/\beta^2}{W^2 + 1} \tag{A-17}$$

If the initial conditions of equations (A-1), (A-2), and (A-3); i.e., r_o, V_{ho}, and $\dot{r}_o$, represent a circular orbit and r_c is arbitrarily defined as r_o, then

$$\rho_o = \rho = 1 \tag{A-18}$$

$$\beta_o = \rho_o^{-\frac{1}{2}} = \beta = 1 \tag{A-19}$$

$$\alpha_o = \alpha = 0 \tag{A-20}$$

The required derivatives then become

$$\frac{d\beta}{d\rho_o} = -\frac{2}{W^2 + 1} \tag{A-21}$$

$$\frac{d\beta}{d\beta_o} = \frac{W^2 - 3}{W^2 + 1} \tag{A-22}$$

$$\frac{d\beta}{d\alpha_o} = -\frac{2\,W}{W^2 + 1} \tag{A-23}$$

$$\frac{d\alpha}{d\rho_o} = \frac{2\,W}{W^2 + 1} \tag{A-24}$$

$$\frac{d\alpha}{d\beta_o} = \frac{4\ W}{W^2 + 1} \tag{A-25}$$

$$\frac{d\alpha}{d\alpha_o} = \frac{W^2 - 1}{W^2 + 1} \tag{A-26}$$

$$\frac{d\rho}{d\rho_o} = \frac{W^2 + 3}{W^2 + 1} \tag{A-27}$$

$$\frac{d\rho}{d\beta_o} = \frac{4}{W^2 + 1} \tag{A-28}$$

$$\frac{d\rho}{d\alpha_o} = \frac{2W}{W^2 + 1} \tag{A-29}$$

Define as $\overset{*}{M}$ the matrix in equation (A-8), Then

$$\overset{*}{M} = \begin{bmatrix} \frac{W^2 + 3}{W^2 + 1} & \frac{4}{W^2 + 1} & \frac{2W}{W^2 + 1} \\ -\frac{2}{W^2 + 1} & \frac{W^2 - 3}{W^2 + 1} & -\frac{2W}{W^2 + 1} \\ \frac{2W}{W^2 + 1} & \frac{4W}{W^2 + 1} & \frac{W^2 - 1}{W^2 + 1} \end{bmatrix} \tag{A-30}$$

$\overset{*}{M}$ is a nonsingular matrix whose inverse is as follows:

$$\overset{*}{M}^{-1} = \begin{bmatrix} \dfrac{W^2+3}{W^2+1} & \dfrac{4}{W^2+1} & -\dfrac{2W}{W^2+1} \\ -\dfrac{2}{W^2+1} & \dfrac{W^2-3}{W^2+1} & \dfrac{2W}{W^2+1} \\ -\dfrac{2W}{W^2+1} & -\dfrac{4W}{W^2+1} & \dfrac{W^2-1}{W^2+1} \end{bmatrix} \tag{A-31}$$

Therefore, since

$$(\delta\rho,\ \delta\beta,\ \delta\alpha)^T = \overset{*}{M}(\delta\rho_o,\ \delta\beta_o,\ \delta\alpha_o)^T$$

then

$$(\delta\rho_o,\ \delta\beta_o,\ \delta\alpha_o)^T = \overset{*}{M}^{-1}(\delta\rho,\ \delta\beta,\ \delta\alpha)^T$$

APPENDIX B

TRANSFER ANGLE SOLUTION FOR A MINIMUM IMPULSE TO A DESIRED APSIS RADIUS

A solution, originally derived by Tim Brand, CSDL, for the transfer angle from impulse to target for a minimum-impulse trajectory to achieve a desired apsis radius is developed in this appendix. Application of optimal control theory to this problem shows that the minimum impulse control vector is parallel to the velocity vector at the impulse. Hence, flightpath angle γ is unchanged by the impulse. In terms of horizontal and radial velocity components after the impulse

$$\begin{aligned} V_h^+ &= V^+ \cos \gamma^+ = V^+ \cos \gamma \\ \dot{r}^+ &= V^+ \sin \gamma^+ = V_h^+ \tan \gamma \end{aligned} \tag{B-1}$$

where the superscript + indicates postimpulse elements.

The basic transfer equation from impulse to target relates the postimpulse horizontal velocity to the transfer angle Θ and the target radius r_T as follows:

$$V_h^+ \left(- \frac{1}{2} (W^2 + 1) \, \delta\rho + 1 + \tan \gamma \, W \right) = \frac{\mu}{r V_h^+} \tag{B-2}$$

where $W = \cot \frac{\Theta}{2}$

and $\delta\rho = 1 - r/r_T$

Likewise, the velocity at the target V_T is expressed as

$$V_T \quad \frac{1}{2} (W^2 + 1) \, \delta\rho + 1 = \frac{\mu}{r_T \, V_T} \tag{B-3}$$

Since $r V_h^+ = r_T \, V_T$, $\dfrac{V_T}{V_h^+} = \dfrac{r}{r_T}$ (B-4)

Substituting this result into equations (B-2) and (B-3) and solving for W yields

$$W = \tan \gamma / \delta\rho \tag{B-5}$$

from which

$$\Theta = \pi + \tan^{-1}\left[2W/(W^2 - 1)\right] \qquad \text{(B-6)}$$