NONLINEAR TEMPERATURE DEPENDENT FAILURE ANALYSIS OF FINITE WIDTH COMPOSITE LAMINATES

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**Abstract:** See page iv
FORWARD

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NONLINEAR TEMPERATURE DEPENDENT FAILURE ANALYSIS OF FINITE WIDTH COMPOSITE LAMINATES

ABSTRACT

A quasi-three dimensional, nonlinear elastic finite element stress analysis of finite width composite laminates including curing stresses is presented.

Cross-ply, angle-ply, and two quasi-isotropic graphite/epoxy laminates are studied. Curing stresses are calculated using temperature dependent elastic properties that are input as percent retention curves, and stresses due to mechanical loading in the form of an axial strain are calculated using tangent modulii obtained by Ramberg-Osgood parameters. It is shown that curing stresses and stresses due to tensile loading are significant as edge effects in all types of laminate studies.

The tensor polynomial failure criterion is used to predict the initiation of failure. The mode of failure is predicted by examining individual stress contributions to the tensor polynomial. Failure is predicted to always initiate at the free edge, but not always at ply interfaces. The location and mode of failure is shown to be laminate dependent.
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1. INTRODUCTION

The fabrication of laminated fibrous composites involves curing the fiber-matrix system at an elevated temperature. Cure temperatures for resin matrix systems vary from 350°F for epoxies to 650°F for polyimides. The mismatch of thermal expansion coefficients between fiber and matrix (or, alternatively, the orthotropic properties of the lamina) coupled with the large temperature drop from the maximum cure temperature can result in relatively high residual curing stresses in laminates at room temperature. These thermal stresses are often large enough to cause transverse microcracking or ply delamination prior to the application of mechanical load. Residual thermal stresses are also present in metal matrix composites such as boron-aluminum, in which their effect is manifested as yielding of the matrix material.

The development of residual stresses in composites does not have a direct counterpart in homogeneous, isotropic media and there are relatively few studies of the subject reported in the literature. All the studies are based upon the assumption that the total strain is the sum of two distinct parts: the mechanical strain which is related to the stresses through the constitutive equation, and the "free" thermal strain which, in itself, does not cause stress in the laminate.

The purpose of this study is to analyze the thermo-mechanical response of resin matrix composites and to predict the occurrence and mode of first failure in finite width laminates. Previous researchers have proposed various methods for calculating residual stresses, and there have been a few studies of stress-strain response to mechanical
load which included residual stresses. Most previous studies typically perform the residual thermal analysis and the mechanical load analysis separately assuming linear elastic behavior. The principal of superposition is used to predict the combined effect of mechanical load and curing stresses. The present study treats the thermal and mechanical behavior separately, but does not make the assumption of linear elasticity. The residual stress field therefore cannot be superposed on the mechanical load, but is used as an initial condition. Special attention is given to the influence of edge effects on the stress field and the occurrence of first failure.

The finite element program NONCOM2 [1,2,3] was modified for this analysis. The efficiency and capability of the program were increased so as to handle a larger number of nodes (with a choice for an in-core or an out-of-core equation solver) and a detailed failure analysis using the tensor polynomial failure criterion for predicting first failure. The modified program is designated NONCOM3. Results were obtained for cross-ply, angle-ply and quasi-isotropic laminates of T300/5208 graphite epoxy.
2. LITERATURE REVIEW

Most previous theoretical studies are lamination theory solutions. They are based upon the classical plate theory assumptions and, therefore, valid only in interior regions away from the free edges of finite width laminates; they yield only laminate stresses. However, failure in laminates is often observed to initiate at the free edges [4,5] and, therefore, the stress distribution there is of paramount interest. The importance of edge effect during thermal loading of graphite-polyimide was clearly demonstrated by Herakovich, Mills and Davis [5].

Tsai [6] presented a thermoelastic formulation for calculating laminate thermal stresses in 1965. This study presents the basic lamination theory development for calculating residual stresses. A micromechanical procedure for calculating residual thermal stresses was outlined by Hashin [7]. One of the earlier reported analytical predictions of residual stresses using lamination theory is a study by Chamis [8], in which he analyzed laminates of different material systems, stacking sequences and fiber volume fractions. Extensive experimental studies were conducted at the IIT Research Center by Daniel and Liber [9]. They reported thermal stresses based upon measured strains and temperature dependent constitutive relations.

Herakovich [10] was apparently the first to consider thermal edge effects in composites. He analyzed laminates of boron-epoxy and aluminum using finite elements, and considered stress distributions due to thermal and mechanical loads. The analysis included interlaminar
stresses but was linear elastic with constant room temperature properties. A nonlinear elastic finite element analysis, which included thermal effects and temperature dependent properties was conducted by Renieri and Herakovich [1], but residual stress predictions formed only a limited part of the study and the finite element mesh used was quite coarse. Their basic formulation will be used in the present analysis with a much finer mesh, an improved equation solver and a failure criterion added to the analysis capability.

Hahn and Pagano [11] pointed out the necessity for the inclusion of terms corresponding to the stress and temperature dependence of properties. They developed a 'total strain' theory, in which the strains and stresses are calculated using temperature dependent elastic properties at the temperature of interest.

Daniel, Liber and Chamis [12] developed a technique to measure residual strain by embedding strain gages between plies in laminates. They used this technique for measuring curing strains in boron/epoxy and S glass/epoxy, and calculated stresses using temperature dependent constitutive relations. Thermal cycling suggested that residual stresses during the curing process were primarily due to thermal mismatch between adjacent plies.

Chamis and Sullivan [13] outlined a procedure for nonlinear analysis of laminates with residual thermal stresses. The laminate was loaded in increments, using stresses calculated in the most recent load step to calculate elastic moduli for the next load step. Micromechanics was used to predict lamina properties, which were used in the lamination
theory analysis.

Hahn [14] concluded that the stress free temperature in laminates is lower than the cure temperature. The method outlined in [11] was used to calculate residual strains which were compared to experimentally determined strains. Daniel and Liber [15] investigated the effect of stacking sequence on residual stresses in graphite/polyimide laminates. The strains were determined experimentally, and the stresses, calculated using constitutive relations, were found to be close to the transverse strength.

Wang and Crossman [16] studied edge effects due to thermal loading on some specific laminates. They predict a peculiar behavior for a [±45]s laminate, with the existence of 'stiff' tensile and 'soft' compressive zones in the laminate.

A report by Chamis [17] summarized work done at the NASA Lewis Research Center on angle ply laminates over a period of eight years. The effect of curing stresses on laminate warpage and fracture was studied experimentally and analytically using lamination theory.

Pagano and Hahn [18] used the procedure described in [11] to calculate residual thermal stresses, and studied their effect on failure in laminates. The curing stresses were found to influence first failure in laminates greatly, often reducing the applied load to failure by about half. They note that the interlaminar normal stress $\sigma_z$ is significant in some stacking sequences, especially at free edges, and that this would result in failure initiating at loads much less than their calculated values. Their analysis is based on lamination theory and thus
does not treat the free edge problem in any detail.

Farley and Herakovich [19], using a finite element analysis, compared boundary layer stress distributions due to mechanical, thermal, and moisture loads in finite width laminates. Each type of load was analyzed separately; the study concentrated on the response of laminates to different moisture gradients in the boundary layer.

Kim and Hahn [20] published results of acoustic emissions of laminates subject to mechanical loads. Curing stresses were included in the lamination theory development for predicting stress at which first failure occurred.
3. THEORETICAL BACKGROUND

The problem under consideration is the stress analysis of symmetric laminates, including thermal and free edge effects. In this study the nonlinear analysis for both mechanical and thermal loading is performed using an incremental procedure. The loading is approximated by a finite number of load steps and each step is treated as a linear problem. The applied load, whether mechanical or thermal, is assumed to be steady and uniform across the laminate.

3.1 General Formulation

A typical balanced, symmetric laminate is shown in Fig. 1. The behavior of the laminate can be assumed to be independent of the x coordinate if b and H are small compared to L. As shown by Pipes and Pagano [21] the linear strain displacement relations can be integrated and manipulated to yield the following displacement field over the cross-section of the laminate.

\[ u = -(C_1 z + C_2)y + (C_4 y + C_5 z + C_6)x + U(y,z) \]

\[ v = (C_1 z + C_2)x - C_4 \frac{x^2}{2} + V(y,z) \]  (3.1)

\[ w = -C_7 xy + C_4 y - C_5 \frac{x^2}{2} + C_8 + W(y,z) \]

The displacement field has the following symmetry: with respect to the x-y plane,
FIGURE 1. TYPICAL LAMINATE GEOMETRY
u(x,y,z) = u(x,y,-z)
\quad \text{(3.2a)}
\begin{align*}
v(x,y,z) &= v(x,y,-z) \\
w(x,y,z) &= -w(x,y,-z)
\end{align*}

with respect to the x-z plane,
\begin{align*}
v(x,y,z) &= -v(x,-y,z) \\
w(x,y,z) &= w(x,-y,z)
\end{align*}

It has been experimentally observed \cite{22} that at z=±H
\begin{align*}
u(x,y,±H) &= -u(x,-y,±H) \quad \text{(3.3)}
\end{align*}

As the thickness of the laminate is small, it can be assumed that
\begin{align*}
u(x,y,z) &= -u(x,-y,z) \quad \text{(3.4)}
\end{align*}

These symmetries simplify the displacement field to
\begin{align*}
u &= C_6 x + U(y,z) \\
v &= V(y,z) \\
w &= W(y,z)
\end{align*}

The analysis can now be restricted to one quarter of the cross-section (Fig. 2) with the following boundary displacement constraints:
\begin{align*}
V(0,z) &= 0 \\
W(y,0) &= 0
\end{align*}

The following stress boundary conditions complete the boundary value problem.
FIGURE 2. BOUNDARY CONDITIONS ON THE QUARTER SECTION OF THE LAMINATE
\[
\tau_{zx}(x,y,\pm H) = \tau_{zy}(x,y,\pm H) = \sigma_z(x,y,\pm H) = 0
\]

\[
\tau_{xy}(x,\pm b,z) = \tau_{zy}(x,\pm b,z) = \sigma_y(x,\pm b,z) = 0
\]

(3.7)

The individual laminae are orthotropic, having a stress strain relation with 9 independent constants. When referred to the laminate axis, the stress strain relation transforms to (Appendix A)

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
\]

(3.8)

3.1.1 Finite Element Formulation

This boundary value problem is cast in the finite element framework. The cross section is subdivided into triangular elements, and the displacement field is assumed to vary linearly within each element. The elemental displacement field is represented in terms of the nodal coordinates and the nodal displacements. The total potential energy, consisting of the strain energy and the potential of external forces, is written for each element in terms of the nodal displacements and forces. The potential energy is then minimized with respect to the nodal displacements to obtain a linear set of equations relating nodal displacements to nodal forces through the element "stiffness matrix".
These elemental stiffness matrices are assembled to form a system of equations in the unknown nodal displacements. The system of equations is solved after imposition of boundary conditions. The strains and stresses in each element are calculated from the displacements of the element nodes, the strain displacement relations and the constitutive equations.

