BIVARIATE NORMAL CONDITIONAL AND RECTANGULAR PROBABILITIES: A COMPUTER PROGRAM WITH APPLICATIONS

Ram Swaroop, James D. Brownlow, George R. Ashworth and William R. Winter

May 1980
INTRODUCTION

In applications involving univariate data where estimates and confidence intervals are required, the normal distribution is commonly employed. This distribution is mainly utilized because the probabilities under a normal curve are readily available. In contrast, use of multivariate probabilities in p-variate normal data are less frequent, primarily because probabilities for the multivariate normal case are generally not available. Except for very special cases, the probabilities for sections of p-dimensional space require extensive computations, since the canonical multivariate normal density changes with every change in correlation coefficient parameters. Even the probability computation in the bivariate normal case (p = 2) with only one value for the correlation coefficient over arbitrary sections of the (x, y) plane is not easy. Probability computations, therefore, in p > 2 dimensions are correspondingly much more difficult. (Ref. 1)

In many applications, problems are posed which not only require the probabilities over a section of p-dimensional space, but also the conditional probabilities of r (r < p) variables when the remaining (p - r) variables are either fixed, or are within designated intervals. For example, in aircraft target tracking studies, it is of interest to know the probability of X deviations from the target when Y deviations are considered within designated bounds. In aircraft performance studies it is important to know the distribution of the pilot's cardiac R-R intervals either under an assigned difficult aircraft maneuver or under the dynamic flight conditions.

The results on conditional and marginal distributions of r variables when the (p - r) remaining variables assume fixed values are well established. (Ref. 1) Similar results, when the remaining (p - r) variables assume values within specified ranges involve complexities and are discussed in this report.

In this study, results on bivariate normal distributions (p = 2) are reviewed. Various derivations and properties of bivariate normal conditional probabilities are derived. A computer program for conditional probabilities for all assigned values is included. From conditional and marginal probabilities, the rectangle probabilities are then obtained. Examples are presented to illustrate the use of the program. The program listing is appended to this report.

SYMBOLS

\[ A_y \] lateral acceleration
\[ A_z \] vertical acceleration
\[ c \] a constant with fixed numerical value
\[ \exp(x) \] exponential function at x
F(s) conditional distribution of X at X = s Given Y is in interval (a, b)

f(u, v) general bivariate normal density

f(x), f(y) standard normal densities

f(x, y) standard bivariate normal density at X = x, Y = y

f(x|a<Y<b) conditional density of X at X = x given Y is in interval (a, b)

f(x|Y=y) conditional density of X at X = x given Y = y

f(x|Y<t, ρ<0) conditional density of X at X = x given Y is less than t and correlation is negative

f(x|Y>-t, ρ>0) conditional density of X at X = x given Y is greater than -t and correlation coefficient ρ is positive

G_p(s, t) double integral with two arguments s and t with a fixed value of correlation coefficient ρ

G_t(x) conditional density of X at X = x given Y is in interval (-t, t)

G_t(x|ρ>0) conditional density of X at X = x when correlation coefficient ρ is positive and Y is in interval (-t, t)

G_t(x|ρ<0) conditional density of X at X = x when correlation coefficient ρ is negative and Y is in interval (-t, t)

p, r dimension of multivariate data or distribution

Pr[a<Y<b] probability that variable Y is in interval (a, b)

Pr[c<X<d, a<Y<b] joint probability that variable X is in interval (c, d) and variable Y is in interval (a, b)

Pr[X<h, Y<k] probability that X is less than h and Y is less than k

U, V, X, Y random variables

u, v, x, y, t specific values of random variables

V_c forward velocity

α fixed positive constant less than 1

μ_c mean of forward velocity V_c
BIVARIATE NORMAL DISTRIBUTION

A bivariate normal distribution of a random vector \((U, V)\) is characterized by parameters: \(\mu_u, \mu_v, \sigma_u, \sigma_v\), and \(\rho\). The density function

\[
f(u, v) = \left[2\pi \sigma_u \sigma_v \sqrt{1 - \rho^2}\right]^{-1} \exp \left(-\left\{ \frac{(u - \mu_u)^2}{2\sigma_u^2} + \frac{(v - \mu_v)^2}{2\sigma_v^2} - 2\rho \frac{(u - \mu_u)(v - \mu_v)}{\sigma_u \sigma_v} \right\}\right)
\]

is defined over the entire \((u, v)\) plane. When the variables \(U\) and \(V\) are standardized, by defining the new variables

\[
X = \frac{U - \mu_u}{\sigma_u}, \quad Y = \frac{V - \mu_v}{\sigma_v}
\]

the density function of \((X, Y)\) reduces to the canonical bivariate normal density

\[
f(x, y) = \left(2\pi \sqrt{1 - \rho^2}\right)^{-1} \exp \left[\frac{- (x^2 - 2\rho xy + y^2)}{2(1 - \rho^2)}\right]
\]

defined over the entire \((x, y)\) plane. The parameter \(\rho\) is called a correlation coefficient and takes values in the interval \((-1, 1)\). Without any loss of generality, this canonical density \(f(s, y)\) is considered in this study.
The density function \( f(x, y) \) exhibits certain properties. It is symmetric in opposite quadrants since

\[
f(x, y) = f(-x, -y)
\]

and

\[
f(x, -y) = f(-x, y)
\]

