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Inversion and Approximation of Laplace Transforms

Mission Planning and Analysis Division

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Lyndon B. Johnson Space Center
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SHUTTLE PROGRAM

INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

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## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2.0</td>
<td>INVERSION AND APPROXIMATION</td>
<td>1</td>
</tr>
<tr>
<td>3.0</td>
<td>EXAMPLES</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>APPENDIX - THE $L_n$ FUNCTIONS AND PROPERTIES</td>
<td>22</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>Plot of first four orthonormal functions</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>$\sum_{n=1}^{1} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>$\sum_{n=1}^{2} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>$\sum_{n=1}^{3} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>$\sum_{n=1}^{4} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>$\sum_{n=1}^{5} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>$\sum_{n=1}^{6} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>$\sum_{n=1}^{10} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>$\sum_{n=1}^{14} A_n^2$ for $F(t) = e^{-t} \cos (\pi t)$</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>Approximations of $F(t) = e^{-t} \cos (\pi t)$</td>
<td>17</td>
</tr>
<tr>
<td>11</td>
<td>$\sum_{n=1}^{N} A_n^2$ for $F(t) = \frac{1}{t} (e^{-t} - e^{-2t}) - e^{-2t}$</td>
<td>18</td>
</tr>
<tr>
<td>12</td>
<td>Approximation of $F(t) = \frac{1}{t} (e^{-t} - e^{-2t}) - e^{-2t}$</td>
<td>19</td>
</tr>
</tbody>
</table>
13 \[ \sum_{n=1}^{N} A_n \] for \( F(t) = \frac{1}{2 \sqrt{\pi t^3}} \exp \left( -\frac{1}{4t} \right) \) \[ \ldots \ldots \] 20

14 Approximation of \( F(t) = \frac{1}{2 \sqrt{\pi t^3}} \exp \left( -\frac{1}{4t} \right) \) \[ \ldots \ldots \] 21
1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programmable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let \( f(s) \) be a Laplace transform and \( F(t) \) its exact inverse. \( \hat{F}(t) \) will be the approximate inverse, given by

\[
\hat{F}(t) = A_1 L_1(st) + A_2 L_2(st) + \cdots + A_N L_N(st)
\]

where the \( L_n(st) \) are the new orthonormal functions (described below and in the appendix). The \( A_n \) values are the Fourier coefficients and are given by

\[
A_n(s) = \int_{0}^{\infty} F(t) L_n(st) dt
\]

\( s \) is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

\[
E(s) = \int_{0}^{\infty} (\hat{F}(t) - F(t))^2 dt = \int_{0}^{\infty} F(t)^2 dt - \sum_{n=1}^{N} A_n^2(s) \geq 0.
\]

To minimize the integral square error, \( s \) is chosen such that

\[
C = \sum_{n=1}^{N} A_n^2(s) \text{ is maximum}
\]
The new orthonormal functions are shown below.

\[ L_n = n^\alpha e^{-st} + n^\beta e^{-2st} + n^\gamma e^{-3st} + \ldots + n^\mu e^{-nst} \]  

(5)

The values of \( n^\alpha \) are chosen such that

\[ \int_0^\infty l_n l_m dt = 0 \quad \text{for } n \neq m \]

(6)

\[ = 1 \quad \text{for } n = m \]

The first 10 orthonormal functions are listed below.

\[ L_1 = \sqrt{2}e^{-st} \quad s > 0 \]

\[ L_2 = \sqrt{4}e^{(-2e^{-st} + 3e^{-2st})} \]

\[ L_3 = \sqrt{6}e^{(3e^{-st} - 12e^{-2st} + 10e^{-3st})} \]

\[ L_4 = \sqrt{8}e^{(-3e^{-st} + 30e^{-2st} - 60e^{-3st} + 35e^{-4st})} \]

\[ L_5 = \sqrt{10}e^{(5e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st})} \]

\[ L_6 = \sqrt{12}e^{(-4e^{-st} + 18e^{-2st} - 56e^{-3st} + 126e^{-4st} - 120e^{-5st} + 66e^{-6st})} \]

\[ L_7 = \sqrt{14}e^{(7e^{-st} - 168e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st} - 5544e^{-6st} + 1716e^{-7st})} \]

\[ L_8 = \sqrt{16}e^{(-8e^{-st} + 252e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st} + 36036e^{-6st} - 24024e^{-7st} + 6435e^{-8st})} \]

\[ L_9 = \sqrt{18}e^{(8e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st} - 168168e^{-6st} + 181056e^{-7st} - 102960e^{-8st} + 24310e^{-9st})} \]
\[ L_{10} = \sqrt{205}(-10e^{-st} + 495e^{-2st} - 7920e^{-3st} + 60060e^{-4st} - 252252e^{-5st}
+ 630630e^{-6st} - 960960e^{-7st} + 875160e^{-8st} - 437580e^{-9st}
+ 92378e^{-10st}) \]

Figure 1 shows plots of the first four, \( L_n \).