3.2 Mechanical Loading

Let the laminate in Fig. 1 be loaded with a uniform strain $\xi$ in the x direction. The displacement field over an element at a cross section $x=x_1$ becomes

$$w = a_1 + a_2y + a_3z + \xi x_1$$
$$v = a_4 + a_5y + a_6z$$
$$u = a_7 + a_8y + a_9z$$

When the parameters $a_1$-$a_9$ are functions of the nodal coordinates and displacements. As the laminate behavior is independent of the x coordinate, $x_1$ is arbitrary. Because the displacement field is assumed to vary linearly over each element and the strain displacement relations are linear, the strains over each element are constant. The elemental strains can be written in the form:
where $A_k$ is the area of the element, $u_i, v_i, w_i$, ($i = 1, 2, 3$) are the $u,v,$ and $w$ displacements of the nodes 1, 2 and 3, respectively, and $a, b, c, d, e, g$ are known constants involving nodal coordinates.

The potential energy of the element is then expressed in terms of the nodal displacements and forces. Minimization with respect to the nodal displacements yields the following set of equations:

$$
\begin{align*}
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{pmatrix}_{k} &= \frac{1}{A_k} \begin{pmatrix}
av_1 + cv_2 + ev_3 \\
bw_1 + dw_2 + gw_3 \\
bv_1 + dv_2 + gw_3 + aw_1 + cw_2 + ew_3 \\
bu_1 + du_2 + gu_3 \\
au_1 + cu_2 + eu_3
\end{pmatrix}_{k} \\
\end{align*}
$$

(3.10)
where $[K]$ is the 9x9 element stiffness given in Appendix B.

The stiffness matrices of all the elements are superposed to obtain the global stiffness matrix. Boundary conditions are imposed as follows (Fig. 2):

Displacement Boundary Conditions:

$V=0$ along $y=0$ and $W=0$ along $z=0$

This is achieved by constraining all the nodes on the line $y=0$ against displacement in the $y$ direction, and those on $z=0$ against displacement in the $z$ direction. Due to the assumed linear variation of the displacement field, constraining two adjacent nodes also constrains the line joining them.

Traction Boundary Conditions:

$T_1=0$ on $z=H$ and $y=b$

The traction boundary conditions are imposed by applying statically equivalent nodal forces. The surfaces at $z=H$ and $y=b$ are stress free and equivalent nodal forces are therefore zero.

The reader is referred to [1] for a more detailed discussion of the finite element formulation.

3.3 Thermal Loading

The basic assumption in the thermal formulation is that the total strain can be written as a sum of a stress related mechanical strain and a free thermal strain.

The displacement field over the element has the same form as (3.5) but the uniform strain $\xi$, instead of being known, is treated as an
additional unknown that is common to all the elements. It is equivalent
to the total laminate strain during the thermal loading.

The mechanical strain $\{\varepsilon\}^0$ is used to calculate the strain energy
of the element.

$$\{\varepsilon\}^0 = \{\varepsilon\}^o - \{\varepsilon\}^T \quad (3.12)$$

where $\{\varepsilon\}^o$ is the total strain, and $\{\varepsilon\}^T$ the thermal strain. In terms
of the displacement field and coefficients of expansion, the mechanical strain in the $k^{th}$ layer is:

$$
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{pmatrix}_k =
\begin{pmatrix}
\xi - (m^2 \alpha_1 + n^2 \alpha_2) \Delta T \\
(ax_1 + cv_2 + ev_3)/A_k - (n^2 \alpha_1 + m^2 \alpha_2) \Delta T \\
(bw_1 + dw_2 + gw_3)/A_k - \alpha_3 \Delta T \\
(bv_1 + dv_2 + cv_3 + aw_1 + cw_2 + ew_3)/A_k \\
(bu_1 + du_2 + gu_3)/A_k \\
\xi(au_1 + cu_2 + eu_3)/A_k + 2mn(\alpha_1 - \alpha_2) \Delta T
\end{pmatrix}_k \quad (3.13)
$$

Minimizing the potential energy of the element with respect to
the nodal displacements and the unknown strain $\xi$, results in the
following set of equations
where \([K]\) is the 10x10 element stiffness given in Appendix B.

The global stiffness matrix is obtained by the superposition of the element stiffness matrices. Boundary conditions are imposed as in the case of the mechanical load. There is one additional equation in the thermal problem for determining the uniform unknown strain \(\xi\). It is equivalent to the force equilibrium equation for the thermal load

\[
\sum_{k=1}^{n} f_k = F = 0
\]  

(3.15)

The system of equations is solved for displacements and, as in the case of the mechanical loading, the strains and stresses calculated.

The thermal response is assumed to be linear elastic in this study. The stress state resulting from some temperature change from
To $T_f$ is given by

$$\{\sigma\} = \int_{T_i}^{T_f} [C(\tau)] \{\frac{d\varepsilon^\sigma}{d\tau}(\tau)\} d\tau \quad (3.16)$$

As exact mathematical forms for $[C]$ and $\{\frac{d\varepsilon^\sigma}{d\tau}\}$ are not known, continuous integration cannot be performed, and for an incremental solution (3.16) must be evaluated as a summation.

$$\{\sigma\} = \sum_{i=1}^{n\text{steps}} \{\Delta\sigma_i\} \quad (3.17)$$

Consider the $i$th load step in which the laminate is subject to a temperature change from $T_1$ to $T_2$. By the incremental strain theory, the increment of stress is given by

$$\{\Delta\sigma_i\} = [C(T)] \{\Delta\varepsilon^\sigma(T)\} \quad (3.18)$$

However, as pointed out by Hahn and Pagano [11] this equation does not take into account the temperature dependence of elastic properties. Their total strain theory gives the expression for the incremental stress to be

$$\{\Delta\sigma_i\} = [C(T)] \{\Delta\varepsilon^\sigma(T)\} + \left(\frac{d}{dT} [C(T)]\right) \varepsilon^\sigma(T) \Delta T \quad (3.19)$$

The second term of (3.19) is difficult to evaluate accurately in view of the limited data available for $C(T)$. Further, for properties which do not exhibit large temperature dependence, the second term will be small for small $\Delta T$. Thus the incremental stresses were approximated as
\[ \{\Delta \sigma_i\}_{T_2} \approx [C(T_m)]\{\Delta \varepsilon^0(T_m)\} \quad (3.20) \]

where \( T_m \) is some intermediate temperature between \( T_1 \) and \( T_2 \), chosen to be the mean in this study. The temperature dependence of properties is included in the formulation through the term \( C(T) \).

3.4 Nonlinear Analysis

3.4.1 Mechanical Loading

The nonlinear analysis is carried out in an incremental fashion using data obtained in the previous load step to calculate the material constants for the current load step.

Ramberg Osgood parameters \[23\] are used to represent the nonlinear stress strain relations. Typically

\[ \varepsilon = \frac{\sigma}{E} + k_i \sigma^ {n_i} \quad i=1,2 \quad (3.21) \]

The tangent modulus is defined as

\[ \bar{E} = \frac{d\sigma}{d\varepsilon} = \frac{E}{k_i E n_i \sigma_i^{n_i-1} + 1} \quad (3.22) \]

The stress at load step \( i \) is

\[ \sigma_i = \sum_{j=1}^{i} \Delta \varepsilon_j E_j \quad (3.23) \]

and the tangent modulus for the \( i+1 \) step is

\[ \bar{E}_{i+1} = \frac{E}{k_i E \left( \sum_{j=1}^{i} \Delta \varepsilon_j E_j \right)^{n_i-1} + 1} \quad (3.24) \]
The tangent modulus for each elastic constant is calculated assuming that there is no interaction between the various stresses during nonlinear behavior.

As indicated in Fig. 3 there is some error in following the stress-strain curve. This error can be minimized by iterating the solution in every load step, or by using smaller load steps, as is done in this study.

3.4.2 Thermal Loading

Temperature dependent elastic properties are used for the analysis of thermal loading. The elastic moduli $E_{11}$, $E_{22}$, $G_{12}$, etc., Poisson's ratios $v_{12}$, $v_{23}$, etc., and strengths $X$, $Z$, $S_{13}$, etc., are input at various temperatures, in the form of percent retentions, as shown in Fig. 4. In a given thermal load step, the mean temperature $T_m$ is found and the property linearly interpolated between the nearest higher and lower temperatures ($T_1$, $T_2$) for which properties have been input. These interpolated values are then used to evaluate the stiffness matrices. The analysis accuracy improves with a larger number of input points, since the retention curves for elastic properties and strengths are approximated to greater accuracy.

3.5 Failure Criterion

The finite element analysis provides a three dimensional state of stress, presenting a unique opportunity to study stress interaction and failure.

Tsai and Wu [24] proposed that the failure surface be represented in the form of a tensor polynomial
Actual Stress Strain Curve
Ramberg Osgood approximation
$\Delta_1, \Delta_2$
errors in load steps 1, 2

FIGURE 3. DETERMINATION OF TANGENT MODULI WITH RAMBERG-OSGOOD APPROXIMATIONS
FIGURE 4. TYPICAL PERCENT RETENTION CURVE
The $F_{ij}$ is a second order tensor and $F_{ijkl}$ a fourth order tensor. The numerical values of the terms are obtained from the material principal strengths. The tensors simplify greatly for orthotropic material. The transformations of the nonzero terms in these tensors, in the contracted notation, are given in Appendix C. The strength parameters $F_{12}$, $F_{23}$ and $F_{13}$ require special biaxial loading tests unlike all other parameters which can be obtained from tensile, compressive and shear tests. Failure is predicted to occur when the value of the polynomial is equal to or greater than 1.0. The failure mode can be predicted by comparing the individual contributions of the stress components to the polynomial [25].

\[ F_{ij} \sigma_{ij} + F_{ijkl} \sigma_{ij} \sigma_{kl} = 1 \] 

(3.25)
4. PRELIMINARY STUDIES

4.1 Mesh Size

The present analysis is conducted at the lamina level, (treating the fiber matrix system as a continuum) and not at the micromechanical level. The finite element method discretizes the domain being analyzed. Using finer grids, one can get a better representation of gradients and hence better results. The problem is deciding on the appropriate size of elements for the problem being studied.

