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CORNELL UNIVERSITY  
Center for Radiophysics and Space Research  
ITHACA, N.Y.

FINAL TECHNICAL REPORT  
to the  
National Aeronautics and Space Administration  
for  
NASA Grant NSG 7126  
"Studies in Occultation Astronomy"  
February 1, 1975 to March 31, 1979

Principal Investigator: Professor Joseph Veverka
FINAL TECHNICAL REPORT

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Report prepared by:

Prof. Joseph Veverka
Principal Investigator
May 1980
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ABSTRACT

This final report summarizes the major scientific results of the research carried out under NASA Grant NSG-7126.
I. INTRODUCTION

NASA Grant NSG-7126 supported a wide range of work in occultation astronomy under the direction of Professors J. Veverka and J. Elliot. The effort at Cornell was terminated in 1979, after Professor Elliot accepted a new position at the Massachusetts Institute of Technology.

2. MAJOR RESULTS

A) Observations of the 8 April 1976 Occultation of ε Gem by Mars

The results of this study were published in three papers:


Copies of these three papers are included in Appendix 1.

B) Studies in Occultation Techniques

Two major papers resulted from this aspect of our investigation:


Texts of these two papers appear in Appendix 2.

C) Studies of the March 1974 Occultation of Saturn by the Moon

The results of this phase of our effort were published in two papers:


These two papers are reproduced in Appendix 3.

In addition we have refined our analysis of the Titan observations obtained during this occultation. The initial analysis was published in Icarus 26, 387-407 (1975). The conclusions of our re-analysis are summarized in the following section.

D) Re-Analysis of the 1974 Lunar Occultation of Titan

Observations of the March 30, 1974, occultation of Titan by the moon are an important data set with which to compare models of Titan's atmosphere. Elliot et al. (1975, Icarus 26, 387-407) fitted the data assuming zero phase angle and
using a Minnaert scattering law. They concluded that strong limb darkening was present and that Titan's diameter, D, \( \gtrsim 5800 \) km.

We have extended the analysis with the following objectives:

1) to determine if assuming the proper occultation geometry modifies Elliot's conclusions,
2) to fit a variety of model light curves to the data,
3) to compare the occultation observations with observed limb-darkening laws for Jupiter and Saturn,
4) to determine the constraints that the observations might place on the wavelength dependence of acceptable limb-darkening models;
5) to investigate systematic effects and other sources of error.

Our conclusions are:

1) Elliot's results are almost unaffected by proper inclusion of the event geometry. The inferred diameter of Titan changes by only 15 km, although the residuals of the fit drop dramatically.

2) Of the many homogeneous models we tried, the Minnaert law, \( I(\mu, \mu_0) = (\mu \mu_0)^k/\mu \) gave the lowest residuals, with formal error \( k = 1.64 \pm 0.35 \). All models which gave low residuals required very strong limb darkening (\( k \leq 1 \)).
3) Observations of $I(\mu)$ for Saturn and Jupiter were used to produce model light curves which were fit to the observations. The observed limb darkening corresponds closely to a Lambert surface for Jupiter ($k = 1$), and slightly smaller $k$ for Saturn. Neither model gave a good fit to the occultation data.

4) Observations of the geometric albedo ($P_\lambda$) and phase coefficient ($\beta_\lambda$) of Titan allow the parameters $g$ and $\omega_0$ in the Henyey-Greenstein scattering model to be determined uniquely. For each wavelength observed, the corresponding $g$ and $\omega_0$ were used to generate a model light curve which was then fit to the data. In each case, the corresponding limb darkening was much less than for the best Minnaert fit. Additionally, the trend of increasing limb darkening with increasing wavelength implied by the $\beta_\lambda$ observations was not detectable in the individual channel data. No simple homogeneous model is capable of fitting the observations of $P_\lambda$, $\beta_\lambda$, and the occultation results. However, if the atmosphere of Titan is modelled as a patchy haze layer over a uniform cloud deck, all of the observations can be reconciled, with a suitable choice of model parameters, corresponding to a haze coverage of about 95 percent of the satellite, with clear patches over the remaining 5 percent. Although the model works reasonably well, it is difficult to imagine how clear patches could be maintained in a photochemical smog.

5) All of the preceding discussion assumes that the lunar limb is a straightedge over the 2 km horizontal distance subtended by Titan at the moon. A careful study shows that lunar
limb topography can in principle lower the fit residuals drastically, even if Titan is assumed to be a uniformly bright disk. However, the required lunar limb slopes are unrealistically steep, and individual channels of data do not give consistent results. Furthermore, if lunar topography were important, a detectable timing difference would be present for the separate observing stations. No such timing difference was found. We conclude that lunar topography can be very important in some observations, but that it is not responsible for the extreme limb darkening inferred from the observations.

An extensive analysis of the power spectrum of the residuals of the light curve shows that the light curve is quite asymmetric about its half-intensity point; more than can be explained by event geometry alone. We find that there is only a 10 percent chance of this being due to noise alone, and we conclude that the more likely explanation is that Titan's atmosphere is not homogeneous. If Titan were truly a Lambert surface, there is only a 7 percent chance that noise would simulate the much stronger limb darkening we obtain in our fits.

Thus, our principle conclusions are:

- Titan is strongly limb darkened, with $D \geq 5800$ km.
- There is internal evidence in the data that Titan's atmosphere is inhomogeneous.
- The observations are inconsistent with any simple homogeneous model atmosphere which matches the $P_{\lambda}$ and $\beta_{\lambda}$ observations of Titan.
APPENDIX 1


Occultation of Epsilon Geminorum by 7 Ears: Evidence for Atmospheric Tides?

Abstract. Temperature profiles of the mariner atmosphere have been derived from airborne observations of 7 Ears in the 1976 occultation of Epsilon Geminorum. Within the altitude range from 50 to 200 kilometers, these profiles show peak-to-peak variations of 10 K, with a vertical scale of 20 kilometers and represent evidence for strong tidal forcing in the mariner atmosphere. However, more information is necessary to conclusively rule out a radiative explanation for the temperature variations.

The occultation of 7 Ears by Geminorum (visual magnitude = +3.1, spectral class G3B) on 8 April 1976 was observed with the 91-cm telescope aboard the National Aeronautics and Space Administration's Kuiper Airborne Observatory. The first occultation observations made with this facility (1). High-quality light curves for both immersion and emergence were obtained simultaneously at three wavelengths (0.37, 0.45, and 0.75 μm) with a time resolution of 4 μsec. A similar occultation of 7 Ears by Jupiter on 13 May 1977 yielded temperature and number density profiles of the jovian atmosphere (2), as well as a measurement of its H2 abundance (3). From our 7 Ears occultation data we have obtained temperature, pressure, and number density profiles of the mariner atmosphere and information about its composition from a differential refractivity measurement. Our calculation matched no more than 10 percent Ar (and N2) is mixed with pure CO2 (4) and is consistent with the low abundance of Ar and N2 obtained by the Viking entry probe (5). In addition, we have obtained the wavelength dependence of the extinction of the mariner atmosphere from the first observation of the central flash—a bright feature in the light curve that was recorded when 7 Ears aligned with the center of Mars (6). We report here temperature profiles deduced from the occultation data, which were obtained above the mariner cloud tops at 277 K, 237 K, and 186 K (emission) and 280 K, 214 K (emission) and cover an altitude range of about 50 to 200 km above the mean surface. These results are compared with those of Viking (1) and with theoretical predictions of thermal tides in the mariner atmosphere (8). The details of our observation, data analysis procedures, and other results are given elsewhere (6).

Temperature profiles were obtained from the occultation light curves with a new inversion procedure (7). This method is mathematically equivalent to standard inversion techniques (2, 8) but has the advantage that error bars (calculated from the known noise in the light curve) can be assigned to the temperature profiles. Thus we can use confidence in separating these features in the temperature profiles that are caused by the mariner atmosphere from the variations arising from the noise in the light curve. The temperature profiles obtained are given in Figs. 1 and 2. The profiles are for the light curves at 0.45 μm, the channel with the best signal-to-noise ratio; the profiles from other two channels agree within their errors. The upper boundary condition for the inversion is determined by an isothermal fit to the upper part of the light curve, indicated by the shaded region at the top of Figs. 1 and 2. The temperatures and their error bars are obtained from the inversion calculation. For each temperature point the error bars represent ±1 standard deviation from the mean temperature obtained from the light curve. The upper boundary condition for the inversion calculation is the light curve with the noise in the light curve. Neighboring temperature points agree better than the error bars because the noise affecting the points is correlated (7). The inversion calculation is terminated when the uncertainty in the upper bound line of the light curve is somewhat smaller than the mean for inversion. A temperature difference is to be expected from differences in solar energy absorption, such as the subsolar latitude is 97°, close to the emersion latitude of 97°N, but 10° from the emersion latitude of 97°S. The mean temperatures for both profiles (as well as the mean density) can be defined by comparing the mean temperatures observed from Mars (6) and Viking entry data (9).

A striking similarity between the occultation temperature profiles and the Viking profile is the wavelet vertical scale of the wavelet from the primary occultation and three pressure scale heights and a peak-to-peak amplitude about 1 km at a density of 1022 cm⁻³, roughly 600 scale heights above the 5-μbar pressure level. The wavelength and amplitude are in agreement with the general character of tidal waves predicted by Zark (10) for clear (not dusty) conditions. Detailed comparison of temperature profiles, including species information, with predictions ... at wavelengths for several reasons. The details of profiles depend upon the amount and distribution of traces of dust in the atmosphere, and these factors are unknown. There may be significant additional forcing due to boundary layer compressibility. Below 50 km in Zark's treatment. The large amplitude of the tides probably leads to the statistics and, as can be seen in the turbulence. Zark pointed out that such turbulence would influence the structure of the tide but is uncertain difficult to observe. There are other possible explanations for the thermal structure. Albritton's detailed radiative equilibrium calculations (11) showed that occultations of temperature with high might occur near these levels on Mars because the concentration of land solar absorption by photo dissociation products was negligible. Albritton's discussion temperature variations in Earth's atmosphere caused by the stratification of photodissociation products due to flow "sweeping." Finally, aerosols have been observed by the Viking orie... at heights as great as 40 km on Mars (2), and stratification layers could lead to varying radiative heating with height. Piersma's thermal layer structure to a slowly varying (not tidal) large-scale flow being probably not a possibility, because radiative relaxation times are less than 1 day (8).

The use of interference between our data and Zark's predictions, namely, the isothermal (not wary) thermal tides in the mariner atmosphere. However, radiative damping increases rapidly with height at these levels, and Zark remarked that its influence is difficult to predict accurately.
Martian Occultation of $\epsilon$ Gem as Observed from the C. E. Kenneth Mees Observatory

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Ground-based observations of the occultation of $\epsilon$ Gem by Mars on April 8, 1976 have been reduced in the manner of French et al. (Icarus 32, 155-202 (1977)) to yield the scale height and temperature profiles of the Martian atmosphere for number densities between $10^9$ and $10^{10}$ cm$^{-2}$. The deduced variations in temperature are remarkably similar to those obtained by Elliot et al. (Astrophys. J. 217, 601-679 (1977)) and to the in situ measurements from the Viking landers.

I. INTRODUCTION

$\epsilon$ Geminarum (m, = +3.1, Sp. G8 I b) was occulted by Mars on 8 April 1976. Several observations of this occultation have been reported both by ground-based observatories (Wasserman et al., 1977; Texas-Arizona Occultation Group, 1977) and by observers with the 91-cm telescope on the Kitt Peak Airborne Observatory (Elliot et al., 1976, 1977). In this paper we present the observations obtained with the 60-cm telescope of the C. E. Kenneth Mees Observatory. From the immersion and emergence light curves we have determined accurate times of half-light, and temperature profiles of the Martian atmosphere were obtained by numerical inversion under the assumption that the density gradients are parallel to the gravity gradients (Elliot and Veverka, 1976). The correctness of this assumption has been challenged (Young, 1976; Jokipii and Hubbard, 1977). The present observations strongly support the model in so far as they produce differential temperature profiles remarkably similar to those produced from the airborne observations of the same event although the two observations probed regions more than 400 km apart in the Martian atmosphere. Similar, wavelike structures were observed during the entry of Viking 1 on 23 July 1976 (Nier et al., 1976), and Viking 2 on 3 September 1976 (Self and Kirk, 1976).

II. OBSERVATIONS

Light curves of the occultation were obtained with one channel of our two-color photometer attached to the Cassegrain focus of the C. E. Kenneth Mees Observatory (latitude = $+42^{\circ}$42'40", longitude = $77^{\circ}$24'5). The photometer channel had a central wavelength of 4500 Å and a bandwidth of 100 Å FWHM; identical filters were used by Elliot et al. (1977), Wasserman et al. (1977), and Groth et al. (1978). The photometer was operated in a pulse-counting mode and the total photodetector count was recorded, without interruption, every 10 sec. For the present analysis, data were averaged at 100-msec resolution. The data system clock was synchronized to within 2 msec of UT by comparison with signals from W.W.V. 4 hr before the event. Immediately following the observation the clock was again checked against W.W.W. and found to have remained within 2 msec of UT.

At the time of the occultation, both Mars and $\epsilon$ Gem were contained within an aperture 150 arcsec in diameter. Continuous data recording began at 00:50 UT and ended at 01:10 UT. Figures 1a and 1b show the immersion and emergence events at 0.1-sec resolution. The data are considerably more noisy than those obtained from the Airborne Observatory, indicating one of the advantages of using that platform for occultation observations. In addition, the present immersion light curve is considerably noisier than the immersion curve. Throughout the observation the photometric quality of the signal degraded appreciably. As with the airborne observa-
tions, spikes in the light curve are much less pronounced than for previous occultations by Neptune and Jupiter (Elliot and Veverka, 1976). Nevertheless, the principal spike in the airborne data as given at 0457-0525 UTC in Fig. 2 of Elliot et al. (1977) also appears in our ground-based data. This feature was also noted in the ground-based observations of Wasserman et al. (1977).

III. RESULTS

The first step in the reduction process was to make a least-squares fit to an occultation curve assuming an isothermal atmosphere after the method of Baum and Code (1963). Parameters fit to this for both immersion and emersion are the mean photon count rates from Mars and from a Gem, the half-light time, and the scale height divided by the perpendicular component of the Martian velocity. The results of this fit are given separately in Table I. The mean count rates so obtained defined the zero and unit stellar intensity levels which were used in the numerical inversion of the light curves. The half-light time is a useful quantity for occultation astrometry, and has been used to deduce the oblateness of the Martian atmosphere (Taylor, 1976). We attribute little significance to the isothermal fit value of the scale height; it has been shown (Wasserman and Veverka, 1973) that such fits can give badly erroneous mean scale heights.

We have calculated the temperature versus altitude for the Martian atmosphere in the manner discussed by Elliot et al. (1977). In so doing the following assumptions are made: (i) the density gradients in the atmosphere are parabolic to the local gravity gradient; (ii) the atmosphere is in hydrostatic equilibrium; and (iii) ray-crossing is not severe. Under these assumptions, the desired profiles were obtained using the inversion technique of French et al. (1978).

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<td>Results of Isothermal Fits to the Data of Figs. 1a and b</td>
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<td>Parameter*</td>
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<tr>
<td>Immersion</td>
</tr>
<tr>
<td>$H/s_1 (sec)$</td>
</tr>
<tr>
<td>$a_s (sec^{-1})$</td>
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<tr>
<td>t (half-light time)</td>
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<tr>
<td>Emission</td>
</tr>
<tr>
<td>$H/s_2 (sec)$</td>
</tr>
<tr>
<td>$a_s (sec^{-1})$</td>
</tr>
<tr>
<td>t (half-light time)</td>
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* The parameters are the mean count rates $n_s$ and $a_s$ from Mars and +Gem, respectively; half-light time $t_h$, the atmospheric scale height $H$ and the apparent velocity $v_s$ of the star perpendicular to the level of Mars.

The errors given are formal errors from the fit.

The temperature profiles for immersion and emersion are shown in Figs. 2a and b, respectively. The suboccultation coordinates on Mars are $18^\circ 33^\prime 32^\prime W, and 33^\circ 51^\prime W$, respectively. The error bars have a total length of two standard deviations and reflect the uncertainty due to the known amount of shot noise present in the signal. The hatched region in the immersion profile corresponds to the region of the isothermal fit used to initiate the inversion, as discussed by French et al. (1978). Such a region is not shown for the immersion profile; because of the significantly higher noise level, the least-squares fitting routine was unable to find an isothermal fit unless the entire light curve was used. The formal errors are large; however, they do encompass the mean value of about 145 K found by Elliot et al. (1977), Wasserman et al. (1977), Viking 1 (Nier et al., 1977), and Viking 2 (Seiff and Kirk, 1978). The temperature profiles obtained by Elliot et al. (1977) are shown for comparison.

Because relative temperature variations as a function of altitude are more accurate than absolute temperature profiles (Elliot et al., 1977), we have fitted linear temperature gradients to the temperature profiles of Figs. 2a and b. The deviations from the least-squares fits to a constant temperature gradient are shown in Figs. 2a and b, respectively. Also shown on these diagrams are the temperature variations obtained from the airborne data. The profiles are aligned in altitude relative to the altitude corresponding to the half-light point of each light curve. The remarkable similarity is readily apparent, especially for the immersion event. Pronounced wavelike structures are evident in both immersion and emersion profiles. We conclude that turbulence in the Martian atmosphere is of minor importance in interpreting the +Gem occultation light curves as opposed to the viewpoint of the Texas-Arizona Occultation Group (1977). This result fortifies the similar conclusion reached by Wasserman et al. (1977). We also conclude that the consistency of the results obtained by the present and previously reported work, both airborne and ground-based, confirms that wavelike temperature variations in the Martian atmosphere extend horizontally over distances of at least several hundred kilometers.

The similarity of the temperature profiles obtained by several workers implies that the light curves of the occultation are similar. A cross-correlation analysis of the residuals in the unprocessed light curves will be presented as part of a
detailed comparison, in preparation, of available observations of the \( \varepsilon \) Gemocronin occultation.

ACKNOWLEDGMENTS

We wish to thank J. L. Elliot, who suggested that we observe the occultation. This work was supported in part by NASA Grant NGR-74-029. R. A. Berg and R. Hardwick assisted in translating the data tapes and G. Housmade made some preliminary sorting of the data.

REFERENCES


The occultation of α Geminorum by Mars on 1976 April 8 was observed at three wavelengths and 4 ms time resolution with the 91 cm telescope aboard NASA's G. K. P. Kuiper Airborne Observatory. Since most of the Earth's atmosphere was below the telescope, scintillation noise in the light curves was greatly reduced from that encountered by ground-based observers. Temperature, pressure, and number-density profiles of the Martian atmosphere were obtained for both the immersion and emergence events. Within the altitude range 20-80 km above the mean surface the temperature is ~145 K, and the profiles exhibit wave-like structures with a peak-to-peak amplitude of 35 K and a vertical scale of about 20 km. The ratio of the refractivity of the atmosphere at 4500 Å and 7500 Å, determined from the time shift of the light curves for these wavelengths, is consistent with the atmospheric composition measured by Viking 1, 15 weeks later. From the "central flash"—a bright feature in the light curve midway between immersion and emergence—we find an optical depth at 4500 Å of 3 ± 1.7 per km atm (0.23 per equivalent Martian air mass) for the atmosphere about 25 km above the mean surface, near the south polar region. This large value and its weak wavelength dependence rule out Rayleigh scattering as the principal cause of the observed extinction.

