BIVARIATE NORMAL CONDITIONAL AND RECTANGULAR PROBABILITIES:
A COMPUTER PROGRAM WITH APPLICATIONS

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May 1980
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NASA
National Aeronautics and
Space Administration
1980
INTRODUCTION

In applications involving univariate data where estimates and confidence intervals are required, the normal distribution is commonly employed. This distribution is mainly utilized because the probabilities under a normal curve are readily available. In contrast, use of multivariate probabilities in p-variate normal data are less frequent, primarily because probabilities for the multivariate normal case are generally not available. Except for very special cases, the probabilities for sections of p-dimensional space require extensive computations, since the canonical multivariate normal density changes with every change in correlation coefficient parameters. Even the probability computation in the bivariate normal case (p = 2) with only one value for the correlation coefficient over arbitrary sections of the (x, y) plane is not easy. Probability computations, therefore, in p > 2 dimensions are correspondingly much more difficult. (Ref. 1)

In many applications, problems are posed which not only require the probabilities over a section of p-dimensional space, but also the conditional probabilities of r (r < p) variables when the remaining (p - r) variables are either fixed, or are within designated intervals. For example, in aircraft target tracking studies, it is of interest to know the probability of X deviations from the target when Y deviations are considered within designated bounds. In aircraft performance studies it is important to know the distribution of the pilot's cardiac R-R intervals either under an assigned difficult aircraft maneuver or under the dynamic flight conditions.

The results on conditional and marginal distributions of r variables when the (p - r) remaining variables assume fixed values are well established. (Ref. 1) Similar results, when the remaining (p - r) variables assume values within specified ranges involve complexities and are discussed in this report.

In this study, results on bivariate normal distributions (p = 2) are reviewed. Various derivations and properties of bivariate normal conditional probabilities are derived. A computer program for conditional probabilities for all assigned values is included. From conditional and marginal probabilities, the rectangle probabilities are then obtained. Examples are presented to illustrate the use of the program. The program listing is appended to this report.

SYMBOLS

- $A_y$ lateral acceleration
- $A_z$ vertical acceleration
- $c$ a constant with fixed numerical value
- $\exp(x)$ exponential function at $x$
F(s)  conditional distribution of X at X = s Given Y is in interval (a, b)
f(u, v)  general bivariate normal density
f(x), f(y)  standard normal densities
f(x, y)  standard bivariate normal density at X = x, Y = y
f(x|a<Y<b) conditional density of X at X = x given Y is in interval (a, b)
f(x|Y=y) conditional density of X at X = x given Y = y
f(x|Y<t, ρ<0) conditional density of X at X = x given Y is less than t and correlation is negative
f(x|Y>-t, ρ>0) conditional density of X at X = x given Y is greater than -t and correlation coefficient ρ is positive
Gp(s, t) double integral with two arguments s and t with a fixed value of correlation coefficient ρ
gt(x) conditional density of X at X = x given Y is in interval (-t, t)
gt(x|ρ>0) conditional density of X at X = x when correlation coefficient ρ is positive and Y is in interval (-t, t)
gt(x|ρ<0) conditional density of X at X = x when correlation coefficient ρ is negative and Y is in interval (-t, t)
p, r  dimension of multivariate data or distribution
Pr[a<Y<b] probability that variable Y is in interval (a, b)
Pr[c<X<d, a<Y<b] joint probability that variable X is in interval (c, d) and variable Y is in interval (a, b)
Pr[X<h, Y<k] probability that X is less than h and Y is less than k
U, V, X, Y  random variables
u, v, x, y, t specific values of random variables
Vc  forward velocity
α  fixed positive constant less than 1
μc  mean of forward velocity Vc
BIVARIATE NORMAL DISTRIBUTION

A bivariate normal distribution of a random vector \((U, V)\) is characterized by parameters: \(\mu_u, \mu_v, \sigma_u, \sigma_v\) and \(\rho\). The density function

\[
f(u,v) = \left[ \frac{2\pi \sigma_u \sigma_v \sqrt{1 - \rho^2}}{2(1 - \rho^2)} \right]^{-1} \exp \left( -\frac{1}{2(1 - \rho^2)} \left[ (u - \mu_u)/\sigma_u \right]^2 - 2\rho \left[ (u - \mu_u)/\sigma_u \right] \left[ (v - \mu_v)/\sigma_v \right] + \left[ (v - \mu_v)/\sigma_v \right]^2 \right)
\]

is defined over the entire \((u, v)\) plane. When the variables \(U\) and \(V\) are standardized, by defining the new variables

\[
x = \frac{U - \mu_u}{\sigma_u}, \quad y = \frac{V - \mu_v}{\sigma_v}
\]

the density function of \((X, Y)\) reduces to the canonical bivariate normal density

\[
X = \frac{U - \mu_u}{\sigma_u}, \quad Y = \frac{V - \mu_v}{\sigma_v}
\]

defined over the entire \((x, y)\) plane. The parameter \(\rho\) is called a correlation coefficient and takes values in the interval \((-1, 1)\). Without any loss of generality, this canonical density \(f(s, y)\) is considered in this study.
The density function \( f(x, y) \) exhibits certain properties. It is symmetric in opposite quadrants since
\[
f(x, y) = f(-x, -y)
\]
and
\[
f(x, -y) = f(-x, y)
\]
Further, \( f(x, y) \) is constant over all the ellipses
\[x^2 - 2\rho xy + y^2 = c(1 - \rho^2)\]
for every value of \( x \). (Fig. 1) The intercepts made by these ellipses on the \( x \) and \( y \) axes are equal. If \( \rho \) is positive, the major axis of the ellipse is along the 45° line with a length of \( 2\sqrt{c(1 + \rho)} \); and the minor axis is along the 135° line with a length of \( 2\sqrt{c(1 - \rho)} \). If \( \rho \) is negative, the major axis is along the 135° line with a length of \( 2\sqrt{c(1 - \rho)} \); the minor axis along the 45° line has a length of \( 2\sqrt{c(1 + \rho)} \). (Ref. 2) The ellipse
\[x^2 - 2\rho xy + y^2 = (1 - \rho^2) \log 1/(1 - \alpha)^2\]
for all \( 0 < \alpha < 1 \), contains the \( \alpha \) proportion of the \( (X, Y) \) distribution. (Ref. 3)