Further, \( f(x, y) \) is constant over all the ellipses

\[
x^2 - 2\rho xy + y^2 = c(1 - \rho^2)
\]

for every value of \( x \). (Fig. 1) The intercepts made by these ellipses on the \( x \) and \( y \) axes are equal. If \( \rho \) is positive, the major axis of the ellipse is along the 45° line with a length of \( 2\sqrt{c(1 + \rho)} \); and the minor axis is along the 135° line with a length of \( 2\sqrt{c(1 - \rho)} \). If \( \rho \) is negative, the major axis is along the 135° line with a length of \( 2\sqrt{c(1 - \rho)} \); the minor axis along the 45° line has a length of \( 2\sqrt{c(1 + \rho)} \). (Ref. 2) The ellipse

\[
x^2 - 2\rho xy + y^2 = (1 - \rho^2) \log (1 - \alpha^2)
\]

for all \( 0 < \alpha < 1 \), contains the \( \alpha \) proportion of the \((X, Y)\) distribution. (Ref. 3)

The marginal distributions of \( X \) and \( Y \) are standard normal with the covariance between \( x \) and \( y \) equal to \( \rho \). When \( \rho = 0 \), then

\[
f(x, y) = (\sqrt{2\pi})^{-1} \exp(-x^2/2) (\sqrt{2\pi})^{-1} \exp(-y^2/2)
\]

\[
= f(x) \cdot f(y)
\]

which is a product of standard normal densities, implying that \( \rho = 0 \) if and only if \( X \) and \( Y \) are independent. When \( \rho \neq 0 \), bivariate normal probabilities \( \Pr(X < h, Y < k) \) for a few selected values of \( h \) and \( k \) are available from tables and graphs. (Ref. 4, 5) For general values of \( h \) and \( k \) approximation and interpolation methods are used.

DERIVATION OF CONDITIONAL DENSITIES

Conditional Density of \( X \) Given \( Y = y \). It was stated earlier that if a random vector \((X, Y)\) has a bivariate normal distribution, then the marginal distribution of either \( X \) or \( Y \) is normal with mean 0 and variance 1. The conditional distribution of \( X \) for a fixed value of \( Y = y \), however, is normal with mean \( \rho y \) and variance \( (1 - \rho^2) \). The conditional density \( f(x|Y = y) \) is derived below.
Figure 1. Marginal, conditional densities and ellipses of constant densities from bivariate normal density.
\[ f(x | Y = y) = \frac{f(x, y)}{f(y)} \]

\[
= \frac{(2\pi \sqrt{1 - \rho^2})^{-1} \exp \left[ -\left( \frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right) \right]}{(\sqrt{2\pi})^{-1} \exp(-y^2/2)}
\]

\[
= \left( \sqrt{2\pi(1 - \rho^2)} \right)^{-1} \exp \left\{ \left[ \frac{x^2 - 2\rho xy + y^2 - (1 - \rho^2)y^2}{2(1 - \rho^2)} \right] \right\}
\]

\[
= \left( \sqrt{2\pi(1 - \rho^2)} \right)^{-1} \exp \left[ -\left( \frac{x^2 - 2\rho xy + \rho^2 y^2}{2(1 - \rho^2)} \right) \right]
\]

which is the density of a normal distribution with mean \( \rho y \) and variance \((1 - \rho^2)\) and is shown in Figure 1.

**Conditional Density of** \( X \) **Given** \( a < Y < b \). The conditional density of \( X \) given \( a < Y < b \) is not normal and is derived as follows.

\[
f(x | a < Y < b) = \frac{(2\pi \sqrt{1 - \rho^2})^{-1} \int_a^b \exp \left[ -\left( \frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right) \right] dy}{}\]

\[
= \left[ \phi(b) - \phi(a) \right]^{-1} \left( \frac{2\pi \sqrt{1 - \rho^2}}{} \right)^{-1} \]

\[
\int_a^b \exp \left\{ \left[ \frac{y^2 - 2\rho xy + \rho^2 x^2 + x^2(1 - \rho^2)}{2(1 - \rho^2)} \right] \right\} dy
\]

\[
= \left( \sqrt{2\pi} \right)^{-1} \exp \left( \frac{x^2}{2} \right) \left[ \phi(b) - \phi(a) \right]^{-1} \]

\[
\int_a^b \left[ \sqrt{2\pi(1 - \rho^2)} \right]^{-1} \exp \left[ -(y - \rho x)^2/2(1 - \rho^2) \right] dy
\]

\[
= f(x) \left[ \phi(b) - \phi(a) \right]^{-1} \left\{ \phi \left( \frac{b - \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{a - \rho x}{\sqrt{1 - \rho^2}} \right) \right\}
\]
where
\[ f(x) = \left(\sqrt{\frac{\pi}{2}}\right)^{-1} \exp\left(-\frac{x^2}{2}\right) \]
is a standard normal density and
\[ \phi(t) = \int_{-\infty}^{t} f(x) \, dx \]
is the standard normal distribution function.

This conditional density is neither normal, nor symmetric. However, in special cases discussed below, symmetry is identifiable.

**Symmetry in Conditioning -t < Y < t.** With -t < Y < t, the conditional density of X at specific values of x and -x are

\[ g_t(x) = f(x| -t < Y < t) \]
\[ = f(x) \left[ \phi(t) - (-t) \right]^{-1} \left\{ \phi \left( \frac{t - \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{-t - \rho x}{\sqrt{1 - \rho^2}} \right) \right\} \]

\[ g_t(-x) = f(-x| -t < Y < t) \]
\[ = f(-x) \left[ \phi(t) - \phi(-t) \right]^{-1} \left\{ \phi \left( \frac{t + \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{-t + \rho x}{\sqrt{1 - \rho^2}} \right) \right\} \]

The symmetry of a standard normal density shows that \( f(-x) = f(x) \). With the asymmetry of distribution function \( \phi(t) = 1 - \phi(-t) \), it is seen that

\[ \phi \left( \frac{t + \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{-t + \rho x}{\sqrt{1 - \rho^2}} \right) = 1 - \phi \left( \frac{-t - \rho x}{\sqrt{1 - \rho^2}} \right) - \left\{ 1 - \phi \left( \frac{t - \rho x}{\sqrt{1 - \rho^2}} \right) \right\} \]
\[ = \phi \left( \frac{t - \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{-t - \rho x}{\sqrt{1 - \rho^2}} \right) \]

Thus \( g_t(-x) = g_t(x) \), showing that for -t < Y < t the conditional density of X is symmetric in x, as shown in figure 2.

