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Inversion and Approximation of Laplace Transforms

Mission Planning and Analysis Division
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Lyndon B. Johnson Space Center
Houston, Texas
SHUTTLE PROGRAM

INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

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<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2.0 INVERSION AND APPROXIMATION</td>
<td>1</td>
</tr>
<tr>
<td>3.0 EXAMPLES</td>
<td>6</td>
</tr>
<tr>
<td>APPENDIX - THE L_n FUNCTIONS AND PROPERTIES</td>
<td>22</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Plot of first four orthonormal functions</td>
</tr>
<tr>
<td>2</td>
<td>$\sum_{n=1}^{2} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>3</td>
<td>$\sum_{n=1}^{2} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>4</td>
<td>$\sum_{n=1}^{3} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>5</td>
<td>$\sum_{n=1}^{4} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>6</td>
<td>$\sum_{n=1}^{5} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>7</td>
<td>$\sum_{n=1}^{6} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>8</td>
<td>$\sum_{n=1}^{10} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>9</td>
<td>$\sum_{n=1}^{14} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>10</td>
<td>Approximations of $F(t) = e^{-t} \cos (\pi t)$</td>
</tr>
<tr>
<td>11</td>
<td>$\sum_{n=1}^{N} A_n^2$ for $F(t) = \frac{1}{t} (e^{-t} - e^{-2t}) - e^{-2t}$</td>
</tr>
<tr>
<td>12</td>
<td>Approximation of $F(t) = \frac{1}{t} (e^{-t} - e^{-2t}) - e^{-2t}$</td>
</tr>
</tbody>
</table>
13 \sum_{n=1}^{N} A_n \text{ for } F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp \left(-\frac{1}{4t}\right) 

14 \text{ Approximation of } F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp \left(-\frac{1}{4t}\right)
1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programmable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let \( f(s) \) be a Laplace transform and \( F(t) \) its exact inverse. \( \hat{F}(t) \) will be the approximate inverse, given by

\[
\hat{F}(t) = A_1 L_1(st) + A_2 L_2(st) + \cdots + A_N L_N(st)
\]

where the \( L_n(st) \) are the new orthonormal functions (described below and in the appendix). The \( A_n \) values are the Fourier coefficients and are given by

\[
A_n(s) = \int_0^\infty F(t) L_n(st) dt
\]

(2)

s is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

\[
E(s) = \int_0^\infty (\hat{F}(t) - F(t))^2 dt = \int_0^\infty F(t)^2 dt - \sum_{n=1}^N A_n^2(s) \geq 0.
\]

(3)

To minimize the integral square error, s is chosen such that

\[
C = \sum_{n=1}^N A_n^2(s) \text{ is maximum}
\]

(4)
The new orthonormal functions are shown below.

\[ L_n = n_1 e^{-st} + n_2 e^{-2st} + n_3 e^{-3st} + \ldots + n_n e^{-nst} \]  

(5)

The values of \( n_{ij} \) are chosen such that

\[ \int_0^\infty l_n l_m dt = 0 \text{ for } n \neq m \]

(6)

\[ l_n l_m = 1 \text{ for } n = m \]

The first 10 orthonormal functions are listed below.

\[ L_1 = \sqrt{2} se^{-st} \quad s > 0 \]
\[ L_2 = \sqrt{3} (e^{-st} + 3e^{-2st}) \]
\[ L_3 = \sqrt{5} (3e^{-st} - 12e^{-2st} + 10e^{-3st}) \]
\[ L_4 = \sqrt{8} (3e^{-st} + 30e^{-2st} - 60e^{-3st} + 35e^{-4st}) \]
\[ L_5 = \sqrt{10} (e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st}) \]
\[ L_6 = \sqrt{12} (4e^{-st} + 12e^{-2st} - 56e^{-3st} + 126e^{-4st} - 120e^{-5st} + 46e^{-6st}) \]
\[ L_7 = \sqrt{15} (4e^{-st} - 168e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st} - 5544e^{-6st} + 1716e^{-7st}) \]
\[ L_8 = \sqrt{16} (6e^{-st} - 210e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st} + 36036e^{-6st} - 220740e^{-7st} + 64350e^{-8st}) \]
\[ L_9 = \sqrt{15} (8e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st} - 166368e^{-6st} + 182160e^{-7st} - 162960e^{-8st} + 24310e^{-9st}) \]
Figure 1 shows plots of the first four, $L_n$.