Lamination theory results are accurate in the interior of the laminate. The elements in that region can be much larger than those near the free edge where large stress gradients exist and a finer mesh is necessary. The effect of mesh size was studied by using various meshes for a [90/0]_s laminate that was loaded with the same strain of 0.1 percent, keeping all the material properties constant. It was observed that not only do the stresses in the region near the free edge change, but the maximum value of the tensor polynomial used to predict failure changes with mesh size. A linear elastic analysis also predicts different first failure location, for the same laminate and the same loading, depending on the mesh used. Table 1 shows the location and maximum value of the tensor polynomial for various meshes (Appendix E1, E2, E3, E4, E5) for a tensile load of 0.1 percent strain. The meshes were generated using the mesh generation code developed by Bergner et al [26].

The stress distribution is also a function of mesh. For example,


TABLE 1

INFLUENCE OF MESH SIZE ON FIRST FAILURE IN A [90/0]_s LAMINATE

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Elements</th>
<th>Failure Location on Free Edge</th>
<th>Tensor Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>124</td>
<td>center of top layer</td>
<td>.238776</td>
</tr>
<tr>
<td>E2</td>
<td>230</td>
<td>center of top layer</td>
<td>.238876</td>
</tr>
<tr>
<td>E3</td>
<td>326</td>
<td>near interface in top layer</td>
<td>.256435</td>
</tr>
<tr>
<td>E4</td>
<td>598</td>
<td>near interface in top layer</td>
<td>.266231</td>
</tr>
<tr>
<td>E5</td>
<td>878</td>
<td>near interface in top layer</td>
<td>.280495</td>
</tr>
</tbody>
</table>

Linear Elastic; Applied Strain Loading
\( \sigma_z \) for a \([90/0]_s\) laminate exhibits singular behavior, with a large tensile value, at the free edge [25,27]. However, if the grid used is not fine enough, it is compressive rather than large tensile (Fig. 5).

In Gr/E laminates, there are approximately 20-25 filaments through the thickness in each ply. Fig. 6 shows the smallest elements in the grid drawn on the photograph of a typical Gr/E ply. In the finest mesh used in this study, there are 16 elements through each ply for four ply laminates and 32 elements through the thickness for two ply laminates. Therefore, per element, there is just over one filament in the thickness direction. For a laminate aspect ratio of 25, the number of filaments calculated to be in the smallest element is approximately 3.75, assuming a fiber volume fraction of 0.5. This mesh (Fig. E5) was modified so that it could also be used for a four layered laminate (Fig. 7).

4.2 Averaging Finite Element Results

The finite element formulation used in this investigation yields constant values for stresses over each element. Two adjacent elements, in general, give different values for the stress at points on their common boundary. In order to eliminate the discontinuity of the stresses, most finite element analyses use an averaging technique.

The following averaging scheme is used in this study (Fig. 8). The interlaminar stresses \( \sigma_z, \tau_{yz}, \tau_{xz} \) must be continuous throughout the laminate. At a point A, these stresses are averaged over the elements 11, 12, 13, 14. The laminate stresses \( \sigma_x, \sigma_y, \tau_{xy} \) may be discontinuous across laminate interfaces, but within each ply, they
FIGURE 5. VARIATION OF $\sigma_z$ AT THE INTERFACE OF A $[90/0]_s$ GR/E LAMINATE WITH MESH SIZE
FIGURE 6. TYPICAL GRAPHITE/EPOXY PLY WITH SMALLEST ELEMENTS SUPERIMPOSED
FIGURE 8. AVERAGING SCHEME
are continuous. At a point B they are averaged over elements 15 and 16.

4.3 Linear Elastic Analysis

The tensor polynomial predicts failure to occur when it attains the value 1.0. Suppose that for an applied strain $\xi$, the stress state is determined and the tensor polynomial calculated as

$$F_{ij} \sigma_{ij} + F_{ijkl} \sigma_{ij} \sigma_{kl} = \alpha + \beta = \gamma \quad (4.1)$$

Let failure occur at a strain of $k \xi$, i.e.

$$k \alpha + k^2 \beta = 1 \quad (4.2)$$

This quadratic equation is solved for $k$, and the strain at first failure determined.

The stresses and tensor polynomial were evaluated for various laminates loaded with an axial strain of 0.1 percent. Based on these results, 'k' was calculated for the element which had the highest value of the tensor polynomial function for each laminate.

These values are presented in Table 2, and were used to estimate the mechanical load for first failure, and the number of load increments in the nonlinear analysis.

4.4 Stress Free Temperature

During manufacture, laminates are heated to some maximum elevated temperature; however, bonding usually takes place at some lower temperature. At that temperature, the laminate is in a stress free
TABLE 2
LINEAR ELASTIC PREDICTIONS OF FIRST FAILURE

<table>
<thead>
<tr>
<th>Laminate</th>
<th>Strain at First Failure*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0/90]_s$</td>
<td>0.327956</td>
</tr>
<tr>
<td>$[90/0]_s$</td>
<td>0.375117</td>
</tr>
<tr>
<td>$[\pm 10]_s$</td>
<td>0.253761</td>
</tr>
<tr>
<td>$[\pm 15]_s$</td>
<td>0.22304</td>
</tr>
<tr>
<td>$[\pm 30]_s$</td>
<td>0.481329</td>
</tr>
<tr>
<td>$[\pm 45]_s$</td>
<td>0.476636</td>
</tr>
<tr>
<td>$[\pm 60]_s$</td>
<td>0.456682</td>
</tr>
<tr>
<td>$[\pm 75]_s$</td>
<td>0.423255</td>
</tr>
<tr>
<td>$[90/0/\pm 45]_s$</td>
<td>0.318849</td>
</tr>
<tr>
<td>$[\pm 45/0/90]_s$</td>
<td>0.149065</td>
</tr>
</tbody>
</table>

* Mesh E6
state. This stress free temperature $T_o$ is the reference temperature for calculating the residual stresses. $T_o$ depends on the material system of the laminate, and the cure cycle used. Tsai [6] suggested that $T_o$ be experimentally determined from a [$\pm\theta$] unsymmetrical laminate which warps on cooling. The temperature at which the laminate becomes flat on reheating is $T_o$. T300/5208 is cured at 350°F, but the suggested values for $T_o$ vary widely. Renieri and Herakovich [1] used a value of 270°F, while Chamis always uses the highest temperature attained in the cure cycle as the value for $T_o$. A stress free temperature of 250°F is suggested in [18,20]. Hahn [14] reheated warped unsymmetrical laminates, but found values of $T_o$ varying from 250 to 300 degrees. $T_o$ was chosen to be 250°F for the present analysis.

4.5 Load Steps for Thermal Load

A study was conducted to evaluate the effect of cooling the laminate in different load steps. A [90/0]$_s$ laminate was chosen because it exhibits the maximum mismatch in expansion coefficients and material properties. This laminate was analyzed assuming the cooling from $T_o$ to room temperature in 1, 2, 3, 4, 5, 6, 8, and 10 load steps, and the resulting distributions of $\sigma_x$ plotted. Typical variation of the stress is presented in Fig. 9. The largest value of the tensor polynomial was also determined for these case studies. The results presented in Table 3 show that the maximum value decreased with increasing number of load steps. The location of the largest tensor
FIGURE 9. TYPICAL $\sigma_x$ CURING STRESS DISTRIBUTIONS AS A FUNCTION OF THE NUMBER OF THERMAL LOAD STEPS
TABLE 3
EFFECT OF LOAD STEPS ON THE TENSOR POLYNOMIAL
[90/0]_s LAMINATE

<table>
<thead>
<tr>
<th>No. of Load Steps</th>
<th>Maximum Value of Tensor Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.8767522</td>
</tr>
<tr>
<td>2</td>
<td>.7699715</td>
</tr>
<tr>
<td>3</td>
<td>.7252781</td>
</tr>
<tr>
<td>4</td>
<td>.6949252</td>
</tr>
<tr>
<td>5</td>
<td>.6705358</td>
</tr>
<tr>
<td>6</td>
<td>.6497218</td>
</tr>
<tr>
<td>8</td>
<td>.6132286</td>
</tr>
<tr>
<td>10</td>
<td>.5822445</td>
</tr>
</tbody>
</table>
polynomial value was independent of the number of load steps. As seen from Fig. 9, the stress distribution appears to converge with increasing number of load steps. In this finite element analysis, the stiffness matrix must be recalculated for each load step. Using a grid with 896 elements therefore involves an enormous amount of expensive computation. It was decided to cool the laminate in 6 load steps of -30°F each, a compromise between satisfactory convergence and computer cost.
5. STRESS AND FAILURE ANALYSIS OF LAMINATES

Cross-ply, angle-ply, and two quasi-isotropic graphite/epoxy laminates were analyzed in this study. In order to obtain the total stress state in the laminate, the process of cooling to room temperature was modeled in thermal load steps with temperature dependent properties and the nonlinear analysis of subsequent mechanical loading was modeled as a number of linear elastic load steps. Stress distributions were plotted at the strain at which the first element was predicted to fail using the tensor polynomial failure criterion. Damage is predicted to initiate at this strain. This study does not predict the ultimate failure strain, but does predict the mode of first failure. Each load step for the mechanical load was taken to be 0.05 percent strain, the choice guided by the linear elastic predictions for the strain at first failure in each laminate and computer cost.

5.1 Cross-Ply Laminates

5.1.1 Stress Distributions

The mismatch of the expansion coefficients between adjacent layers is maximum in these laminates and results in very high curing stresses. For the purpose of comparison, distributions of non-zero laminate stresses are presented in Figures 10-13 for the following three cases:

1. residual thermal stresses.