The marginal distributions of \( X \) and \( Y \) are standard normal with the covariance between \( x \) and \( y \) equal to \( \rho \). When \( \rho = 0 \), then
\[
f(x, y) = (\sqrt{2}\pi)^{-1} \exp(-x^2/2) (\sqrt{2}\pi)^{-1} \exp(-y^2/2)
\]
\[= f(x) \cdot f(y)\]
which is a product of standard normal densities, implying that \( \rho = 0 \) if and only if \( X \) and \( Y \) are independent. When \( \rho \neq 0 \), bivariate normal probabilities \( \Pr(X < h, Y < k) \) for a few selected values of \( h \) and \( k \) are available from tables and graphs. (Ref. 4, 5) For general values of \( h \) and \( k \) approximation and interpolation methods are used.

**DERIVATION OF CONDITIONAL DENSITIES**

**Conditional Density of \( X \) Given \( Y = y \).** It was stated earlier that if a random vector \( (X, Y) \) has a bivariate normal distribution, then the marginal distribution of either \( X \) or \( Y \) is normal with mean 0 and variance 1. The conditional distribution of \( X \) for a fixed value of \( Y = y \), however, is normal with mean \( \rho y \) and variance \( (1 - \rho^2) \). The conditional density \( f(x | Y = y) \) is derived below.
Figure 1. Marginal, conditional densities and ellipses of constant densities from bivariate normal density.
\[ f(x | Y = y) = \frac{f(x, y)}{f(y)} \]

\[
= \frac{\left(2\pi \sqrt{1 - \rho^2}\right)^{-1} \exp \left\{ -\frac{(x^2 - 2\rho xy + y^2)}{2(1 - \rho^2)} \right\}}{(\sqrt{2\pi})^{-1} \exp \left( -\frac{y^2}{2} \right)}
\]

\[
= \left[\frac{2\pi(1 - \rho^2)}{\sqrt{2\pi}}\right]^{-1} \exp \left\{ - \left[ \frac{x^2 - 2\rho xy + y^2 - (1 - \rho^2)y^2}{2(1 - \rho^2)} \right] \right\}
\]

\[
= \left[\frac{2\pi(1 - \rho^2)}{\sqrt{2\pi}}\right]^{-1} \exp \left( -\frac{(x^2 - 2\rho xy + \rho^2 y^2)}{2(1 - \rho^2)} \right)
\]

which is the density of a normal distribution with mean \( \rho y \) and variance \( 1 - \rho^2 \) and is shown in Figure 1.

Conditional Density of \( X \) Given \( a < Y < b \). The conditional density of \( X \) given \( a < Y < b \) is not normal and is derived as follows.

\[
f(x | a < Y < b) = \int_a^b \frac{\left(2\pi \sqrt{1 - \rho^2}\right)^{-1} \exp \left\{ -\frac{(x^2 - 2\rho xy + y^2)}{2(1 - \rho^2)} \right\}}{(\sqrt{2\pi})^{-1} \int_a^b \exp \left( -\frac{y^2}{2} \right) dy} dy
\]

\[
= \left[\frac{\phi(b) - \phi(a)}{\frac{2\pi \sqrt{1 - \rho^2}}{\sqrt{2\pi}}}\right]^{-1} \cdot \int_a^b \exp \left\{ -\frac{(x^2 - 2\rho xy + \rho^2 x^2 + x^2(1 - \rho^2))}{2(1 - \rho^2)} \right\} dy
\]

\[
= \left(\frac{\sqrt{2\pi}}{\sqrt{2\pi(1 - \rho^2)}}\right)^{-1} \exp \left( \frac{x^2}{2} \right) \left[\frac{\phi(b) - \phi(a)}{\sqrt{1 - \rho^2}}\right]^{-1} \cdot \int_a^b \exp \left( -\frac{(y - \rho x)^2}{2(1 - \rho^2)} \right) dy
\]

\[
= f(x) \left[\frac{\phi(b) - \phi(a)}{\sqrt{1 - \rho^2}}\right]^{-1} \left\{ \phi \left( \frac{b - \rho x}{\sqrt{1 - \rho^2}} \right) - \phi \left( \frac{a - \rho x}{\sqrt{1 - \rho^2}} \right) \right\}
\]
where
\[ f(x) = (\sqrt{2\pi})^{-1} \exp\left(-x^2/2\right) \]
is a standard normal density and
\[ \phi(t) = \int_{-\infty}^{t} f(x) \, dx \]
is the standard normal distribution function.

This conditional density is neither normal, nor symmetric. However, in special cases discussed below, symmetry is identifiable.

**Symmetry in Conditioning \(-t < Y < t\).** With \(-t < Y < t\), the conditional density of \(X\) at specific values of \(x\) and \(-x\) are

\[ g_t(x) = f(x| -t < Y < t) \]

\[ = f(x) \left[ \phi(t) - (-t) \right]^{-1} \left\{ \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

\[ g_t(-x) = f(-x| -t < Y < t) \]

\[ = f(-x) \left[ \phi(t) - \phi(-t) \right]^{-1} \left\{ \phi\left(\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t + \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

The symmetry of a standard normal density shows that \(f(-x) = f(x)\). With the asymmetry of distribution function \(\phi(t) = 1 - \phi(-t)\), it is seen that

\[ \phi\left(\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t + \rho x}{\sqrt{1 - \rho^2}}\right) \]

\[ = 1 - \phi\left(\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right) - \left\{ 1 - \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

\[ = \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right) \]

Thus \(g_t(-x) = g_t(x)\), showing that for \(-t < Y < t\) the conditional density of \(X\) is symmetric in \(x\), as shown in figure 2.