The values of the $n_{ai}$ coefficients are given by

$$n_{ai} = (-1)^{n+i} \frac{\sqrt{2n_i}}{i!(i-1)!(n-1)!} (n+1-i)!$$

or

$$n_{ai} = (-1)^{n+i} \frac{\sqrt{2n_i}}{i!(i-1)!} \Pi_{j=1}^{i} (n^2 - j^2)$$

where $n_{a1} = (-1)^{n+1} \sqrt{2n_i} n$

The recursion relationship for the $n_{ai}$ is given by

$$n_{ai} = (-1)^{n+i} \sqrt{2n_i} n$$

$$n_{ai} = \frac{n^2 - (i-1)^2}{i(i-1)} n_{a1} - 1$$

The recursion relationship for the $L_n$ is given by

$$U_n = 2 \frac{2n - 1}{n(n-1)}$$

$$V_n = \frac{n(n-1)}{2n - 1}$$
This is the relationship that should be used to compute the $L_n$ in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of $f(t)$. In terms of the Laplace transform, $F(s)$, they are given by

$$A_n = \sum_{i=1}^{n} r_{ai}f(is)$$  \hspace{1cm} (17)

Note that as $n$ increases, so does the magnitude of the $a_{ai}$, which has an oscillating sign. This can cause serious roundoff error problems in computing the $A_n$. It is speculated that the maximum value of $n = N$ be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set $f(s) = \frac{1}{s + 1}$

Let $s = 1$ and compute the $A_n$. Theoretically

$$A_1 = \frac{1}{\sqrt{2}}$$

$$A_n = 0 \quad \text{for} \quad n > 1$$

[Also see theorem 15 in the appendix.]
Due to roundoff error, the theoretical values will not be achieved for \( N \) large.

Perhaps a better way of computing the \( A_n \) (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes \( C \).

\[
C = 0
\]

\[
\text{DO } \text{c } n = 1, N
\]

\[
A_n = f(n a)
\]

\[
\text{IF } (n \text{EQ.1}) \text{ GOTO b}
\]

\[
\delta = 1
\]

\[
\text{DO } \text{a } i = 1, n - 1
\]

\[
A_n = \frac{\frac{(n - i)}{(n + 1 - i)(n - i)}}{\delta \{n - i\} n a}
\]

\[
\delta = -\delta
\]

\[
A_n = \sqrt[n]{a} \pi a_{n}
\]

\[
C = C + A_n^2
\]

\[
\text{PRINT } C
\]
Note that \( s \) should be chosen such that \( G \) is maximum.

All the \( L_n \) approach zero as \( t \) approaches infinity. Therefore, the approximations work well only when \( F(t) \to 0 \) as \( t \) approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when \( F(t) \) does not decay to zero is shown below. Let

\[
g(s) = \frac{1 - e^{-2s}}{s^2}
\]

Apply the final value theorem.

\[
G(\infty) = \lim_{s \to 0} s g(s) = 2
\]

So instead of inverting \( g(s) \), invert

\[
f(s) = g(s) - \frac{2}{s}
\]

Now \( F(t) \to 0 \) as \( t \) approaches infinity and \( G(t) = F(t) + 2 \). Thus

\[
\dot{G}(t) = 2 + s \ddot{F}(t)
\]

3.0 EXAMPLES

As the first example, let

\[
f(s) = \frac{s + 1}{(s + 1)^2 + m^2}
\]