2. nonlinear analysis of mechanical load at first failure (including residual stresses).
FIGURE 10. THERMO-MECHANICAL $\sigma_x$ STRESSES IN A [0/90]$_s$ LAMINATE
FIGURE 11. THERMO-MECHANICAL $\sigma_x$ STRESSES IN A $[90/0]_S$ LAMINATE
FIGURE 12. THERMO-MECHANICAL $\sigma_y$ STRESSES IN A $[0/90]_s$ LAMINATE
FIGURE 13. THERMO-MECHANICAL $\sigma_y$ STRESSES IN A [90/0]$_s$ LAMINATE
3. stresses obtained from a linear elastic analysis of pure mechanical load, scaled to the first failure strain as predicted by the nonlinear analysis.

The laminate stresses $\sigma_x$ and $\sigma_y$ for the $[0/90]_s$ and $[90/0]_s$ laminates are shown in Figures 10-13. As a result of cooling, the laminate shrinks and $\sigma_x$ is compressive in the $0^\circ$ layers and tensile in the $90^\circ$ layers, while $\sigma_y$ is tensile in the $0^\circ$ layers and compressive in the $90^\circ$ layers. The stress magnitudes are equal for both layers and stacking sequences. The signs are opposite, thus satisfying equilibrium. With the application of an axial strain load, stress reversal occurs for $\sigma_x$ in the $0^\circ$ layers, but the $90^\circ$ layers experience increased stress magnitude (Figs. 10, 11). Inclusion of thermal stresses is shown to have a significant influence on the overall state of stress at first failure. Comparison of the linear elastic and thermo-mechanical results for $\sigma_x$ at first failure shows that the $[0/90]_s$ laminate is preferred with $\sigma_x$ in the $0^\circ$ layer being more than three times that in the $[0/90]_s$ laminate at first failure.

Edge effects are seen to be present for these laminate stresses. The axial stress ($\sigma_x$) is higher in the boundary layer of the $90^\circ$ layers and lower in the $0^\circ$ layers. The transverse stress ($\sigma_y$) decreases to zero at the free edge for both laminates and layers as required by the boundary conditions. Careful examination of the figure indicates that the boundary layer for thermal and thermo-mechanical loading is generally three to eight times that for the linear elastic analysis.

Interlaminar normal stress distributions are presented in
Figures 5 and 14 for the three loading cases. Moment and force equilibrium of the free body diagram in Figure 15 requires that

\[ \sum M = 0 \Rightarrow \int_{z_1}^{z_b} \sigma_y zdz = \int_{0}^{b} \sigma_z ydy \]

\[ \sum F_x = 0 \Rightarrow \int_{-b}^{b} \sigma_z dy = 0 \]

Thus, the \( \sigma_z \) distribution along the 0/90 interface should be equivalent to a pure couple which balances the moment due to \( \sigma_y \). Since the sign of \( \sigma_y \) changes when the stacking sequence is reversed, the direction of the \( \sigma_z \) couple should also be reversed. As indicated in Figure 14, this condition is satisfied in principle by the \( \sigma_z \) distributions for the two stacking sequences, for both thermal and thermo-mechanical loading. The results in Figure 5 for linear elastic loading of a [90/0]s laminate also indicate satisfaction of these equilibrium requirements.

As mentioned in section 4.1, the \( \sigma_z \) distribution near the free edge is very dependent on mesh size. The general character of the distribution is such that the equilibrium requirement are not grossly violated for any of the meshes studied in this investigation, however the magnitude \( \sigma_z \) at the free edge varied from -80 psi for the coarse mesh to +145 psi for the finest mesh for a linear elastic analysis and axial strain of 0.1 percent. These results confirm those of Wang and Dickson [27] that \( \sigma_z \) attains a tensile value at the free edge for
FIGURE 14. $\sigma_z$ AT THE INTERFACE NEAR THE FREE EDGE IN $[0/90]_s$ AND $[90/0]_s$ LAMINATES
FIGURE 15. PARTIAL FREE BODY DIAGRAMS OF A LAMINATE
a [90/0]_{s} as well as the [0/90]_{s} laminate.

The results in Figure 14 were obtained with the finest finite element mesh from Figure 5 (i.e. E5). They show that \( \sigma_z \) in the [0/90]_{s} laminate is tensile with singular behavior for both thermo and thermo-mechanical loading. It is also apparent that the thermal effects dominate the boundary layer stress distribution for the [0/90]_{s} laminate. Reversing the stacking sequence does not result in a mirror image of the stress distributions. Figures 5 and 14 both indicate a second reversal of the gradient of \( \sigma_z \) near the free edge. The linear elastic results (Figure 5) predict a tensile \( \sigma_z \) near the free edge whereas the thermal and thermo-mechanical results show that thermal effects and nonlinear behavior have a beneficial effect on the magnitude of \( \sigma_z \) at the free edge of a [90/0]_{s} laminate.

It is also apparent from Figure 14 that the significance of thermal effects is laminate dependent. Thermal effects dominate the \( \sigma_z \) distribution in [0/90]_{s} laminates, but mechanical effects are more dominate in the [90/0]_{s} laminate. Boundary layer width is, however, essentially the same for both types of loading and both laminates extending over approximately 15-20 percent of the laminate width. The width of the boundary layer in the [90/0]_{s} laminate is essentially the same for all three types of loading (Figures 5 and 14).

Through-the-thickness variations of \( \sigma_z \) and \( \sigma_x \) for the residual stress state are compared to the distribution obtained by Wang and Crossman [16] using a linear elastic thermal analysis in Figures 16 and 17. Though the shape of the stress distributions is approximately
FIGURE 16. RESIDUAL $\sigma_z$ THROUGH-THE-THICKNESS DISTRIBUTIONS NEAR THE FREE EDGE OF $[0/90]_s$ AND $[90/0]_s$ GR/E LAMINATES
FIGURE 17. RESIDUAL $\sigma_x$ THROUGH-THE-THICKNESS DISTRIBUTIONS NEAR THE FREE EDGE OF $[0/90]_s$ AND $[90/0]_s$ GR/E LAMINATES
the same, there is significant difference in the magnitude of the stresses. The maximum value of $\sigma_z$ in a $[0/90]_s$, for example, is predicted to be 2.01 ksi by this analysis compared to a value of 5.4 ksi from [16]. This difference can be attributed to the incremental analysis using temperature dependent elastic properties. It should also be observed from Fig. 16 that, in all cases, the maximum positive value of $\sigma_z$ occurs within a layer and not at the ply interface.

5.1.2 Failure Analysis

The curing stresses in cross-ply laminates are very high. In a $[0/90]_s$ laminate, the stresses resulting from cooling the laminate in six load steps were high enough for the tensor polynomial to predict failure. (In fact, cracks are sometimes observed at the free edge of cross-ply laminates [8,20].) For the purpose of this analysis, the $[0/90]_s$ laminate was cooled in eight load steps in order to reduce the step size and, thereby, eliminate the prediction of failure. All other laminates were cooled in six thermal increments. With the application of mechanical load, first failure was predicted to occur at a strain of 0.05 percent in the $[0/90]_s$ laminate and at 0.15 percent in the $[90/0]_s$ laminate.

The tensor polynomial is plotted along the interface and through the thickness for both laminates in Figures 18-20. Failure for both laminates was predicted to initiate in the $90^\circ$ ply at the free edge. Figs. 18 and 19 show the variation of the tensor polynomial along the interface in the $90^\circ$ ply, as determined from the curing stresses and subsequent mechanical loading. The curing stresses are predicted to
FIGURE 18. TENSOR POLYNOMIAL ALONG THE INTERFACE IN THE 90° LAYER FOR A [0/90]_s LAMINATE
FIGURE 19. TENSOR POLYNOMIAL ALONG THE INTERFACE IN THE 90° LAYER FOR A [90/0]_s LAMINATE
FIGURE 20. TENSOR POLYNOMIAL THROUGH-THE-THICKNESS AT THE EDGE OF $[0/90]_s$ AND $[90/0]_s$ LAMINATES
make a major contribution to initiation of failure in regions close to the free edge in both laminates. Through-the-thickness variation (Fig. 20) shows the effect of curing stresses to be significant in the 90° layers of both laminates. In the 0° layers the tensor polynomial has a negative value, which is possible when using the tensor polynomial failure criterion. The maximum value of the polynomial occurs within the 90° layer for both laminates, not at the 0/90 interface. Thus, first failure is predicted to occur within the layer and not at the interface.

The tensor polynomial for the element which was first to fail was analyzed in detail; the individual contributions from each of the contributing stresses are presented in Table 4. The table shows that while $\sigma_z$ make the largest contribution to the polynomial at failure for both laminates, the contribution from $\sigma_3$ is very significant in the [0/90]$_s$ laminate. It may be said that the [0/90]$_s$ laminate fails in a mixed $\sigma_2$-$\sigma_3$ mode but the [90/0]$_s$ laminate fails primarily due to $\sigma_2$, i.e. transverse tension. The [0/90]$_s$ laminate experiences first failure at one-third the failure strain of the [90/0]$_s$. Since the [90/0]$_s$ is predicted to fail at a higher applied strain, it is preferred over the [0/90]$_s$ for tensile loading.

5.2 Angle-Ply Laminates

5.2.1 Stress Distributions

The angle-ply laminates studied were the [±10]$_s$, [±15]$_s$, [±30]$_s$, [±45]$_s$ [±60]$_s$ and [±75]$_s$. The thermal mismatch between adjacent plies
TABLE 4
FAILURE MODE ANALYSIS OF CROSS-PLY LAMINATES

<table>
<thead>
<tr>
<th>Laminate</th>
<th>$F_{22}^2$</th>
<th>$F_{22}^2$</th>
<th>$F_{33}^2$</th>
<th>$F_{33}^2$</th>
<th>$F_2^2 + F_{22}^2$</th>
<th>$F_{33}^2 + F_{33}^2$</th>
<th>$\varepsilon$ at First Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0/90]_S</td>
<td>.4898</td>
<td>.1186</td>
<td>.3367</td>
<td>.6561</td>
<td>.6084</td>
<td>.3928</td>
<td>.0005</td>
</tr>
<tr>
<td>[90/0]_S</td>
<td>.6360</td>
<td>.2000</td>
<td>.1531</td>
<td>.0116</td>
<td>.8360</td>
<td>.1647</td>
<td>.0015</td>
</tr>
</tbody>
</table>
is not as severe as that in cross-ply laminates except for the \([\pm 45]_s\) laminate. Thus, the residual stresses are in general lower. It is interesting to note that, in the material principal coordinates, the residual stresses in the cross-ply and the \([\pm 45]_s\) laminate are the same, except at the free edges. This is of course expected because of the tensor property of the coefficient of thermal expansion.