The conditioning, \(-t < Y < t\), with positive and negative values of correlation coefficient \(\rho\) also show symmetry of \(g_t(x)\). It is to be noted that

\[ g_t(x|\rho > 0) = f(x) [\phi(t) - \phi(-t)]^{-1} \]

\[ \left\{ \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]
Conditional density

Standard normal density

Figure 2. Conditional density of $X$ given $-t < Y < t$ ($t = 1.000$, probability = 0.6826) where $(X, Y)$ is bivariate normal with $\rho = 0.9000$, and standard normal density.
\[ g_t(x|\rho < 0) = f(x)\left[\phi(t) - \phi(-t)\right]^{-1} \cdot \left\{ \phi\left(\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right) - \phi\left(\frac{-t + \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

By the symmetry of \( f(x) \), the asymmetry of \( \phi(t) \), and the arguments given earlier, it is seen that \( g_t(x|\rho > 0) = g_t(x|\rho < 0) \). The graph of such a density is shown in figure 2.

**Symmetry when \( -\infty < Y < t \) and \( -t < Y < +\infty \).** In these cases it is to be noted that \( \phi(-\infty) = 0 \), \( \phi(\infty) = 1 \). Thus the conditional densities of \( X \) are

\[ g_t(x|\rho > 0) = f(x|Y < t) \]

\[ = f(x)\left[\phi(t)\right]^{-1} \cdot \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) \]

\[ g_{-t}(x|\rho > 0) = f(x|-t < Y) \]

\[ = f(x)\left[1 - \phi(-t)\right]^{-1} \left\{ 1 - \phi\left(\frac{-t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

\[ = f(x)\left[\phi(t)\right]^{-1} \left\{ \phi\left(\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

\[ g_{-t}(-x|\rho > 0) = f(-x|-t < Y) \]

\[ = f(x)\left[\phi(t)\right]^{-1} \left\{ \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

Thus \( g_t(x) = g_{-t}(-x) \), showing that a one-sided conditioning on \( Y \) yields the same density for \( X \) as does the conditioning on the other side for the opposite \( x \). Further, for negative and positive values of \( \rho \), it is to be noted that

\[ g_t(x|\rho > 0) = f(x)\left[\phi(t)\right]^{-1} \left\{ \phi\left(\frac{t - \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

and

\[ g_t(x|\rho < 0) = f(x)\left[\phi(t)\right]^{-1} \left\{ \phi\left(\frac{t + \rho x}{\sqrt{1 - \rho^2}}\right) \right\} \]

\[ = g_{-t}(x|\rho > 0) \]
Therefore, if the conditioning on \( Y \) and the sign of the correlation coefficient are reversed, the density remains invariant. An example of these densities is shown in figure 3.

**DERIVATION OF CONDITIONAL DISTRIBUTIONS**

**Conditional Distribution Function of \( X \) Given \( Y = y \).** The distribution function from the conditional density

\[
 f(x|Y = y) = \left[\sqrt{2(1 - \rho^2)}\right]^{-1} \exp\left[ -\frac{(x - \rho y)^2}{2(1 - \rho^2)} \right]
\]

derived earlier, is easily obtainable via the normal distribution function with mean \( \rho y \) and variance \( 1 - \rho^2 \). It is to be observed from figure 1, that mean \( \rho y \) is a function of the correlation \( \rho \) and the specific conditioned value of \( y \), but the variance depends only on \( \rho \) and is invariant for all values of \( y \). Thus the width of any \( \alpha \) level confidence interval remains the same irrespective of the conditioned values of \( y \).

In applications, the conditioning of variable \( Y \) is seldom a fixed value. The conditioning is usually in a range \( a < Y < b \), and the formulae for this case are different from the results for \( Y = y \).

**Conditional Distribution of \( X \) Given \( a < Y < b \).** The conditional density

\[
 f(x|a < Y < b) = f(x) \left[\phi(b) - \phi(a)\right]^{-1} \left\{ \frac{\phi((b - \rho x)/\sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} - \frac{\phi((a - \rho x)/\sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} \right\}
\]

where

\[
 f(x) = \left(\sqrt{2\pi}\right)^{-1} \exp\left( -\frac{x^2}{2} \right)
\]

and

\[
 \phi(t) = \int_{-\infty}^{t} f(x) dx
\]

was derived earlier. A general expression for the distribution function

\[
 F(s) = \int_{-\infty}^{s} f(x|a < Y < b) dx
\]

\[
 = \left[\phi(b) - \phi(a)\right]^{-1} \int_{-\infty}^{s} f(x) \left\{ \frac{\phi((b - \rho x)/\sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} - \frac{\phi((a - \rho x)/\sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} \right\} dx
\]

for all the values of \( s \) involves integration of the expression which is the product of the normal density and distribution function in the appropriate range of the \( x \) values. Specifically, for the computation of \( F(s) \), the
Figure 3. Conditional density of $X$ given $-\infty < Y < t$ ($t = 1.00$, probability = 0.8413) where $(X, Y)$ is bivariate normal with $\rho = 0.6000$, and standard normal density.
value of double integrals such as

\[ G_p(s,t) = \int_{-\infty}^{s} \exp\left(-x^2/2\right) \left[ \int_{-\infty}^{t} \frac{(t - \rho x) \sqrt{1 - \rho^2}}{\exp(-u^2/2)} \, du \right] \, dx \]

for all values of \( s, t \) and \( \rho \) are required. In terms of these functions, it is easily seen that

\[ F(s) = \left\{ 2\pi [\phi(b) - \phi(a)] \right\}^{-1} \left[ G_p(s,b) - G_p(s,a) \right] \]

A closed analytical expression for \( G_p(s,t) \) is not available and for specific values, numerical methods may be employed. However, in cases where symmetry occurs, the numerical computations for a smaller range of values are needed. In order to calculate \( F(s) \) for all values of \( s, a, b \) and \( \rho \), a computer program using quadratures was developed at DFRC and is given in the Appendix.