(18)

The exact inverse is

\[
F(t) = e^{-t} \cos (\omega t)
\]

(19)
Figures 2 through 9 show the values of

\[ C = \sum_{n=1}^{N} a_n^2 \]  \hspace{1cm} (20)

versus \( s \) for values of \( N \) from 1 to 14. The maximum value that \( C \) can obtain (neglecting roundoff errors) is 0.27306 since

\[ \int_{0}^{\infty} F(t)^2 dt = 0.27300 \]  \hspace{1cm} (21)

It is seen that each value of \( N \) has its own optimum value of \( s \), and the choice of \( s \) can greatly influence the accuracy of the fit.

Figure 10 shows plots of \( F(t), F(t)^2, \) and \( \frac{dF(t)}{dt} \). For \( N = 3 \) the optimum value of \( s \) was \( s = 2.2 \). In this case

\[ A_1 = 0.33378 \text{ 959410} \]
\[ A_2 = 0.28719 \text{ 57089} \]
\[ A_3 = -0.21431 \text{ 58487} \]

\[ 3F(t) = -3.145e^{-2.2t} + 11.921e^{-4.4t} - 7.805e^{-6.6t} \]  \hspace{1cm} (22)

The approximate Laplace transform is thus seen to be

\[ 3f(s) = \frac{3.345}{s + 2.2} \cdot \frac{11.921}{s + 4.4} - \frac{7.805}{s + 6.6} \]  \hspace{1cm} (23)

For \( N = 4 \) the optimum value of \( s \) is 0.3 and

\[ A_4 = 0.186\text{ 59 67315} \]
\[ A_2 = 0.36507 \text{ 61727} \]
\[
A_3 = 0.22334 \quad 71.754 \quad A_4 = -0.14742 \quad 23756
\]
\[
A_5 = -0.11047 \quad 63746 \quad A_6 = 0.11492 \quad 41046
\]
\[
6F(t) = -1.916e^{-0.9t} + 43.527e^{-1.8t} - 252.178e^{-2.7t} + 554.831e^{-3.6t} - 517.636e^{-4.5t} + 174.488e^{-5.4t}
\]  
(28)

From figure 8 it is seen that \( N = 10 \) and \( s = 0.65 \) will give an excellent fit. For this case

\[
A_1 = 0.14940 \quad 23073 \quad A_2 = 0.31134 \quad 50651
\]
\[
A_3 = 0.31771 \quad 10153 \quad A_4 = 0.03684 \quad 84058
\]
\[
A_5 = -0.18661 \quad 71768 \quad A_6 = -0.03590 \quad 26947
\]
\[
A_7 = 0.10334 \quad 31558 \quad A_8 = -0.02596 \quad 24761
\]
\[
A_9 = -0.03656 \quad 09018 \quad 110 = 0.04125 \quad 86606
\]

and

\[
10F(t) = -0.822e^{-0.65t} + 69.854e^{-1.3t} - 1195.825e^{-1.95t} + 10 \quad 132.374e^{-2.6t} - 44 \quad 280.068e^{-3.25t} + 110 \quad 0.528e^{-3.9t} - 169 \quad 0.633e^{-4.55t} + 142 \quad 526.134e^{-5.2t} - 68 \quad 134.644e^{-5.85t} + 13 \quad 742.171e^{-6.5t}
\]  
(25)

For the next example

\[
f(s) = \ln\left(\frac{s + 2}{s + 1}\right) - \frac{1}{s + 2}
\]  
(26)

The exact inverse is
\[ F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t} \]  

(27)

Figure 11 shows plots of \( C = \sum_{n=1}^{N} a_n \) versus \( s \) for \( N = 4, 8, \) and \( 12 \). It is clear that for \( s = 0.5 \), only four terms are needed to give an excellent fit.