Subject headings: occultations—planets: Mars—stars: individual

I. INTRODUCTION

The occultation of α Geminorum (m = +3.1, G8 IV) by Mars on 1976 April 8 was observed at three wavelengths with the 91 cm telescope aboard the Kuiper Airborne Observatory. A highlight of these observations was the discovery of the "central flash" when α Gem was directly behind the center of Mars (Elliot, Dunham, and Church 1976). The records of the central flash yielded unexpected data on extinction in the Martian lower atmosphere—a new application for stellar occultation observations.

From the immersion and emergence light curves we have obtained temperature, pressure, and number-density profiles for the Martian atmosphere under the assumption that the density gradients are parallel to the gravity gradient. In the context of the β Scorpior occultation by Jupiter, the validity of this assumption has been disputed, and no evidence exists to settle the issue conclusively (Young 1976; Elliot and Veverka 1976; Jokipii and Hubbard 1977). The α Gem occultation presents a unique opportunity to compare the structure and composition of the Martian upper atmosphere, obtained under the gravity-gradient assumption, with the in situ measurements made during the entry of Viking 1. A significant aspect of our analysis is the use of a new inversion technique (French, Elliot, and Gierasch 1977) that assigns error bars to the temperature, pressure, and number-density profiles.

II. OBSERVATIONS

Light curves of the occultation were obtained with our three-channel photometer (Elliot, Veverka, and Goguen 1975) attached to the best Cassagrain focus of the 91 cm telescope aboard NASA's Kuiper Airborne Observatory (KAO). From the predictions by Taylor (1976a), the flight path was planned so that the apparent velocity of α Gem was strictly perpendicular to the limb of Mars. This course was chosen to facilitate the analysis of the differential refractivity measurements (see §IV) but also resulted in the discovery of the "central flash" (see §V).

At the time of the occultation, the Martian subsolar latitude was +12° and the planetocentric longitude of the Sun (Ls) was 341°. Immersion occurred at about 0330 local Martian time above the suboccultation point 27° S and 321° W longitude. The suboccultation point for emersion was 28° N, 152° W longitude, and the event occurred at about 1330 local solar time.

According to the inertial navigation system on board the KAO, the occultation was located at altitude 33°26'24" N and longitude 69°43'20" W at immersion and latitude 36°04'31" N and longitude 69°43'23" W at emersion. The times for immersion (00h05m19s48 ± 0.040 UTC) and emersion (01h02m34s01 ± 0.040 UTC) are defined by the "half-light" times obtained by fitting an isothermal light curve to the data (Baum and Code 1933). Errors in the telescope coordinates, owing to uncalibrated internal errors in the inertial navigation system, are probably less than 2'. For both events the altitude of the plane was 21.4 km above sea level; its ground speed of 9.22 km s⁻¹, on a heading of 337°, was only 1% of the the shadow velocity (21.9 km s⁻¹; Taylor 1976a).

The center wavelengths and passbands of the three photometric channels are given in Table I. The photomultiplier for channel 1 had an S-2 photocathode response while those for channels 2 and 3 had an S-20 response. Each was contained in an uncooled, rf-shielded housing. Because of the large photon fluxes incident on the photomultipliers, voltage-to-frequency converters (Dunham and Elliot 1977) were used instead of pulse amplifiers. The alignments of the photomultipliers and their respective field lenses were carefully adjusted to make the photometer response independent of the position of a source within the entrance aperture of the photometer. Deviations from an ideal flat response could cause errors in the light curve in the event of poor telescope tracking. In fact, throughout the observations the telescope tracking was excellent, as confirmed by watching the image of Mars on a television screen that monitored a portion of the light received from a beam splitter within the photometer.

The data were recorded as a continuous series of 4 ms integrations, made simultaneously in all three channels, with a data-recording system described previously (Elliot, Veverka, and Goguen 1975). The data system clock was synchronized with time signals from radio station WWV when the KAO made its closest approach to Boulder, Colorado, a few hours before the occultation.

At the time of the occultation, both Mars and α Gem were contained within an aperture 90° in diameter. Continuous data recording began at 0030 and ended at 0116 UTC. A light curve of the entire event at 4500 Å is shown in Figure 1, at 1 s time resolution.

**Figure 1.** Light curve of the occultation obtained at 4500 Å; each point represents a 1 s integration. The central flash was produced by radially symmetric refraction when α Gem was directly behind the center of the planet.
where the central flash occurs exactly at mid-occultation. Figures 2 and 3 show light curves for immersion and emersion at 0.1 s time resolution. Spikes in the light curve are much less pronounced than for previous occultations by Neptune and Jupiter (Elliot and Veverka 1976), at least partially because of a large projected diameter of e Gem at Mars (~6 km; de Vegt 1976).

Since our observations represent the first optical photometry performed with the airborne telescope, we shall briefly assess the photometric quality of the data. As seen in Figure 1, the baseline of the light curve is stable, showing little drift. The fourth column of Table 1 gives the ratio of the counting rate from e Gem (n_e) to that from Mars (n_m) and the background (n_b, sky, dark counts, and e-to-f offset). For all channels, n_e > n_b. The fifth column gives the rms noise in each channel for a 1 s integration, expressed as a fraction of the counting rate from e Gem. We denote this rms noise by σ_e and compute it from several seconds of data before the occultation, when any variation in the data would be due to the noise only. N is the mean counting rate for the jth integration bin of duration Δt (4 ms), N the number of integration bins, and ̄ the average counting rate for all N integration bins, then

\[ \sigma_e = \left( \frac{\Delta t}{N} \right)^{1/2} \left[ 1 + \frac{2}{N} \sum_{j=1}^{N} (n_j - ̄)^2 \right]^{1/2} \]  

We consider two sources as likely causes of the rms noise level σ_e: photon noise (shot noise), and terrestrial scintillation. We believe that photon noise, and not scintillation noise, makes the dominant contribution for the following reasons. The level of scintillation noise from Mars and e Gem for a telescope at 12.5 km altitude predicted by equation (2.1.6) of Young (1974) yields σ_e = 0.004 for our channel 2, which is less than the value calculated from the observations (Table 1). Furthermore, the rms scintillation noise decreases exponentially with the altitude of the telescope above sea level. Hence Young's equation would also predict that a light curve for this occultation obtained from a ground-based telescope of similar aperture would have a rms noise level σ_g = 0.02 owing to scintillation alone. This would imply that the light curves obtained from ground-based telescopes would be about 3 times noisier than the airborne data, which appears to be confirmed (Wasserman, Millis, and Williamon 1977; Texas-Arizona Occultation Group 1977). Later, we hope to expand upon this necessarily brief discussion of noise encountered in airborne photometry.

III. TEMPERATURE, PRESSURE, AND NUMBER-DENSITY PROFILES

a) Method

When e Gem was occulted by Mars, the process that caused the starlight to dim was differential refraction by the Martian atmosphere. From the light curves of Figures 2 and 3 we can obtain temperature, pressure, and number-density profiles for the Martian atmosphere if the following assumptions are satisfied: (i) the density gradients in the atmosphere are parallel to the local gravity gradient (i.e., perpendicular to the limb); (ii) the atmosphere is in hydrostatic equilibrium; and (iii) ray crossing is not severe (see § V of Elliot and
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VEVERKA 1976). Under these assumptions we can obtain the desired profiles through the inversion technique of French, Elliot, and Gierasch (1977), which is similar to the "standard" inversion method (Kovalevsky and Elliot 1969; Hubbard et al. 1972; Wasserman and Veverka 1973), but has the advantage of assigning error bars to the temperature profiles. In the following discussion we use the notation for the occultation geometry shown in Figure I of French, Elliot, and Gierasch (1977).

From the observations we obtain the starlight intensity \( I(h) \) at a wavelength \( \lambda \) as a function of the time \( t \). The intensity is normalized by the unocculted stellar intensity so that \( I(h) \) begins at 1.0 and drops to 0.0 for an immersion event. Invoking the assumptions of no ray crossing and no density gradients parallel to the limb, we can write an implicit equation for the time \( t(h) \) at which the symmetrical path of the starlight on the occultation curve has reached a level \( h \) deeper into the atmosphere at a previous time \( t(h) \):

\[
\Delta h = t_{1}(h)^{(a-b)} \, \phi_{h}(t) \Delta h. \tag{2}
\]

In equation (2), \( v_{1} \) is the apparent velocity of the star perpendicular to the limb of Mars. For the same shell of atmosphere of thickness \( \Delta h \), the refraction angle \( \phi_{h}(t) \) changes by \( \Delta \phi_{h}(t) \):

\[
\Delta \phi_{h}(t) = \frac{\Delta h}{D} \int_{\Delta h}^{\Delta h} \left[ 1 - \phi_{h}(t') \right] d t', \tag{3}
\]

where \( D \) is the Earth-Mars distance. The function \( \phi_{h}(t) \) is equal to the function \( \phi_{h}(t) \), from which we can derive information about the Martian atmosphere. In addition to its dependence on refraction by the Martian atmosphere, \( \phi_{h}(t) \) contains noise that propagates into the values of \( \Delta h \) and \( \Delta \phi_{h}(t) \). If the noise that affects \( \phi_{h}(t) \) is Gaussian white noise (photonic noise, for example), then \( \Phi_{h}(t) \) and \( \Delta \phi_{h}(t) \) are the rms errors for \( \Delta h \) and \( \Delta \phi_{h}(t) \), and are given by

\[
\Phi_{h}(t) = v_{1} \left[ \int_{\Delta h}^{\Delta h} \Phi_{h}(t') d t' \right]^{1/2}, \tag{4}
\]

\[
\Phi_{h}(t) = \frac{\Phi_{h}}{D}, \tag{5}
\]

where the integrand is the variance of \( \phi_{h}(t') \) for the time interval \( t' \).

We can write an equation for the number-density profile, \( n(h) \):

\[
n(h) = \left( \frac{1}{\pi} \right)^{2} \frac{a}{(b)^{1/3}} \int_{\Delta h}^{\Delta h} \left( h' - h \right)^{-1/3} d h', \tag{6}
\]

\[
\Delta h = \frac{2}{a} \left( \frac{b}{a} \right)^{1/3} \Phi_{h} \tag{7}
\]

1 About eq (11) given by Baum and Code (1953) does not include the lateral focusing effects of a spherical planetary atmosphere. French et al. (1977) show that the resulting error introduced into the values of \( n(h), p(h), \) and \( H(h) \) is negligible.
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The altitude scales for these and subsequent figures were obtained from a preliminary astrometric solution (Taylor 1976).

The zero points for the immersion and emersion altitude scales are 3401 ± 5 km and 2604 ± 7 km from the center of Mars. From these values our altitude scales can be related to the altitude above the true surface for any model of the surface figure. The accuracy of the relative altitude scales for the same event but different channels depends on the integration of occultation light curve (eq. (2)); these should have errors of only 1–2 km. For clarity we have not plotted error bars on Figures 4 and 5, but the scatter of the values for the three different channels gives a good indication of the magnitude of the errors.

The top of the profile is the largest, then they decrease, reaching their minimum value, ~35 K, for number densities which correspond to an altitude of ~60 km before increasing again. The uncertainty in the baseline (a) of the occultation curve also causes an additional error in the profiles for low altitudes. We have ended our plots at levels where we believe the error caused by baseline uncertainty about equals the error caused by shot noise in the light curve.

Figures 6 and 7 show the temperature profiles for immersion and emersion obtained from equation (6). The behavior of the errors in these profiles is similar to that for the number-density profiles discussed above.

In Figures 8 and 9 we have plotted temperature versus number-density profiles for our light curves at 4500 Å; these had the lowest noise level (see Table 1). The shaded portion of the figure corresponds to altitudes greater than 5 km, the region of the isothermal fit. The portions of the light curves required to generate these profiles are the segments shown in Figures 2 and 3. The error bars have been calculated from the light curve noise as described in the previous section. Since the noise that affects neighboring points in the profile is correlated, the random scatter of neighboring points is much less than the absolute error in temperature for each point (indicated by the error bars). The profiles show wave-like variations, with peak-to-peak amplitudes of ~35 K and a vertical scale of 20 km. These variations clearly exceed any that could be attributed to noise.

Temperature versus altitude profiles for all three channels are shown in Figures 10 and 11. The errors for each channel are proportional to the rms noise for that channel (Table 1; French, Elliot, and Gierasch 1977), so that the error bars for channel 1 are about twice as large as those for channel 2 (see Figs. 7, 8); the errors for channel 3 are comparable to those for channel 2. The profiles mutually agree within their error bars.

The temperature variations on these profiles show better agreement than the absolute temperatures, because the variations are not sensitive to the large initial errors which affect the profile for several scale heights. Since the errors in successive points are correlated, short-scale temperature gradients are more reliable than implied by the error bars, which more clearly reflect the uncertainty in the positioning of the profiles.

To see how well the short-scale temperature variations agree among the profiles for our three channels, we have removed a linear temperature fit from each profile. We write the temperature \( T(h) \) in the following form:

\[
T(h) = T + \frac{dT}{dh} (h - h_0) + \Delta T(h), \tag{10}
\]

where \( T \) is the mean temperature and \( \frac{dT}{dh} \) is the mean temperature gradient over an altitude interval that has a mean altitude \( h \). The quantity \( \Delta T(h) \) is the difference between \( T(h) \) and the linear function in brackets (eq. 10).

For each temperature profile we fit by least-squares for \( T \) and \( \frac{dT}{dh} \) over the altitude interval 55–80 km for the immersion profiles and 72–80 km for the emersion profiles. The (unweighted) average of the \( T \)s obtained from the three immersion profiles was 143 ± 11 K, and the average \( T \) for the emersion profiles was 146 ± 9 K. The mean temperature gradients obtained from the fits were 0.4 ± 0.1 K km⁻¹ and 0.3 ± 0.3 K km⁻¹ for immersion and emersion, respectively.
used for the fit—particularly its relation to the phase of the wave-like structures. Hence from this analysis we conclude only that our "mean" temperatures for immersion and emission are not significantly different and that we see no large-scale gradients greater than \( \pm 0.7 \) K km\(^{-1}\).

After subtracting the linear fits from the temperature profiles, we have plotted the temperature residuals \( \Delta T(\theta) \) in Figures 12 and 13. The agreement among the three profiles is excellent, with an average rms dispersion of 2-3 K except near the end points. The wave-like structures appear in both figures. The main difference among the profiles is the nearly isotothermal characteristic of the emission profiles above about 70 km. We note that the immersion and emission profiles are nearly identical in their region of overlap if the emission profile is displaced 17 km upward. The altitude difference is significant, since the difference in the zero points of the immersion and emission altitude scales should be not more than 12 km (see previous discussion). Further support for this conclusion is found from examination of Figures 8 and 9 where we see that a temperature maximum occurs at a number density of 6 x 10\(^4\) cm\(^{-3}\) for immersion; for emission, however, the temperature maximum nearest to this occurs at a number density of 3.5 x 10\(^4\) cm\(^{-3}\). This comparison is independent of the altitude scales and shows that the phase of the wave-like temperature variations, relative to the number density of the atmosphere, differs for the regions of the atmosphere probed by the immersion and emission events.

We emphasize that our model and results can be compared directly with the profiles of Figures 12 and 13 only after a linear temperature fit is subtracted (eq. [10]).

**IV. REFRACTIVITY DISPERSION AND ATMOSPHERIC COMPOSITION**

From our light curves we can determine the ratio of the refractivity of the Martian atmosphere at two wavelengths, and from this measurement place limits on the amount of gases other than CO\(_2\) in the atmosphere. The precision of this technique is inferior to that of the methods used by Viking I (Nier et al. 1976; Owen and Biemann 1976), but it is comparable with that of other remote sensing methods used to determine the helium fraction of Jupiter's atmosphere (Hunten and Veverka 1976). Hence a comparison of our results with the more accurate measurements of Viking is an important test of the occultation method for determining the composition of planetary atmospheres.

During the occultation, refractive dispersion of the gases that compose the Martian atmosphere causes the light curve \( \mathcal{G}(t) \) at wavelength \( \lambda_1 \) to be delayed by a time \( \tau \) relative to the light curve \( \mathcal{G}(t) \) obtained at wavelength \( \lambda_2 \), so that

\[
\mathcal{G}(t + \tau) = \mathcal{G}(t).
\]

The delay \( \tau \) is related to the refractivities of the atmosphere, \( \eta_1 \) and \( \eta_2 \), at wavelengths \( \lambda_1 \) and \( \lambda_2 \) by the relation (derived from eqs. [6]-[9] of Elliot et al. 1974),

\[
\tau = \left( \frac{\eta_2 - \eta_1}{\eta_1} \right) \int_{1}^{\infty} \left( 1 - \mathcal{R}(t) \right) dt. \tag{12}
\]

The refractivity ratio \( \eta_2 / \eta_1 \) for the atmosphere is the ratio of the sum of the refractivities of its constituent gases. If \( \mathcal{R}(t) \) and \( \mathcal{R}(t) = (1 - \mathcal{R}(t)) \) are the fractions by number of argon and CO\(_2\) in a CO\(_2\)-argon atmosphere, then

\[
\eta_2 = \mathcal{R}(t) \eta_2 (\mathcal{C}(\lambda_2, \lambda_2)) + f(\eta_2 (\mathcal{A}(\lambda_2, \lambda_2)) \eta_1 (\mathcal{A}(\lambda_1, \lambda_1)), \lambda_1, \lambda_2, \lambda_3), \tag{13}
\]

where the \( \eta \)'s are the refractivities of argon and CO\(_2\) at wavelengths \( \lambda_1 \) and \( \lambda_2 \). Fortunately, modern laboratory measurements of the refractivities of argon and CO\(_2\) are available (Old, Gentil, and Peck 1971; Peck and Fisher 1964).

To use equation (13) we must also know the mean wavelengths of our three photometric channels, which are determined by combining the spectrum of a Gem and the transmission profiles of the interference filters used. The mean wavelengths for channels 2 and 3 are nearly equal to the center wavelengths of the interference filters (Table 1), but the ultraviolet spectrum of a Gem is steep, causing a significant and as yet undetermined shift of the mean wavelength of channel 1 to a larger value. Hence, if the light curve of channel 1 was not used in the present analysis.