The values of the \( a_i \) coefficients are given by

\[ a_i = (-1)^{n+1} \frac{\sqrt{2 \pi} n}{(n+1)!} \frac{(n+1-i)!}{i!(i-1)!(n-i)!} \] (7)

or

\[ a_i = (-1)^{n+1} \frac{\sqrt{2 \pi} n}{(n+1)!} \frac{n-i}{i!(n-i)!} \prod_{j=1}^{i-1} (n^2 - j^2) \] (8)

where \( a_1 = (-1)^{n+1} \sqrt{2 \pi} n \) (9)

The recursion relationship for the \( a_i \) is given by

\[ a_i = (-1)^{n+1} \sqrt{2 \pi} n \]

\[ a_i = - \frac{n^2 - (i-1)^2}{i(i-1)} a_{i-1} \]

\[ n = 2, 3, \ldots, N \] (10)

The recursion relationship for the \( L_n \) is given by

\[ V_n = 2 \frac{2n-1}{\sqrt{n(n-1)}} \]

\[ n = 2, 3, \ldots, N \] (12)

\[ V_n = \frac{n(n-1)}{2n-1} \]

\[ n = 2, 3, \ldots, N \] (13)
This is the relationship that should be used to compute the \( L_n \) in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of \( f(t) \). In terms of the Laplace transform, \( F(s) \), they are given by

\[
A_n = \sum_{i=1}^{n} \lambda_i f(is) \quad (17)
\]

Note that as \( n \) increases, so does the magnitude of the \( \lambda_n \), which has an oscillating sign. This can cause serious roundoff error problems in computing the \( A_n \). It is speculated that the maximum value of \( n = N \) be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set\(^4\)

\[
f(s) = \frac{1}{s + 1}
\]

Let \( s = 1 \) and compute the \( A_n \). Theoretically

\[
A_1 = \frac{1}{\sqrt{2}}
\]

\[
A_i = 0 \quad \text{for } n > 1
\]

\(^4\)Also see theorem 15 in the appendix.
Due to roundoff error, the theoretical values will not be achieved for \( N \) large.

Perhaps a better way of computing the \( A_n \) (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes \( C \).

\[
C = 0
\]

\[
\text{DO } c \ n = 1, N
\]

\[
A_n = f(n, \alpha)
\]

\[
\text{IF (n.EQ.1) GOTO b}
\]

\[
\delta = 1
\]

\[
\text{DO } a \ i = 1, n - 1
\]

\[
A_n = \frac{((n - 1)}{(n + 1 - i)(n - i)} A_{n-1} = \delta f((n - i)\alpha)
\]

\[
a \ \delta = -\delta
\]

\[
b \ \delta_n = \sqrt{2} \alpha \ n A_n
\]

\[
C = C + A_n^2
\]

\[
\text{PRINT } C
\]
Note that \( a \) should be chosen such that \( C \) is maximum.

All the \( L_n \) approach zero as \( t \) approaches infinity. Therefore, the approximations work well only when \( F(t) \to 0 \) as \( t \) approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when \( F(t) \) does not decay to zero is shown below. Let

\[
g(s) = \frac{1 - e^{-2s}}{s^2}
\]

Apply the final value theorem.

\[
G(\infty) = \lim_{s \to 0} sg'(s) = 2
\]

So instead of inverting \( g(s) \), invert

\[
f(s) = g(s) - \frac{2}{s}
\]

Now \( F(t) \to 0 \) as \( t \) approaches infinity and \( G(t) = F(t) + 2 \). Thus

\[
gG(t) = 2 + gF(t)
\]

3.0 EXAMPLES

As the first example, let

\[
f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2}
\]

The exact inverse is

\[
F(t) = e^{-t} \cos (\pi t)
\]
Figures 2 through 9 show the values of

\[ C = \sum_{n=1}^{N} A_n^2 \]  

versus \( s \) for values of \( N \) from 1 to 14. The maximum value that \( C \) can obtain (neglecting roundoff errors) is 0.27300 since

\[ \int_{0}^{\infty} F(t)^2 dt = 0.27300 \]  

(21)

It is seen that each value of \( N \) has its own optimum value of \( s \), and the choice of \( s \) can greatly influence the accuracy of the fit.