In this case,

\[ 4F(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \]  

(28)

Figure 11 shows plots of \( F(t) \) and \( 4F(t) \). There is no visible difference between \( F(t) \) and \( 4F(t) \).

Let

\[ g(s) = \ln\left(\frac{s + 2}{s + 1}\right) \]  

(29)

Then from equations 26 and 28, \( g(s) \) is approximated by

\[ 4g(s) = -\frac{0.00261}{s + 0.5} + \frac{0.17292}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \]  

(30)

For \( s > 0 \), \( 4g(s) \) is an excellent approximation of \( g(s) \), as seen below.
\begin{tabular}{|c|c|c|}
\hline
$s$ & $g(s)$ & $\eta g(s)$ \\
\hline
0 & 0.69315 & 0.69304 \\
0.1 & 0.64663 & 0.64660 \\
1 & 0.40547 & 0.40547 \\
2 & 0.28768 & 0.28769 \\
5 & 0.15415 & 0.15415 \\
10 & 0.08701 & 0.08701 \\
\hline
\end{tabular}

Note

$$g(t) = -0.00261e^{-0.5t} + 0.17392e^{-t} + 0.66316e^{-1.5t} + 0.16647e^{-2t}$$ \hspace{1cm} (31)

where

$$G(t) = \frac{1}{t}(e^{-t} - e^{-2t})$$ \hspace{1cm} (32)

Note \(G(0) = 1\) and \(\eta G(0) = 0.99994\).

For the final example

$$f(s) = e^{-\sqrt{s}}$$ \hspace{1cm} (33)

which has an exact inverse of

$$F(t) = \frac{1}{\sqrt{t}} \exp \left(-\frac{1}{4t}\right)$$ \hspace{1cm} (34)

Figure 13 shows plots of \(G = \sum_{n=1}^{N} A_n^2\) versus \(s\) for values of \(N = 6, 10, \text{ and } 14\).

For \(N = 6\) the optimum value of \(s = 0.8\), and
\[ 6F(t) = 1.4951e^{-0.8t} - 15.6761e^{-1.6t} + 83.8937e^{-2.4t} \\
- 204.8870e^{-3.2t} + 232.5783e^{-4.4t} - 97.7713e^{-4.8t} \] (35)

As seen from figure 14, \( 6F(t) \) is a very good approximation of \( F(t) \), which is remarkable since \( F(t) \) is a complicated function of time that is very dissimilar to a power series in \( e^{-0.8t} \).
\[
\begin{align*}
\mathcal{Z}(s) &= \frac{s + 1}{(s + 1)^2 + \eta^2} \\
\eta &= \text{MAX}(171)
\end{align*}
\]

Figure 2. \( \sum_{n=1}^{\infty} \mathcal{Z}_n \) for \( F(t) = e^{-t} \cos(\pi t) \)

\[
\begin{align*}
\mathcal{Z}(s) &= \frac{s + 1.5}{(s + 1.5)^2 + \eta^2} \\
\eta &= 0.234
\end{align*}
\]

Figure 3. \( \sum_{n=1}^{\infty} \mathcal{Z}_n \) for \( F(t) = e^{-t} \cos(\pi t) \)
Figure 4. $\sum_{n=1}^{\infty} \alpha_n$ for $F(t) = e^{-t} \cos(\pi t)$.

Figure 5. $\sum_{n=1}^{\infty} \beta_n$ for $F(t) = e^{-t} \cos(\pi t)$.
Figure 1. \( \Sigma_{n=1}^{\infty} f_n \) for \( f(t) = e^{-t} \cos(nt) \).

Figure 2. \( \Sigma_{n=1}^{\infty} f_n \) for \( f(t) = e^{-t} \sin(nt) \).
Figure 3. $\sum_{\ell=1}^{10} \phi_\ell$ for $F(t) = e^{-t} \cos(\pi t)$.