The highest absolute value of each stress component was normalized and plotted versus the ply angle. Figs. 21 and 22 show the variation of the laminate and the interlaminar stresses for thermal and mechanical loading, respectively. The thermal mismatch in angle-ply laminates is maximum at 45°, and all thermal stresses (Fig. 21) attain their maximum values at 45°, except for \(\sigma_x\) which attains its maximum at 30°. This is because the stress state not only depends on the curing strain (thermal mismatch), but also on the elastic modulus and \(E_x\) decreases sharply as the ply angle increases from zero, tapering off at larger angles [29].

The maximum value of the individual stress components occur at different fiber angles for mechanical loading (Fig. 22). The magnitude of \(\sigma_x\) is large at low angles, with its maximum at 0°, while \(\tau_{xz}\) attains its maximum at 15°. Three components, \(\sigma_z\), \(\tau_{yz}\), \(\sigma_y\), attain their maximum value at 30° and \(\tau_{xy}\) attains its maximum at 45°. These results show that there are fundamental differences between thermal and mechanical loading of angle-ply laminates.

The distribution of curing stresses is roughly the same in all angle-ply laminates, the difference being in the magnitudes of
a) Intralaminar Stresses

\[
\begin{align*}
|\sigma_x|_m &= 2.6 \text{ KSI} \\
|\tau_{xy}|_m &= 3.06 \text{ KSI} \\
|\sigma_y|_m &= 0.227 \text{ KSI}
\end{align*}
\]

b) Interlaminar Stresses

\[
\begin{align*}
|\sigma_z|_m &= 0.262 \text{ KSI} \\
|\tau_{yz}|_m &= 0.02 \text{ KSI} \\
|\tau_{xz}|_m &= 3.3 \text{ KSI}
\end{align*}
\]

FIGURE 21. NORMALIZED MAXIMUM CURING STRESSES IN \([\pm \theta]_s\) LAMINATES
FIGURE 22. NORMALIZED MAXIMUM STRESSES AT FIRST FAILURE IN $[\pm \theta]$ LAMINATES FOR THERMO-MECHANICAL LOADING

**a) Intralaminar Stresses**

- $|\sigma_x|_m = 53.5$ KSI
- $|\tau_{xy}|_m = 14.5$ KSI
- $|\sigma_y|_m = 0.354$ KSI

- $|\sigma_x|_m = .123$ KSI
- $|\sigma_z|_m = .647$ KSI
- $|\tau_{xz}|_m = 10.9$ KSI

**b) Interlaminar Stresses**
different ply orientations. The curing stresses in the $[\pm 45]_s$ are
typical and are presented, Fig. 23 showing the lamina stresses and
Fig. 24, the interlaminar stresses. The stresses $\sigma_y$, $\tau_{xy}$, and $\tau_{yz}$
are seen to approach zero as required by the stress free boundary
conditions. As in cross-ply laminates, the curing stresses exhibit
edge effects, with the presence of a boundary layer for $y/b$ greater
than 0.9. Fig. 24 shows that the boundary layer for $\tau_{yz}$ at the $\pm 45$
interface is approximately twice that of $\sigma_z$ at the midplane.

Though-the-thickness variation of $\sigma_x$ and $\tau_{xz}$ near the free edge
for a $[\pm 45]_s$ laminate are compared to distributions obtained by Wang
and Crossman [15] in Fig. 25. As in the cross-ply laminates (Fig. 17)
the present solution predicts much lower stresses. Both components
of stress exhibit sharp gradients in the vicinity of the interface.

5.2.2 Failure Analysis

The interlaminar distribution of the tensor polynomial, at first
failure, as determined for thermo-mechanical loading, is shown in
Fig. 26 for various fiber angles. This figure demonstrates that the
edge effects are dominant at small angles of orientation, and that edge
stress concentrations decrease with increasing angle. At large angles,
failure is first predicted at the free edge, but elements in the
interior have large values for the tensor polynomial, hence the entire
laminate is close to failure. The tensor polynomial exhibits a small
negative value in the laminate interior for low fiber angles. This
is acceptable in the failure criterion, and signifies that the region
is well below failure. These results indicate that the laminate fails
FIGURE 24. RESIDUAL $\sigma_z$ AND $\tau_{yz}$ STRESSES IN A $[\pm 45]$z GR/E LAMINATE
FIGURE 25. THROUGH-THE-THICKNESS DISTRIBUTIONS FOR THE RESIDUAL $\sigma_x$ AND $\tau_{xz}$ ALONG THE FREE EDGE IN A $[\pm 45]_s$ GR/E LAMINATE
FIGURE 26. TENSOR POLYNOMIAL ALONG THE INTERFACE OF $[\pm \theta]_s$ LAMINATES AT 1ST FAILURE FOR THERMO-MECHANICAL LOADING
in an edge mode for small fiber angles and in a laminate mode for large fiber angles.

Thermal stresses in angle-ply laminates are an edge effect. This is clearly seen from Fig. 27, where the tensor polynomial has been plotted through the thickness along the free edge for several fiber angles, for the curing stresses as well as for the stress state existing at first failure. The presence of the free edge and dissimilarity of material causes additional stress gradients at the interface. Failure is predicted to initiate at the interface for low angles, shift to the midplane at $45^\circ$, and shift back to the interface for angles greater than $45^\circ$. The maximum value of the tensor polynomial for thermal loading occurs at $\theta=45^\circ$ where the property mismatch is largest.

The stress state of the element where first failure was predicted was transformed into the material coordinate system and the individual terms of the tensor polynomial evaluated and presented in Table 5. The tensor polynomial is completely dominated by $\tau_{13}$ for the $10^\circ$ and $15^\circ$ degree laminates, and the mode of failure is therefore predicted to be transverse shear. With increasing angle, the contribution of $\tau_{13}$ decreases while that from $\tau_{23}$ and $\tau_{12}$ increase and the mode of failure continues to be transverse shear up to $\theta=30^\circ$. At $45^\circ$, the polynomial is dominated by the $\sigma_2$ terms, though there is significant contribution from $\tau_{12}$ which decreases with further increase in fiber angle. The failure mode for angles equal to or greater than $45^\circ$ is predicted to be transverse tension.

Failure was predicted to initiate at the free edge for all
FIGURE 27. THROUGH-THE-THICKNESS TENSOR POLYNOMIAL DISTRIBUTIONS FOR CURING STRESSES AND STRESSES AT FIRST FAILURE IN ANGLE-PLY LAMINATES
<table>
<thead>
<tr>
<th>Laminate</th>
<th>$F_1 \sigma_1$</th>
<th>$F_{11} \sigma_1^2$</th>
<th>$F_2 \sigma_2$</th>
<th>$F_{22} \sigma_2^2$</th>
<th>$F_3 \sigma_3$</th>
<th>$F_{33} \sigma_3^2$</th>
<th>$F_{44} \tau_{23}$</th>
<th>$F_{55} \tau_{13}$</th>
<th>$F_{66} \tau_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[±10]s</td>
<td>0.0273</td>
<td>0.0576</td>
<td>-0.0715</td>
<td>0.0025</td>
<td>0.0079</td>
<td>0.0000</td>
<td>0.0494</td>
<td>0.9016</td>
<td>0.0216</td>
</tr>
<tr>
<td>[±15]s</td>
<td>0.0231</td>
<td>0.0409</td>
<td>-0.1043</td>
<td>-0.0054</td>
<td>-0.0101</td>
<td>0.0001</td>
<td>0.1145</td>
<td>0.8808</td>
<td>0.0443</td>
</tr>
<tr>
<td>[±30]s</td>
<td>0.0163</td>
<td>0.0204</td>
<td>-0.1241</td>
<td>0.0076</td>
<td>-0.0729</td>
<td>0.0026</td>
<td>0.3305</td>
<td>0.5382</td>
<td>0.2750</td>
</tr>
<tr>
<td>[±45]s</td>
<td>0.0027</td>
<td>0.0005</td>
<td>0.6070</td>
<td>0.1823</td>
<td>0.0315</td>
<td>0.0005</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1793</td>
</tr>
<tr>
<td>[±60]s</td>
<td>-0.0003</td>
<td>0.0000</td>
<td>0.6770</td>
<td>0.2266</td>
<td>0.0302</td>
<td>0.0005</td>
<td>0.0311</td>
<td>0.0048</td>
<td>0.0300</td>
</tr>
<tr>
<td>[±75]s</td>
<td>-0.0002</td>
<td>0.0000</td>
<td>0.7135</td>
<td>0.2517</td>
<td>0.0115</td>
<td>0.0001</td>
<td>0.0179</td>
<td>0.0006</td>
<td>0.0048</td>
</tr>
</tbody>
</table>
laminates studied. In the $10^\circ$ and $15^\circ$ laminates first failure initiated at 0.003 percent strain, in the $45^\circ$ at 0.0045 percent strain and at 0.004 for the $30^\circ$, $60^\circ$ and $75^\circ$ laminates. The strains at which first failure is predicted is the same for some laminates because the strain was applied in load steps of 0.05 percent. Linear elastic results in [25] indicated that the $[\pm 15]_s$ laminate was the most critical and that the strain to failure increased with increasing fiber angle.

5.3 Quasi-Isotropic Laminates

5.3.1 Stress Distributions

The quasi-isotropic laminates analyzed were the $[\pm 45/0/90]_s$ and the $[90/0/\pm 45]_s$. Laminate and interlaminar stresses for residual thermal and first failure under thermo-mechanical loading are presented in Figs. 28-33. Axial ($\sigma_x$) and transverse ($\sigma_y$) stresses in the $90^\circ$ layer of both laminates are shown in Figs. 28 and 29, respectively. The results show a strong edge effect in $\sigma_x$ which is tensile, and thus leads to early transverse tension failure at the free edge. The width of the boundary layer for thermo-mechanical loading is significantly larger than that for thermal loading. This is believed to be a manifestation of nonlinear material behavior. The residual stresses are shown to make a significant contribution to the stress state that exists when failure initiates.