For the Martian atmosphere, the refractivity ratio \( \eta_2 / \eta_1 = 1 \), for \( \lambda_1 = 4500 \) A, \( \lambda_1 = 7500 \) A was determined by the following procedure. A portion of the light curve \( \mathcal{G}(t) \) (at 4500 A) containing one or more spikes was selected for analysis, and for a test value of \( \eta_2 / \eta_1 = 1 \), the times delays \( \tau \) were computed for the 4 ms integration bin with the aid of equation (12). The delays \( \tau \) were applied to the light curve \( \mathcal{G}(t) \) (at 7500 A) to produce \( \mathcal{G}(t + \tau) \). Then the sum of the squared differences, \( \sum (\mathcal{G}(t) - \mathcal{G}(t + \tau))^2 \), was computed, and the increments of \( \tau \) over the interna selected for analysis. The computer was directed to try smaller values of \( \eta_2 / \eta_1 = 1 \) in increments of 0.0005 within the range \( -0.0005 \leq \eta_2 / \eta_1 - 1 \leq -0.0010 \). The test value that produced the minimum sum of squared differences was chosen as the best estimate of \( \eta_2 / \eta_1 - 1 \) for that portion of the light curve.

The above procedure was applied to several regions of the immersion and emission light curves that contained obvious spikes, and the resulting refractivity ratios are given in Table 2A. The error in their mean was computed from the internal consistency of the individual values. Next the procedure was applied to essentially the entire immersion light curve, from the beginning of the main intensity drop to the last major group of spikes, and to a corresponding interval of the emission light curve. The refractivity ratios so obtained are given in Table 2B.

Since the two approaches to finding the refractivity are based on essentially the same data, we might expect somewhat better agreement between the two results. However, both are consistent with pure CO\(_2\),
for which $vI/n - 1 = -0.0209 (\pm 0.0169$ for pure argon). Version (B) of our analysis has the advantage of including all the data but probably has a greater
error, since the spikeless portions of the curves contribute noise but no refractivity information. Hence we prefer version (A) of our analysis, since it presumably has a smaller error. This value ($-0.0206 \pm 0.0006$) has been plotted in Figure 14, along with curves for the refractivity ratios for various gases. Our measurement corresponds to an argon fraction by number, $N/\lambda(\text{Ar})$, equal to $10^2 \pm 20\%$, $10^3 \pm 10\%$. We note that this value applies to any combination of nitrogen and argon, since the refractivity ratios for these two gases are nearly equal (see Fig. 14). The amounts of argon and nitrogen found by Bailey are $17\%$ and $27\%$. (Owen and Biemann 1976), consistent with our result.

V. ATMOSPHERIC EXTINCTION AND THE CENTRAL FLASH

The discovery of the central flash, the bright feature in the occultation light curve midway between immersion and emersion, presented an unexpected opportunity to determine an average for extinction in the Martian atmosphere at lower altitudes than probed by the immersion and emersion events. We shall show that the flash was formed through symmetric refraction by the Martian atmosphere when e Geminorum was directly behind the center of Mars; our calculations show that this light passed through the atmosphere about 25 km above the mean surface. The integrated density along this path equals about 4 Martian air masses. The most abrupt variation in our observations of the central flash is the sharp rise at 00:59:56 UT (Fig. 16), which has a duration $\Delta \nu = 1.5 \times 10^{10}$. This establishes a minimum distance scale for large intensity variations within the Martian shadow of $\Delta x = 10$ km. Since this is much greater than the Freuden scale (0.2 km), we believe that a geometrical optics treatment is adequate

for a preliminary analysis. From the optics and the astrometry, we estimate the mean optical depth of the atmosphere at the altitude sampled by the flash, and, by comparing the integrated flux of the central flash at different wavelengths, we determine the wavelength dependence of the atmospheric extinction.

a) Intensity Profile of the Central Flash

i) Spherical Planet

For an occultation by a spherical planet with an isothermal atmosphere, Baum and Code (1953) have derived an implicit equation for the occulted stellar intensity which is valid only near the limb of the planet's shadow (i.e., the immersion and emersion events). Hence to obtain an intensity profile for the central flash, we must write an equation for the stellar intensity that is valid throughout the shadow. To do this we use the derivation of Baum and Code (1953), with two additional effects included: (i) the curvature of the limb perpendicular to the line of sight, which causes the convergence of the light rays at the center of the shadow, producing the central flash; and (ii) the additional contribution of light from the "emersion limb." For a point a distance $\rho$ from the center of the shadow, we define the "immersion limb" to be the point on the shadow limb closest to $\rho$ and the "emersion limb" to be the most distant. We shall derive an equation for the normalized stellar intensity, $I(\rho)/I(0)$, which can be obtained when $I(0)$ is specified.

If the atmosphere of the planet has a scale height $H$, then $I(\rho)$ is given by the equation derived by French (1977):

$$I(\rho) = 2H\rho \phi_1 \left[ \frac{1}{\rho} - 2 - \ln \left( \frac{1}{\rho} - 1 \right) \right].$$

(14)

The functions $\phi_1$ and $\phi_2$ are the intensities from the immersion and emersion limits that would be obtained by neglecting the effects of limb curvature and are given implicitly by Baum and Code's equation (11):

$$p_0 - p = H \left[ \frac{1}{\rho} - 2 \right] + \ln \left( \frac{1}{\rho} - 1 \right).$$

(15)

$$p_0 + p = H \left[ \frac{1}{\rho} - 2 \right] + \ln \left( \frac{1}{\rho} - 1 \right).$$

(16)

Here $p_0$ is the "half-light" radius of the shadow and is defined by $p_0 = p$ when $\phi_0 = \phi_1$. Near the limb of the shadow, $\rho \approx H$ and $\phi_0 \approx \phi_1$. Making these approximations in equation (14), we find that $4\rho_0 = H$, a result equivalent to Baum and Code's equation (11). Near the center of the shadow $\rho \ll H$ and $\phi_0 \approx \phi_1 = 1$. From equation (14) we see that $4\rho_0 \approx 2H$—the intensity of the central flash falls as $\rho^{-2}$ away from its point of perfect geometrical focus in the center of the shadow. The maximum intensity of the central flash is not

FIG. 14—Refraction ratio for the Martian atmosphere. The refractivity ratio [4O1]/[7500 A] - 1 for various gases is plotted, along with the value measured from the e Geminorum occultation. The result indicates an atmosphere with 10^7 (\pm 20%) to 10^3 (\pm 10%) argon and/or nitrogen, a value consistent with the composition found by Bailey (Owen and Biemann 1976). The refractivity ratio determined for the Jovian atmosphere from the b Sco occultation (Elliot et al. 1974) has also been plotted. For purposes of comparison with the present result, the value of [4O1]/[7500 A] - 1 actually measured for Jupiter has been scaled to [4O1]/[7500 A] - 1.
infinite and will depend on the radius of the occulted star and diffraction, two effects not considered in the present model. If the stellar radius is the dominant effect and \( r_s \) is its projected radius at the distance of the planet from its shadow, then the maximum intensity at the center of the shadow is \( 4 \pi r_s^2 \) for a uniformly bright stellar disk. The minimum intensity in the shadow occurs near \( r = r_s/2 \) and is approximately 16 \( \pi r_s^2 \).

For the present occultation, this value is 0.016.

ii) Oblate Planet

Since Mars is significantly oblate, we have extended the model of the previous section to include this effect. The limb is assumed to be an ellipse, and consequently the radius of curvature varies from point to point along the limb. The focus of perfect focusing is no longer confined to the point \( r = 0 \), but forms a curve known as the evolute of the ellipse. To illustrate the situation, we refer to the ray optics diagram in Figure 15, where normals to the ellipse have been drawn at equal intervals along its perimeter; the evolute is seen as a concave diamond shape. However, the density of lines in Figure 15 is not a true indicator of the intensity throughout the shadow, since the decreasing intensity of each ray with increasing distance from the limb has not been illustrated.

If \( r_e \) and \( r_n \) are the equatorial and polar radii of the planet, the \( (x, y) \)-coordinates of the limb obey the equation for an ellipse,

\[
\frac{x^2}{r_e^2} + \frac{y^2}{r_n^2} = 1,
\]

and the \( (x, y) \)-coordinates of the corresponding evolute are obtained from the following implicit equation (Beyer and Selby 1976):

\[
(x^2 + y^2) = (x^2 - r_y y^2).
\]

To calculate the intensity \( I(x, y) \) at a point \((x, y)\) within the shadow of an oblate planet, we use the following procedure. For each point \((x, y)\) on the observer's path through the shadow, we find the points on the limb \((x, y)\) whose normals intersect \((x, y)\). After finding the distance \( d \) between \((x, y)\) and \((x, y)\), we calculate the intensity at \((x, y)\) from the equation (11) for an atmosphere of scale height \( H \). The intensity is then enhanced by the factor \( r_y/\theta \), where \( r_y \) is the radius of curvature of the ellipse at \((x, y)\) and \( d \) is the distance between \((x, y)\) and the center of curvature of the ellipse at \((x, y)\). Then the intensities from all points \((x, y)\) on the limb whose normals intersect \((x, y)\) are added to obtain \( I(x, y) \).

The procedure outlined involves exactly the same steps that were used to obtain the analytic solution \( I(x) \) for a spherical planet (eq. [14]). Further details of the methods used for calculating \( I(x, y) \) are given by French (1977).

In this model we have smoothed the central flash along the \((x, y)\)-path (i.e., in one dimension only) by a triangular function that approximates the strip brightness distribution of \( e \) Gem. If the atmosphere has a total optical depth through the path traversed by each ray, we can include this extinction effect by multiplying the entire profile by \( e^{-\tau} \). Some of the light removed by extinction will in fact be scattered by the Martian atmosphere. Unless this scattering is strongly peaked at angles of a few arcsec (an unlikely possibility), this effect will not be important for our analysis.

b) Atmospheric Extinction

In the upper frame of Figure 16, we present light curves of the central flash at 0.1 s time resolution for all three wavelengths (Table I). The values of the background intensity, \( u_b \), and \( e \) Gem intensity, \( u_n \), used to obtain \( I(x) \) (eq. [9]) were the means of the values found for immersion and emersion. The lower frames of Figure 16 contain three model profiles of the central flash generated by the procedure described in the previous section. The path of the telescope relative to the evolute for each profile is shown in Figure 17. The shape of each profile was determined by five parameters: (i) \( e \) = \((r_e - r_n)/r_n\), the ellipticity of the model planet; (ii) \( L_0 \), the closest approach of the model planet to the center of the planet; (iii) \( \phi \), the angle between the telescope path and the shadow equation; (iv) \( H \), the scale height of the model planet's atmosphere; and (v) \( \tau \), the optical depth of the atmosphere along the path traversed by the light rays, which is assumed to be the same for all rays. For all our model profiles we chose \( H \) = 8 km and varied \( \tau \) to find the profile intensity scale that appeared to best fit the data.

For case A we adjusted all parameters except \( H \) to achieve the best agreement with the data.
the main features of the central flash can be reproduced: the broad wing at the left occurs when the path passes near a cusp of the evolute, and the two sharp peaks occur when the boundaries of the evolute are crossed.

For case B we set $\theta$ and $\epsilon$ equal to the values indicated by the preliminary astrometric solution (Taylor 1976), and adjusted $\rho_0$ and $\tau$ for best agreement with the data. The last profile was obtained by fixing $\varphi$ at its value from the astrometric solution and fixing $\epsilon = 0.05$, the ellipticity of the Martian surface (Christensen 1973). Again $\rho_0$ and $\tau$ were varied to achieve the best fit to the data.

The values for $\rho_0$, $\varphi$, and $\epsilon$ selected for the three model profiles would seem to bracket most reasonable possibilities, and these cases reproduce the main features of the data. Case A matches the data best, but the others use a more realistic value for $\varphi$. We feel that the average optical depth $\tau$ along the path should lie somewhere between the extremes of cases A and C. For a definite value we chose the mean for these two cases, with error bars that include both extremes: $\tau = 0.90 \pm 0.45$. This is the total optical depth at 4500A along a slant path through the atmosphere sampled by the flash.

In order to relate this extinction to a definite column density of Martian atmosphere, we first write an equation for the number density $n(h)$ of the Martian atmosphere at the height $h$, probed by the central flash,

$$n(h) = \frac{\rho(h)S}{\text{ext}(h)[2(R_h + h)]^2}$$  \hspace{1cm} (19)

We have assumed an isothermal atmosphere of scale height $H$, and the other quantities in equation (19) have been defined in \S III. To evaluate equation (19), we note that at the center of the shadow the refraction angle of the light which forms the central flash is $\theta(h) = (R_h + h)/D$, and we make the approximation $h/D \ll 1$. For a CO$_2$ atmosphere, $\rho_0(4500\text{A}) = 4.55 \times 10^{14}$, and setting $H = 8\text{ km}$, we find $n(h) = 1.75 \times 10^{10}$ cm$^{-2}$. The main uncertainty in $n(h)$ is the scale height $H$, which must vary around the limb of the planet, for a $\pm 20\%$ variation in scale height $n(h)$ will be uncertain by $\pm 10\%$.

The column density $N(h)$ of atmosphere traversed by the light that forms the central flash is given by

$$N(h) = m(h)[2(R_h + h)H]^{-2} = \frac{Hm(h)}{\text{ext}(h)}$$  \hspace{1cm} (20)

From equation (20) we find $N(h) = 0.27$ km atm$^{-2}$, where again the main uncertainty enters through the scale height $H$. This value of $N(h)$ is equivalent to about 4 Martian air masses if we assume one Martian air mass to be 0.670 km atm (Young 1969). Hence the optical depth of $0.90 \pm 0.45$ is about $3.3 \pm 1.7$ per km atm, or about $0.23 \pm 0.12$ per Martian air mass.

\begin{itemize}
  \item Case A $\rho_0 = 14^\circ$ $\rho_0 = 250.0$ km
  \item Case B $\rho_0 = 3.7^\circ$ $\rho_0 = 45.3$ km
  \item Case C $\rho_0 = 28^\circ$ $\rho_0 = 10.0$ km
\end{itemize}

Finally, we write an equation for the altitude of the atmosphere probed by the central flash:

$$h = H \ln \left[ \frac{n_0}{\rho(h)} \right]$$  \hspace{1cm} (21)

where $n_0$ is the number density at the mean surface and $H$ is an average scale height between the surface and $h$. For $H = 10\text{ km}$ and surface number density of $2.1 \times 10^{14}$ cm$^{-2}$, we find $h = 25\text{ km}$. For different values of $n_0$ and $H$ that occur around the planet, $h$ would lie in the range 20–30 km.

c Wavelength Dependence of the Extinction

We can also determine the wavelength dependence of the optical depth by comparing the light curves of the central flash at different wavelengths. If $\tau_{i\lambda}$ is the optical depth of the Martian atmosphere for the $i$th channel (Table 1) and $f_i$ is the normalized flux for that channel (eq. [9]), then the following equation holds:

$$f_i(t) \exp\left[+i(\tau_{i\lambda}/\lambda_0)\right] = f_0(t) \exp\left[+i(\tau_{0\lambda}/\lambda_0)\right]$$

$$= f_0(t) \exp\left[+i(\tau_{i\lambda}/\lambda_0)\right],$$  \hspace{1cm} (22)

where $f(t)$ is a function that accounts for the fact that each wavelength samples a slightly different altitude because of the variation of refractive index with wavelength ($\rho_{\lambda}$). For these calculations we let $f(t) = 1$. We adjusted the value of the optical depth difference, $\tau_{i\lambda} - \tau_{0\lambda}$, to minimize the squared difference between $f_i(t)$ and $f_0(t)$ within the time interval containing the central flash (from 00:59:46.9 to 01:00:02.9 UT; see Fig. 16) and obtained the value $\tau_{3800\text{A}} - \tau_{4500\text{A}} = 0.05 \pm 0.03$. The same procedure applied to $\tau_{1\lambda}$ and $\tau_{2\lambda}$ yielded the optical depth difference $\tau_{4500\text{A}} - \tau_{5300\text{A}} = 0.13 \pm 0.02$.

We have plotted these results in Figure 18, where we see that the relative extinction at different wavelengths is determined much better than the absolute value. The wavelength dependence of the optical depth is weaker than calculated for Rayleigh scattering, and the lower bounds on the optical depths greatly exceed the Rayleigh scattering value. We conclude that other extinction processes—by haze, dust, or high-level water-vapor clouds—which are decoupled at the 25 km level of the Martian atmosphere at the time of our observations.

Although our model explains several features of the central flash, the best model profile does not fit the data within the uncertainties of the random noise on the light curves. To extend this analysis, values of $\rho_0$ and $\varphi$ can be fixed when the final astrometric solution is available and the isothermal assumption can be replaced by a more realistic representation of horizontal and vertical temperature gradients. Also, since each segment of the central flash originates from a different region of the limb, it may be possible to obtain regional, rather than global, extinction information. The present astrometric solution (Taylor
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**VI. DISCUSSION**

The \( \theta \) Gem occulation occurred just 15 weeks before Viking I entry experiments, which measured the structure of the Martian atmosphere within the same altitude range probed by the occultation. Before Viking I landed on Mars, some of the results of Sections III and IV were issued in a report (Elliot et al. 1976a). These results served to assure the Viking Project that the entry dynamics had been configured for a proper model Martian atmosphere. The number densities probed by the occultation event corresponded to the critical level for the aerodynamic braking of an entry probe.

Now for the first time we can compare temperature profiles obtained from a stellar occultation with in situ measurements. The preliminary Viking I temperature profile (Nier et al. 1976) shows a mean temperature of \(-130 \, \text{K}\), slightly cooler than our values (\(-145 \, \text{K}\)), in the altitude range 50–80 km, while Wavell temperature profiles, with a vertical scale of 20 km, appear on both our emission and temperature profiles as well as the Viking I temperature profile. The pressure profiles from the occultation are comparable to that of the Viking I occultation, but the \( \theta \) Gem occultation by Jupiter in Figure 12 of Elliot and Veerika (1976), we see that at any given time the atmosphere is about 67% of the refraction is in the shape of a "squashed cylinder" (refraction cylinder) with its long axis along the line of sight. For the \( \theta \) Gem occultation, the length of the cylinder is 2.8, and its diameter about 6 km (the projected diameter of \( \theta \) Gem at Mars) when the occultation begins. The axis of the cylinder perpendicular to the limb decreases by the factor \( \phi \) (normalized occultation flux) as the occultation proceeds. Since the atmosphere is a gas, the center line, the motion of the refraction cylinder parallel to the limb was only a few km. The refraction cylinder extended almost exactly along a parallel of Martian latitude, and its length was about 6° of longitude.

Large-amplitude waves with long vertical and horizontal scales in the atmosphere have been observed by the Pioneer Venus Orbiter. This work is in progress.

Another check on our assumptions is to compare the direct measurements of atmospheric composition made by Viking I. Within the errors of the present occultation, the occultation result (see Fig. 14 and IV) agreed with the composition found by Viking. Hence we must conclude that the models used to represent horizontal refractivity gradients (turbulence, for an example) are not correct at the time of the occultation. The large-scale refractivity gradients have small horizontal refractivity gradients.