Figure 4. $\sum_{\ell=1}^{10} \phi_\ell$ for $F(t) = e^{-t} \cos(\pi t)$.
Figure 10: Approximation of \( F(t) \) as \( e^{-t} \cos(t) \).

\[
f(s) = \frac{s + 1}{(s + 1)^2 + 1}
\]
\[ f(s) = \ln \left( \frac{s + 2}{s + 1} \right) - \left( \frac{1}{s + 2} \right) \]
Figure 12. Approximation of $F(t) = \frac{1}{2}(e^{-t} - e^{-2t}) - e^{-2t}$. 

Approximation of $F(t)$ is given by $F(t) = \frac{0.002}{4} + \frac{1.28}{2} + 0.66316e^{-1.5t}$ for $t = 0.5$. 

$F(t) = (e^{-t} - e^{-2t})/t - e^{-2t}$ 

The integral $\int F(t) - F(t) = 0$ 

$f(s) = \ln\left(\frac{5+2}{5+1}\right) - \frac{1}{2}$. 

$t$ values range from 0 to 3.2.
Figure 14.- Approximation of \( f(t) = \frac{1}{2\sqrt{\pi\sigma^3}} \exp\left(-\frac{1}{2\sigma^2}t^2\right) \).
APPENDIX

THE L_n FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:
The scalar product of \( f(t) \) and \( g(t) \) will be defined by

\[
(f,g) = \int_0^\infty f(t)g(t)dt
\]  

(1)

Definition 2:
Define \( L_n(st) \) by

\[
L_n(st) = \sum_{i=1}^{n} r_i e^{-ist}
\]  

(2)

where

\[
r_i = (-1)^{i+1} n! \frac{(n + i - 1)!}{i!(i-1)!(n - i)!}
\]  

(3)

Alternatively

\[
r_i = (-1)^{i+1} n! \frac{n}{i!(i-1)!} \prod_{j=1}^{i-1} (n^2 - j^2)
\]  

(4)

where

\[
r_i = (-1)^{i+1} n!\frac{n}{i!
\]  

(5)
Lemma 1:

For \( n > 1 \)

\[
\sum_{i=1}^{n} \frac{n^{a_i}}{x + i} = \sqrt{2sn} \frac{(x - 1)(x - 2) \cdots (x - (n - 1))}{(x + 1)(x + 2) \cdots (x + (n - 1))(x + n)}
\]  \hspace{1cm} (6)

Corollary A:

\[
\int_{0}^{\infty} l_n(st)dt = (-1)^{n+1} \sqrt{2sn} \frac{2}{\sqrt{2ns}}
\]  \hspace{1cm} (7)

or

\[
\sum_{i=1}^{n} \frac{n^{a_i}}{i} = (-1)^{n+1} \sqrt{2sn} \frac{1}{n}
\]  \hspace{1cm} (8)

Corollary B:

\[
L_n(0) = \sqrt{2ns}
\]  \hspace{1cm} (9)

or

\[
\sum_{i=1}^{n} n^{a_i} = \sqrt{2ns}
\]  \hspace{1cm} (10)
Theorem 1:

The system of functions \( L_n(st) \) are orthonormal. That is

\[
(L_n, L_m) = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases}
\]

Definition 4:

The generating function \( g(z, t) \) is defined as

\[
g(z, t) = 1 - \frac{1}{\sqrt{1 + \frac{e^{-st}}{(1 - z)^2}}} = g(1/z, t) \tag{12}
\]

Theorem 2:

Expansion of \( g(z, t) \) into Maclaurin's series gives

\[
g(z, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n-1}{\sqrt{2n\pi}} L_n(st) z^n \quad z^2 \leq 1 \tag{13}
\]

\[
g(z, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n-1}{\sqrt{2n\pi}} \frac{1}{z^n} L_n(st) \quad z^2 \geq 1 \tag{14}
\]

Theorem 3:

The difference equation satisfied by \( L_n(st) \) is
Theorem 4:

The differential equation satisfied by \( L_n \) is

\[
(e^{st} - 1)L_n + s e^{st} L_n + s^2 n^2 L_n = 0
\]

Also of interest is

\[
L_n = (-1)^{n+1} \frac{\sqrt{2} \text{BES}}{(n-1)!} \frac{d^{n-1}}{d(e^{st})^{n-1}} \left[ e^{-nst} (1 - e^{-st})^{n-1} \right]
\]

Theorem 5:

\[
e^{-nst} = \frac{2}{s} n!(n-1)! \sum_{i=1}^{n} \frac{\sqrt{1}}{(n+1)!(n-1)!} L_i(st)
\]

Definition 5:

\[
\int_{0}^{\infty} F(t)^2 dt
\]

be finite.

Let \( \Phi(t) \) be an approximation of \( F(t) \). The integral square error is defined by
Theorem 6:

The best approximation of $F(t)$ in the integral square error sense (E minimized) is given by

$$N\, F(t) : \sum_{n=1}^{N} A_n L_n(st)$$

where

$$A_n(s) = \int_{0}^{\infty} F(t) L_n(st) dt$$

The integral square error is now given by

$$E = \int_{0}^{\infty} (NF(t) - F(t))^2 dt$$

$$E = \int_{0}^{\infty} F(t)^2 dt - \sum_{n=1}^{N} A_n^2 \geq 0$$

$E$ is minimized by choosing $s$ such that $\sum_{n=1}^{N} A_n^2$ is maximum.

Theorem 7, completeness theorem:

If

$$\sum_{n=1}^{N} A_n^2$$

is maximum.
\[ \int_{0}^{\infty} F(t)^2 dt \]

is finite, and the Laplace transform of \( F(t) \), \( f(s) \) exists, then

\[ E \to 0 \text{ as } N \to \infty. \]

Theorem 8:

Let the Laplace transform of \( F(t) \) be

\[ f(s) = \int_{0}^{\infty} F(t)e^{-st} dt \quad (23) \]

Then

\[ A_n(s) = \sum_{i=1}^{n} n^a_i f(is) \quad (24) \]

Theorem 9:

\( N^F(t) \) can be written as

\[ N^F(t) = \sum_{n=1}^{N} N^{b_n} e^{-n^a t} \quad (25) \]

where

\[ N^{b_n} = 2^\infty \left[ N^{b_1} + N^{b_2} + N^{b_3} + \cdots + N^{b_n} f^\infty \right] \quad (26) \]
and where

\[ N_{bij} = N_{bji} = \frac{1}{2^s} \sum_{k=1}^{N} k^a_i k^a_j \quad (a_m = 0 \text{ for } m > k) \]  

or

\[ N_{bij} = N_{bji} = \frac{(-1)^{i+j}}{2(i+j)} \frac{1}{1!} \frac{1}{(i-1)!} \frac{(N+i)!(N+j)!}{j!} \frac{(N-i)!}{(N-j)!} \]  

Lemma 2:

\[ \sum_{i=1}^{N} \frac{N_{bij}}{x+i} = \frac{(-1)^{N-j}}{2} \frac{(N+j)!}{(j-1)!} \frac{1}{(N-j)!} \frac{1}{(x-j)} \]  

Theorem 10:

\[ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{N^n_n}{1+n} \]  

Theorem 11:

\[ \sum_{n=1}^{N} a^n_n = \sum_{n=1}^{N} N^n_n f(ns) \]  

where \( N^n_n \) was given by equation 26.
Theorem 12:

\[ N^F(0) = \sum_{n=1}^{N} \sqrt{2n^3} A_n \]  

(32)

Theorem 13:

\[ \int_0^\infty G(t)L_m(st)L_n(st)dt = \sum_{j=1}^{m} n^j n^j \]  

(33)

where

\[ n^j = \sum_{i=1}^{n} n^i \delta[(1 + j)s] \]  

(34)

where \( g(s) \) is the Laplace transform of \( G(t) \).