**Rectangle Probabilities.** The region \((c < x < d, a < Y < b)\) is a rectangle in the \((x, y)\) plane. Thus the joint probability \( \Pr(c < X < d, a < Y < b) \) for real values of \( a, b, c \) and \( d \) corresponds to a rectangle probability. The appended computer program can be used to calculate all such rectangle probabilities. The procedure is to identify first that

\[ \Pr[c < X < d, a < Y < b] = \Pr[c < X < d|a < Y < b] \Pr[a < Y < b] \]

\[ = [F(d) - F(c)] \Pr[a < Y < b] \]

\[ = [F(d) - F(c)] [\phi(b) - \phi(a)] \]

for all values of \( c < d \) and \( a < b \), and then use the computer program with the proper inputs.

**COMPUTER PROGRAM INPUTS AND OUTPUTS**

The computer program developed at DFRC computes the conditional density and distribution function as outputs for specified values of \( x \) given the end points of the interval of the conditioning variable \( Y \), and the correlation coefficient \( \rho \). Thus the inputs to the program are specific \( x \) values, end points of the \( Y \) interval and the \( \rho \) value. The output has two options. Either the density or distribution function, or both may be obtained by stating the options in the program.

The rectangle probabilities are to be obtained by finding the conditional probabilities. The computer program with its options is explained in the Appendix.
EXAMPLES

The following examples illustrate the use of the program and tables shown in the Appendix to calculate various probabilities.

The data for the examples are taken from a Closed Circuit Television (CCTV) experiment. In this experiment, two pilots, A and B, landed an aircraft with the help of an airborne television camera and video monitor. Each pilot made ten (10) touchdowns under visual flight regulations, and eighteen (18) touchdowns utilizing the closed circuit television monitor. The summary of data from the twenty-eight (28) touchdowns is given in Table I. For this illustration the data parameters are vertical acceleration, $A_z$, forward velocity, $V_c$ and lateral acceleration $A_y$.

### TABLE I. SUMMARY OF 28 TOUCHDOWN DATA OF CCTV EXPERIMENT

<table>
<thead>
<tr>
<th>Pilot</th>
<th>Parameter (Units)</th>
<th>Mean $\mu$</th>
<th>S.D. $\sigma$</th>
<th>Correlation Between</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$A_z$ (G)</td>
<td>1.313</td>
<td>.2021</td>
<td>$(A_z, V_c) = .2481$</td>
</tr>
<tr>
<td></td>
<td>$V_c$ (MPH)</td>
<td>60.25</td>
<td>1.3089</td>
<td>$(A_z, A_y) = -.0715$</td>
</tr>
<tr>
<td></td>
<td>$A_y$ (G)</td>
<td>.023</td>
<td>.1227</td>
<td>$(V_c, A_y) = .2807$</td>
</tr>
<tr>
<td>B</td>
<td>$A_z$ (G)</td>
<td>1.294</td>
<td>.1044</td>
<td>$(A_z, V_c) = -.2569$</td>
</tr>
<tr>
<td></td>
<td>$V_c$ (MPH)</td>
<td>62.04</td>
<td>1.8747</td>
<td>$(A_z, A_y) = -.2199$</td>
</tr>
<tr>
<td></td>
<td>$A_y$ (G)</td>
<td>-.007</td>
<td>.0801</td>
<td>$(V_c, A_y) = -.1993$</td>
</tr>
</tbody>
</table>

The variables $(A_z, V_c, A_y)$ are assumed to follow a multivariate normal distribution. Thus any two variables follow a bivariate normal distribution and any single variable, a univariate normal distribution, as shown in figure I. Further, all the values in these data are considered to be parameter values.

**Example 1.** Computation of a 95% confidence interval of forward velocity ($V_c$) given vertical acceleration ($A_z$) mean is within ± one standard deviation ($\sigma$).

It is desired in this example to determine a 95% confidence interval for aircraft forward velocity ($V_c$), in miles per hour, at the point of touchdown, given the pilot's average vertical acceleration ($A_z$), in G's, within ± one standard deviation. The 95% confidence interval end points for $V_c$ given $A_z$ mean is within ± $\sigma$ are obtained by solving for $t$ from the equation
\[ .95 = \Pr[-t < \frac{(V_c - \mu_c)}{\sigma_c} < t | -1 < \frac{(A_z - \mu_z)}{\sigma_z} < 1] \]
\[ = \Pr[-t < X < t | -1 < Y < 1] \]

and identifying the interval as \((-t\sigma_c + \mu_c, t\sigma_c + \mu_c)\).

The solution of the equation for pilot A data of \(\mu_c = 60.25, \sigma_c = 1.3089, \mu_z = 1.313, \sigma_z = 0.2021\) and correlation \((A_z, V_c) = -.2481\), yields the value of \(t = 1.91666\). The 95% confidence interval, therefore, becomes

\[(57.7413, 62.7587)\]

This shows that if in pilot A data, the aircraft's vertical acceleration at touchdown is within \(+1.3 \pm 0.2\) G's, he has a 95% chance of landing the aircraft between 58 and 63 MPH.

For pilot B data, from table I, the \(t\) value computes to be 1.9136. Thus the 95% confidence interval is

\[(57.7453, 62.7547)\]

indicating if pilot B's vertical acceleration data at touchdown is within \(+1.3 \pm 0.1\) G's, he also has a 95% chance of landing the aircraft between 58 and 63 MPH.

Example 2. Computation of the probability that the forward velocity \((V_c)\) and \((A_y)\) are both within \(\pm \sigma\) of each variable. The probability of \(V_c\) and \(A_y\) being within \(\pm \sigma\) of each respective mean is an example of rectangle probability. In this example, the probability that simultaneously, \(V_c\) and \(A_y\), will be within one standard deviation of each variable's respective mean is to be computed.