**VII. CONCLUSIONS**

The Martian atmosphere probed by our occultation has been measured in terms of the \( \phi \) Gem occultation by Veerika and Testa (1976a) and in preparation. Elliot, J. L., Dunham, E., and Church, C. J., 1976, Space Sci., 34, 185, 255


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185

Elliot, J. L., Dunham, E., and Church, C. J., 1976, Space Sci., 32


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185


Elliot, J. L., and Veerika, J. 1976, Earth, 27, 359

Elliot et al. 1976

REFERENCES


Church, C. J., 1976, Astr. J., 46, 473

Dunham, E., 1976, Space Sci., 34, 185


OCCULTATION OF α GEMINORUM BY MARS

Ming, A. O., Hansen, W. B., Stith, A., McElroy, M. B.,
Siemer, N. W., Dicke, R. J., Knight, T. C. D., and
41, 89.
Pitt, B. 1965, private communication.

Wasserman, L. H., Millis, R. L., and Willamson, R. M. 1977,
Young, A. T. 1974, in Methods of Experimental Physics:
Pres). p. 95.

C. Church, E. Dunham, J. L. Elliot, R. G. French, P. J. Gierasch, Carl Sagan, and J. Veverka: Laboratory
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APPENDIX 2


Analysis of Stellar Occultation Data

Effects of Photon Noise and Initial Conditions

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A new inversion technique for obtaining temperature, pressure, and number density profiles of a planetary atmosphere from an occultation light curve is described. This technique employs an improved boundary condition to begin the numerical inversion and permits the computation of errors in the profiles caused by photon noise in the light curve. We present our assumptions about the atmosphere, optics, and noise and develop the equations for temperature, pressure, and number density and their associated errors. By inverting in equal increments of altitude, we use the data points as the initial condition for the inversion. The importance of the boundary condition used is stressed, and a new initial condition is given. Numerical results are presented for the special case of inversion of a noisy isothermal light curve. From these results, simple relations are developed which can be used to predict the noise quality of an occultation. It is found that fractional errors in temperature profiles are comparable to those of pressure and number density profiles. An example of the inversion method is shown. Finally, we discuss the validity of our assumptions. In an appendix we demonstrate that minimum fractional errors in scale height determined from the inversion are comparable to those from an isothermal fit to a noisy isothermal light curve.

I. INTRODUCTION

Observations of stellar occultations by planets have been used to obtain temperature and number density profiles of the occulting planet’s atmosphere, through the numerical inversion of the light curve (Kovalevsky and Link, 1969; Hubbard et al., 1972; Vaililson et al., 1973; Veverka et al., 1974). An objection to this technique for reducing occultation data is that there has been no way of associating quantitative uncertainties with the results of the inversion. The uncertainties are of two types. First, the assumptions concerning the planetary atmosphere necessary to do the inversion calculation may not be valid (Young, 1976; Elliot and Veverka, 1976; Jokipii and Hubbard, 1977). Second, the photometric errors affecting the light curve—photon noise, terrestrial oscillation, and baseline uncertainties, for example—must necessarily propagate into the temperature and number density profiles, but these effects have not been evaluated with any generality (Hubbard et al., 1972; Wasserman and Veverka, 1973; Hunt and Veverka, 1976). In this paper we present a method for calculating the errors due to photon noise in the temperature, pressure, and number density profiles, making the usual assumptions necessary for the inversion of an occultation light curve. To accomplish this task we have recast the inversion equations into a simplified form, more appropriate for treating photon noise. Our method explicitly demonstrates the importance of the initial condition used for the inversion calculation, long recognized as critically influencing a large portion of the profiles. We describe a new procedure for establishing this initial condition, and evaluating the uncertainty that it produces.

Assuming the basic validity of the inversion method, our final prescription for obtaining number density, pressure, and temperature profiles, and their errors is directly applicable to light curves whose dominant photometric error is photon noise—such as the airborne observations of the α Gem occultation (Elliot et al., 1977b). For this class of light curves, the formulas developed here can be used to estimate, in advance of the occultation, the errors to be expected in temperature, pressure, and number density. For light curves containing significant amounts of other photometric noise, the errors calculated on the basis of our photon noise model are lower limits for the actual errors.

II. A NEW METHOD FOR INVERSION OF LIGHT CURVES

A. NOTATION AND ASSUMPTIONS

The occultation geometry for an immersion event is illustrated in Figure 1. Starlight, incident from the left side of the figure, is refracted by a planetary atmosphere and received by a telescope a distance D from a spherical planet of radius Rp. As seen by the telescope, the apparent velocity of the occulted star perpendicular to the limb of the planet is v, and the starlight varies in intensity due to the process of differential refraction. A light ray whose unrefracted path makes the closest approach A to the center of the planet is refracted through an angle θ(0), which has a negative algebraic sign for the angles indicated in the figure. The z-axis lies parallel to the direction of the light rays and coordinate r is the distance from the center of the planet. The atmosphere has a refractivity r, the index of refraction minus unity. The shell notation of the figure will be defined later, when it is used for computations.
To obtain the number density, pressure and temperature profiles of the planetary atmosphere from the occultation light curve, we make the following assumptions:

Atmosphere
(i) The approximate scale height $H$ of the atmosphere satisfies the condition $H < R_e$.
(ii) The refractivity of the atmosphere $n(r)$ is a function of $r$ only.
(iii) The atmosphere has uniform composition.
(iv) The atmosphere is in hydrostatic equilibrium.
(v) Rayleigh scattering and other forms of extinction are negligible when compared to the decrease in intensity due to differential refraction. This will be true if the distance $D$ is large (Baum and Code, 1953).

Optics
(vi) The occulted star is a point source.
(vii) Diffraction effects are negligible (i.e., the ray optics approximation is used).
(viii) There is no ray-crossing (i.e., $d\theta(h)/dh < -1/D$; see Section V of Elliot and Veverka, 1976).
(ix) Bending angles are small (i.e., $|\theta| < 1$).

Noise
(x) The light curve received by the telescope contains photon noise.

B. FUNDAMENTAL EQUATIONS

$$H/R_e < 1, \text{ it can be shown that}$$

$$n(h) = \frac{2}{\pi (2R_e)^3 N_a} \int (h')^3 \nu dh(h').$$

(1)

Then the number density $n(h)$ (cm$^{-3}$) is given by

$$n(h) = \xi (h) \nu_{\text{str}},$$

(2)

where $\xi$ is the number of photons per unit wavelength. The refractivity of the atmosphere at standard temperature and pressure for the wavelength of the observation. The pressure, $p(h)$, can be found by integrating the equation of hydrostatic equilibrium

$$dp(h) = -g(h) dh/N_a,$$

(3)

where $g$ is the mean molecular weight of the atmosphere and $N_a$ is Avogadro's number. The gravitational acceleration $g$ is assumed to be constant, but an altitude dependence could be included here if necessary. Integration of (3) yields

$$n(h) = \frac{4g\xi}{3\pi (2R_e)^3 \nu_{\text{str}} \nu_{a}} \times \int (h')^3 \nu dh(h').$$

Finally, the temperature $T(h)$ can be determined by combining (2), (4) and the perfect gas law:

$$T(h) = N_a p(h)/\mu g,$$

(5)

where $R$ is the universal gas constant.

The scale height $H(h) = R T(h)/\mu g$ can be found from (3):

$$H(h) = \frac{2}{3} \int (h'^3 - h^3) dh(h'^3)/.$$n

(6)

We note that $H(h)$ is independent of the atmospheric composition. Since the scale height is the ratio of two differently weighted integrals of the same variable, much of the noise cancels out, and we shall find in Section III that a consequence of this is that temperature profiles can have smaller fractional errors than number density and pressure profiles.

To evaluate the integrals in (6), we will write them as summations. In Fig. 1 we have divided the atmosphere into shells of thickness $\Delta h$. For the $i$th shell we define the following quantities: $A_i$ and $m_i$, the altitude of its lower and upper boundaries; $\Delta h_i$, the increment in refraction angle for the ($i - 1$)th and $i$th rays; $\Delta h$, the amount of time between the receipt of the ($i - 1$)th and $i$th rays by the telescope; and $\phi_i$, the average of the normalized stellar flux received by the telescope during the time interval $\Delta h_i$.

The function $\Delta h_i$ is a function of the light curve flux $\phi_i$ of a two-step process. Starting at a reference level $h_0$, we can find an implicit equation for the time $\Delta h_i$.

$$\Delta h_i = \frac{h_i - h_{i-1}}{|\nu_i|} = r_i \int_{h_{i-1}}^{h_i} \phi_i dh_i.$$n

(7)

After finding $\Delta h_i$ from (7), we can now find the desired increase in refraction angle $\Delta h_i$:

$$\Delta h_i = \Delta h_{i-1} - \Delta h_i / D.$$n

(8a)

C. ERRORS CAUSED BY PHOTON NOISE

The noise-free occultation flux $\phi_i$ is not directly obtained from the observations, since photon noise (shot noise) is present in the light curve. Let the quantity $\nu_i$ be the variance of the shot noise integrated for one second, computed from the Poisson statistics obeyed by photons. For a one second interval when $\phi(i)$ has a constant value $\phi$, the average number of detected photons will be $\phi \pm n_0$, where $n_0$ and $n_0$ are the rates of photons detected per second from the unocculted star and the background. $\nu_i$ is convenient to use units of seconds because the time scale for an occultation is $H_i/R$, which is typically of order one second.) For Poisson statistics the variance is equal to the mean, and after normalizing by $n_0$,

$$\epsilon(\phi) = (\phi/n_0)^{1/2}$$

(9)

The value of $\epsilon(\phi)$ for two limiting cases will be of interest later:

$$\epsilon(\phi) = (\phi/n_0)^{1/2}$$

(10)

$$\epsilon(\phi) = (\phi/n_0)^{1/2}$$

(11)

Finally, the variance in the flux from the $i$th shell is given by

$$\nu_i = \phi_i / \Delta h_i.$$n

(12)

This equation is the fundamental relation for propagating errors from the light curve into number density, pressure, and temperature profiles. In this formulation, the fluxes $\phi_i$ are taken to be a set of independent random variables, the times $|\nu_i|$, defined by $\Delta h = n_0 D$, are taken to be known exactly, but there is an uncertainty in the location of the shell boundaries which correspond to known times on the light curve.

D. AN IMPROVED INITIAL CONDITION

There are two kinds of initial errors in the evaluation of the integrals in (1) and (4). First, there is truncation error, since in practice the limits of the integral must be finite. Second, the bending angle is so small at the onset of the occultation that noise fluctuations will completely dominate intensity variations due to the atmosphere. The first requires that the numerical inversion begin early enough to include all true variation, while the second indicates that starting too early will introduce additional errors. The problem we are left with can be stated as follows: Can we find an initial condition which utilizes all of the data in the light curve and which allows an accurate estimate of the errors due to random noise?

In previous work, three approaches have been used. The first is to begin the summation at some value for which $n \equiv 1$ and $\Delta h = 0$ (Wasserman and Veverka, 1973). The difficulty with this method is that an unnecessarily large error is introduced into the computation if the inversion is begun too soon, and information is lost if the-
version is begun too late. Additionally, as is shown in Section V, the results are sensitive to the starting point used. A second method has been to generate a sequence of $\Delta t$'s for the initial part of the light curve by assuming that the atmosphere is isothermal with the scale height determined from a least-squares fit to the refractivity $p_4$, $\Delta t$'s are then used to start the inversion until the flux has fallen to --0.98. The objection to this approach has been that small uncertainties in the flux of the first point used in the inversion lead to large errors in estimated $\Delta t$. A final method has been to specify an initial temperature at an arbitrary altitude (Vapillon et al., 1973).

We have developed an initial condition which utilizes an isothermal fit to the initial part of the light curve only. The procedure has the following features: (a) truncation errors are eliminated; (b) information further down on the light curve does not bias the initial isothermal fit; (c) the fit utilizes uncorrelated random variables whose errors can be determined directly; and (d) the arbitrary features of the other methods are avoided. The method assumes that the upper and lower baselines ($\phi = 1$ and 0 levels) have already been determined by some means. First, $d(h_j)/dh$ is determined from the data using (7) and (8), between $h = h_{max}$ and $h = \Delta h$. The altitude $h_{max}$ corresponds to some point on the light curve which precedes the onset of the occultation, and $h_j$ corresponds to a point on the light curve where the flux has clearly diminished. The procedure used to determine $h_j$ is described below.

The function

$$
\frac{dh}{d(h)} = \alpha u_{29} e^{-u_{29} h} - \alpha \Delta h \frac{dh}{dh}
$$

(13)

is fit to the data in the least-squares sense. The form of (13) is chosen so that $a$ and $c$ are uncorrelated random variables whose errors, $s(a)$ and $s(c)$, can be determined from the fit. The scale height of the initial fit is given by $H_i = 1/a$, and $c = H_i \Delta t (h_a)/dh$. Since the fit is performed on data which are equally spaced in altitude rather than in time, the formal errors in the fit have to be modified to account for the interpolation involved in converting data from the time to the space domain. If data are analyzed at very high time resolution, $\Delta t$, so that the condition $n \Delta t < \Delta h$ is satisfied, then many data points in time will be included in each $\Delta h$, and the effects of interpolation will be negligible. In any case, we require $\Delta h < H_i$; there must be many shells per scale height in order for the numerical approximations to the integrals to be valid.) On the other hand, practical considerations often require $\delta \Delta h > \Delta h$ in order that $\Delta h < H_i$, and in this case it is important to correct for the formal errors, $s(a)_{\text{form}}$ and $s(c)_{\text{form}}$. We have derived the following correction factor:

$$
\sigma_{\text{cor}} = \left( \frac{3 \beta \Delta h}{2 \Delta h} \right)^{1/2} \sigma_{\text{form}}
$$

(14)

In computing this factor, we have assumed that the initial fit is halved before $\phi < 1$. We emphasize that this correction is appropriate only if $\beta \Delta h > \Delta h$.

Finally, the value of $h_a$ is chosen so that the fitted scale height $H$ is established with an error comparable to that associated with the inversion of later sections of the light curve. In practice this means experimenting with a range of $h_a$ and selecting the largest value for which $H_a(h_a)$ has settled down to a slowly varying function. With very noisy data, this point can be far down the light curve. Examples are discussed in Sections III and V.

**E. Number Density, Pressure, and Temperature Profiles and Their Associated Errors**

We are now in a position to write the full expressions for $n(h)$, $p(h)$ and $H(h)$ and their associated errors. Let

$$
L_v(h) = \frac{1}{2} \left( k^{-1} - h^{-1} \right)^{a_{01} e^{-u_{29} (h_0 - h)} \Delta h}
$$

(15)

and

$$
\Sigma_v(h) = \frac{1}{2} \left( k^{-1} - h^{-1} \right)^{a_{01} e^{-u_{29} (h_0 - h)} \Delta h}
$$

(16)

where $a_{01}$ is the index of the shell $h$ between $h = h_0$ and $h = h_a + \Delta h$. In (16), we have reintegrated the factor $k^{-1} - h^{-1}$ over the numerical approximations to the integrals to be valid.) On the other hand, practical considerations often require $\delta \Delta h > \Delta h$, and in this case it is important to correct for the formal errors, $s(a)_{\text{form}}$ and $s(c)_{\text{form}}$. We have derived the following correction factor:

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(15)

and

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\Sigma_v(h) = \frac{1}{2} \left( k^{-1} - h^{-1} \right)^{a_{01} e^{-u_{29} (h_0 - h)} \Delta h}
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III. THE NATURE OF THE ERRORS

In order to gain insight into the nature of fractional errors in number density, pressure, and scale height, we have obtained solutions to (21), (22), and (23) for the special case of an isothermal atmosphere with a known noise level. (We stress that these equations are valid for any atmosphere which satisfies the assumptions in Section 11A.) Since \( \sigma'(\phi) \) is a function of both background and stellar fluxes [Eqns. (9)], we have treated two limiting cases: background flux much greater than stellar flux, \( n_b \gg n_s \); and background flux much less than stellar flux throughout the light curve, or, in the limit, \( n_b = 0 \).

Figure 2 illustrates our results for the case \( (n_b/H)^{1/2} \phi = 0.01 \), where \( \phi \) is evaluated at \( \phi = 1.0 \). Fractional errors in number density, pressure, scale height, \( \Delta H \), and the error in altitude difference \( (h_b - h)/H \) are shown as functions of depth in the atmosphere and of normalized flux of the light curve. \( E_b \) has been computed for the case \( (n_b/\Delta H)^{1/2} \phi = 0.01 \).

Note that, in this figure, the light curve is plotted against depth in the atmosphere, rather than against time. The upper branch of each curve corresponds to the case, \( n_b \gg n_s \), and the lower branch to the case, \( n_b = 0 \).

The fractional errors in Fig. 2 are computed using the results of Appendix A, without utilizing an initial fit to the data and assuming that the inversion was begun five scale heights above half-light. We determined the point \( h_b \) in (15) at which the scale height error in the initial fit is equal to the scale height error, \( \sigma[H(h_b)] \) for the case shown. This occurs at \( \phi = 0.00 \) and \( h_b = 0.41 H \), for an isothermal atmosphere. If the initial fit is extended to a deeper level (smaller \( h_b \)), the fractional errors will be smaller than those shown in Fig. 2; if \( h_b \) is larger than 0.41\( H \), the fractional errors in Fig. 2 will be slight underestimates for the first scale height or so below \( h_b \). The choice of \( h_b \) in a practical case is discussed in Section V.

Figure 2 can be best understood by examining the nature of the error in \( H(h_b) \).

In (10) there is an uncertainty in \( \Delta h_b \) and also in \( (h_b - h)^{1/2} \), which has been preintegrated in this equation. From (7), (8), and (12) we find

\[
\sigma(\Delta H) = \sigma(\Delta h_b) \left( \frac{\Delta h_b}{\Delta h} \right) (1 - \phi). \tag{27}
\]

Equation (27) illustrates that \( \sigma(\Delta H)/\Delta H \) is largest near \( \phi = 1 \), when noise causes much more variation in apparent flux than does the slight bending due to the atmosphere. As \( \phi \) approaches zero, so does the fractional error in \( \Delta H \). This is shown in Fig. 2 as \( E_b \).

Equation (26) gives

\[
\frac{\sigma(h_b - h)}{H} = \frac{\sigma(n_b H^{1/2})}{H} \left( \sum_{m=-\infty}^{\infty} \frac{\sigma(\phi_H^{1/2})}{H} \right). \tag{28}
\]

This is shown in Fig. 2 as \( E_b \) with \( h_b \) taken as \( 5H \). For constant \( \sigma(\phi) \), \( E_b \) increases rapidly as \( \phi \to 0 \), and this is reflected in the corresponding increase in \( E_n \), \( E_p \), and \( E_E \). For \( n_b = 0 \), \( \sigma(\phi) \) is proportional to \( \phi^{1/2} \) [see Eq. (10)], and as \( \phi \to 0 \), \( E_n \), \( E_p \), and \( E_E \) approach constant limiting values.