The conditioning, -t < Y < t, with positive and negative values of correlation coefficient \( \rho \) also show symmetry of \( g_t(x) \). It is to be noted that

\[ g_t(x|\rho > 0) = f(x) \left[ \phi(t) - \phi(-t) \right]^{-1} \cdot \left\{ \phi \left( \frac{t - \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{-t - \rho x}{\sqrt{1 - \rho^2}} \right) \right\} \]
Figure 2. Conditional density of $X$ given $-t < Y < t$ ($t = 1.000$, probability = 0.6826) where $(X, Y)$ is bivariate normal with $\rho = 0.9000$, and standard normal density.
\[
g_t(x|\rho < 0) = f(x)[\phi(t) - \phi(-t)]^{-1} \cdot \left\{ \phi\left[\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right] - \phi\left[\frac{-t + \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

By the symmetry of \(f(x)\), the asymmetry of \(\phi(t)\), and the arguments given earlier, it is seen that \(g_t(x|\rho > 0) = g_t(x|\rho < 0)\). The graph of such a density is shown in figure 2.

Symmetry when \(-\infty < Y < t\) and \(-t < Y < +\infty\). In these cases it is to be noted that \(\phi(-\infty) = 0\), \(\phi(\infty) = 1\). Thus the conditional densities of \(X\) are

\[
g_t(x|\rho > 0) = f(x|Y < t) = f(x)[\phi(t)]^{-1} \cdot \phi\left[\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right]
\]

\[
g_{-t}(x|\rho > 0) = f(x|-t < Y)
\]

\[
= f(x)[1 - \phi(-t)]^{-1} \cdot \left\{ 1 - \phi\left[\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

\[
= f(x)[\phi(t)]^{-1} \cdot \left\{ \phi\left[\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

\[
g_{-t}(-x|\rho > 0) = f(-x|-t < Y)
\]

\[
= f(x)[\phi(t)]^{-1} \cdot \left\{ \phi\left[\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

Thus \(g_t(x) = g_{-t}(-x)\), showing that a one-sided conditioning on \(Y\) yields the same density for \(X\) as does the conditioning on the other side for the opposite \(x\). Further, for negative and positive values of \(\rho\), it is to be noted that

\[
g_t(x|\rho > 0) = f(x)[\phi(t)]^{-1} \cdot \left\{ \phi\left[\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

and

\[
g_t(x|\rho < 0) = f(x)[\phi(t)]^{-1} \cdot \left\{ \phi\left[\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right] \right\}
\]

\[= g_{-t}(x|\rho > 0)\]
Therefore, if the conditioning on $Y$ and the sign of the correlation coefficient are reversed, the density remains invariant. An example of these densities is shown in figure 3.

DERIVATION OF CONDITIONAL DISTRIBUTIONS

**Conditional Distribution Function of $X$ Given $Y = y$.** The distribution function from the conditional density

$$f(x|Y = y) = \left[\frac{1}{2} \left(1 - \rho^2\right)\right]^{-1} \exp\left[-(x - \rho y)^2/2(1 - \rho^2)\right]$$

derived earlier, is easily obtainable via the normal distribution function with mean $\rho y$ and variance $(1 - \rho^2)$. It is to be observed from figure 1, that mean $\rho y$ is a function of the correlation $\rho$ and the specific conditioned value of $y$, but the variance depends only on $\rho$ and is invariant for all values of $y$. Thus the width of any $\alpha$ level confidence interval remains the same irrespective of the conditioned values of $y$.

In applications, the conditioning of variable $Y$ is seldom a fixed value. The conditioning is usually in a range $a < Y < b$, and the formulae for this case are different from the results for $Y = y$.

**Conditional Distribution of $X$ Given $a < Y < b$.** The conditional density

$$f(x|a < Y < b) = f(x)\phi(b) - \phi(a)\left[\frac{1}{\sqrt{1 - \rho^2}}\right] - \phi\left(a - \rho x\right)\left[1 - \frac{1}{\sqrt{1 - \rho^2}}\right]$$

where

$$f(x) = (\sqrt{2\pi})^{-1} \exp(-x^2/2)$$

and

$$\phi(t) = \int_{-\infty}^{t} f(x) dx$$

was derived earlier. A general expression for the distribution function

$$F(s) = \int_{-\infty}^{S} f(x|a < Y < b) dx$$

$$= \left[\phi(b) - \phi(a)\right]^{-1} \int_{-\infty}^{S} f(x) \left\{\phi\left(b - \rho x\right)/\sqrt{1 - \rho^2}\right\} - \phi\left(a - \rho x\right)/\sqrt{1 - \rho^2} dx$$

for all the values of $s$ involves integration of the expression which is the product of the normal density and distribution function in the appropriate range of the $x$ values. Specifically, for the computation of $F(s)$, the
Figure 3. Conditional density of $X$ given $-\infty < Y < t$ ($t = 1.00$, probability = 0.8413) where $(X, Y)$ is bivariate normal with $\rho = 0.6000$, and standard normal density.
value of double integrals such as

\[ G_p(s,t) = \int_{-\infty}^{\infty} \exp\left(-x^2/2\right) \left[ \int_{-\infty}^{\infty} \frac{(t - \rho x)/\sqrt{1 - \rho^2}}{\exp(-u^2/2)} du \right] dx \]

for all values of \( s, t \) and \( \rho \) are required. In terms of these functions, it is easily seen that

\[ F(s) = \left\{ 2\pi \left[ \Phi(b) - \Phi(a) \right] \right\}^{-1} \left[ G_p(s,b) - G_p(s,a) \right] \]

A closed analytical expression for \( G_p(s,t) \) is not available and for specific values, numerical methods may be employed. However, in cases where symmetry occurs, the numerical computations for a smaller range of values are needed. In order to calculate \( F(s) \) for all values of \( s, a, b \) and \( \rho \), a computer program using quadratures was developed at DFRC and is given in the Appendix.