This rectangle probability can be obtained by finding

\[ \Pr[-1 < \frac{(V_c - \mu_c)}{\sigma_c} < 1 , -1 < \frac{(A_y - \mu_y)}{\sigma_y} < 1] \]
\[ = \Pr[-1 < X < 1 | -1 < Y < 1] \Pr[-1 < Y < 1] \]

From univariate tables, \(\Pr[-1 < Y < 1] = \phi(1) - \phi(-1) = .6826\) and is not affected by the correlation coefficients. In order to obtain \(\Pr[-1 < X < 1 | -1 < Y < 1]\), the values of the correlation coefficients are needed.

The correlation coefficient \((V_c, A_y)\) for pilot A data is equal to -0.2807. The computer program output, therefore, for this correlation yields

\[ \Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c , \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554 \]

Thus, for pilot A there is a 48% chance that simultaneously at touchdown, the aircraft's forward velocity will be within \(60 \pm 1.3\) MPH and the lateral
acceleration is within 0 ±0.1 G's. Conversely, the probability is 0.52 that both variables will not simultaneously be within one standard deviation of their respective means. Similarly, for pilot B with the correlation \((V_C, A_y)\) equal to 0.1993, the program yields

\[
\Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47078
\]

which represents a 0.47 probability that the forward velocity will be within 62 ±1.9 MPH and lateral acceleration is within 0 ±0.08 G's.

**Example 3.** Computation of the probability of forward velocity \((V_c)\) and lateral acceleration \((A_y)\) being within ±σ of each variable, given vertical acceleration is equal to its mean \((A_z = \mu_z)\). This rectangle probability can be obtained as in Example 2, except in this case the vertical acceleration \((A_z)\) is set equal to the variable's mean value \((\mu_z)\). The probability in other words, is a function of a conditional correlation coefficient which is different from the coefficient given in the table.

For the pilot A data, this conditional coefficient is equal to 0.3809 and the program output yields

\[
\Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .48391
\]

which represents a 0.48 probability that the forward velocity will be within 60.25 ±1.309 MPH, and lateral acceleration within 0.023 ±0.1227 G's given that vertical acceleration is 1.313 G's.

For pilot B, the conditional correlation coefficient is equal to -0.2807 and the corresponding rectangle probability is

\[
\Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554
\]

This represents a 0.48 probability that the forward velocity will be within 62.0 ±1.9 MPH and lateral acceleration is within 0 ±0.08 G's given the vertical acceleration is +1.294 G's.

*Dryden Flight Research Center*
*National Aeronautics and Space Administration*
*Edwards, California, March 17, 1980*
APPENDIX

The program to compute the conditional density and distribution function for specified values of $x$ given the conditioning on variable $Y$. 
PROGRAM MAIN(OUTPUT)

C**** ILLUSTRATIVE USE OF THE ENCLOSED COMPUTER
C**** PROGRAMS TO COMPUTE VARIOUS
C**** PROBABILITIES ASSOCIATED WITH THE EXAMPLES
C**** GIVEN IN THE TEXT OF THIS PAPER...
C**** BY BROWNLOW, SDC/ISI

5 PRINT I, RPROB(-1.,1.,-1.,1.,-2907)
PRINT 1, RPROB(-1.,1.,-1.,1.,-1993)
PRINT 1, RPROB(-1.,1.,-1.,1.,-3809)

10 PRINT1, TINV(.95,-1.,1.,-2491)
PRINT 1, TINV(.95,-1.,1.,-2569)

15 FORMAT(*F10.5)
END
FUNCTION CD(X)
C****
C**** CONDITIONAL DENSITY FUNCTION
C**** CD(X\(\text{A}<X<\text{B}\)) = 1/\sqrt{2\pi}\exp(X^2)*
C**** (\phi((B-R\times X)/(\sqrt{1-R^2})) - \phi((A-R\times X)/(\sqrt{1-R^2})))
C**** / (\phi(B) - \phi(A))
C**** WHERE R = COEFFICIENT OF CORRELATION BETWEEN
C**** X AND Y
C****
C**** PHI(T) = INTEGRAL F(X) DX
C**** -INF
C**** AND F(X) = 1/\sqrt{2\pi}\exp(X^2)
C**** BY BRJWNLOW, SDC/ISI
C****
C**** COMMON/PARAM/A,B,R,SQR
C**** CD = .39894228/\sqrt{\exp(X^2)}* (\phi((B-R\times X)/SQR) -
C**** .\phi((A-R\times X)/SQR))/ (\phi(B)-\phi(A))
C****
RETURN
END
FUNCTION FINT2 737/4  OPT=1  FTN 4.2+75060

FUNCTION FINT2(F,A,B)
C****  INTERNAL OF THE FUNCTION F FROM A TO B
C****  BY GAUSSIAN-LEONARDO QUADRATURE.  95 POINT FORM
C****  REQUIRES 95 EVALUATIONS OF F(X).
C****
C****  F MUST BE DECLARED EXTERNAL IN
C****  THE CALLING PROGRAM.
C****  BY MACNOW, SOYISI
C****
C****  DOUBLE PRECISION KDOT(48), WEIGHT(48), ANSWER,A,B
C****  EQUIVALENCE (ARY(I,1),RO3T(1)), (ARY(I,2),WEIGHT(1))
C****
C****  SET UP ROJTS AND WEIGHTS...
C****
C****
DATA ((ARY(I,J),J=1,2),I=1,18) /
  0.0431  29.51  3.049  731112000  0.0332  611187  1336593  639387000
  0.6129  74904  64729  944940000  0.3244  71537  14094  623540000
  0.1135  58501  10655  920110000  0.3234  38225  66575  928429000
  0.1493  37146  54939  941999000  0.3220  62047  94030  253669000
  0.17309  58930  67191  602729000  0.3203  44662  31932  663213000
  0.21003  13104  63557  203633000  0.0132  37533  9411  005353000
  0.24174  11251  63393  312380000  0.0318  93307  70727  168558000
  0.27314  81120  91347  141973000  0.3131  64255  96851  352313000
  0.30436  54443  54495  393024000  0.3101  03325  86313  637423000
  0.33520  35223  92625  422616000  0.3067  13761  23669  149014000
  0.36595  58614  72113  635031000  0.3039  91154  20327  535743000
  0.39579  76493  28909  603235000  0.2936  35411  36325  385389000
  0.42424  83084  37300  243355000  0.32945  10899  99157  909790000
  0.45473  94221  67743  008536000  0.32899  45141  50355  236543000
  0.48349  79739  23996  354708000  0.28689  74110  63085  385643000
  0.51109  41771  54667  673546000  0.27979  0076  16548  334443000
  0.53399  81033  24137  436227000  0.27421  29627  26029  242623000