From the form of (1), (4), and (9), it is evident that the deduced values of \( n(h) \), \( P(h) \), and \( H(h) \) depend on the nature of the entire light curve prior to that point. This reflects the simple fact that a grazing ray at altitude \( h \) is refracted not only by the atmosphere at \( h \), but also by the atmosphere along the rest of its path. We can see the manner in which the prior data are weighted by plotting the integrals \( \Delta I \) (15) with \( m = 1 \) and \( m = 0 \), for the case of an isothermal atmosphere. These are used in determining \( P(h) \) and \( n(h) \), whose ratio is proportional to the scale height. The integrals are shown in Fig. 3, where the curves have been normalized to unity at their maxima, and where altitude, \( z \), is measured in scale heights above \( h \). The main contribution to the integral for \( n(h) \)

![Fig. 2. Fractional errors in \( n(h) \), \( P(h) \), \( H(h) \), \( M(h) \), and altitude \( h = h_b/H \), computed for an isothermal atmosphere. \( E_n \), \( E_p \), \( E_E \), and \( E_N \) were determined for the case \( (n_b/\Delta H)^{1/2} \phi = 0.01 \), and \( h_b = 0.1H \), and \( E_E \) for the case \( (n_b/\Delta H)^{1/2} \phi = 0.01 \), where \( \phi \) is evaluated as \( \phi = 1 \). The upper branches correspond to the limit of background flux much greater than stellar flux, \( n_b \gg n_s \), and the lower branches to the case, \( n_b = 0 \). The light curve, below, allows fractional errors at different altitudes to be related to normalized flux, \( n_b \), and time from half-light, \( t_b \), along the light curve.](image-url)
comes from a fairly narrow altitude range centered at about $h + 0.5H$, while the main contribution to the integral for $p(h)$ is centered at $h + 1.5H$, with significant contribution coming as high as $h + 4.5H$. In other words, the data used to determine $n(h)$ are, in a sense, more "local" than those for $p(h)$.

The fractional errors in $n(h)$ and $p(h)$ are determined from similar weighted averages of the errors in $(h - l)\xi$ and $\Delta \eta$ of prior data. As the inversion proceeds, $E_n$ decreases rapidly because it weighs a fairly narrow range of prior data whose fractional errors in $\Delta \eta$ are decreasing rapidly; $E_p$ diminishes somewhat later because it weighs most strongly a range of prior data at higher altitude, whose fractional errors in $\Delta \eta$ are still large. In this region the error in $(h - l)\xi$ does not contribute strongly. If $n_o = 0$, the fractional errors $E_n$, $E_p$, and $E_{\phi}$ will decrease as the inversion is continued, because $E_{\phi} = 0$ and $E_n$ is rising only slowly. If $n_o \gg n_o$, $E_n$ rises from its minimum rapidly, in response to the rapid rise in $E_p$; $E_n$ and $E_p$ rise more slowly because of their weaker dependence on errors in $(h - l)\xi$.

It is important to note that $E_H$ is comparable to $E_n$ and $E_p$; as long as all of the assumptions of the method are satisfied, temperature profiles are as accurate as number density and pressure profiles. Additionally, the flux level corresponding to minimum errors is low, due to the weighting functions of Fig. 3. Even for the case $n_o \gg n_o$, the fractional error in $H$ does not rise to double its minimum value until $\phi = 0.001, k/H = 4.0$, and $v_o(l - l)H = 105$. The lesson is that inversions may be continued to deeper levels than previously suspected before errors due to photon noise become large. Inverting in shells of $\Delta \eta$ rather than $\Delta \eta$ is advantageous in this context because the inversion need not be terminated at the first point at which the apparent flux is negative. Ultimately, uncertainty in the lower baseline will be a more important source of error than photon noise.

**Analysis of Stellar Occultation Data**

It is useful to have a simple way to predict the quality of an occultation from the fundamental quantities $n, H, \phi$, and $\epsilon(\phi)$. The calculations of fractional errors for an isothermal atmosphere, described above, can serve as a guide to the magnitude of the minimum errors we can expect. Expected fractional errors can be obtained by scaling the solutions for $E_n, E_p, E\phi$, and $E_n$ in Fig. 2 according to the relations

\begin{align}
\frac{\epsilon(n)}{n} = \left(\frac{E_H}{n}\right)^{1/2} \\
\frac{\epsilon(p)}{p} = \left(\frac{E_H}{p}\right)^{1/2} \\
\frac{\epsilon(H)}{H} = \left(\frac{E_H}{H}\right)^{1/2} \\
\frac{\epsilon(\phi)}{\phi} = \left(\frac{E_H}{\phi}\right)^{1/2}
\end{align}

For the case $n_o \gg n_o$, the upper branch solutions in Fig. 2 should be used. If $n_o = 0$, the lower branch solutions are appropriate. In (29)–(33), $\epsilon(\phi)$ has been evaluated at $\phi = 1.0$. Expected minimum fractional errors and the flux levels at which they occur are:

For $n_o \gg n_o$:

\begin{align}
\frac{\epsilon(n)}{n} = \left(\frac{E_H}{n}\right)^{1/2} &\leq 0.42 \\
\frac{\epsilon(p)}{p} = \left(\frac{E_H}{p}\right)^{1/2} &\leq 0.93 \\
\frac{\epsilon(H)}{H} = \left(\frac{E_H}{H}\right)^{1/2} &\leq 0.56
\end{align}

For $n_o = 0$:

\begin{align}
\frac{\epsilon(n)}{n} = \left(\frac{E_H}{n}\right)^{1/2} &\leq 0.13 \\
\frac{\epsilon(p)}{p} = \left(\frac{E_H}{p}\right)^{1/2} &\leq 0.09 \\
\frac{\epsilon(H)}{H} = \left(\frac{E_H}{H}\right)^{1/2} &\leq 0.07
\end{align}
### Table I

Proton Noise Errors for Occultations

<table>
<thead>
<tr>
<th>Occultation</th>
<th>(\sigma(H)/H)</th>
<th>(\sigma(H)/H)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha) Sgr</td>
<td>0.0004</td>
<td>0.0007</td>
<td>Other noise sources definitely more important than shot noise in determining minimum errors</td>
</tr>
<tr>
<td>by Jupiter</td>
<td>0.012</td>
<td>0.023</td>
<td>Errors due to photon noise consistent with mutual agreement of temperature profiles</td>
</tr>
<tr>
<td>(\alpha) Geminorum</td>
<td>0.003</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>by Mars</td>
<td>0.008</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>SAO 158967</td>
<td>0.009</td>
<td>0.010</td>
<td></td>
</tr>
<tr>
<td>by Uranus</td>
<td>0.012</td>
<td>0.013</td>
<td></td>
</tr>
</tbody>
</table>

* Evaluated for \(\phi = \frac{H}{L} = 1.000\).

In (37)-(39), the inequality signs indicate that, for \(n_0 = 0\), the fractional errors had not reached their limiting values before \(\phi = 0.001\), the point at which calculations were stopped. This is evident in Fig. 2.

As an example of the use of these equations, we have estimated the minimum fractional errors in scale height due to shot noise for several occultations. The results are given in Table I. It should be recognized that \(\sigma(H)/H\) is not an intrinsic property of a particular occultation, and its value will depend on how the occultation is observed. For example, the telescope aperture and filter bandwidth are important factors.

### V. Application of the Inversion Method: An Example

To illustrate several important features of the new inversion method, we have utilized the equations developed in Section II to invert an isothermal light curve with added random noise. For this case we let \(H = 10\) km, \(v_0 = 10\) km \(\text{sec}^{-1}\), \(\sigma(H)/H = 0.02\), and \(\Delta H = 1\) km. These parameters give a noise factor of \(\sigma(H)/H = 0.02\), comparable to that of the airborne observations of the occultation of \(\alpha\) Gem by Mars. We assumed that \(n_0 = 0\), so that \(\sigma(H)/H\) can be regarded as constant throughout the occultation [see Eq. (11)]. The light curve is shown in Fig. 4. The solid line is the noise-free isothermal light curve computed from the well-known equation of Baum and Code (1953).

Two methods of numerical inversion were used to deduce the scale height. The first utilizes the full theory developed above, together with the new initial condition, an isothermal fit to the initial data in the light curve. The second method does not utilize the new initial condition, and in effect assumes that \(H = 0\) at the onset of the inversion.

Figure 5 presents the results of the inversions. For the case shown on the left, an isothermal fit to the initial data, using (12), gave \(H = 10.58\) km \(\pm 1.50\). The region of the light curve used in the fit is shown in Fig. 4, and begins six scale heights above half-light, and ends at \(\phi = 0.60\). The error bars on the temperature profile were computed from (25), and have a total length of two standard deviations. At all altitudes, the error bars are consistent with the actual scale height of 10 km. Errors in altitude are not shown, but they can be estimated from Fig. 2. The characteristics of \(E_y\) in Fig. 2 are reflected in the errors of the temperature profile: the errors are initially very large, and eventually slow. An additional 56 sec of data would minimize at about three scale heights below what had to be included in the inversion the half-light level, after which they rise before the deduced temperature profile.

![Fig. 4](image-url)  
**Fig. 4.** Isothermal light curve with scale height \(H = 10\) km and noise figure \((\sigma(H)/H) = 0.02; it is assumed that \(n_0 \gg n_0\). The solid line is the noise-free light curve. The results of inversion of the noisy light curve are shown in Fig. 5. The inversion was terminated at \(n_0 = 3\) km, although the curve is shown above only to \(n_0 = 3\) km.**

![Fig. 5](image-url)  
**Fig. 5.** Scale height profiles obtained from inversion of the light curve shown in Fig. 4. The true scale height is 10 km. The profile on the left was obtained using the new method described in this paper. The error bars are consistent with the true scale height. Errors in altitude are not shown, but the one standard deviation values are about \(\pm 0.5\) km at the half-light altitude and \(\pm 1.5\) km at the lower end of profile. On the right, profiles were obtained by assuming \(H = 0\) at the onset of the inversion. The inversion starting times used for Cases A, B, C, and D are shown in Fig. 4, along with the region of the light curve used in the initial fit.
could be extended a scale height deeper in the atmosphere. At that depth, for the parameters of this example, it can be seen from Fig. 2 that $\sigma(\phi)/H$ would have increased only to 0.075.

It is important to recognize that the noise in successive points is correlated. For this reason, adjacent points agree better than implied by the error bars. In other words, the error bars refer to the gross positioning of the profile, rather than to the much smaller uncertainties in local temperature variations.

On the right of Fig. 5, we show the results of inversion of the same light curve, but without utilizing an isothermal fit to the initial data. The inversions were begun at four different starting times, corresponding to six, five, four, and three scale heights above half-light; see Fig. 4 for the corresponding positions on the light curve. Case D, the inversion utilizing the most data, converges most slowly to the correct scale height, $H$, illustrating that errors tend to diverge as the inversion is begun earlier and earlier. Cases B and C give results comparable to the inversion utilizing the initial fit. Case D illustrates the problem of beginning the inversion too late. Although the flux has dropped only to $\phi = 0.95$ at the onset of the inversion, the temperature profile does not converge to within 10% of the actual scale height until $\phi = 0.30$.

We have obtained four substantially different temperature profiles from the same initial data, by changing only the starting point of the inversion and assuming that $H = 0$ initially. Lacking a priori knowledge of the true temperature profile, there would be no clear best choice among the curves. In contrast, the new inversion method is less sensitive to the starting time, does not assume a zero temperature at the onset of inversion, and provides a quantitative measure of the error at all altitudes due to a known source of noise. An important consideration in the use of the new method is the choice of how much of the light curve should be included in the initial isothermal fit. In the parlance of (15), what governs the choice of $h_d$? If only the first part of the curve is used, (27) shows that the data are noise dominated, and the errors in the fit [$\sigma(\phi)$ and $\sigma(\phi)$] will be enormous; they will propagate down through the entire light curve, as seen in (25). On the other hand, if too much of the light curve is used in the initial fit, information about detailed temperature variation will be masked. Additionally, errors in $a$ and $c$ may actually grow as more data are fit, because the assumption that the atmosphere is isothermal may become less realistic. The method may be unreliable if the isothermal approximation is strongly violated.

Although we have found that, for an isothermal atmosphere, it is optimum to fit to the $\phi = 0.60$ level, this can only serve as a rough guide in practical use, because there is no guarantee that the sampled atmosphere is isothermal. Our strategy is to fit successively larger sections of the data until the deduced scale height, $H_d$, is insensitive to changes in $h_d$ which are small compared to $H$. In Table II, we give the results of a variety of fits to the initial data in this example to show how the change in $H_d$, as the region of the fit is varied. By this means, the initial fit serves the function of damping the erratic initial behavior of the inversion and of rendering the inversion insensitive to its starting point. It provides an estimate of errors due to photon noise, and by stopping the fit as early as possible, the atmospheric temperature profile can be determined by the inversion over a significant altitude range.

### Table II

<table>
<thead>
<tr>
<th>$h_d/H$</th>
<th>$\sigma(\phi)$</th>
<th>$H_d$, (km)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>1.1</td>
<td>11.98 ± 2.93</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>10.88 ± 1.59</td>
</tr>
<tr>
<td>5.0</td>
<td>2.5</td>
<td>10.68 ± 3.56</td>
</tr>
<tr>
<td></td>
<td>2.4</td>
<td>8.52 ± 5.79</td>
</tr>
<tr>
<td>4.0</td>
<td>2.0</td>
<td>7.23 ± 2.02</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>6.23 ± 4.10</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>6.7 ± 1.3</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>6.21 ± 1.51</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0</td>
<td>12.33 ± 2.60</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>12.67 ± 1.99</td>
</tr>
</tbody>
</table>

* True value of scale height = 10 km.

**VI. DISCUSSION**

We have used the new inversion method to obtain number density, pressure, and temperature profiles of the Martian atmosphere from airborne observations of the occultation of $\alpha$ Gem on 3 April 1976 (Elliot et al., 1977). Three light curves were obtained simultaneously, each probing the same region of the Martian atmosphere, but each containing a different sample of photon noise. Hence the profiles obtained from different light curves should agree within their error bars—which they do—providing an internal consistency check of our method for computing errors.

Further analysis of the temperature profiles obtained from the $\alpha$ Gem occultation showed that temperature variations, having a vertical scale of order one scale height, are determined much more precisely than the mean temperature, in the presence of photon noise (see Figs. 12 and 13 in Elliot et al., 1977). The present analysis could be extended to understand quantitatively the reliability of temperature information over various length scales, perhaps by a power spectrum analysis of temperature variations.

Fractional errors in $n(\phi)$, $p(\phi)$, and $H(\phi)$ due to photon noise minimize at very low fluxes, and at this point other sources of uncertainty become important. Baseline instability and uncertainty remain to be studied in detail. Additionally, as the inversion is extended into the noise-dominated tail of a light curve, statistical fluctuations will give solutions $\Delta\phi$, in (7) even if the flux from the star is zero. This would have the effect of producing a spurious high-temperature tail on the deduced profile. In practice, inversions must be terminated for other reasons before this effect becomes important.

To some extent, each of the assumptions in Section III A is only approximately satisfied. On the basis of scintillation theory, several authors (Young, 1976; Jokipii and Hubbard, 1977) have concluded that the assumption of spherical symmetry is so strongly violated as to invalidate the inversion method. By comparing the light curves obtained by several observers of the $\alpha$ Gem occultation, it may be possible to place some limits on the degree of spherical symmetry of the Martian atmosphere, and hence to test this important assumption. This work is being pursued.

**VII. CONCLUSIONS**

With the new method of analysis of occultation light curves described above, we are able to place error bars on temperature, pressure, and number density profiles due to photon noise in the data, given the assumptions stated in Section III A. An improved initial condition renders the inversion less sensitive to its starting point, and thus is less arbitrary than previous methods. An important result is that temperature profiles are shown to be as reliable as refractivity, number density, and pressure profiles. The manner in which prior data are weighted is responsible for large initial fractional errors in $n(\phi)$, $p(\phi)$, and $H(\phi)$, but also accounts for the fact that errors in these quantities due to photon noise minimize at very low flux levels. Consequently, as long as baselines are stable, the inversion can be extended deep in the atmosphere; by inverting in increments $\Delta\phi$, rather than $\Delta\phi$, the inversion need not be
halted when the first negative data point is reached.

We have computed the errors in \( n(h), \)
\( \Phi(h), \) and \( H(h) \) for an isothermal light curve with a known level of photon noise. The noise quality of an occultation can be estimated by calculating the isothermal solution to the appropriate value of \( \sigma(H)/H(H) \). Fractional errors due to photon noise are only a few percent when \( \sigma(H)/H(H) = 0.01 \), a reasonable value for high-quality observations. Additionally, we have shown that minimum fractional errors in scale height determined from the inversion are comparable to those from an isothermal fit to a noisy isothermal light curve.

**APPENDIX A**

We derive here the quantities used to determine \( \sigma(H)(h)), \sigma(H)(H), \) and \( \sigma^2(H)(h), \) given in Section 11E. From (18),

\[
\sigma^2(H)(h)) = \sum_{i=1}^{2} \left( \frac{\partial H(h)}{\partial \phi_i} \right)^2 \sigma^2(\phi_i).
\]

The variance in \( \phi_i, \) \( \sigma^2(\phi_i) \) is given by (12).

\[
\Gamma_i = \frac{(1 - \phi_i)}{(1 + \phi_i)} \left( \sum_{j=1}^{2} \frac{\partial H(h)}{\partial \phi_j} \right) \sigma^2(\phi_i) \quad \text{(A2)}
\]

Then

\[
2 \{{H}_h(h) = \sum_{i=1}^{2} \Gamma_i \} \quad \text{(A3)}
\]

and

\[
\sum_{i=1}^{2} \frac{\partial H(h)}{\partial \phi_i} = \left\{ \frac{\partial H(h)}{\partial \phi_i} \right\}
\]

From (A2) and the relation \( \Delta h = v(\phi_i) \Delta t, \) we find

\[
\Delta h = (1 + \phi_i) \Delta t
\]

where the derivatives in \( \Delta h \) have been defined in (A4)–(A6). Letting \( u = a(h - h), \) we find from (15)

\[
\Delta a(u)/\Delta t = [(u + u)/a(h - h) - I(u)(h), \quad \text{(A11)}
\]

where

\[
I(u)(h) = (\alpha/a)(u)^2
\]

\[+ \frac{1}{2} \alpha^2(e^2 - e) \text{erfc (u)}
\]

\[
\text{(A13)}
\]

**APPENDIX B**

Isothermal curve fitting has often been used to obtain an estimate of the mean scale height of the atmosphere sampled by the light curve. The merits and hazards of this practice have been the subject of considerable debate, and we wish to show only that, for the high background case \( n_0 \gg n_s, \) the error in an isothermal fit to a noisy isothermal light curve is approximately equivalent to the minimum error given by the inversion method. In other words, even in the most favorable case for an isothermal fit, the error is not significantly smaller than that given by the inversion method.