Figure 10 shows plots of \( F(t), zF(t), \) and \( 6F(t) \). For \( N = 3 \) the optimum value of \( s \) was \( s = 2.7 \). In this case

\[
A_1 = 0.33378 \ 95910 \\
A_2 = 0.28719 \ 57089 \\
A_3 = -0.24531 \ 58487 \\
zF(t) = -3.845e-2.2t + 11.921e-4.4t - 7.805e-6.6t
\]  

(22)

The approximate Laplace transform is thus seen to be

\[ zF(s) = \frac{-3.845}{s + 2.2} - \frac{11.921}{s + 4.4} + \frac{7.805}{s + 6.6} \]  

(23)

For \( N = 6 \) the optimum value of \( s \) is 0.3 and

\[
A_1 = 0.18679 \ 62215 \\
A_2 = 0.26576 \ 41275
\]
\[ A_3 = 0.22334 \quad 71.754 \quad A_4 = -0.14742 \quad 23756 \]
\[ A_5 = -0.11047 \quad 63746 \quad A_6 = 0.11492 \quad 41046 \]
\[ eF(t) = -1.916e^{-0.9t} + 43.527e^{-1.8t} - 252.178e^{-2.7t} \]
\[ + 654.831e^{-3.6t} - 517.636e^{-4.5t} + 174.486e^{-5.4t} \]
\[ \text{(28)} \]

From figure 8 it is seen that \( N = 10 \) and \( s = 0.65 \) will give an excellent fit. For this case:

\[ A_1 = 0.14940 \quad 23073 \quad A_2 = 0.31134 \quad 50651 \]
\[ A_3 = 0.31771 \quad 10153 \quad A_4 = 0.03684 \quad 84058 \]
\[ A_5 = -0.18611 \quad 71768 \quad A_6 = -0.03590 \quad 26947 \]
\[ A_7 = 0.10394 \quad 31198 \quad A_8 = -0.02596 \quad 24761 \]
\[ A_9 = -0.03655 \quad 09318 \quad A_{10} = 0.04125 \quad 86606 \]

and

\[ 10F(t) = -0.822e^{-0.65t} + 69.854e^{-1.3t} - 1195.825e^{-1.95t} \]
\[ + 10 \quad 133.374e^{-2.6t} - 44 \quad 280.063e^{-3.25t} + 110 \quad 050.528e^{-3.9t} \]
\[ - 169 \quad 940.633e^{-4.55t} + 142 \quad 526.134e^{-5.2t} \]
\[ - 68 \quad 134.684e^{-5.85t} + 13 \quad 742.171e^{-6.5t} \]
\[ \text{(25)} \]

For the next example,

\[ f(a) = \ln\left(\frac{a + 2}{a + 1}\right) - \frac{1}{a + 2} \]
\[ \text{(26)} \]

The exact inverse is
\[ F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t} \]  \tag{27}

Figure 11 shows plots of \( C = \sum_{n=1}^{N} a_n \) versus \( s \) for \( N = 4, 8, \) and 12. It is clear that for \( s = 0.5 \), only four terms are needed to give an excellent fit. In this case

\[ 4F(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \]  \tag{28}

Figure 11 shows plots of \( F(t) \) and \( 4F(t) \). There is no visible difference between \( F(t) \) and \( 4F(t) \).

Let

\[ g(s) = \ln \left( \frac{s + 2}{s + 1} \right) \]  \tag{29}

Then from equations 26 and 28, \( g(s) \) is approximated by

\[ 4g(s) = -\frac{0.00261}{s + 0.5} + \frac{0.17292}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \]  \tag{30}

For \( s > 0 \), \( 4g(s) \) is an excellent approximation of \( g(s) \), as seen below.
<table>
<thead>
<tr>
<th>s</th>
<th>g(s)</th>
<th>ηg(s)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.69315</td>
<td>0.69304</td>
</tr>
<tr>
<td>0.1</td>
<td>0.64663</td>
<td>0.64660</td>
</tr>
<tr>
<td>1</td>
<td>0.40547</td>
<td>0.40547</td>
</tr>
<tr>
<td>2</td>
<td>0.28768</td>
<td>0.28769</td>
</tr>
<tr>
<td>5</td>
<td>0.15415</td>
<td>0.15415</td>
</tr>
<tr>
<td>10</td>
<td>0.08701</td>
<td>0.08701</td>
</tr>
</tbody>
</table>

Note

\[ ηG(t) = -0.00261e^{-0.5t} + 0.17292e^{-t} + 0.66316e^{-1.5t} + 0.16647e^{-2t} \] (31)

where

\[ G(t) = \frac{1}{t} (e^{-t} - e^{-2t}) \] (32)

Note \( G(0) = 1 \) and \( ηG(0) = 0.99994 \).

For the final example,

\[ f(s) = e^{-\sqrt{s}} \] (33)

which has an exact inverse of

\[ F(t) = \frac{1}{\sqrt{\pi t^3}} \exp \left( -\frac{1}{4t} \right) \] (34)

Figure 13 shows plots of \( C = \sum_{n=1}^{N} \frac{A_n^2}{n^2} \) versus \( s \) for values of \( N = 6, 10, \) and 14.