Rectangle Probabilities. The region \((c < x < d, a < Y < b)\) is a rectangle in the \((x, y)\) plane. Thus the joint probability \(Pr(c < X < d, a < Y < b)\) for real values of \(a, b, c\) and \(d\) corresponds to a rectangle probability. The appended computer program can be used to calculate all such rectangle probabilities. The procedure is to identify first that

\[ Pr[c < X < d, a < Y < b] = Pr[c < X < d | a < Y < b] Pr[a < Y < b] \]

\[ = [F(d) - F(c)] Pr[a < Y < b] \]

\[ = [F(d) - F(c)] \left[ \Phi(b) - \Phi(a) \right] \]

for all values of \( c < d \) and \( a < b \), and then use the computer program with the proper inputs.

**COMPUTER PROGRAM INPUTS AND OUTPUTS**

The computer program developed at DFRC computes the conditional density and distribution function as outputs for specified values of \( x \) given the end points of the interval of the conditioning variable \( Y \), and the correlation coefficient \( \rho \). Thus the inputs to the program are specific \( x \) values, end points of the \( Y \) interval and the \( \rho \) value. The output has two options. Either the density or distribution function, or both may be obtained by stating the options in the program.

The rectangle probabilities are to be obtained by finding the conditional probabilities. The computer program with its options is explained in the Appendix.
EXAMPLES

The following examples illustrate the use of the program and tables shown in the Appendix to calculate various probabilities.

The data for the examples are taken from a Closed Circuit Television (CCTV) experiment. In this experiment, two pilots, A and B, landed an aircraft with the help of an airborne television camera and video monitor. Each pilot made ten (10) touchdowns under visual flight regulations, and eighteen (18) touchdowns utilizing the closed circuit television monitor. The summary of data from the twenty-eight (28) touchdowns is given in Table I. For this illustration the data parameters are vertical acceleration, \( A_z \), forward velocity, \( V_c \) and lateral acceleration \( A_y \).

<table>
<thead>
<tr>
<th>TABLE I. SUMMARY OF 28 TOUCHDOWN DATA OF CCTV EXPERIMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pilot</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

The variables \( (A_z, V_c, A_y) \) are assumed to follow a multivariate normal distribution. Thus any two variables follow a bivariate normal distribution and any single variable, a univariate normal distribution, as shown in figure 1. Further, all the values in these date are considered to be parameter values.

Example 1. Computation of a 95% confidence interval of forward velocity \( (V_c) \) given vertical acceleration \( (A_z) \) mean is within ± one standard deviation (\( \sigma \)). It is desired in this example to determine a 95% confidence interval for aircraft forward velocity \( (V_c) \), in miles per hour, at the point of touchdown, given the pilot's average vertical acceleration \( (A_z) \), in G's, within ± one standard deviation. The 95% confidence interval end points for \( V_c \) given \( A_z \) mean is within ± \( \sigma \) are obtained by solving for \( t \) from the equation.


\[
.95 = \Pr[-t < (V_c - \mu_c)/\sigma_c < t| -1 < (A_z - \mu_z)/\sigma_z < 1]
\]

\[
= \Pr[-t < X < t| -1 < Y < 1]
\]

and identifying the interval as \((-t\sigma_c + \mu_c, t\sigma_c + \mu_c)\).

The solution of the equation for pilot A data of \(\mu_c = 60.25, \sigma_c = 1.3089,\)
\(\mu_z = 1.313, \sigma_z = 0.2021\) and correlation \((A_z, V_c) = -.2481\), yields the value
of \(t = 1.91666\). The 95% confidence interval, therefore, becomes

\[
(57.7413, 62.7587)
\]

This shows that if in pilot A data, the aircraft's vertical acceleration
at touchdown is within \(+1.3 \pm 0.2\) G's, he has a 95% chance of landing the
aircraft between 58 and 63 MPH.

For pilot B data, from table I, the \(t\) value computes to be 1.9136.
Thus the 95% confidence interval is

\[
(57.7453, 62.7547)
\]

indicating if pilot B's vertical acceleration data at touchdown is within
\(+1.3 \pm 0.1\) G's, he also has a 95% chance of landing the aircraft between 58
and 63 MPH.

Example 2. Computation of the probability that the forward velocity \((V_c)\)
and \((A_y)\) are both within \(\pm \sigma\) of each variable. The probability of \(V_c\) and \(A_y\)
being within \(\pm \sigma\) of each respective mean is an example of rectangle proba-
bility. In this example, the probability that simultaneously, \(V_c\) and \(A_y\),
will be within one standard deviation of each variable's respective mean is
to be computed.

This rectangle probability can be obtained by finding

\[
\Pr[-1 < (V_c - \mu_c)/\sigma_c < 1, -1 < (A_y - \mu_y)/\sigma_y < 1]
\]

\[
= \Pr[-1 < X < 1 | -1 < Y < 1] \Pr[-1 < Y < 1]
\]

From univariate tables, \(\Pr[-1 < Y < 1] = \phi(1) - \phi(-1) = .6826\) and is not
affected by the correlation coefficients. In order to obtain \(\Pr[-1 < X < 1 |
-1 < Y < 1]\), the values of the correlation coefficients are needed.