Assume that the isothermal light curve equation of Baum and Code, Eq. (B1), is being fit to an isothermal light curve with known noise \( \sigma(\phi), \) but unknown scale height, \( H, \) and midtime, \( t_0. \) We assume that the baselines are known exactly.

\[
\phi(t - t_0)/H = (1/\phi - 2) + \ln(1/\phi - 1)
\]

\[
\text{(B1)}
\]

Then the error in the values of \( H \) and \( \phi \) given by the fit can be computed from standard formulas (Clifford, 1973):

\[
\sigma(H) = (\alpha/a)(H)^{1/2}
\]

\[
\text{(B2)}
\]

\[
\sigma(\phi) = (\alpha/a)^{1/2}
\]

\[
\text{(B3)}
\]

By comparison, the minimum fractional error in scale height given by the inversion method for the same conditions is given from Section IV as

\[
\sigma(H)/H = 1.54(\alpha/a)^{1/2}
\]

\[
\text{(B13)}
\]

\[
\sigma(\phi)/\phi = 1.96(\alpha/a)^{1/2}
\]

\[
\text{(B14)}
\]
We conclude that the inversion method has minimum fractional errors comparable to those of an isothermal fit for the case of a noisy isothermal light curve. The error in the estimate to the midtime, $t_m$, from the isothermal fit, is

$$\sigma(t_m) = 3.55(H/\kappa)^{1/4}.$$  \hspace{1cm} (B15)

This gives a fundamental limit to the accuracy of the "half-light" time, a quantity which is useful for occultation astronomy (Taylor, 1976).

Finally, the correlation coefficient has the value $r_m = 0.217$. This very low value means that the formal error in $t_m$ will be nearly the same whether or not $H$ is a free parameter in the fit, and vice versa.

ACKNOWLEDGMENTS

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REFERENCES


Uranus occults SAO158687

On March 10, 1977, Uranus will occult SAO158687, a star of late spectral type with a visual magnitude of $m = 9.5$ (see 13). The occultation will be visible from land areas surrounding the Indian Ocean. From high-quality light curves of this event, one can obtain temperature, pressure, and number density profiles of the Uranus atmosphere; information about its composition; and by combining occultation timing from several stations the precise diameter and albedo of the planet can be found. For the atmospheres of Jupiter and Mars, these quantities have been obtained from recent stellar occultations by these planets. This experience can be used to predict what can be learned from the forthcoming occultation by Uranus.

Although Uranus is several magnitudes brighter than SAO 158687 at visual wavelengths, the strong methane absorption bands in the far red and infrared spectrum of Uranus make the intensity of both objects nearly equal at these wavelengths. To see how to use the methane absorption bands to best advantage, we obtained the spectra of Uranus and SAO 158687 (Fig. 1). These spectra are shown to have relative instrumental interferences of the two objects and have therefore not been calibrated on an absolute scale. The resolution of the spectra is 10 Å and the cutoff near 9,000 Å is due to a cutoff in the instrumental response. The contribution from the sky background and dark current is negligible.

Fig. 1 Relative intensity of Uranus and SAO 158687. Each spectrum was obtained using a 10 Å bandwidth. The intensity scales are linear in the instrumental system. The two spectra on a common scale are expanded version of the original scan. The deep methane band near 8,400 Å is the telluric band of Uranus, the common absorption feature near 3,000 Å is the telluric Fraunhofer A band of molecular oxygen.

The instrument response of the light curves should be limited by photon statistics. The latter give a fundamental limit for the signal-to-noise ratio of the light curves. We define $w(t)$ to be the flux from SAO 158687 observed during the occultation, normalized to the unocculted intensity of the star. Hence $w(t)$ begins at 1.0, and decreases to 0.0 as the occultation proceeds. If $w(t)$ is the r.m.s. error in $v(t)$ due to photon statistics for a 1-s integration, then $w(t)$ is equal to the square root of the total photons detected from Uranus and the star, divided by the photons that would have been detected from the unocculted star

$$w(t) = \frac{\sqrt{\text{Flux(Photons)}/2}}{\text{Flux(Photons)}}$$

where $\lambda$ is the average value of $w(t)$ during a 1-s interval, $d$ is the area (cm$^2$) of the telescope, $a_5$ the past_CONV of the filter (in Å), and $q$ is the fraction of the photons incident on the Earth's atmosphere that are ultimately detected by the photometer.

The factor $q$ includes the transmission of the Earth's atmosphere, telescope optics, photometer optics and the quantum efficiency of the photometer. The quantity $c_5$, has the dimensions of $\text{photons}^{-1}$, the actual r.m.s. error in $\lambda$ for an arbitrary integration time of $t$ is found by dividing $w(t)$ by $\lambda c_5$. In writing equation (1) we assume that the background counting rate from the photometer itself and the sky are negligible. This may not be entirely true, since a quarter Moon will be only 17° away during the occultation.

Values of $w(t)$ (Table 1) were computed for $\lambda = 1 - q = 0.04$, $d = 6.6 \times 10^{-5}$ cm$^2$ (6-inch telescope) and $t = 1/2$. From Table 1 we see that low photon noise errors can be achieved by choosing appropriate passbands for observation. Such low errors could be realized in ideal conditions, but in practice changes in transparency, guider errors, severe calibration errors and other problems could increase the effective noise in the signal above the level of photon noise.

From the photon noise levels in Table 1, we calculate the r.m.s. error expected for the temperature profiles obtained from the occultation light curves, if we assume that the gravity gradient model applies to the Uranus atmosphere. From the improved procedure of French et al. for obtaining temperature profiles from occultation light curves, the minimum r.m.s. error in the scale height of $T/H$ is

$$H \leq \frac{1}{\sqrt{2}} \left( \frac{1}{H} \right)^{1/4} (7)$$

where $H$ is the apparent velocity of SAO 158687 perpendicular to the limb of Uranus. The scale height is related to the temperature $T$ by $T = NH$, where $g$ is the gravitational acceleration, $R$ the universal constant and $\mu$ is the mean molecular weight of the atmosphere. At the occultation level we expect $T = 430 K$ (ref. 12), hence $H = 65$ km (ref. 12). For this occultation $T = 430 K$, and the mean molecular weight of $\mu$ in Table 1 (0.48), the fractional error in the scale height is only $0.5$%. The uncertainty in $H$ at the occultation level will probably be dominated by uncertainties in the mean molecular weight.

The level of the atmosphere probed by the occultation corresponds to a number density given approximately by the number density in 'ball light' for an ideal isothermal atmosphere:

$$n = \frac{n(0)}{n(0)} = \frac{1}{2} \left( \frac{273 K}{T} \right)^{1/2} \left( \frac{M}{\mu} \right)^{1/2} = 6 \times 10^{19} \text{cm}^{-3}$$

where $n$ is the number density of hydrogen, $n(0)$ is the number density at 1 atm. Atmospheric conditions, the mixing ratio of $\mu$ and $H$. For solar composition the mixing ratio of $\mu$ would be $7 - 10^{19}$ (ref. 13), which would leave $H$ and $H$ as the major constituents. It may be possible to obtain a value for the helium number fraction $H = H(H)$, from light curves at two wavelengths (6,200 and 8,000 Å), for example, as was done for Jupiter, and for the $A$ fraction in the case of Mars. $A$, however, is difficult to predict the expected accuracy of this measurement, since the accuracy will depend on the number and intensity of the 'spikes' in the light curve.

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Table 1 Photon noise errors for selected passbands

<table>
<thead>
<tr>
<th>Center wavelength (Å)</th>
<th>Photometric limit (A)</th>
<th>Photo. error (A)</th>
<th>r.m.s. photon noise error for 1-s integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>6,200</td>
<td>400</td>
<td>0.14</td>
<td>0.2</td>
</tr>
<tr>
<td>7,700</td>
<td>500</td>
<td>0.19</td>
<td>0.3</td>
</tr>
<tr>
<td>8,600</td>
<td>400</td>
<td>1.2</td>
<td>0.6</td>
</tr>
<tr>
<td>9,000</td>
<td>350</td>
<td>1.8</td>
<td>0.9</td>
</tr>
</tbody>
</table>

1 Photons cm$^{-1}$ Å$^{-1}$ outside the Earth's atmosphere.
APPENDIX 3


Lunar Occultation of Saturn
II. The Normal Reflectances of Rhea, Titan, and Iapetus

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An inversion procedure to obtain the reflectance of the central region of a satellite's disk from lunar occultation data is presented. The scheme assumes that the limb darkening of the satellite depends only on the radial distance from the center of the disk. Given this assumption, normal reflectances can be derived that are essentially independent of the limb darkening and the diameter of the satellite. The procedure has been applied to our observations of the March 1974 lunar occultation of Tethys, Dione, Rhea, Titan, and Iapetus. In the V passband we derived the following normal reflectances: Rhea (0.97 ± 0.02), Titan (0.24 ± 0.03), Iapetus, bright face (0.60 ± 0.14). For Tethys and Dione the values derived have large uncertainties, but are consistent with our result for Rhea.

I. INTRODUCTION

The 1974 March occultations of Tethys, Dione, Rhea, Titan, and Iapetus were observed simultaneously with the 224 and 61 cm telescopes at Mauna Kea Observatory (Elliott et al., 1975; hereafter Paper I). Lightcurves were obtained in three colors with the 224 cm telescope and in a single broadband channel with the 61 cm telescope. The center wavelengths and passbands of the four channels are given in Table I. For later reference, Fig. 1 shows these filter passbands superimposed on a portion of the spectral reflectance curve of Titan given by McCord et al. (1974).

Photocounting systems were used on all four channels, and all light curves were recorded on digital magnetic tape as a series of 10 msec integrations of photon counts. Altogether 19 occultation light curves were obtained. An example of the data is shown in Fig. 2, where the occultation curve for Titan obtained at 7400 Å is shown at the full time resolution of 10 msec. A detailed description of the instrumentation, observations and circumstances of the occultations is given in Paper I. One particularly fortunate circumstance was the position of Iapetus in its orbit, only 3° from western elongation. Hence at the time of the occultation we were getting almost a full view of the bright face of this unique satellite.

In Paper I we fit model light curves to the data and, for each satellite, found solutions constraining the diameter and limb darkening parameter $k$ assuming a Minnaert law ($f(\alpha) = f_0 \cos^{\alpha-1} \alpha$) for the limb darkening. Only for Titan was the signal-to-noise ratio great enough to constrain both the diameter and the exponent $k$. In this paper we introduce a novel method for obtaining the normal reflectance of the satellites from the lunar occultation light curves.

II. METHOD OF ANALYSIS

When a satellite of large angular diameter is occulted by the moon, diffraction effects are not important (Paper I), and the observed light curve will be that of an illuminated disk occulted by a uniformly moving straightedge (Fig. 3). In the

---

Table I

<table>
<thead>
<tr>
<th>Channel No.</th>
<th>Telescope</th>
<th>Waveband (Å)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>224 cm</td>
<td>6200</td>
</tr>
<tr>
<td>2</td>
<td>224 cm</td>
<td>4900</td>
</tr>
<tr>
<td>3</td>
<td>224 cm</td>
<td>7400</td>
</tr>
<tr>
<td>4</td>
<td>61 cm</td>
<td>5500</td>
</tr>
</tbody>
</table>

* Full width at half maximum.

The center wavelength of this filter is ill-defined because of the broad passband.

---

Fig. 1. Normalized spectral reflectivity for Titan (after McCord et al., 1971). Occultation light curves were obtained for each of the four wavelength intervals shown (cf. Table I).

---

Fig. 2. Occultation light curve for Titan obtained with the 224 cm telescope at Mauna Kea Observatory. Each data point represents a 0.01 sec integration.
LUNAR OCCULTATION OF SATURN. II.

Fig. 3. Model satellite for computing central reflectances. When a satellite is occulted by the
lunar limb, the observed light curve is the strip brightness distribution of the satellite. If we assume
the light intensity is symmetric about its center, the average reflectance for each ring of a concentric system can be recovered from the occultation light curve. The reflectance of the central ring, for all practical purposes, equals the normal reflectance of the satellite's visible disk.

From Fig. 3 we see that we can find the value of $b_i$ from the intensity difference $(I_i-I_e)$ and the area of the third ring intersected by the third strip. In general, any $b_i$ can be computed from the drop in light intensity across the $i$th ring, the areas of the rings intersected by the $i$th strip and the $b_i$ values for the rings exterior to the $i$th ring. However, for direct evaluation of errors in the $b_i$ we shall find it expedient to eliminate the dependence of each $b_i$ on the $b_i$ values of exterior rings. We express each $b_i$ in terms of the light curve intensities $I_i$:

$$b_i = \sum_{n=1}^{N} c_n f_{n}$$

where $N$ is the number of rings used to construct the model satellite. The coefficients $c_n$, which depend on the areas defined by ring and strip boundaries, are given in Table II for model satellites consisting of 2, 3, 4, and 5 rings.

To obtain the intensities $I_i$ required by (1), we must normalize the light curves and integrate the result over a time interval $t_i$ centered about the time $t_o$. Following Paper I, the instantaneous photon counting rate is denoted by $n(t)$, and the desired intensity $I_i$ is given by

$$I_i = \frac{1}{(1/\Delta)} \int_{t_i-\Delta}^{t_i+\Delta} (1/n) \times [n(t) - (a + b(t - t_o))] dt$$

where $a$ is the unocculted satellite counting rate and $(a + b(t - t_o))$ is the background counting rate. Values of the constants $a$, $b$, $t_o$, and $a$ were obtained from Tables II and VI of Paper I.

Since photon noise (not scintillation) is the principal source of noise in these occultation light curves, the intensities $I_i$ are uncorrelated random variables, and the variances $\sigma^2(b_i)$ are given by the usual equation for error propagation (Young, 1962):

$$\sigma^2(b_i) = \sum_{n=1}^{N} c_n^2 \sigma^2(I_n)$$

where $\sigma^2(I_n)$ are the variances of the intensities $I_n$, arising from photon noise. To reduce errors in $b_i$ caused by photon noise, we can sum the light curves of all channels. Additionally, if the assumption of radial symmetry of the brightness distribution is correct, the summed light curve can be folded about the midpoint $t_o$, forming a new set of intensities $I'_i$:

$$I'_i = [I_i + (1 - L_o)]$$

where photon noise is eliminated and the background is corrected for. We now demonstrate, that to the accuracy of the data, the brightness distribution on each satellite disk is independent of color and is radially symmetric. Thus we are justified in summing the light curves for our four channels and in folding the summed light curves about their midtimes.

III. TESTING THE ASSUMPTIONS

For each satellite we must test the two assumptions on which (4) is based. This procedure is best illustrated using the Titan light curves since these data have the highest signal-to-noise ratio. First, we shall verify that at each wavelength the Titan light curves can be folded...
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about the midtime \( t \). Using (1) and (2) we calculate \( \delta \) for each of the four light curves using \( N = 3 \) and \( \Delta t = 0.34 \) sec. As discussed below, a three-ring model (\( N = 3 \)) is a good choice for the Titan data. For \( \Delta t = 0.34 \) sec the outer diameter of the third ring corresponds to 5075 km at the distance of Titan. The values of \( b_1, b_2, \) and \( b_3 \) calculated for Titan using (a) the first half, (b) the second half, and (c) the folded light curve are given in Table III. The \( b \)'s obtained for cases (a), (b), and (c) agree within their errors (except perhaps for ring 2, where the difference between the results for the two halves of the light curve may be significant). We conclude that in general we are justified in using the folded data in our Titan analysis. Similar calculations show that this assumption is also valid for the other satellites.

Next, we verify that in the case of the Titan data the light curves from all four

channels may be summed to increase the signal-to-noise ratio.

The values of \( b_1 \) (\( \Delta t = 1, 2, 3 \)) computed for the folded light curves are plotted as a function of wavelength in Fig. 4. Within the error bars no color dependence is evident. Thus, in the case of Titan the light curves for all four channels may be summed (cf. Table III). Similarly, the validity of this procedure can be verified for the other satellites.

Thus we conclude that for all satellites we can use the summed and folded light curves in our analysis.

IV. SENSITIVITY OF METHOD TO OTHER ASSUMPTIONS

In our discussion of the Titan data in Section III we used \( N = 3 \) (three rings) and \( \Delta t = 0.34 \) sec (corresponding to a diameter of 5075 km for the three rings at the distance of Titan) in (2). We now demonstrate that the calculated \( b \)'s are not very sensitive to the specific choice of these parameters.

Values of \( b_1 \) for Titan calculated from the summed and folded light curves for \( \Delta t = 0.34 \) sec and for \( N = 3, 4, \) and 5 are compared in Table IV. We see that the addition of a fourth and fifth ring does not affect the values of \( b_1 \) and \( b_2 \); even then value of \( b_1 \) is not affected significantly. Thus we conclude that our choice of \( N = 3 \) has not biased the values of \( b_1 \) and \( b_2 \) derived from our analysis.

In calculating the normal reflectances of the satellites, we define \( b' \) to be the fraction of the total light reflected by the satellite per square kilometer for the \( n \)th ring. The \( b' \)s are related to the ring brightnesses \( b \) for an \( N \) ring model satellite and the satellite diameter \( d \) by the equation

\[
b' = \frac{b}{b_1} \left[ 2N/d \right]^2.
\]

In writing (5) we have indicated the functional dependence of the \( b' \) on the assumed diameter \( d \). This dependence exists because, by their definition, the \( b \)'s depend on the choice of \( \Delta t \), and \( d \) is determined by the choice of \( \Delta t \) through the relation

\[
d = 2N/\sqrt{\Delta t \lambda/2000000}.
\]

where \( \lambda (\text{sec}) \) is the radial rate of the lunar limb (Table VII of Paper I); \( \Delta t \) (km) the topocentric distance to the satellite and 2000000 the number of arc seconds in a radian.