Various interlaminar stresses are plotted at different interfaces of both laminates in Figs. 30 and 31. As indicated in the figure, the edge effects extend further into the interior for thermo-mechanical loading than they do for purely thermal loading as was the case for
FIGURE 28. $\sigma_x$ IN THE 90° LAYER OF $[\pm 45/0/90]_s$ AND $[90/0/\pm 45]_s$ LAMINATES
Figure 29. $\sigma_y$ in the 90° layer of $[\pm 45/0/90]_S$ and $[90/0/\pm 45]_S$ laminates
FIGURE 30. INTERLAMINAR STRESSES IN A \([\pm45/0/90]_s\) LAMINATE
FIGURE 31. INTERLAMINAR STRESSES IN A [90/0/±45]₀ LAMINATE
FIGURE 32. THROUGH-THE-THICKNESS $\sigma_x$ DISTRIBUTION AT THE EDGE OF $[\pm 45/0/90]_s$ AND $[90/0/\pm 45]_s$ LAMINATES
Figure 33. Through-the-thickness $\sigma_z$ distribution at the edge of $[\pm45/0/90]_{S}$ and $[90/0/\pm45]_{S}$ laminates.
laminate stresses. Though \( \tau_{yz} \) tends to zero at the boundary, \( \tau_{xz} \) and \( \sigma_z \) are singular at the free edge. This was the case at all interfaces except the 90/0 interface where \( \sigma_z \) reverses sign from its large negative value, tending to zero or some tensile value at the free edge. Such a behavior was also predicted for a [90/0]_s laminate (Figs. 5 and 14).

Through-the-thickness distributions of \( \sigma_x \) and \( \sigma_z \) are presented for thermal and thermo-mechanical loading in Figs. 32 and 33. The \( \sigma_x \) stress distributions show the unloading of the 0\(^\circ\) layers with the application of mechanical load. However, this positive feature of thermal stress is offset by the fact that the thermal stress has the same sign as the stress due to mechanical in the 90\(^\circ\) layers and, therefore, contributes to early failure in that layer. The \( \sigma_z \) distributions show that the signs of the stresses due to thermal and thermo-mechanical are the same. The interlaminar normal stresses are predominately compressive for these two stacking sequences and, therefore, do not contribute to delamination. The only exception being near the midplane of the \([\pm45/0/90]_s\) laminate. It would appear from these results that, of the two, the [90/0/\(\pm45\)]_s is the preferred laminate for tensile loading.

5.3.2 Failure Analysis

The distribution of the tensor polynomial in the 90\(^\circ\) layer adjacent to the 0/90 interface of both laminates is shown in Fig. 34 and the through-the-thickness distributions near the free edge are shown in Fig. 35. Both figures show results for residual thermal and thermo-
FIGURE 34. TENSOR POLYNOMIAL IN THE 90° LAYER ALONG THE 0/90 INTERFACE IN $[\pm 45/0/90]_S$ AND $[90/0/\pm 45]_S$ LAMINATES
FIGURE 35. THROUGH-THE-THICKNESS TENSOR POLYNOMIAL DISTRIBUTION IN $[\pm 45/0/90]_S$ AND $[90/0/\pm 45]_S$ LAMINATES
mechanical loading. The 90° layer was chosen for illustration because, as indicated in Fig. 35, the polynomial attains its largest value in this layer. The distributions in Fig. 34 show that the polynomial attains its maximum at the free edge. A boundary layer effect is very evident with the width at the boundary layer for thermo-mechanical loading being several times that for thermal loading. This figure also shows that thermal effects make a significant contribution to the tensor polynomial, and that the boundary layer effects are much stronger in the \([±45/0/90]_s\) laminate.

The through-the-thickness distributions in Fig. 35 indicate that, for both laminates, the maximum value of the polynomial occurs within the 90° layer as opposed to the 0/90 interface. In the \([90/0/±45]_s\) laminate, the maximum value is just below the midpoint of the layer thickness. In the \([±45/0/90]_s\), the maximum value is at the center of the 90° layer, the midplane.

The terms making significant contributions to the tensor polynomial at failure (in material coordinates) are presented in Table 6 for the element that was the first to fail in each laminate. These results show that there is significant difference in the mode of first failure of the two quasi-isotropic laminates. The \([±45/0/90]_s\) laminate fails in a mixed \(σ_2-σ_3\) mode whereas the \([90/0/±45]_s\) laminate fails in a predominately \(σ_2\) mode. The influence of the higher interlaminar normal stress in the \([±45/0/90]_s\) is shown to lead to failure at a lower applied strain.
TABLE 6
FAILURE MODE ANALYSIS OF QUASI-ISOTROPIC LAMINATES

<table>
<thead>
<tr>
<th>Laminate</th>
<th>$F_2\sigma_2$</th>
<th>$F_{22}\sigma_2^2$</th>
<th>$F_3\sigma_3$</th>
<th>$F_{33}\sigma_3^2$</th>
<th>$F_{2\sigma_2}^2+F_{22}\sigma_2^2$</th>
<th>$F_{3\sigma_3}^2+F_{33}\sigma_3^2$</th>
<th>ε at First Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\pm 45/0/90]_S$</td>
<td>.5228</td>
<td>.1351</td>
<td>.2992</td>
<td>.0443</td>
<td>.6579</td>
<td>.3435</td>
<td>.001</td>
</tr>
<tr>
<td>$[90/0/\pm45]_S$</td>
<td>.6606</td>
<td>.2157</td>
<td>.1174</td>
<td>.0068</td>
<td>.8762</td>
<td>.1242</td>
<td>.0015</td>
</tr>
</tbody>
</table>
6. CONCLUSIONS

The present analysis has concentrated on the evaluation of residual thermal stresses induced during curing of composites and thermo-mechanical stresses due to combined thermal/mechanical loading. Particular attention was given to the influence that boundary layer effects near the free edge have on the initiation of failure in angle-ply, cross-ply and quasi-isotropic laminate as predicted by the tensor polynomial failure criterion. The following conclusions can be drawn as a result of the study.

1. Mesh size has a significant effect on the stress values obtained from a finite element investigation of the stress distribution near the edge of a finite width laminated composite.

2. Thermal effects are significant in the boundary layer of laminated composites.

3. The boundary layer stress distribution in cross-ply laminates is very dependent on the stacking sequence. For linear elastic material behavior, both $[0/90]_s$ and $[90/0]_s$ laminates exhibit tensile interlaminar normal stress at the free edge.

4. Failure in a $[0/90]_s$ laminate initiates at approximately one-third the initial failure strain of a $[90/0]_s$ laminate under tensile load.
5. Failure of cross-ply laminates initiates at the free edge within the 90° layer, not at the 0/90 interface.

6. Failure of a [0/90]_s laminate is a mixed mode in \( \sigma_2 - \sigma_3 \) whereas the [90/0]_s laminate fails primarily due to transverse tension (\( \sigma_2 \)).

7. The stress behavior in angle-ply laminates is fundamentally different for thermal and mechanical loading. The [±45]_s laminate is most critical for thermal loading, but the [±15]_s laminate is most critical for tensile loading.

8. Angle-ply laminates with small fiber angles fail due to interlaminar shear whereas laminates with large fiber angles fail due to transverse tension.

9. Failure of angle-ply laminates initiates at the free edge. For small and large fiber angles, failure initiates at the ±\( \theta \) interface. For intermediate angles, failure initiates at the midplane.

10. Two modes of failure are predicted for angle ply laminates, an edge mode for fiber angles equal to or less than 30° and a laminate mode for angles equal to or greater than 45°.

11. Failure in the [±45/0/90]_s and [90/0/±45]_s laminates initiates at the free edge near the center of the 90° layer(s). The [±45/0/90]_s fails at a lower applied strain in a mixed \( \sigma_2 - \sigma_3 \) mode whereas the
[90/0/±45]_s laminate fails primarily in a transverse tension mode.
BIBLIOGRAPHY


APPENDIX A

CONSTITUTIVE RELATIONS
APPENDIX A
CONSTITUTIVE RELATIONS

The constitutive relationship for an orthotropic material in the principal material directions can be written

\[
\{\sigma\}_1 = [C](\{\varepsilon\}_1 - (\alpha)_1 \Delta T)
\]

where

\[
[C] = \begin{bmatrix}
    C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
    C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
    C_{33} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & C_{44} & 0 & 0 \\
    0 & 0 & 0 & 0 & C_{55} & 0 \\
    0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix}
\]

\[
\{\sigma\}_1 = \begin{bmatrix}
    \sigma_1 \\
    \sigma_2 \\
    \sigma_3 \\
    \tau_{23} \\
    \tau_{13} \\
    \tau_{12}
\end{bmatrix}, \quad \{\varepsilon\}_1 = \begin{bmatrix}
    \varepsilon_1 \\
    \varepsilon_2 \\
    \varepsilon_3 \\
    \gamma_{23} \\
    \gamma_{13} \\
    \gamma_{12}
\end{bmatrix}
\]
For a rotation about the 3 (z) axis (Fig. 1), the constitutive relationship becomes

\[ \{\sigma\} = [\bar{C}](\{\varepsilon\} - \{\alpha\} \Delta \tau) \]

where

\[ [\bar{C}] = \begin{bmatrix}
\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\
\bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\
\bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\
\bar{C}_{44} & \bar{C}_{45} & 0 \\
\bar{C}_{55} & 0 \\
\bar{C}_{66} \\
\end{bmatrix} \]

Symmetric

\[ \{\sigma\} = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy} \\
\end{bmatrix} \quad \text{and} \quad \{\varepsilon\} = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy} \\
\end{bmatrix} \]
and the various matrix and vector terms as functions of the principal material properties are given below (m=cos\(\phi\), n=sin\(\phi\)).