The correlation coefficient \((V_c, A_y)\) for pilot A data is equal to \(-0.2807\). The
computer program output, therefore, for this correlation yields

\[
\Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554
\]

Thus, for pilot A there is a 48% chance that simultaneously at touchdown,
the aircraft's forward velocity will be within \(60 \pm 1.3\) MPH and the lateral
acceleration is within \( 0 \pm 0.1 \) G's. Conversely, the probability is 0.52 that both variables will not simultaneously be within one standard deviation of their respective means. Similarly, for pilot B with the correlation \((V_c, A_y)\) equal to .1993, the program yields

\[
Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \quad \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47078
\]

which represents a 0.47 probability that the forward velocity will be within 62 \( \pm 1.9 \) MPH and lateral acceleration is within \( 0 \pm 0.08 \) G's.

**Example 3.** Computation of the probability of forward velocity \( (V_c) \) and lateral acceleration \( (A_y) \) being within \( \pm \sigma \) of each variable, given vertical acceleration is equal to its mean \( (A_z = \mu_z) \). This rectangle probability can be obtained as in Example 2, except in this case the vertical acceleration \( (A_z) \) is set equal to the variable's mean value \( (\mu_z) \). The probability in other words, is a function of a conditional correlation coefficient which is different from the coefficient given in the table.

For the pilot A data, this conditional coefficient is equal to .3809 and the program output yields

\[
Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \quad \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .48391
\]

which represents a 0.48 probability that the forward velocity will be within 60.25 \( \pm 1.309 \) MPH, and lateral acceleration within \( .023 \pm 0.1227 \) G's given that vertical acceleration is 1.313 G's.

For pilot B, the conditional correlation coefficient is equal to -.2807 and the corresponding rectangle probability is

\[
Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \quad \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554
\]

This represents a 0.48 probability that the forward velocity will be within 62.0 \( \pm 1.9 \) MPH and lateral acceleration is within \( 0 \pm 0.08 \) G's given the vertical acceleration is +1.294 G's.
APPENDIX

The program to compute the conditional density and distribution function for specified values of \( x \) given the conditioning on variable \( Y \).
PROGRAM MAIN

C***** ILLUSTRATIVE USE OF THE ENCLOSED COMPUTER
C***** PROGRAMS TO COMPUTE VARIOUS
C***** PROBABILITIES ASSOCIATED WITH THE EXAMPLES
C***** GIVEN IN THE TEXT OF THIS PAPER...
C*****
C***** BY BROWNLOW, SDC/ISI
C*****

5 PRINT I, RPROB(-1.,1.,-1.,1.,-2907)
10 PRINT 1, RPROB(-1.,1.,-1.,1.,-1993)
15 PRINT 1, RPROB(-1.,1.,-1.,1.,-3809)

C*****

PRINT1, TINV(.95,-1.,1.,-2491)
PRINT 1, TINV(.95,-1.,1.,-2569)
1 FORMAT(*F10.5)
END
FUNCTION CD(X)
C**** CONDITIONAL DENSITY FUNCTION
C**** CD(X\A<Y\B) = 1./SQRT(2*PI*EXP(X*X)) * 
C**** \{ PHI( (B-R*X)/(SQRT(1-R*R)) ) - PHI( (A-R*X)/(SQRT(1-R*R)) ) \} 
C**** /\{ PHI(B) - PHI(A) \}
C**** WHERE R = COEFFICIENT OF CORRELATION BETWEEN
C**** X AND Y
C****
C**** PHI(T) = INTEGRAL F(X) DX
C**** -INF
C****
C**** AND F(X) = 1./SQRT(2*PI*EXP(X*X))
C**** BY BRWNLOW, SDC/ISI
C****
C**** COMMON/PARAM/A, B, R, SQR
CD = .39894228/SQRT(EXP(X*X)) * ( PHI( (B-R*X)/SQR) - 
* PHI((A-R*X)/SQR) )/(PHI(B)-PHI(A))
C**** RETURN
END
FUNCTION FINT2 73/74  OPT=1 FNT 4.2+75060