Figure 5 shows the values of \( b' \) calculated from the summed and folded Titan data for \( N = 3 \), as a function of \( \Delta t \) (or diameter). Since the basic time resolution of the data is 10 msec, we have incremented \( \Delta t \) by 20 msec per ring in Fig. 4. For \( N = 3 \) this corresponds to diameter increments of about 400 km. Considering the error bars, we see that \( b' \) is independent of the choice of \( \Delta t \). The value of \( b' \) is weakly sensitive to \( \Delta t \) while \( b' \) is very sensitive to the choice of this parameter. Since the main purpose

| TABLE III |
| Normalized Surface Brightnesses \( b \) for Titan |

<table>
<thead>
<tr>
<th>Channel No.</th>
<th>First half</th>
<th>Last half</th>
<th>Folded</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( b_1 ) (ring 1, center)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.058 ± 0.010</td>
<td>0.061 ± 0.007</td>
<td>0.060 ± 0.006</td>
</tr>
<tr>
<td>2</td>
<td>0.066 ± 0.012</td>
<td>0.068 ± 0.011</td>
<td>0.067 ± 0.008</td>
</tr>
<tr>
<td>3</td>
<td>0.055 ± 0.018</td>
<td>0.044 ± 0.015</td>
<td>0.050 ± 0.010</td>
</tr>
<tr>
<td>4</td>
<td>0.046 ± 0.012</td>
<td>0.035 ± 0.009</td>
<td>0.033 ± 0.009</td>
</tr>
<tr>
<td>Data added</td>
<td>0.053 ± 0.007</td>
<td>0.035 ± 0.003</td>
<td>0.035 ± 0.004</td>
</tr>
<tr>
<td>b. ( b_1 ) (ring 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.058 ± 0.006</td>
<td>0.041 ± 0.004</td>
<td>0.051 ± 0.004</td>
</tr>
<tr>
<td>2</td>
<td>0.049 ± 0.003</td>
<td>0.044 ± 0.007</td>
<td>0.046 ± 0.003</td>
</tr>
<tr>
<td>3</td>
<td>0.056 ± 0.017</td>
<td>0.045 ± 0.009</td>
<td>0.062 ± 0.009</td>
</tr>
<tr>
<td>4</td>
<td>0.058 ± 0.007</td>
<td>0.044 ± 0.006</td>
<td>0.051 ± 0.005</td>
</tr>
<tr>
<td>Data added</td>
<td>0.055 ± 0.004</td>
<td>0.045 ± 0.003</td>
<td>0.050 ± 0.003</td>
</tr>
<tr>
<td>c. ( b_1 ) (ring 3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.017 ± 0.003</td>
<td>0.025 ± 0.002</td>
<td>0.020 ± 0.002</td>
</tr>
<tr>
<td>2</td>
<td>0.022 ± 0.004</td>
<td>0.025 ± 0.003</td>
<td>0.023 ± 0.003</td>
</tr>
<tr>
<td>3</td>
<td>0.020 ± 0.005</td>
<td>0.024 ± 0.005</td>
<td>0.022 ± 0.003</td>
</tr>
<tr>
<td>4</td>
<td>0.000 ± 0.004</td>
<td>0.025 ± 0.003</td>
<td>0.023 ± 0.003</td>
</tr>
<tr>
<td>Data added</td>
<td>0.020 ± 0.002</td>
<td>0.024 ± 0.002</td>
<td>0.022 ± 0.001</td>
</tr>
</tbody>
</table>

| TABLE IV |
| Normalized Surface Brightnesses \( b \) for Titan |

<table>
<thead>
<tr>
<th>No. of rings</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 ) (ring 1, center)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.055 ± 0.004</td>
<td>0.050 ± 0.003</td>
<td>0.022 ± 0.003</td>
<td>0.0017 ± 0.0017</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.055 ± 0.004</td>
<td>0.050 ± 0.003</td>
<td>0.0224 ± 0.0021</td>
<td>0.0017 ± 0.0017</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.055 ± 0.004</td>
<td>0.050 ± 0.003</td>
<td>0.0227 ± 0.0021</td>
<td>0.0002 ± 0.0015</td>
<td>0.0015 ± 0.0010</td>
</tr>
</tbody>
</table>

* Extra rings added.
a particular satellite diameter. Figure 7 shows the $b'_r$ values calculated from the summed and folded Titan light curves for $N = 3, 4,$ and 5 models where $2N\Delta t$ is approximately constant. Because of the finite time resolution of the data, it was not possible to have these models fit exactly the same satellite diameter. It is important to note that the profiles and the $b'_r$ values for the three models agree within their error bars, showing that the method is independent of the number of rings we fit to a given satellite diameter.

It is also interesting to see that the error bars for the $N = 4$ and $N = 5$ models are considerably larger than those for the $N = 3$ model. This is the major reason that the $N = 3$ model was chosen as the final one for Titan.

Similar conclusions result from the application of this analysis to other satellite light curves with the exception that the $N = 2$ model is optimum for these satellites.

V. NORMAL REFLECTANCE

We have demonstrated that reliable values of $b_r$ of the brightness of the central portion of an occulted satellite, can be calculated with the method outlined in Section II. We now show that the value of $b_r$ can be used to obtain an accurate estimate of the normal reflectance of the satellite surface.

Let $r_s(a, \theta, \lambda)$ be the average reflectance of the satellite's nth ring at a wavelength $\lambda$ when the solar phase angle is $\alpha$ and the orbital longitude of the satellite is $\theta$, and define $r(a, \theta, \lambda)$ relative to a perfectly reflecting, plane Lambert ring, whose surface normal points toward the Sun. This definition implies that if the satellite were replaced by a Lambert disk (not a sphere), then $r(a, \theta, \lambda)$ would equal 1.00 for all rings.

Since within the accuracy of our data the derived $b'_r$s are independent of wavelength for the colors observed, we will assume that they apply to the $V$ passband as well. Let $V'(a, \theta)$ be the visual magnitude of the satellite as a function of solar phase angle $\alpha$ and orbital longitude $\theta$, but referred to the mean opposition distance of Saturn from the Earth ($R = 3.54$ AU) and the Sun ($\Delta = 9.54$ AU). Then:

$$r_s(a, \theta, \lambda) = \frac{V'(a, \theta)}{2.5 \log(10) R^2} \times 10^{2.5 \log(10) A - 2A},$$

where $b'_r$ is given in terms of $b_r$ by (5), $1.50 \times 10^4$ is the number of kilometers in 1 AU, $R$, and $\Delta$ are mean opposition distance in AU, and $V = -26.77$ (Gehrels et al., 1961).

This reflectance, $r_s(a, \theta, \lambda)$, which we shall call the central reflectance of the satellite, is the average reflectance of the central portion of the disk enclosed by ring 1. Computed values of $r_s(a, \theta, \lambda)$ for the five occulted satellites are given in Table V.

Strictly these values apply only to the satellite faces visible at the time of the occultations. Next we estimate $r_s(a, \theta, \lambda)$, the average normal reflectance of the entire satellite surface in the $V$ passband, averaged over orbital longitude.
LUNAR OCCULTATION OF SATURN. II.

From (7), we can write

\[ r_1(\theta, \phi, V) = b_1(\theta, \phi, V) \]

\[ r_1(0^\circ, \phi, V) = b_1(0^\circ, \phi, V) \]

\[ \times \frac{\sin(1 - \gamma)(1 - \gamma)}{10^{4.4V - 10\gamma}} \]

where \( \theta \) denotes an orbital longitude at which the satellite achieves its mean brightness (averaged over \( \phi \)), and \( V(0^\circ, \phi) \) is the mean opposition magnitude which from now on will be denoted by \( V(0^\circ) \).

Given the likely assumption that the degree of limb darkening over the disk does not change strongly between \( a = 0^\circ \) and \( a = 9^\circ \), it follows from the definition of \( b_1 \) that the ratio of the \( b_1 \) terms in the above expression can be taken to be unity. In effect, if we represent the limb darkening by a Minnaert law we are assuming that in \( k \) would not change for values of \( a \) between \( 0^\circ \) and \( 9^\circ \). Hence

\[ r_1(0^\circ, \phi, V) = r_1(\theta, \phi, V) 10^{-4.4V} \]

where \( \Delta V = V(0^\circ) - V(\theta, \phi) \)

Finally we must relate the central reflectance which refers to the average reflectance of the finite spherical cap enclosed within the first ring to the average normal reflectance of the material: \( r_1(V) \). To convert \( r(0^\circ, \phi, V) \) to \( r_1(V) \) we assume that the limb darkening follows a Minnaert law with an exponent \( k \), since the correction does not depend strongly on the value of \( k \).

For a satellite divided into \( N \) rings, we find

\[ r_1(V) = r_1(0^\circ, \phi, V) \times \frac{\sin(1 - \gamma)(1 - \gamma)}{10^{4.4V - 10\gamma}} \]

where \( m = 2k \) and \( \gamma = (1 - 1/\gamma) \). The bracketed correction is close to unity for most practical cases of interest. For example for \( N = 3 \), it equals 1.03 for \( k = 1.0 \); for \( N = 2 \) it equals 1.07 for \( k = 1.0 \). For \( k = 0.5 \) the correction term is unity, independent of \( \gamma \).

The average normal reflectances for Tethys, Dione, and the bright face of Iapetus, computed from (10) are given in Table VI. The corresponding values for Tethys and Dione have been omitted due to the large errors in the central reflectances of these two satellites (Table V). Note that the values of \( k \) indicated in Table VI have been assumed, but that the results are only weakly dependent on this assumption (see above).

In computing the errors in \( r_1(V) \) given in Table VI, we have included: (i) the shot noise errors in \( b_1 \) (Table V), (ii) an estimated uncertainty of \( \pm 0.05 \) for \( V_0 \), (iii) errors in the mean opposition magnitude (Noland et al., 1974), (iv) changes in \( b_1 \) (about \( \pm 75 \% \)) for different assumed satellite diameters over the range for \( k \) (Figs. 5 and 6), and (v) uncertainties in the cap correction factor [Eq. (10)] resulting from the uncertainties in \( k \). For each satellite the errors were added quadratically. Although this is not a statistically rigorous procedure, we believe that it yields a realistic estimate of the true uncertainties in the normal reflectances.

**CONCLUSIONS**

We have demonstrated that the normal reflectance of the central portion of a satellite's disk can be estimated from a lunar occultation light curve and that this estimate does not depend strongly on the actual diameter or on the limb darkening law. The normal reflectances derived in this paper for Tethys, Dione, Tethys, Dione, and the bright face of Iapetus are high, and \( \pm 2 \) consistent with values expected for an ice-covered surface. We note that our value of the normal reflectance of the bright face of Iapetus (0.00 \( \pm 0.14 \)) is at best only marginally consistent with the value of about 0.35 derivable from the radiometric analysis of Morrison et al. (1975). (cf. Veverka et al., 1978 for details).

For Titan we conclude that the normal reflectance in the \( P \) passband is 0.24 \( \pm 0.03 \). We recall that in Paper I we showed that the disk of Titan is highly limb darkened and that the satellite's diameter is certainly \( \geq 8000 \) km, and possibly may exceed 6200 km (cf. Fig. 14 of Paper I). We also note that our data place limits on the color dependence of the limb darkening of Titan.

If the Minnaert law is used as reference form for the limb darkening, we found that the difference between the value of \( k \) for any two of the wavelength bands that we used (cf. Fig. 1) is \( \leq 0.1 \) (cf. Table V of Paper I).

**ACKNOWLEDGMENTS**

We appreciate the observing time which was made available to us by the Lunar and Planetary Institute and thank the ESO staff for their assistance. D. Morrison, T. Jonas, and J. Hart provided helpful discussions. This research was supported by NASA Grant NGL 33-010-062 and NSF 71219.

**REFERENCES**

**Elliott, L. L., Veverka, J., and Cooker, J. (1973).**


**Gladman, T., Croft, T., and O'Brien, D. (1965).**

Wavelength dependence of polarization, III. The lunar star. *Icarus* 6, 243-52.

**McCrea, T. B., and Jessen, T. V., and Elliott, J. H. (1971).**


The two faces of Iapetus. *Icarus* 24, 17-27.


Lunar occultation of Saturn. II. How big is Iapetus? *Icarus* 33, 281-316.

From (7), we can write
\[ r_0(\theta, \theta, V) = b_0(\theta, \theta, V) \times \left[ \frac{\Omega(\theta, \theta, V) - \Omega(\theta, \theta, V)}{\Omega(\theta, \theta, V) - \Omega(\theta, \theta, V)} \right]. \tag{8} \]
where \( \theta \) denotes an orbital longitude at which the satellite achieves its mean brightness (averaged over \( \theta \)), and \( \Omega(\theta, \theta, V) \) is the mean opposition magnitude which from now on will be denoted by \( \bar{V}(\theta) \).

Given the likely assumption that the degree of limb darkening over the disk does not change strongly between \( a = 0^\circ \) and \( a = 5^\circ \), it follows from the definition of \( b_0 \) that the ratio of the \( b_i \) terms in the above expression can be taken to be unity. In effect, if we represent the limb darkening by a Mieaert law we are assuming that for \( a = 0^\circ \) and \( a = 5^\circ \) the limb darkening follows a Mieaert law with an exponent \( k \), since the correction does not depend strongly on the value of \( k \).

For a satellite divided into \( N \) rings, we find
\[ r_i(\theta, \theta, V) = r_0(\theta, \theta, V) \times \left[ \frac{\Omega(\theta, \theta, V) - \Omega(\theta, \theta, V)}{\Omega(\theta, \theta, V) - \Omega(\theta, \theta, V)} \right], \tag{10} \]
where \( N = 2 \), and \( \gamma = (1 - 1/N)^2 \). The bracketed correction is close to unity for most practical cases of interest. For example, for \( N = 3, \) it equals 1.03 for \( k = 1.0; \) for \( N = 2 \) it equals 1.07 for \( k = 1.0. \) For \( k = 0.5 \) the correction term is unity, independent of \( N \).

The average normal reflectances for Rhea, Titan, and the bright face of Iapetus, computed from (10) are given in Table VI. The corresponding values for Tethys and Dione have been omitted due to the large errors in the \( c_1 \)-reflections of these two satellites (Table V). Note that the values of \( k \) indicated in Table VI have been assumed, but that the results are only weakly dependent on this assumption (see above).

In computing the errors in \( r_i(\theta, \theta, V) \) given in Table VI, we have included (i) the shot noise errors in \( b_i \) (Table V), (ii) an estimated uncertainty of \( \pm 0.005 \) for \( \Omega(\theta, \theta, V) \), (iii) errors in the mean opposition magnitudes (Noland et al., 1974), (iv) changes in \( b_i \) (about \( \pm 0.1 \)) for different assumed satellite diameters over the range for \( k \) (Figs. 5 and 6), and (v) uncertainties in the cap correction factor [Eq. (10)] resulting from the uncertainties in \( k \). For each satellite the errors were added quadratically. Although this is not a statistically rigorous procedure, we believe that it yields a realistic estimate of the true uncertainties in the normal reflectances.

CONCLUSIONS

We have demonstrated that the normal reflectance of the central portion of a satellite's disk can be estimated from a lunar occultation light curve and that this estimate does not depend strongly on the actual diameter or the limb darkening law. The normal reflectances derived in this paper for Tethys, Dione, Rhea, and the bright face of Iapetus are high, and are consistent with values expected for frosted covered surfaces. We note that our value of the normal reflectance of the bright face of Iapetus (0.00 \( \pm 0.14 \)) is at best only marginally consistent with the value of about 0.35 derived from the radiometric analysis of Morrison et al. (1975). (cf. Veverka et al., 1975 for details).

For Titan we conclude that the normal reflectance in the \( V \) passband is 0.24 \( \pm 0.03 \). We recall that in Paper I we showed that the disk of Titan is highly limb darkened and that the satellite's diameter is certainly \( \geq 5800 \) km, and possibly may exceed 6200 km (cf. Fig. 14 of Paper I). We also note that our data place limits on the color dependence of the limb darkening of Titan.

<table>
<thead>
<tr>
<th>Satellite</th>
<th>Mieaert</th>
<th>Mean</th>
<th>Av normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>limb darkening</td>
<td>opposition</td>
<td>reflectance for 0° solar phase angle, ( r(\theta) )</td>
</tr>
<tr>
<td>Rhea</td>
<td>1.0 ( \pm 0.5 )</td>
<td>0.97 ( \pm 0.01 )</td>
<td>0.97 ( \pm 0.02 )</td>
</tr>
<tr>
<td>Titan</td>
<td>1.25 ( \pm 0.25 )</td>
<td>0.97 ( \pm 0.01 )</td>
<td>0.97 ( \pm 0.02 )</td>
</tr>
<tr>
<td>Iapetus</td>
<td>0.75 ( \pm 0.25 )</td>
<td>1.24 ( \pm 0.02 )</td>
<td>0.80 ( \pm 0.14 )</td>
</tr>
</tbody>
</table>

\* The errors in \( k \) are guesses.
\* Noland et al. (1974).
\* Bright hemisphere.

If the Mieaert law is used as reference form for the limb darkening, we find that the difference between the value of \( k \) for any two of the wavelength bands that we used (cf. Fig. 1) is \( \leq 0.1 \) (cf. Table V of Paper I).

ACKNOWLEDGMENTS

We appreciate the observing time which was made available to us by the Mauna Kea Observatory and thank the IMO staff for their assistance. D. Morrison, T. Jones, and J. Bart provided helpful discussions. This research was supported by NASA Grants NGR 33-010-062 and NSG 7126.

REFERENCES


Lunar Occultation of Saturn

III. How Big Is Iapetus?

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By combining both the orbital lightcurve of Iapetus and data obtained during the March 30, 1974, occultation of the satellite by the Moon, we obtain information about the brightness distribution on the bright face of Iapetus and derive an accurate value for the satellite's radius. From the observed orbital lightcurve, we find that the trailing face of Iapetus must consist predominantly of a single bright material with an effective limb-darkening parameter of \( k = 0.62 \pm 0.04 \). Given this result, the occultation observations imply a radius of 718.2±1.5 km. If the purely albosed model proposed by Morrison et al. represents the surface of Iapetus accurately (as far as the relative albosed distribution is concerned), then the center of Iapetus is 724±90 km. Both estimates are consistent with the radiometric radius of 733 (±50, −75) km derived by Morrison et al. Combining our results with the value of 0.60±0.14 for the normal reflectance (in \( V \)) of the material at the center of the bright face derived by Elliot et al., we find that the normal reflectance of the dark side material is 0.11±0.02. These values are higher than the corresponding values of 0.35 and 0.09 quoted by Morrison et al.

1. INTRODUCTION

Various attempts have been made to determine the size of Iapetus. Methods based on measuring the apparent size of the satellite in a telescope are uncertain, since Iapetus subtends, at best, only 0.3 arcsec. The best measurements of this type, the photometer measurements quoted by Dallus (1970), give a radius of 640±200 km (see Elliot et al., 1975).

More recently, Morrison and his collaborators used the "radiometric method" described by Morrison (1973) to determine the radius. Their original value was 850±100 km, but subsequent refinements of the technique and the observations led to their corrected value of 795 (±25, −75) km (Morrison et al., 1975).

In this paper we present an independent determination of the radius based on observations of an occultation of Iapetus by the Moon on March 30, 1974. These observations have already been published by Elliot et al. (1975). It is noted in that paper that for Iapetus, the signal-to-noise ratio achieved during the occultation did not allow independent determinations of (i) the absolute radius, and (ii) the brightness distribution across the disk. Elliot et al. (1975) derived two radii for Iapetus from their data; the first, 094 ± 58 km, assuming no limb darkening; the second, 785 ± 70 km, assuming limb darkening according to Lambert's law. In the present paper we exploit the fact that the large periodic brightness variations that Iapetus exhibits as it orbits Saturn constrain possible brightness distributions on its surface, and attempt to fit models of Iapetus that satisfy both the observed orbital lightcurve, and the occultation observations made on March 30, 1974.

For our purposes finding a suitable "model" amounts to finding an absolute radius and a brightness distribution on the surface which are consistent with the observations.

In agreement with previous investigators (e.g., Cook and Franklin, 1970; Morrison et al., 1975) we find that the observed lightcurve requires a "two-faced" model of Iapetus—that is, a model in which the leading face of the satellite is almost completely covered with low-albedo material, while the trailing side is almost completely covered with bright material. Furthermore, the orbital lightcurve places useful constraints on the degree of limb darkening of the bright material, a fact which is crucial to our analysis since it was the bright face of Iapetus which was occulted on March 30, 1974. (Iapetus was occulted only 16 hours after its western elongation, so that its bright face was turned almost exactly toward the Earth.)