\[
\{ \alpha \} = \begin{pmatrix}
\alpha_x \\
\alpha_y \\
\alpha_z \\
0 \\
0 \\
\alpha_{xy}
\end{pmatrix}
\]

\[
\alpha_{11} = m^{4}c_{11}+2m^{2}n^{2}\left(c_{12}+2c_{66}\right)+n^{4}c_{22}
\]

\[
\alpha_{12} = m^{2}n^{2}\left(c_{11}+c_{22}-4c_{66}\right)+(m^{2}+n^{4})c_{12}
\]

\[
\alpha_{13} = m^{2}c_{13}+n^{2}c_{23}
\]

\[
\alpha_{16} = -mn\left[m^{2}c_{11}-n^{2}c_{22}-\left(m^{2}-n^{2}\right)(c_{12}+2c_{66})\right]
\]

\[
\alpha_{22} = n^{4}c_{11}+2m^{2}n^{2}\left(c_{12}+2c_{66}\right)+m^{4}c_{22}
\]

\[
\alpha_{23} = n^{2}c_{13}+m^{2}c_{23}
\]

\[
\alpha_{26} = -mn\left(n^{2}c_{11}-m^{2}c_{22}\right)+(m^{2}-n^{2})(c_{12}+2c_{66})
\]

\[
\alpha_{33} = c_{33}
\]

\[
\alpha_{36} = mn(c_{23}-c_{13})
\]

\[
\alpha_{44} = m^{2}c_{44}+n^{2}c_{55}
\]

\[
\alpha_{45} = mn(c_{44}-c_{55})
\]

\[
\alpha_{55} = n^{2}c_{44}+m^{2}c_{55}
\]

\[
\alpha_{66} = m^{2}n^{2}\left(c_{11}+c_{22}-2c_{12}\right)+(m^{2}-n^{2})^{2}c_{66}
\]
\begin{align*}
\alpha_x &= m^2 \alpha_1 + n^2 \alpha_2 \\
\alpha_y &= n^2 \alpha_1 + m^2 \alpha_2 \\
\alpha_z &= \alpha_3 \\
\alpha_{xy} &= -2mn(\alpha_1 - \alpha_2)
\end{align*}
APPENDIX B

STIFFNESS MATRIX
APPENDIX B

STIFFNESS MATRIX

Equations (B.1) represent the equilibrium equations for applied strain loading. Equ's (B.2) represent the equilibrium equations in average force loadings. In these equations, \([K]\) is the symmetric elemental stiffness matrix, \(\xi_x\{S\}\) and \(\{T\}\) are force vectors corresponding to the applied strain and temperature change respectively, \(\{F\}\) is the vector of applied forces, and \(\{x\}\) is the vector of unknown nodal displacements.

\[
[K]^{(x)} \{x\}^{(x)} + \xi_x\{S\}^{(x)} = \{F\}^{(x)} \tag{B.1}
\]

\[
(9\times9) \quad (9\times1) \quad (9\times1)
\]

\[
[K]^{(x)} \{x\}^{(x)} - \{T\}^{(x)} = \{F\}^{(x)} \tag{B.2}
\]

\[
(10\times10) \quad (10\times1) \quad (10\times1) \quad (10\times1)
\]

Defining the following terms

\[a = (Z_2 - Z_3)/2\]
\[b = (Y_3 - Y_2)/2\]
\[c = (Z_3 - Z_1)/2\]
\[d = (Y_1 - Y_3)/2\]
\[e = (Z_1 - Z_2)/2\]
\[g = (Y_2 - Y_1)/2\]
\( A^\ell = \text{the area of element } (\ell) \)
\( F^* = \text{average normal force} \)

where \( Y_1 \) through \( Y_3 \) and \( Z_1 \) through \( Z_3 \) are the coordinates of the nodal points of element \( \ell \) in the \( Y-Z \) plane, the element of the matrices of Equ. (B.1) can be defined as follows.

\[
\begin{align*}
K_{11} &= \left( \tilde{C}_{55}b^2 + \tilde{C}_{66}a^2 \right)/A^\ell \\
K_{12} &= \left( \tilde{C}_{55}bd + \tilde{C}_{66}ac \right)/A^\ell \\
K_{13} &= \left( \tilde{C}_{55}bg + \tilde{C}_{66}ae \right)/A^\ell \\
K_{14} &= \left( \tilde{C}_{26}a^2 + \tilde{C}_{45}b^2 \right)/A^\ell \\
K_{15} &= \left( \tilde{C}_{26}ca + \tilde{C}_{45}bd \right)/A^\ell \\
K_{16} &= \left( \tilde{C}_{26}ea + \tilde{C}_{45}bg \right)/A^\ell \\
K_{17} &= \left( \tilde{C}_{36}ba + \tilde{C}_{45}ba \right)/A^\ell \\
K_{18} &= \left( \tilde{C}_{36}da + \tilde{C}_{45}bc \right)/A^\ell \\
K_{19} &= \left( \tilde{C}_{36}ga + \tilde{C}_{45}be \right)/A^\ell \\
K_{22} &= \left( \tilde{C}_{55}a^2 + \tilde{C}_{66}c^2 \right)/A^\ell \\
K_{23} &= \left( \tilde{C}_{55}d^2 + \tilde{C}_{66}ce \right)/A^\ell \\
K_{24} &= \left( \tilde{C}_{26}ac + \tilde{C}_{45}db \right)/A^\ell \\
K_{25} &= \left( \tilde{C}_{26}c^2 + \tilde{C}_{45}d^2 \right)/A^\ell \\
K_{26} &= \left( \tilde{C}_{26}ec + \tilde{C}_{45}dg \right)/A^\ell \\
K_{27} &= \left( \tilde{C}_{26}ea + \tilde{C}_{45}bg \right)/A^\ell \\
K_{28} &= \left( \tilde{C}_{36}bc + \tilde{C}_{45}da \right)/A^\ell \\
K_{29} &= \left( \tilde{C}_{36}dc + \tilde{C}_{45}de \right)/A^\ell \\
K_{33} &= \left( \tilde{C}_{55}g^2 + \tilde{C}_{66}e^2 \right)/A^\ell \\
K_{34} &= \left( \tilde{C}_{26}ae + \tilde{C}_{45}gb \right)/A^\ell \\
K_{35} &= \left( \tilde{C}_{26}ce + \tilde{C}_{45}bd \right)/A^\ell \\
K_{36} &= \left( \tilde{C}_{26}e^2 + \tilde{C}_{45}g^2 \right)/A^\ell \\
K_{37} &= \left( \tilde{C}_{36}be + \tilde{C}_{45}ga \right)/A^\ell \\
K_{44} &= \left( \tilde{C}_{22}a^2 + \tilde{C}_{44}b^2 \right)/A^\ell \\
K_{45} &= \left( \tilde{C}_{22}ac + \tilde{C}_{44}bd \right)/A^\ell \\
K_{46} &= \left( \tilde{C}_{22}ae + \tilde{C}_{44}bg \right)/A^\ell \\
K_{47} &= \left( \tilde{C}_{44}ba + \tilde{C}_{23}ab \right)/A^\ell \\
K_{48} &= \left( \tilde{C}_{44}bc + \tilde{C}_{23}ad \right)/A^\ell
\end{align*}
\]
\[ K_{38} = \frac{(\tilde{c}_{36}de + \tilde{c}_{45}gc)}{A} \]

\[ K_{39} = \frac{(\tilde{c}_{36}ge + \tilde{c}_{45}ge)}{A} \]

\[ K_{55} = \frac{(\tilde{c}_{22}e^2 + \tilde{c}_{44}d^2)}{A} \]

\[ K_{56} = \frac{(\tilde{c}_{22}ce + \tilde{c}_{44}dg)}{A} \]

\[ K_{57} = \frac{(\tilde{c}_{44}da + \tilde{c}_{23}cb)}{A} \]

\[ K_{58} = \frac{(\tilde{c}_{44}dc + \tilde{c}_{23}cd)}{A} \]

\[ K_{59} = \frac{(\tilde{c}_{44}de + \tilde{c}_{23}cg)}{A} \]

\[ K_{77} = \frac{(\tilde{c}_{33}b^2 + \tilde{c}_{44}a^2)}{A} \]

\[ K_{78} = \frac{(\tilde{c}_{33}bd + \tilde{c}_{44}ac)}{A} \]

\[ K_{79} = \frac{(\tilde{c}_{33}bg + \tilde{c}_{44}ae)}{A} \]

\[ K_{49} = \frac{(\tilde{c}_{44}be + \tilde{c}_{23}ag)}{A} \]

\[ K_{66} = \frac{(\tilde{c}_{22}e^2 + \tilde{c}_{44}d^2)}{A} \]

\[ K_{67} = \frac{(\tilde{c}_{44}ga + \tilde{c}_{23}eb)}{A} \]

\[ K_{68} = \frac{(\tilde{c}_{44}gc + \tilde{c}_{23}eg)}{A} \]

\[ K_{69} = \frac{(\tilde{c}_{44}ge + \tilde{c}_{23}eg)}{A} \]

\[ S_1 = \tilde{c}_{16} a \quad S_2 = \tilde{c}_{16} c \quad S_3 = \tilde{c}_{16} e \]

\[ S_4 = \tilde{c}_{12} a \quad S_5 = \tilde{c}_{12} c \quad S_6 = \tilde{c}_{12} e \]

\[ S_7 = \tilde{c}_{13} b \quad S_8 = \tilde{c}_{13} d \quad S_9 = \tilde{c}_{13} g \]

\[ x_1 = u_1 \quad x_2 = u_2 \quad x_3 = u_3 \]

\[ x_4 = v_1 \quad x_5 = v_2 \quad x_6 = v_3 \]

\[ x_7 = w_1 \quad x_8 = w_2 \quad x_9 = w_3 \]
\[ F_1 = f_x \quad F_2 = f_x \quad F_3 = f_x \]
\[ F_4 = f_y \quad F_5 = f_y \quad F_3 = f_y \]
\[ F_7 = f_z \quad F_8 = f_z \quad F_8 = f_z \]

where \( f \)'s are nodal forces.