DATA ((ARY(I,J),J=1,2),I=19,37) /
  0.05021  64135  51347  163404000  0.02682  68667  25591  762198000
  0.33033  23647  75772  038540000  0.32621  23407  35672  413913000
  0.51392  59401  25495  970336000  0.2567  03630  05349  361499000
  0.04416  34037  34477  105790000  0.04290  06332  22483  610283000
  0.55371  31100  43716  193933000  0.23420  84171  92166  691232000
  0.39256  45368  42171  561344000  0.23494  33790  35260  219240000
  0.71597  93213  4546752253000  0.32737  70626  81329  374010000
  0.73333  65437  44033  1238510000  0.32176  50447  83744  349190000
  0.75943  23411  76947  498703000  0.20117  29398  92191  294983000
  0.79056  43383  73483  217634000  0.20235  57971  54333  324599000
  0.80033  37441  34140  817229000  0.19211  90311  40145  022410000
  0.81840  31017  37931  675539000  0.14660  50795  27411  467300000
  0.83722  35112  29127  126474000  0.17788  25023  16045  297643000
  0.85675  50343  43401  453463000  0.16888  54798  54245  172450000
  0.87138  55059  92634  328748000  0.11576  58293  32952  291381000
  0.88639  43174  32420  416097000  0.15039  87210  26914  393008000
  0.90166  30333  14092  341310000  0.14099  39417  72314  853150000
  0.91937  14231  23791  374925000  0.13132  32295  56961  972037000
  0.92771  24967  22033  533455000  0.12115  16046  71988  319353000

C****
FUNCTION FINT2  73/74  JPT=1  

DATA((ARY(I,J),J=1,2),I=33,43)/
.  0.93937  3397   92753  21693200,
  0.95033  27177  64437  63976000,
  0.96938  92144  48742  33930000,
  0.96932  53294  93244  21217400,
  0.97081  17495  83136  46543000,
  0.98251  72639  30142  67744700,
  0.98908  1363  29523  79941000,
  0.99034  39003  23762  62157200,
  0.99599  18429  37204  29055000,
  0.99935  43758  3181  57772900,
  0.99963  3308  32320  75682400,

C****
C****    INTEGRATION DONE BY TRANSLATING F TO THE
C****    INTERVAL -1 TO 1
C****

ANSWER = 0.00
DJ = 3
DA = 1

C****
DJ = I=1,43
T = ((DJ-DA)*RODT(I) -(D9+DA))/2.00
ANSWER = ANSWER + DEIGHT(I) * FIT
T = ((DJ-DA)*(-RODT(I)) +(D9+DA))/2.00
ANSWER = ANSWER + DEIGHT(I) * FIT
C****
FINT2 = (D1-DA)*ANSWER/2.00
C****
RETURN
END
FUNCTION RECT(A,B,R)

C**** RECTANGLE PROBABILITY...
C**** VOLUME UNDER THE NORMAL BIVARIATE DENSITY, 
C**** -INFX<X<A', -INFY<Y<B'.
C**** BY BRJWLOW, SDC/ISI
C**** COMMON/GPARM/ AA,BB,RR,SQR
EXTERNAL G
C****
AA = A
BB = B
RR = R
SQR = SQR(1-R*R)
C****
RECT = FINT2(G,-15.,A)
C****
RETURN
END
FUNCTION G(X)
C****
C****        CONDITIONAL DISTRIBUTION FUNCTION...
C****
5
C**** CJAMJN/SPAR4/A,B,R,SQR
C****
T = (3-R*X)/SJR
G = EXP(-X*X/2.)*PHI(T)*2.506628275
RETURN
10
END
FUNCTION TINV

FUNCTION TINV(P, A, B, R)

GIVEN A < Y < B FIND T SO THAT -T < X < T AND
P(-T < X < T, A < Y < B) = P

WITH COEFFICIENT OF CORRELATION BETWEEN X AND
Y EQUAL TO R.

P(-T < X < T, A < Y < B) = P(-T < X < T, A < Y < B) / P(A < Y < B)

T IS FOUND BY INTERVAL HALVING.

BY BROWN, SDC/IISI

DENOM = PHI(3) - PHI(A)
TMAX = 10.
TMIN = 0.

DO 1 I = 1, 50
  T = (TMAX + TMIN) / 2.

PCOMPT = RPROB(A, 3, -T, T, R) / DENOM

IF(PCOMPT .GT. P) TMAX = T
IF(PCOMPT .LT. P) TMIN = T
IF(ABS(PCOMPT - P) .LE. 1.0 .E-5) GO TO 2
  CONTINUE

PRINT 100, P, A, B, R

100 FORMAT('COULDN'T FIND T IN 50 ITERATIONS: P=*, F7.4, A=*, F7.4, B=*, F7.4,
R=*, F7.4.')
TINV = (TMIN + TMAX) / 2.
RETURN

2 TINV = T
RETURN
END
FUNCTION RPR3 (A, B, C, D, R)

C**** RECTANGULAR PROBABILITY FOR BIVARIATE
C**** NORMAL DISTRIBUTION...