Having constrained the degree of limb darkening on the bright face, we are able to calculate the radius of the satellite from our occultation data using methods described in detail by Elliot et al. (1975).

2. ORBITAL BRIGHTNESS VARIATIONS

Reproducible light variations of Iapetus with a period equal to the satellite's orbital period about Saturn (79 days) have been observed for about three centuries, proving that the satellite's spin period is synchronous with its orbital period. It is generally argued that the large brightness variations which are observed are due to albedo differences on the surface and not to an irregular shape.

Modern photometric observations of the lightcurve of Iapetus are summarized by Mills (1973), Noland et al. (1974), Morrison et al. (1975), and Veverka (1977). A nominal lightcurve based on the data compiled by Morrison et al. (1975) in which longitude-dependent solar phase effects have been removed is shown in Fig. 1. Its significant characteristics can be summarized as follows:

(a) Maximum brightness occurs at Western Elongation (orbital longitude \( \theta = 270^\circ \)); minimum brightness occurs at Eastern Elongation (\( \theta = 90^\circ \)).

(b) The amplitude in \( V \) is about 1.75 magnitudes, corresponding to a brightness ratio of about 5 to 1 between the bright and dark sides.

(c) The lightcurve is only weakly wavelength dependent.

(d) Plotted on a linear brightness scale, the lightcurve is approximately sinusoidal.

Fig. 1. Nominal lightcurve of Iapetus from Morrison et al. (1975) (their Fig. 5). The curve shown, based on observations by Mills (1973), Franklin and Cook (1974) and Noland et al. (1974), is plotted on a brightness scale. The total amplitude corresponds to 1.75 magnitudes. The important lightcurve parameters \( \Delta m_m \), \( \Delta m_2 \), and \( \Delta m_3 \), defined in the text, are indicated.
Cook and Franklin (1970) and Morrison et al. (1975) have shown that only a limited class of models can reproduce the light curve in Fig. 1. One such model, developed by Morrison et al., is shown in Fig. 2, and is discussed at length below. We note that the shape and amplitude of the lightcurve depend only on the relative albedos of the patches. The absolute albedos given in Fig. 2 are determined by the value of the absolute radius derived by Morrison et al. The albedo pattern shown in Fig. 2 reproduces the nominal observed lightcurve (Fig. 1) precisely.

To demonstrate how sensitive the lightcurve is to the brightness distribution on the surface, it is convenient to define the three lightcurve parameters \( \Delta_m \), \( \Delta_m \), and \( \delta_m \) shown in Fig. 1.

If we represent the average disk-integrated intensity of Iapetus by

\[
I = \left( \frac{I_{\text{max}} + I_{\text{min}}}{2} \right),
\]

the definitions of the above parameters are as follows:

\[
\Delta_m = \text{width of the lightcurve (in degrees of orbital longitude) halfway between } I \text{ and } I_{\text{min}}, \quad \Delta_m = \text{width of the lightcurve halfway between } I \text{ and } I_{\text{min}}, \quad \text{and } \delta_m \text{ is the brightness interval.}
\]

**2. LIGHTCURVE MODELS**

To appreciate the sensitivity of the lightcurve shape to the brightness distribution (including limb darkening) on the surface of Iapetus we begin by considering a family of very simple models which describes Iapetus as a spherical satellite with one uniformly limb-darkened bright hemisphere, and one uniformly limb-darkened dark hemisphere. Although no member of this family fits the lightcurve of Iapetus as well as the "spotted model" shown in Fig. 2, it is very easy to see from such an exercise how the shape of the lightcurve is affected by the degree of limb darkening.

The limb darkening can be described by either one of two convenient expressions. For the observations made very close to opposition we can assume that \( I = 4 \pi = \alpha \) (where \( \alpha = \text{incidence angle and } \epsilon = \text{emission angle} \) and write for the intensity of the scattered light

\[
I(\epsilon) = (1 - \beta) + b \cos \epsilon, \quad (1)
\]

or

\[
I(\epsilon) = B_0 \cos^2 \epsilon. \quad (2)
\]

where \( B_0 \) can be chosen equal to 1. The connection between these two representations is shown in Fig. 3. In general, the Mie/Berger representation (2) gives zero brightness at the limb (except for \( k = 1 \)), while the linear representation (1) gives a nonzero brightness at the limb (except for \( b = 1 \)). It turns out that in practice the exact behavior of \( I(\epsilon) \) near the limb is not important for our purposes. For example, the lightcurve for \( k = 0.7 \) is similar to that for \( b = 0.5 \). Thus our results are essentially independent of the analytic form used to represent the limb darkening.

We note that laboratory measurements (Veverka et al., 1978) show that Mie's law gives a good representation of the limb darkening of most particulate materials near opposition.

We now consider simple models of Iapetus in which the leading hemisphere (seen at Eastern Elongation) is entirely dark, and the trailing hemisphere (seen at Western Elongation) is entirely bright.

The observed photometric characteristics of the leading hemisphere suggest that the surface material is at least as dark, and at least as porous, as the average lunar regolith (Veverka, 1979). We therefore expect that this material, like the lunar regolith, shows no limb darkening at opposition, so that \( k = 0.5 \), or \( L = 0 \).

Our main task is to estimate the amount of limb darkening displayed by the brightside material. The recent discovery of water frost on the bright side of Iapetus by Morrison et al. (1976) and by Fox et al. (1976) suggests that \( k \) for the brightside material may be as high as 1 (Veverka, 1973) if the frost cover on the bright side is complete. If the frost cover is not uniform, or if the texture of the frost is unusually rough, the effective value of \( k \) could be significantly less than unity (Veverka et al., 1978). Fortunately, the observed lightcurve places significant constraints on the degree of limb darkening of the bright material.

Our simple model has a dark hemisphere which shows no limb darkening and a bright hemisphere which is limb darkened according to either (1) or (2). By entering the dark hemisphere at \( \theta = 90^\circ \) and the bright hemisphere at \( \theta = 270^\circ \) we immediately satisfy two key characteristics of the observed lightcurve (the locations of the maximum and minimum). The observed lightcurve amplitude fixes the ratio \( B_0 \) (bright)/\( B_0 \) (dark) once \( k \) (bright) is spec-

![Fig. 2. Model albedo distribution on the surface of Iapetus derived by Morrison et al. (1975). This albedo distribution reproduces the lightcurve in Fig. 1 precisely. Note that the lightcurve shape and amplitude depend only on the relative albedos of the patches, and not on the absolute values. In the original paper the values shown are called geometric albedos, but according to Jones (private communication), they are in fact normal reflectances. According to our data (Section 7) the normal reflectance of the brightest material is about 0.14 and that of the darkest is 0.112.](image-url)
It is also clear that a large value of $k$ (say $k \geq 1$) leads to a small value of $\Delta_{\text{max}}$ and a large value of $\Delta_{\text{min}}$ whereas if the degree of limb darkening is negligible, $\Delta_{\text{min}}$ and $\Delta_{\text{max}}$ are comparable.

Although none of the simple models in Fig. 4 fits the observed parameters exactly, we see that an effective brightness $k$ of about 0.6 to 0.7 is indicated. Values of $k < 0.5$ give unacceptably small values of $\Delta_{\text{min}}$ and $\Delta_{\text{max}}$ while once $k$ exceeds ~0.8, values of $\Delta_{\text{max}}$ become too large. It is clear from Fig. 4 that the bright side of Iapetus must have an effective limb-darkening parameter less than that for a Lambert surface ($k = 1$).

Figure 5 is the equivalent of Fig. 4, but for a linear limb-darkening law. No value of $b$ gives an accurate match with the observed lightcurve parameters. A value of $b = 0.2$ to 0.5 appears to be a reasonable compromise; for $b > 0.7$, $\Delta_{\text{max}}$ becomes too wide, while for $b < 0.1$, $\Delta_{\text{min}}$ becomes too narrow.

To better estimate the range of $k$ (or $b$) consistent with the orbital lightcurve we have compared the observed lightcurve with the calculated lightcurves at 12 points at intervals of 30° in orbital longitude (i.e., at $\theta = 0°, 30°, 60°, \ldots$) and evaluated $\sum = (O - C)$, the sum of the squares of the residuals.

In Fig. 6 we have plotted $\sum$ as a function of $k$. We see that $\Delta$ reaches a minimum around $k = 0.62$; the formal one-sigma uncertainty in $k$ is $\pm 0.05$.

A plot of $\sum$ in terms of $b$ gives a curve similar to that shown in Fig. 6. In this case, the minimum occurs at $b = 0.25$ and the formal one-sigma uncertainty is $\pm 0.15$, values which are in good agreement with those obtained from the analysis in terms of $k$ (cf. Fig. 3).

We stress that none of the above simple modeling methods matches the orbital lightcurve precisely.

“Two-faced” models match the orbital lightcurve closely. To match the lightcurve precisely one must consider an albedo distribution which isn’t precisely “two-faced.” Unfortunately, “spotted” albedo distributions cannot be derived uniquely from the lightcurve (Russell, 1966) but one can produce models which match the observed lightcurve accurately. One such model produced by Morrison et al. (1975) has already been mentioned (Fig. 2). In that model some dark material overlaps onto the bright side of Iapetus and a Lambert surface.

In summary, we conclude that the lightcurve of Iapetus requires an approximately, but not exactly, “two-faced” albedo distribution with the leading side essentially all dark and the trailing side essentially all bright. It is probable that the dark material shows no limb darkening. The best value of the limb-darkening parameter of the bright material is $k = 0.62^{+0.05}_{-0.05}$ (cf. Fig. 6), where we have quoted a conservative 3-sigma uncertainty.
the number of data points of 0.01-sec time resolution in the observed lightcurve across a satellite 700-800 km in radius. The model lightcurves were normalized to a post-occultation value of 1.0 and a post-occultation value of 0.0. An analytic expression for each lightcurve (in terms of a sixth-order Legendre polynomial) was found by least squares.

The observed lightcurve used e-mists of data points at 0.01-sec intervals, normalized so that before the occultation the signal = 1.0 and after the occultation, the signal = 0.0 (cf. Elliot et al., 1975).

Each model lightcurve, in its analytic form, was fitted by least squares to the observed lightcurve to determine the single parameter, R, the duration of the occultation in seconds, and the statistical (one-sigma) uncertainty in R (eq. 2).

In the notation of Elliot et al. (1975), the duration of the occultation, T (sec), is related to the radius of the satellite, R (km), by

\[
R = \frac{1}{2} C \rho \Delta r, \quad R = 306 \text{ km}
\]

where \( \Delta r \) = distance of the occulted satellite from the Earth = 9.14 AU, \( \rho \) = radial rate of advance of the mean lunar limb = 0.3707 arasec/sec, and \( C \) = correction factor to \( \rho = 1.0 \).

The results of such calculations for the models discussed in Section 3 are given in Table I, first for the simple uniform hemisphere models and second for the model of Morrison et al. (1975). The results of this section are not sensitive to whether we specify the limb darkening of the bright material according to (1) or (2). For example, from Fig. 3 we see that for \( b = 0.5 \) corresponds approximately to \( k \approx 0.7 \); Table I shows that the inferred radius for \( k = 0.7 \) is practically identical to that inferred for \( b = 0.5 \). This fact is generally true for corresponding values of \( k \) and \( b \).

We note that the errors quoted in Table I represent statistical errors only, determined from the signal-to-noise characteristics of the occultation data (cf. Elliot et al., 1975).

In Section 3 we concluded that for the bright material \( k = 0.62 \pm 0.05 \); hence, from Table I, we infer that for Iapetus \( R = 718 \pm 5 \text{ km} \), which is consistent with the nominal radiometric value of 335 +50, -75 km quoted by Morrison et al. (1975).

The specific albedo model for the bright side of Iapetus proposed by Morrison et al. (cf. Fig. 2) gives a radius of

\[
R = 724 \pm 60 \text{ km}
\]

Since the albedo distribution on the bright side is asymmetric in this model, the orientation of the spin axis of Iapetus relative to the lunar limb at the time of the occultation had to be considered in this calculation. Note that this value for the radius is consistent with those given by the simple two-facialed models which have \( k \approx 0.64 \) or \( b \approx 0.4 \) (Table I).

Although the Morrison et al. surface albedo model (Fig. 2) does reproduce the orbital lightcurve exactly, it is not unique and hence need not represent the true albedo distribution on Iapetus precisely. However, from the discussion in Section 3 it is clear that the true albedo distribution can differ from that in Fig. 2 only to the extent that the average limb darkening on the bright side remains within the range 0.5 to 0.8. According to Table I, \( k > 0.5 \) implies \( R > 640 \text{ km} \), while \( k < 0.8 \) implies \( R < 827 \text{ km} \). Thus, it appears unlikely that the true radius of Iapetus can differ from that derived for the Morrison et al. albedo model, by more than about \( \pm 100 \text{ km} \). Therefore, conservatively, we adopt 724 \( \pm 100 \text{ km} \) as the nominal lunar occultation radius of Iapetus.

5. RIM MODELS

Since the radiometric radius of Iapetus is slightly larger than our occultation value, we investigated various special models which could conceivably produce larger occultation radii. For example, we considered a series of dark-rim models in which a rim of dark leading-side material is seen as an annulus around the limb when the bright side is viewed at \( \theta = 370^\circ \). We assumed that the dark rim material does not show limb darkening, while the bright material is limb darkened according to \( \cos^{2}\theta \). Once again the lightcurve amplitude fixes the ratio of the normal reflectances of the bright and dark materials.

To evaluate the effects of a dark rim of various radii, we computed lightcurves for Iapetus at various times of the occultation as a function of the rim radius and the rim reflectance. The results of these calculations are presented in Table II.

<table>
<thead>
<tr>
<th>RIM EXTENT (deg)</th>
<th>RIM RADIUS (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>704 \pm 58</td>
</tr>
<tr>
<td>30</td>
<td>780 \pm 65</td>
</tr>
<tr>
<td>40</td>
<td>873 \pm 72</td>
</tr>
<tr>
<td>50</td>
<td>989 \pm 88</td>
</tr>
</tbody>
</table>

* Assuming \( a = 0.5 \) for both the dark and bright materials (see text).
7. NORMAL REFRACTION AND GEOMETRIC ALBEDO

From an independent analysis of the occultation data Elliot et al. (1975), derive 0.60 ± 0.14 for the normal refraction (r_a) of the bright-side material in V. Since k = 0.62 ± 0.16, we have for the geometric albedo of the bright side (Veyerka, 1977)

\[ P_b = \frac{r_a}{k + 1} = 0.60 ^{+0.14}_{-0.14} \]

From the observed lightcurve amplitude we know that

\[ P_a \text{(bright side)}/P_a \text{(dark side)} = 5, \]

so that \[ P_d \text{(dark side)} = 0.11 ^{+0.08}_{-0.09}. \]

As assumed, the dark side shows no limb darkening, then for the dark side

\[ r_a \text{(V)} = P_d = 0.11 ^{+0.08}_{-0.09}. \]

We now compare our values of the normal refractions with those derived by Morrison et al. (1975). According to the caption of Fig. 7 (our Fig. 2) of Morrison et al. the geometric albedo in V for the "bright" and "dark" material are 0.28 and 0.09, respectively. However, T. J. Jones (private communication) informs us that these values refer to normal reflectances and not to geometric albedo. Thus for the bright material the normal reflectance of 0.33 derived by Morrison et al. is to be compared with our value of 0.60 ± 0.14. For the dark material, their value of 0.08 is to be compared to our 0.11 ± 0.09.

Thus although the occultation radius and the radiometric radius agree within their stated error bars, the normal refractions derived by Morrison et al. are at most marginally consistent with our values. Our analysis indicates that both the "bright" and the "dark" materials are brighter than supposed by Morrison et al.

8. THE MEAN DENSITY OF IAPETUS

The mass of Iapetus is only poorly determined from perturbations on Titan (Struve, 1953). The formal result (1.4 ± 0.7 × 10^10 kg) corresponds to about 2% of a lunar mass. Combined with the occultation radius based on the albedo model of Morrison et al. (724 ± 60 km), we obtain a mean density of 0.9 (±0.9, -0.6) g/cm^3, a result consistent with Iapetus being an icy object. However, the uncertainties are too large at present to place any useful constraints on the bulk composition of the satellite.

9. SUMMARY AND CONCLUSIONS

From the orbital lightcurve of Iapetus we find that the bright material on the trailing face of the satellite has an effective limb-darkening parameter of \( k = 0.62^{+0.14}_{-0.09} \). Given this fact, we can use the observations of the March 30, 1974, occultation of Iapetus by the Moon (Elliot et al., 1975) to derive the radius of the satellite. The result is \( R = 718^{+5}_{-5} \) km.

The specific pateful albedo model proposed by Morrison et al. (1975, Fig. 2) fits the observed orbital lightcurve precisely and yields an occultation radius of 724 ± 60 km. We note that as far as fitting the observed orbital lightcurve is concerned it is the relative albedo in Fig. 2 that are important, and not the absolute values. In fact, it appears from the work of Elliot et al. (1975) that the absolute values of the albedo shown in Fig. 2 are all too low. Elliot et al. find the normal reflectance of the bright material to be 0.60 ± 0.14 in V, and not 0.35 as shown in Fig. 2. A value of \( r_a \text{(V)} = 0.60 ^{+0.14}_{-0.14} \) for the bright material implies that for the dark material \( r_a \text{(V)} = 0.11 ^{+0.08}_{-0.09} \) (Section 7) and not 0.08 as shown in Fig. 2.

We note that the values of \( P_a = 0.60 ^{+0.14}_{-0.14} \) and \( k = 0.62^{+0.14}_{-0.09} \) are consistent with a partial water frost cover on the bright side of Iapetus (McCord et al., 1971; Morrison et al., 1976; Fink et al., 1976).

We conclude that if the albedo model of Morrison et al. turns out to represent the surface of Iapetus reliably that is, as far as the relative distribution of albedo is concerned, it not in terms of absolute albedo) then the radius of Iapetus is 724 ± 60 km. This value is consistent with the radiometric radius of 833 (±50, -75 km) derived by Morrison et al. (1975).

We stress that even though the Morrison et al. albedo model does reproduce the observed orbital lightcurve accurately it is not unique and need not represent the actual albedo distribution on the surface of Iapetus. Nevertheless, as we saw in Section 3, the true relative albedo-distribution on the bright face of Iapetus can differ from that shown in Fig. 2 only to the extent that the average limb-darkening parameter \( k \) of the bright face remains within the range 0.5 to 0.8. This in turn implies that the true radius probably lies between 640 and 527 km; in other words the true radius can differ at most by about 100 km from the value of 724 km derived for the Morrison et al. model. Thus, we adopt \( R = 725^{+100}_{-100} \) km as the lunar occultation radius of Iapetus, and note that if the relative albedo distribution shown in Fig. 2 is correct, then the radius lies within 724 ± 60 km.

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REFERENCES