FUNCTION FINT2(F,A,B)
C****  INTEGRAL OF THE FUNCTION F FROM A TO B
C****  BY GAUSSIAN-LEGENDRE QUADRATURE, 95 POINT FORM
9
C****  REQUIRES 95 EVALUATIONS OF F(X).
C****
C****  F MUST BE DECLARED EXTERNAL IN
C****  THE CALLING PROGRAM.
C****  BY JACKNOW, SOEYSI
10
C****
C****  DOUBLE PRECISION KDOT(48), WEIGHT(48), ANSWER,DA, DB
C****  DOUBLE PRECISION ARY(48,2)
C****  EQUIVALENCE (ARY(1,1),RO3T(1)), (ARY(1,2),WEIGHT(1))
15
C****
C****  SET UP RO3TS AND WEIGHTS...
C****
DATA ((ARY(I,J),J=1,2),I=1,15) /
  3.01327 37448 49202 96979000, 0.03255 06114 92363 16524200,
  0.04331 29451 33049 73111200, 0.33325 61187 13869 83938700,
  0.05129 74994 64920 92494700, 0.03324 71537 14094 26735400,
  0.11359 53801 10655 92201100, 0.03324 38225 66575 92842900,
  0.14957 37146 54990 94198900, 0.03325 62047 94030 23669000,
  0.17307 56823 65135 60279400, 0.03323 49562 31142 66321300,
  0.21003 13104 63557 20363300, 0.03132 97533 94411 00533500,
  0.24174 11251 63360 21263300, 0.03118 93307 70727 16855800,
  0.27314 84120 91347 14197300, 0.03131 64525 96851 33231800,
  0.30346 94443 84499 93020400, 0.03101 03325 83613 83742300,
  0.33320 35223 92625 42261500, 0.03067 13761 23669 14901400,
  0.35599 58614 72313 53053100, 0.03029 94154 20872 53734900,
  0.39379 74643 23209 63023500, 0.02939 53441 36325 83598400,
  0.42447 84034 37300 42335500, 0.02944 10899 95657 90979000,
  0.4673 94221 67743 00853600, 0.02899 46141 50555 23654300,
  0.48349 79739 20596 35976800, 0.02849 74110 65085 38564300,
  0.51109 41771 54667 67353600, 0.02797 99076 16948 33444400,
  0.53339 81033 24137 43622700, 0.02741 29627 26029 24232300/
C****
DATA ((ARY(I,J),J=1,2),I=19,37) /
  3.06231 34135 51397 16340400, 0.02628 86867 25591 76219800,
  0.39303 23467 77702 50354000, 0.02620 23407 35612 41913000,
  0.51392 59301 25458 97033600, 0.02537 00360 05349 36149900,
  0.55416 34037 34477 10579900, 0.02490 60332 22483 61032800,
  0.53871 31100 43116 19393300, 0.02420 84117 92104 69123200,
  0.59256 45368 42171 50134400, 0.02349 33790 35226 21942000,
  0.71567 93123 49367 52929000, 0.02273 70647 83129 37430100,
  0.73033 60437 14403 13295100, 0.02196 60444 38744 91599000,
  0.73943 23411 76467 49970300, 0.02117 29398 72491 92193800,
  0.74035 43338 77433 21763400, 0.02035 67941 54333 32549500,
  0.80033 37441 31404 81722900, 0.01951 90311 40145 02214000,
  0.81460 33107 37931 67553900, 0.01866 05795 27411 46734900,
  0.84378 35112 21127 12147900, 0.01778 25023 16045 26033900,
  0.85975 90334 34031 45786300, 0.01683 54798 54245 17245000,
  0.87138 95019 92756 32874800, 0.01597 05299 32952 29138100,
  0.89839 41074 32423 41605700, 0.01503 78210 20911 53830900,
  0.90246 30333 19582 34131300, 0.01409 39417 73314 85315000,
  0.91397 14231 23911 37492500, 0.01312 32295 56961 97203700,
  0.92771 24297 22103 54345500, 0.01215 16046 71988 31953500/

C****
FUNCTION FINT2 73/74 JPT=1

DATA ((ARY(I,J), J=1,2), I=33,43)/
  0.933373 33377 9273 21693200, 0.01116 21020 99838 49359100,
  0.93533 27177 64337 6279600, 0.01016 07705 35003 41575800,
  0.939368 82914 48742 23933000, 0.00914 36712 30793 38653300,
  0.91932 39284 03264 21217000, 0.00912 63769 25948 75921700,
  0.91735 31749 35135 46545300, 0.00709 64707 91153 86526900,
  0.93251 72639 33014 67744700, 0.00605 85455 04235 96168300,
  0.99006 +1263 29523 79948100, 0.00501 42027 42927 51759300,
  0.99234 39003 23762 62159200, 0.00396 45543 38444 68657400,
  0.99593 18429 87204 29055000, 0.00291 07318 17934 94560900,
  0.99335 43758 03161 57772 9000, 0.00192 39507 98966 92173200,
  0.99763 +3339 33230 75632400, 0.00079 67920 55552 01242900/

C****
C**** INTEGRATION DONE BY TRANSLATING F TO THE
C**** INTERVAL -1 TO 1
C****

75    ANSWER = 0.00
    D3 = 3
    DA = 1
C****

83    D1 = 1,43
    T = ((D3-DA)*ROOT(I) + (D3+DA))/2.00
    ANSWER = ANSWER + WEIGHT(I) * FIT(T)
    T = ((D3-DA)*(-ROOT(I)) + (D3+DA))/2.00
    ANSWER = ANSWER + WEIGHT(I) * FIT(T)
C****

85    FINT2 = (D1-DA)*ANSWER/2.00
C****
    RETURN
END
FUNCTION RECT(A,B,R)
C**** RECTANGLE PROBABILITY...
C**** VOLUME UNDER THE NORMAL BIVARIATE DENSITY,
C**** -INFX<A, -INFY<B.
C**** BY BROWNLOW, SDC/ISI
C**** COMMON/SPARM/ AA,B,B,R,SQR
EXTERNAL G
C**** AA = A
   BB = B
   KR = R
   SSR = SQR(1.-R*R)
C**** RECT = FINT2(G,-15.,A)
C**** RETURN
END
FUNCTION G

73/7+ OPT=1

FUNCTION G(X)

C**** CONDITIONAL DISTRIBUTION FUNCTION...

C****

5 C3MWN/SPARK/A,B,R,SQR

C****

10 RETURN

END
FUNCTION TINV

GIVEN $A < Y < B$ FIND $T$ SO THAT $-T < X < T$ AND $P(-T < X < T, A < Y < B) = P$

WITH COEFFICIENT OF CORRELATION BETWEEN $X$ AND $Y$ EQUAL TO $R$.

$P(-T < X < T, A < Y < B) = P(-T < X < T, A < Y < B) / P(A < Y < B)$

$T$ IS FOUND BY INTERVAL HALVING.

BY BROWNLOW, SOC/IST

DENOM = PHI(3) - PHI(A)
TMAX = 10.
TMID = 0.