Theorem 14:

The best approximation to the \( j \)th derivative of \( F(t) \) is

\[ N^F(j)(t) = \sum_{n=1}^{N} j^n n^j L_n(st) \]  

(35)

where
\[ jA_n = \sum_{i=1}^{n} n^{a_i(is)} f(is) - F(+) \sum_{i=1}^{n} n^{a_i(+)s} (36) \]

\[ \left. \frac{dF}{dt} \right|_{t=+0} \sum_{i=1}^{n} n^{a_i(Is)} (3-2) - \left. \frac{d^2F}{dt^2} \right|_{t=+0} \sum_{i=1}^{n} n^{a_i(Is)} (3-3) \]

\[ \left. \frac{d^{j-1}F}{dt^{j-1}} \right|_{t=+0} \sum_{i=1}^{n} n^{a_i} \]

Note

\[ nF(j)(t) \neq \frac{d^jF(t)}{dt^j} \tag{37} \]

For example, if \( j = 1 \), the first derivative, then

\[ jA_n = \sum_{i=1}^{n} n^{a_i(is)} f(is) - \sqrt{\text{sn} \ F(+0)} \tag{38} \]

Note equation 10, corollary D,

\[ \sum_{i=1}^{n} n^{a_i} = \sqrt{\text{sn} s} \]

was used to obtain equation 38. The value of \( F(+) \) can be obtained from the initial value theorem:

\[ F(+) = \lim_{s \to 0} s f(s) \tag{39} \]
If $j = 2$, the second derivative, then

$$2A_n = \sum_{i=1}^{n} n a_i (i s)^2 f(i s) - \sqrt{2 \pi n} n^2 s F(+0) - \sqrt{2 \pi n} \frac{dF}{dt} \bigg|_{t \to 0}$$

(40)

If $j = 3$

$$3A_n = \sum_{i=1}^{n} n a_i (i s)^3 f(i s) - \sqrt{2 \pi n} \frac{n^2}{2} (n^2 + 1) s F(+0)$$

$$- \sqrt{2 \pi n} n^2 s \frac{dF}{dt} + \sqrt{2 \pi n} \frac{d^2F}{dt^2} \bigg|_{t \to 0}$$

(41)

For $j = 4$

$$4A_n = \sum_{i=1}^{n} n a_i (i s)^4 f(i s) - \sqrt{2 \pi n} \frac{n^2}{6} (n^4 + 4n^2 + 1) s^3 F(+0)$$

$$- \frac{1}{2} \sqrt{2 \pi n} n^2 (n^2 + 1) s^2 \frac{dF}{dT} + \sqrt{2 \pi n} n^2 s \frac{d^2F}{dt^2} \bigg|_{t \to 0}$$

$$- \sqrt{2 \pi n} \frac{d^3F}{dt^3} \bigg|_{t \to 0}$$

(42)

Theorem 15:

If

$$f(s) = \frac{A}{s + a}$$

(43)
\[ A_n = (-1)^{n+1} \frac{1}{\sqrt{2\pi n}} \frac{(s-a)(2s-a)\cdots ((n-1)s-a)}{(s+a)(2s+a)\cdots (ns+a)} \] (44)

\[ A_1 = A \frac{1}{\sqrt{2\pi}} \frac{1}{s+a} \] (45)

Note the results for \( A = 1 \) and \( a = 0 \), \( F(t) \) a unit step function. In this case

\[ A_n = (-1)^{n+1} \frac{2}{\sqrt{2\pi n}} \] (46)

Hence

\[ NF(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2\pi n}} L_n(st) \] (47)

From corollary B, \( L_n(0) = \sqrt{2\pi n} \), hence

\[ NF(0) = \begin{cases} 0 & N \text{ even} \\ 2 & N \text{ odd} \end{cases} \] (48)

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.
END

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