5 C**** 
C**** CXX<0
C**** AXY<3
C****
C**** AND THE COEFFICIENT OF CORRELATION BETWEEN

10 C**** X AND Y IS R.
C**** BY BROXHO, SCIC/ISI
C****
C**** RPR3 = (RECT(D,3,R) - RECT(C,3,R)-RECT(D,A,R) + RECT(C,A,R))

15 C**** 
C**** RETURN 
END
FUNCTION PHI 73/74 OPT=1 FTN 4.2+75363

C****
C****
60 RETURN
END
FUNCTION PHI

C****
C****
C****         NORMAL(0,1) DISTRIBUTION FUNCTION
C****         PHI(X) = INTEGRAL OF NORMAL DENSITY
C****         FROM -INFINITY TO X.
C****         BY JANLOL, SOG/ISI
C****
C****  LOGICAL FLAG
C****  IF(X .GT. -10.) GO TO 1
C****    PHI = 0.
C****    RETURN
C****  1 IF(X .LT. 10.) GO TO 2
C****    PHI = 1.
C****    RETURN
C****  2 FLAG = .T.
C****
C****       DETERMINE IF X>0, SERIES EXPANSION IS FOR
C****       POSITIVE VALUES OF X.
C****
C****  IF(X .ST. 0.) GO TO 3
C****     FLAG = .F.
C****
C****       INITIALIZE VALUES FOR PARTIAL SUM OF THE SERIES...
C****     3 Z = ABS(X)
C****       D = 1.
C****       SJM = 0.
C****       TJP = Z
C****       BIT = 1.
C****
C****     4 CONTINUE
C****       SAVE = SUM
C****       SJM = SJM + TJP/8*BIT
C****
C****       CONTINUE TO SUM UNTIL MACHINE UNDEROWS...
C****
C****     5 IF(SAVE .LT. SUM) GO TO 6
C****
C****       UPDATE EXPRESSIONS FOR THE SUM...
C****     6 TJP = TJP+Z*Z
C****       D = D + 2.
C****       BIT = BIT*3
C****     GO TO 4
C****
C****
C****       DEPENDING UPON WHETHER ORIGINAL X>0 OR X<0,
C****       GET APPROPRIATE INTEGRAL VALUE...
C****
C****     5 PHI = SJM/SQR((6.28318533)*EXP(X*X)) *.5
C****     IF(FLAG) RETURN
C****     PHI = 1.-PHI
FUNCTION FRAC(73/74 OPT=1 FTN 4.2+75060)

FUNCTION FRAC(PV,Y1,Y2)

GIVEN A BIVARIATE NORMAL DISTRIBUTION
WITH COEFFICIENT OF CORRELATION RHO
AND Y1 < Y < Y2, FRAC(PV,Y1,Y2) RETURNS
THAT VALUE T, SUCH THAT:

PROB(-T < X < T, Y1 < Y < Y2) = PV

BINARY SEARCH, LIMITED TO A MAXIMUM OF 20 ITERATIONS

BY BROWNLOW, SDC/ISI, 11/79

KJOUNT = 0
TMIN = 0.
TMAX = 10.

1 T = (TMIN+TMAX)/2.

VAL = RECT(-T,T,Y1,Y2)
PRINT 100, VAL,T
100 FORMAT(* F10.3//)

IF WERE WITHIN 1.E-5 OF THE VALUE, WE
HAVE FOUND THE SOLUTION...

IF(ABS(VAL-PV) .LT. 1.E-5) GO TO 2

IF(VAL.LT. PV) TMIN=T
IF(VAL.GT. PV) TMAX = T

CHECK FOR MAXIMUM NUMBER OF ITERATIONS...

IF(KJOUNT .GE. 20) RETURN
KJOUNT = KJOUNT + 1

GJ TO 1

2 FRAC = T

RETURN
FUNCTION CONDEN(x)

C****
C**** CONDITIONAL DENSITY FUNCTION OF X, GIVEN
C**** A<X<B FROM BIVARIATE NORMAL DISTRIBUTION
C**** F(x,y) WITH COEFFICIENT OF CORRELATION RHO.
C****
C**** BY BRJWLJW, SDC/ISI, 11/79
C****

10 COMMON/RHO,A,B
C**** SET UP THE PARAMETERS, PHI IS THE UNIVARIATE
C**** NORMAL DISTRIBUTION FUNCTION.
C****

15 R = SQRT(1.-RHO*RHO)
D = PHI(B)-PHI(A)
T = PHI((3.-RHO*X)/R) - PHI((A-RHO*X)/R)
C****
CONDEN = EXP(-X*X/2.)*T/(D*2.506628275)

20 C****
RETURN
END
FUNCTION G(S,T)
C*** Multivariate normal distribution function.
C*** G(S,T) = double integral of normal
C*** multivariate density function, -Inf to S,
C*** -Inf to T.
C*** Notice that the numerical computations
C*** use the fact that the contribution to the
C*** integral value from -Inf to -15.
C*** is insignificant.
C***
C*** By 3C4NL74, SDC/ISI, 11/79
C***
15 COMMON /PASS1T/ C*** EXTERNAL FC4
C*** FF = T
20 G = FINT2(FUN,-15.,S1/5.,293195308)
C*** RETURN
END
FUNCTION FCN(x)

C***** DENSITY FUNCTION FOR DOUBLE INTEGRAL,
C***** PHI(x)*Z(x), WHERE PHI AND Z ARE THE
C***** NORMAL DISTRIBUTION AND DENSITY FUNCTIONS
C***** RESPECTIVELY.
C*****
C*****
C***** BY S. J. McLean, SUS/ISI, 11/79
C*****
C*****
C***** COMMON/ARG/RHO*A
C***** COMMON/PASS/TIT
C*****
C***** Z(ARG) = EXP(-ARG**2/2)
C*****
C***** J = (TT-RH)*X/SQRT(1.0-RH**2)
C*****
C***** FCN = PHI(J)*Z(X)**2.0*629273
C*****
C***** RETURN
END

30
FUNCTION RECT(x1, x2, y1, y2)

C**** VEC4T4GL: PR0BABILITY FOR BIVARIATE NORMAL
C**** DISTRIBUTION, x1<x2, y1<y2, AND THE
C**** COEFFICIENT OF CORRELATION IS RH0.
C****
C**** BY 
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****
C****