DO 1 I = 1, 50
T = (TMAX + TMIN)/2.

PCOMPT = RPROB(A, B, -T, T, R)/DENOM

IF(PCOMPT .GT. P) TMAX = T
IF(PCOMPT .LT. P) TMIN = T
IFABS(PCOMPT - P) .LE. 1.E-5) GO TO 2
CONTINUE

PRINT 100, P, A, B, R


TINV = (TMIN + TMAX)/2.
RETURN

2 TINV = T
RETURN
END
FUNCTION PR23( A, B, C, D, R)

C**** RECTANGULAR PROBABILITY FOR BIVARIATE
C**** NORMAL DISTRIBUTION...
5  C****
C**** CSXD
C**** AY<3
C****
C**** AND THE COEFFICIENT OF CORRELATION BETWEEN
10  C**** X AND Y IS R.
C**** BY BROKNER, SOC/ISI
C****
PR23 = (RECT(D,3,R) - RECT(C,3,R) - RECT(D,A,R) + RECT(C,A,R))
    * 0.154754743
15  C****
C**** RETURN
END
FUNCTION PHI 73/74 OPT=1 FTN 4.2+75363

C****
C**** RETURN
60 END
FUNCTION PHI

FUNCTION PHI(X)

C****
C****
C****
C****
C****
C****
C****
C****
C****

NORMAL(0,1) DISTRIBUTION FUNCTION

PHI(X) = INTEGRAL OF NORMAL DENSITY

FROM -INFINITY TO X.

BY J.R.JANLOW, SOC/ISI

LOGICAL FLAG

IF(X .GT. -10.) GOTO 1
PHI = 0.
RETURN

IF(X .LT. 10.) GOTO 2
PHI = 1.
RETURN

FLAG = .T.

Determine if X > 0, series expansion is for positive values of X.

IF(X .GE. 3.) GOTO 3
FLAG = .F.

INITIALIZE VALUES FOR PARTIAL SUM OF THE SERIES...

Z = A35(X)
D = 1.
SUM = 0.
TOP = Z
BJT = 1.

CONTINUE
SAVE = SUM
SUM = SUM + TOP/BJT

CONTINUE TO SUM UNTIL MACHINE UNDERFLOWs...

IF(SAVE .EQ. SUM) GOTO 5

UPDATE EXPRESSIONS FOR THE SUM...

TOP = TOP*Z*Z
D = D + 2.
BJT = 4*BJT)
GOTO 4

Depending upon whether original X > 0 or X < 0,

GET APPROPRIATE INTEGRAL VALUE...

PHI = SUM/SQRT(6.233185393)*EXP(X*X)) + .5

IF(FLAG) RETURN

PHI = 1.-PHI

END
FUNCTION FRAC(PV, Y1, Y2)

C**** GIVEnA BIVARIATE NORMAL DISTRIBUTION
C**** WITH COEFFICIENT OF CORRELATION RH0
C**** AND Y1 < Y < Y2, FRAC(PV, Y1, Y2) RETURNS
C**** THAT VALUE T, SUCH THAT:
C**** PR0B(-T < X < T, Y1 < Y < Y2) = PV
C****
C**** BINARY SEARCH, LIMITED TO A MAXIMUM OF 20 ITERATIONS
C****
C**** BY BROWNLOW, SDC/ISI, 11/79
C****

15   KOUNT = 0
20   TMIN = 0.,
20   TMAX = 10.
25   T = (TMIN + TMAX) / 2.
30   VAL = REC(-T, T, Y1, Y2)
30   PRINT 100, VAL, T
30   100 FORMAT(* 2E10.5//)

35   IF(ABS(VAL - PV) .LT. 1.E-5) GO TO 2
35   IF(VAL .LT. PV) TMIN = T
35   IF(VAL .GT. PV) TMAX = T
35   IF(KOUNT .LE. 20) RETURN
35   KOUNT = KOUNT + 1
40   GO TO 1
40   1   FRAC = T
45   RETURN
E40
FUNCTION CONDEN(x)

**FUNCTION CONDEN(x)**

**CONDITIONAL DENSITY FUNCTION OF X, GIVEN**

**AKY<8 FROM BIVARIATE NORMAL DISTRIBUTION**

5 **F(x,y) WITH COEFFICIENT OF CORRELATION RHO.**

**BY BROWNLJW, SDC/ISI, 11/79**

**COMMON/ARG/RHO,A,B**

**SET UP THE PARAMETERS, PHI IS THE UNIVARIATE**

**NORMAL DISTRIBUTION FUNCTION.**

15 **R = SQRT(1.-RHO*RHO)**

**D = PHI(B)-PHI(A)**

**T = PHI((3.-RHO*X)/R) - PHI((A-RHO*X)/R)**

**CONDEN = EXP(-X**2.)*T/(D*2.506628275)**

**RETURN**

**END**
FUNCTION G(S,T)
C**** BIVARIATE NORMAL DISTRIBUTION FUNCTION.
C**** G(S,T) = DOUBLE INTEGRAL OF NORMAL
C**** BIVARIATE DENSITY FUNCTION, -INF TO S,
C**** -INF TO T.
C****
C**** NOTICE THAT THE NUMERICAL COMPUTATIONS
C**** USE THE FACT THAT THE CONTRIBUTION TO THE
C**** INTEGRAL VALUE FROM -INF TO -15.
C**** IS INSIGNIFICANT.
C****
C**** AT BOUNDARY, SDC/ISI, 11/79
C****
15 COMMON/PSAXX/IT
C**** EXTERNAL FN
C**** IT = T
20 C**** G = FN(2(FN*-15.,S1/5.,293195308
C**** RETURN
END
FUNCTION FCN(X)