For Equ. (B.2) the previously defined terms apply plus the following additional terms

\[ K_{110} = C_{16}^a \quad K_{210} = C_{16}^c \quad K_{310} = C_{16}^e \]
\[ K_{410} = C_{12}^a \quad K_{510} = C_{12}^c \quad K_{610} = C_{16}^e \]
\[ K_{710} = C_{13}^b \quad K_{810} = C_{13}^d \quad K_{910} = C_{13}^g \]
\[ K_{1010} = C_{13}^b \]

\[ X_{10} = \bar{e}_x \quad F_{10} = F^* \]

\[ T_1 = (C_{16}^{e_x} + C_{26}^{e_y} + C_{36}^{e_z} + C_{66}^{e_{xy}})a \]
\[ T_2 = (C_{16}^{e_x} + C_{26}^{e_y} + C_{36}^{e_z} + C_{66}^{e_{xy}})c \]
\[ T_3 = (C_{16}^{e_x} + C_{26}^{e_y} + C_{23}^{e_z} + C_{66}^{e_{xy}})e \]
\[ T_4 = (C_{12}^{e_x} + C_{22}^{e_y} + C_{23}^{e_z} + C_{26}^{e_{xy}})a \]
\[ T_5 = (C_{12}^{e_x} + C_{22}^{e_y} + C_{23}^{e_z} + C_{26}^{e_{xy}})c \]
\[ T_6 = (\ddot{c}_{12} \varepsilon_x^T + \ddot{c}_{22} \varepsilon_y^T + \ddot{c}_{23} \varepsilon_z^T + \ddot{c}_{26} \gamma_{xy}) \epsilon \]

\[ T_7 = (\ddot{c}_{13} \varepsilon_x^T + \ddot{c}_{23} \varepsilon_y^T + \ddot{c}_{33} \varepsilon_z^T + \ddot{c}_{36} \gamma_{xy}) \delta \]

\[ T_8 = (\ddot{c}_{13} \varepsilon_x^T + \ddot{c}_{23} \varepsilon_y^T + \ddot{c}_{33} \varepsilon_z^T + \ddot{c}_{36} \gamma_{xy}) \delta \]

\[ T_9 = (\ddot{c}_{13} \varepsilon_x^T + \ddot{c}_{23} \varepsilon_y^T + \ddot{c}_{33} \varepsilon_z^T + \ddot{c}_{36} \gamma_{xy}) \delta \]

\[ T_{10} = (\ddot{c}_{11} \varepsilon_x^T + \ddot{c}_{12} \varepsilon_y^T + \ddot{c}_{13} \varepsilon_z^T + \ddot{c}_{16} \gamma_{xy}) \delta \]

where

\[ \varepsilon_x^T = (m^2 \alpha_1 + n^2 \alpha_2) \Delta T \]

\[ \varepsilon_y^T = (n^2 \alpha_1 + m^2 \alpha_2) \Delta T \]

\[ \varepsilon_z^T = \alpha_3 \Delta T \]

\[ \gamma_{xy}^T = 2mn(\alpha_1 - \alpha_2) \Delta T \]

For moisture analysis the vector \( \{T\} \) is identical except \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are replaced by \( \beta_1, \beta_2 \) and \( \beta_3 \).
APPENDIX C

TENSOR POLYNOMIAL FAILURE CRITERION
APPENDIX C
TENSOR POLYNOMIAL FAILURE CRITERION

The tensor polynomial failure criterion in the contracted tensor notation (for an orthotropic material in the principal material directions) has the form

\[ F_1 \sigma_1 + F_2 \sigma_2 + F_3 \sigma_3 + F_{11} \tau_{11} + F_{22} \tau_{22} + F_{33} \tau_{33} + F_{44} \tau_{44} + F_{55} \tau_{55} + F_{66} \tau_{66} + 2F_{12} \sigma_1 \sigma_2 + 2F_{13} \sigma_1 \sigma_3 + 2F_{23} \sigma_2 \sigma_3 = 1 \]  

(C.1)

where the \( F_i \) and \( F_{ij} \) terms are as previously defined in Chapter 3.

In the \( xyz \) (laminate) coordinate system, the tensor polynomial failure criterion transforms (from the \( 1-2 \) to \( x-y \) by anticlockwise rotation of \( +\theta \)) into

\[ F'_{1x} \tau_{1x} + F'_{1y} \tau_{1y} + F'_{1z} \tau_{1z} + F'_{2x} \tau_{2x} + F'_{2y} \tau_{2y} + F'_{2z} \tau_{2z} + F'_{3x} \tau_{3x} + F'_{3y} \tau_{3y} + F'_{3z} \tau_{3z} + 2F'_{12} \sigma_1 \sigma_2 + 2F'_{13} \sigma_1 \sigma_3 + 2F'_{23} \sigma_2 \sigma_3 = 1 \]

(C.2)

where the \( F' \) terms, as functions of the unprimed \( F \)'s and \( \theta \), are as follows (\( m = \cos \theta \), \( n = \sin \theta \))

\[ F'_1 = m^2 F_1 + n^2 F_2 \]
\[ F'_2 = n^2 F_1 + m^2 F_2 \]
\[ F'_{3} = F_{3} \]
\[ F'_{6} = -2mn(F_{1} - F_{2}) \]
\[ F'_{11} = m^{4}F_{11} + m^{2}n^{2}(F_{66} + 2F_{12}) + n^{4}F_{22} \]
\[ F'_{22} = n^{4}F_{11} + m^{2}n^{2}(F_{66} + 2F_{12}) + m^{4}F_{22} \]
\[ F'_{33} = F_{33} \]
\[ F'_{44} = m^{2}F_{44} + n^{2}F_{55} \]
\[ F'_{55} = n^{2}F_{44} + m^{2}F_{55} \]
\[ F'_{66} = 4m^{2}n^{2}(F_{11} + F_{22} - 2F_{12}) + (m^{2} - n^{2})^{2}F_{66} \]
\[ F'_{16} = -mn[2(m^{2}F_{11} - n^{2}F_{22}) - (m^{2} - n^{2})(2F_{12} + F_{66})] \]
\[ F'_{26} = -mn[2(n^{2}F_{11} - m^{2}F_{22}) + (m^{2} - n^{2})(2F_{12} + F_{66})] \]
\[ F'_{36} = -mn(F_{13} - F_{23}) \]
\[ F'_{45} = mn(F_{44} - F_{55}) \]
\[ F'_{12} = m^{2}n^{2}(F_{11} + F_{22} - F_{66}) + (m^{4} + n^{4})F_{12} \]
\[ F'_{13} = m^{2}F_{13} + n^{2}F_{23} \]
\[ F'_{23} = n^{2}F_{13} + m^{2}F_{23} \]

These are transformations from the right handed 1-2 coordinate system into another right hand coordinate system obtained by an anticlockwise rotation of \( \theta \) about the 3 axis. If a ply is oriented at \( +\theta \) from the laminate axis, the \( F'_{ij} \) are obtained by using the above equations with the sines and cosines of \(-\theta\).
APPENDIX D

USERS GUIDE FOR NONCOM 3

(Available upon request)
FIGURE E1. MESH E1

124 ELEMENTS, 81 NODES
FIGURE E2. MESH E2
FIGURE E3. MESH E3

326 ELEMENTS, 191 NODES
FIGURE E4. MESH E4

598 ELEMENTS, 346 NODES
FIGURE E5. MESH E5

878 ELEMENTS, 490 NODES
APPENDIX F

NONCOM3 FLOW CHART
NONCOM3 FLOW CHART

START

DATA READ IN
R T ELASTIC MODULII
R O PARAMETERS, ETC

IS LOAD THERMAL OR
MOISTURE?

CALCULATE GRID
PROPERTIES

INITIALIZE
VARIOUS ARRAYS

NUMBER EQUATIONS
IMPOSE CONSTRAINTS

IS LOAD THERMAL OR
MOISTURE?

CALCULATE ELASTIC
MODULII FROM R O
PARAMETERS

CALCULATE & TRANSFORM
LAMINA STIFFNESSES

CALCULATE & TRANSFORM
STRENGTH TENSORS

CALCULATE & ASSEMBLE
ELEMENT STIFFNESS
MATRICES

SOLVE SYSTEM OF
EQUATIONS FOR MODAL
DISPLACEMENTS

CALCULATE STRAINS &
STRESSES FROM
DISPLACEMENTS

CALCULATE TENSOR
POLYNOMIAL & CHECK
FOR FAILURE

ARE ALL LOAD STEPS
COMPLETED?

ARE ALL LOAD CASES
COMPLETED?

STOP

DATA READ IN
RETENTION CURVES FOR
TEMPERATURE & MOISTURE.
APPENDIX G

T300/5208 GRAPHITE-EPOXY PROPERTIES
<table>
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<tr>
<th>Curve</th>
<th>Elastic Modulus (MSI)</th>
<th>Elastic Limit (KSI)</th>
<th>$n_1$</th>
<th>$K_1$(PSI$^{-n_1}$)</th>
<th>$\sigma^*$ (KSI)</th>
<th>$n_2$</th>
<th>$K_2$(PSI$^{-n_2}$)</th>
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<td>Elastic Modulus (Msi)</td>
<td>Percent Retention of Room Temp., 0% Moisture Property</td>
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### TABLE G.2 continued

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<th>Room Temp. 0% Moisture Temperature Coefficient (μin/in/°F)</th>
<th>Temperature Coefficient (μin/in/°F)</th>
<th>Percent Weight Gain</th>
<th>Percent Weight Gain</th>
<th>Percent Weight Gain</th>
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<td>0.00     0.83 1.13</td>
<td>0.00 0.83 1.13</td>
<td>0.00 0.83 1.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.193    -     -</td>
<td>0.226 - -</td>
<td>0.226 - -</td>
<td></td>
</tr>
<tr>
<td>α₂</td>
<td></td>
<td>13.8     13.8  -</td>
<td>13.8 - -</td>
<td>13.8 - -</td>
<td></td>
</tr>
<tr>
<td>α₃</td>
<td></td>
<td>13.8     13.8  -</td>
<td>13.8 - -</td>
<td>13.8 - -</td>
<td></td>
</tr>
</tbody>
</table>

α₁ has the value 0.193 from room temperature to 120°F and 0.226 from 120°F to 350°F.
### TABLE G.2 continued

<table>
<thead>
<tr>
<th>Property</th>
<th>Room Temp. 0% Moisture Coefficient (in/in/%Wt)</th>
<th>Moisture Coefficient (in/in/%Wt Gain)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Temperature 70° F</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Percent Weight Gain</td>
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<tr>
<td></td>
<td></td>
<td>0.00  0.83  1.13</td>
</tr>
<tr>
<td>$\beta_1$</td>
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<td>0.0  0.0  0.0</td>
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<tr>
<td>$\beta_2$</td>
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<td>0.0  0.0049  0.0061</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.0</td>
<td>0.0  0.0049  0.0061</td>
</tr>
</tbody>
</table>
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