RECT = G(x2, y2) - G(x1, y2) - G(x2, y1) + G(x1, y1)

RETURN

END
FUNCTION CN4JST(4)

C****  CONDITIONAL DISTRIBUTION FUNCTION OF X GIVEN
C****  A<X<3 FROM BIVARIATE NORMAL DISTRIBUTION
C****  F(X,Y) WITH COEFFICIENT OF CORRELATION RHO.
C****
C****  BY R.H.NLGD, SDC/ISI, 11/79
C****
C****
C****

10  C4JST(4XG/PHI,A+4)
C****  PHI IS THE UNIVARIATE NORMAL DISTRIBUTION
C****  FUNCTION.
C****

15  C4JST=(G(Y,3)-(Y,A))/(PHI(B)-PHI(A)*5.23195308)
C****  RETURN
    END
FUNCTION FINT2
71/7+ JOPCL
FIN 4.2+75069

FUNCTION FINT2(F,*A3)
C****
C**** INTEGRAL OF THE FUNCTION F FROM A TO B
C**** BY GAUSSIAN-LEGENDRE QUADRATURE, 76 POINT FORM
C****
C**** ADD 76 EVALUATIONS OF F(X)
C****
C**** F MUST BE DECLARED EXTERNAL IN
C**** THE CALLING PROGRAM.
C****
C**** BY JOPCL JULIUS 3/13/83

10 DIJILE Prelude (4i1) & WEIGHT(48), ANSWER, DATA, DATA (48,2)
EQUIVALENCE (ARY(1,1) = FINT(1)), (ARY(1,2) = WEIGHT(1))
C****
C****
15 DATA (ARY(I,J), J=1,2), I=1,18) /
C****
C****
C****

25 C****
C****
C****

35 C****
C****
C****

DATA (ary(i,j), j=1,2), i=1,18) /
C****
C****
C****

45 C****
C****
C****

55 C****
C****
C****

DATA (ary(i,j), j=1,2), i=1,18) /
FUNCTION FINT2 73/74  J\=1

0.33373 0.3397 0.3753 2.1693000 0.01116 21.20 9.8328 49.8591000
0.37703 27.177 0.437 0.3275400 0.01016 0.7745 35.008 41.5790000
0.45775 4.2174 0.3745 0.3193000 0.00914 9.5712 3.0763 3.8663000
0.53725 0.3244 0.3206 2.1217430 0.00812 1.8769 2.5698 1.759217.000
0.61745 0.1375 0.4504300 0.00739 5.4707 9.1153 3.6526900
0.69731 0.353 0.3014 5.77644700 0.00658 8.5435 0.4225 96.1668300
0.77723 2.9223 7.9748100 0.00571 4.2027 4.2927 5.17539300
0.85724 9.3903 0.3742 0.2437200 0.00494 4.5543 3.8444 68.5574000
0.93735 1.932 2.3359200 0.00421 0.7318 1.7934 9.4640800
1.01746 5.3753 0.171 5.77724000 0.00353 3.907 8.3946 92.1732000
1.09758 0.3338 0.1210 7.6542900 0.00290 6.7920 0.5352 0.12429000

C###
C### INTEGRATION CUE BY TRANSLATING F TO THE
C### INITIAL -1 TO 1
C###

\[\int_{-1}^{1} f(x) dx \]

\[\int_{-1}^{1} f(T) dT \]

C###
C### DO 1 I=1,43
C### T = ((I-3)/4) * (3.079-0.01)
C### ANSWR = ANSWR + WEIGHT(I) * F(T)
C### 1 ANSWR = ANSWR + WEIGHT(I) * F(T)
C###
C### FINT2 = (I-0.01) * ANSWR/2.00
C###
C### RETURN
C###
FUNCTION PHI

FUNCTION PHI(X)

C****
C**** NORMAL(0,1) DISTRIBUTION FUNCTION
C**** PHI(X) = INTEGRAL OF NORMAL DENSITY
C**** FROM -INFINITY TO X.
C****
C**** LOGICAL FLAG
10 IF(X .GT. -10.) GO TO 1
     P+I = 0.
     RETURN
C****
1 IF(X .LT. 10.) GO TO 2
     PHI = 1.
     RETURN
C****
2 FLAG = .T.
C****
20 DETERMINE IF X>0, SERIES EXPANSION IS FOR
C**** POSITIVE VALUES IF X.:
C****
25 IF(X .GT. 0.) GO TO 3
     FLAG = .F.
C****
30 INITIALIZE VALUES FOR PARTIAL SUM OF THE SERIES...
C****
3 Z = ASIN(X)
     D = 1.
     SJM = 0.
     TJP = Z
     BJT = T.
C****
4 CONTINUE
C**** SAVE = SUM
     SJM = SJM + TJP/BJT
C**** CONTINUE TO SUM UNTIL MACHINE UNDERFLOWS...
C****
45 IF(SAVE .NE. SUM) GO TO 5
C**** UPDATE EXPRESSIONS FOR THE SUM...
C****
4 TJP = TJP#Z
     D = D + Z.
     BJT = BJT#)
     GO TO 4
C****
50 DEPENDING UPON WHETHER ORIGINAL X>0 OR X<0,
C**** GET APPROPRIATE INTEGRAL VALUE...
C****
5 PHI = SJM/SQRT(5.293185308*EXP(X*X)) + .5
     IF(FLAG) RETURN
C****
6 PHI = 1. - PHI
FUNCTION PHI      73/74   OPT=1

C**** RETURN
END
Normal distribution is widely employed in numerous disciplines. Unfortunately, probabilities for the multivariate normal distribution are generally not available. This paper presents some results for the bivariate normal distribution. Computer programs for conditional normal probabilities, marginal probabilities as well as joint probabilities for rectangular regions are given: routines for computing fractile points and distribution functions are also presented. Some examples from a closed circuit television experiment are included.