C***** DENSITY FUNCTION FOR DOUBLE INTEGRAL,
C***** PHI(X)*Z(X), WHERE PHI AND Z ARE THE
C***** NORMAL DISTRIBUTION AND DENSITY FUNCTIONS
C***** RESPECTIVELY.
C*****
C*****
C*****
C*****
C*****

BY S. OKUMURA, SUC/ISI, 11/79

15

C****

CJDYN/ARG/RHO*/A,*
CJDYN/PASS/IT

C*****

Z(ARG) = EXP(-ARG*ARG/2*3)

C*****

J = (IT-RH*IT)/52.51(L2-RH*RH)

C*****

FCN = PHI(J)*Z(X)*2.30529275

C*****

RETURN
END
FUNCTION RECT(x1, x2, y1, y2)
C**** Rectangular probabilities for bivariate normal distribution, x1<x2, y1<y2, and the coefficient of correlation is rho.
C**** By Robert D. Student, SOC/ISI, 11/79
C****
C****
10   RECT = g(x2, y2) - g(x1, y2) - g(x2, y1) + g(x1, y1)
C****
RETURN
END
FUNCTION CONDIST(A)
C****
C**** CONDITIONAL DISTRIBUTION FUNCTION OF X GIVEN
C**** A<y<a FROM BIVARIATE NORMAL DISTRIBUTION
C**** F(x,y) WITH COEFFICIENT OF CORRELATION RHO,
C****
C**** BY R.L.N. CHI, SOC/ISI, 11/79
C****
C****
C**** CJ=(1/2)*G(A+B), A+B
C**** PHI IS THE UNIVARIATE NORMAL DISTRIBUTION
C**** FUNCTION,
C****
15 CONDIST=(G(Y,B) - (Y,A))/((PHI(B) - PHI(A))*5.2d3135308)
C****
RETURN
END
FUNCTION FINT2 71/7+ J=1,2 FIN 4.2+75060

FINT12 FINT2(F,4,3)
C**** 5
C**** INTEGRAL OF THE FUNCTION F FROM A TO B
C**** BY GAUSSIAN-LEGENDRE QUADRATURE, 76 POINT FORM
C**** REQUIRES 76 EVALUATIONS OF F(X).
C****
C**** THE CALLING PROGRAM
C**** INTEGRAL2
C****
C**** DUE TO PREVIOUS REQUESTS, WEIGHT(43), ANSWER,DATA,ARY,ARY(48,2)
C**** EQUIVALENCE (ARY(1,1),J,ARY(1,1)), (ARY(1,2),WEIGHT(1))
C****
C**** DATA ((ARY(J,J),J=1,2),J=1,18) /
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****

DATA ((ARY(J,J),J=1,2),J=1,18) /
FUNCTION FINT (73/7+) \#I=1

C****
C**** INTEGRATION DONE BY TRANSLATING F TO THE
C**** INITIAL -1 TO 1
C****

\$5*F = 0.00

\$5 = 1
\$1 = \$A

C****
I=1,4S
\$T = ((\$O-\$A)**2)\$T(\$I) + (\$A*\$A) / 2.00
\$ANS = \$ANS + \$NIGHT(\$I) * \$T
\$T = ((\$O-\$A)**2 - \$ANS(\$I) + (\$A*\$A) / 2.00
\$ANS = \$ANS + \$NIGHT(\$I) * \$T

C****
FINT = (\$B-\$A) * \$ANS*69/2.00

C**** RETURN
END
PROGRAMSuve PHI(x)

FUNCTION PHI(x)

C****
C**** NORMAL(0,1) DISTRIBUTION FUNCTION
C**** PHI(x) = INTEGRAL OF NORMAL DENSITY
C**** FROM -INFINITY TO x.
C****
C**** LOGICAL FLAG
10 IF(x .GT. -10.) GO TO 1
P+I = 0.
RETURN
C**** 1 IF(x .LT. 10.) GO TO 2
PHI = 1.
RETURN
C**** 2 FLAG = .T.
C**** DETERMINE IF X>0, SERIES EXPANSION IS FOR
C**** POSITIVE VALUES OF X.
C**** IF(X .GT. 0.) GO TO 3
FLAG = .F.
C****
C**** INITIATE VALUES FOR PARTIAL SUM OF THE SERIES...
C****
C**** 3 Z = ABS(X)
   D = 1.
   SJM = 0.
   TJP = Z
   BJT = 1.
C****
C**** CONTINUE
35 SAVE = SUM
   SJM = SJM + TJP/BJT
C**** CONTINUE TO SUM UNTIL MACHINE UNDERFLOWS...
C****
C**** IF(SAVE .EQ. SUM) GO TO 5
C**** UPDATE EXPRESSIONS FOR THE SUM...
C****
C**** TJP = TJP*Z/Z
45 D = D + 2.
   BJT = 3D/BJT
   GO TO 4
C****
C**** DEPENDING UPON WHETHER ORIGINAL X>0 OR X<0,
C**** GET APPROPRIATE INTEGRAL VALUE...
C****
C**** PHI = SUM/SQRT(5.293185308*EXP(X*X)) +.5
50 IF(FLAG) RETURN
C****
C**** PHI = 1.-PHI
C****
FUNCTION PHI 73/74 OPT=1 FTN 4.2+75050 C

**RETURN**

END
Normal distribution is widely employed in numerous disciplines. Unfortunately, probabilities for the multivariate normal distribution are generally not available. This paper presents some results for the bivariate normal distribution. Computer programs for conditional normal probabilities, marginal probabilities as well as joint probabilities for rectangular regions are given: routines for computing fractile points and distribution functions are also presented. Some examples from a closed circuit television experiment are included.