For \( N = 6 \) the optimum value of \( s = 0.8 \), and

10
\[6F(t) = 1.4551e^{-0.8t} - 15.6761e^{-1.6t} + 83.8957e^{-2.4t} - 204.8879e^{-3.2t} + 232.5763e^{-4.4t} - 97.7713e^{-4.8t}\]  (36)

As seen from figure 14, \(6F(t)\) is a very good approximation of \(F(t)\), which is remarkable since \(F(t)\) is a complicated function of time that is very dissimilar to a power series in \(e^{-0.8t}\).
Figure 1 - Plot of first four orthogonal functions.
\[ f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2} \]

\[ s = \text{fig } (\max, 171) \]

Figure 1. $\sum_{n=1}^{\infty} a_n$ for $e^{-t} \cos(\pi t)$

Figure 2. $\sum_{n=1}^{\infty} a_{n}^2$ for $e^{-t} \cos(\pi t)$
Figure 4. $\sum_{n=1}^{\infty} \gamma_n$ for $F(t) = e^{-t} \cos(\pi t)$.

Figure 5. $\sum_{n=1}^{\infty} \gamma_n$ for $F(t) = e^{-t} \cos(\pi t)$.
Figure 1: \( \sum_{n=1}^{\infty} \) for \( f(t) = e^{-t} \cos(nt) \).

Figure 2: \( \sum_{n=1}^{\infty} \) for \( f(t) = e^{-t} \sin(nt) \).
Figure 1: $\sum_{n=0}^{\infty} e^{-t\cos(nt)}$

Figure 2: $\sum_{n=0}^{\infty} e^{-t\cos(nt)}$
Figure 10: Approximation of \( f(t) \) and its Fourier transform.
\[ f(x) = \ln\left(\frac{\hat{\sigma} + \frac{2}{n}}{\hat{\sigma}}\right) - \left(\frac{1}{8} - \frac{1}{n}\right) \]

Figure 11: \[ \sum_{i=1}^{n} \frac{f_{i}^{2}}{n} - 1 \cdot \frac{1}{n} \cdot \frac{1}{\hat{\sigma}^{2}} \cdot \mathbf{1} \]
Figure 12. Approximation of $F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t}$. 

$F(t) = (e^{-t} - e^{-2t})/t - e^{-2t}$ 

$F(t)^2 - F(t) = 0$
Figure 14.- Approximation of \( f(t) = \frac{1}{2\sqrt{\pi t^3}} \exp\left(-\frac{1}{2t}\right) \).

\[
f(t) = 1.18815e^{-0.5t} - 15.5131e^{-1.6t} - 23.6517e^{-2.4t} \\
- 204.9872e^{-3.2t} - 637.5702e^{-4.4t} - 77.7713e^{-1.8t} \quad (s = .8)
\]
APPENDIX
THE $L_n$ FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:
The scalar product of $f(t)$ and $g(t)$ will be defined by

$$ (f,g) = \int_0^\infty f(t)g(t)dt \quad (1) $$

Definition 2:
Define $L_n(st)$ by

$$ L_n(st) = \sum_{n=1}^n n! \cdot (-1)^{n+1} \frac{(n+1-i)!}{i!(i-1)!} \quad (2) $$

where

$$ n! = (-1)^n \cdot i^{n+1} \frac{(n+1-i)!}{i!(i-1)!} \quad (3) $$

Alternately

$$ n! = (-1)^n \cdot i^{n+1} \frac{n}{i!(i-1)!} \frac{1}{j^{i-1}} \cdot \frac{1}{j^{i-1}} \quad (4) $$

where

$$ n! = (-1)^n \cdot i^{n+1} \frac{n}{j^{i-1}} \quad (5) $$
Lemma 1:

For $n > 1$

$$\sum_{i=1}^{n} \frac{n^a_i}{x + i} = \frac{\sqrt{2s_n}}{n} \frac{(x - 1)(x - 2) \cdots (x - (n - 1))}{(x + 1)(x + 2) \cdots (x + (n - 1))(x + n)}$$  \hspace{1cm} (6)

Corollary A:

$$\int_{0}^{\infty} l_n(st)dt = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \frac{1}{n}$$  \hspace{1cm} (7)

or

$$\sum_{i=1}^{n} \frac{n^a_i}{i} = (-1)^{n+1} \sqrt{2s_n} \frac{1}{n}$$  \hspace{1cm} (8)

Corollary B:

$$L_n(0) = \sqrt{2ns}$$  \hspace{1cm} (9)

or

$$\sum_{i=1}^{n} n^a_i = \sqrt{2ns}$$  \hspace{1cm} (10)
Theorem 1:
The system of functions $L_n(st)$ are orthonormal. That is

$$(L_n,L_m) = 0 \quad \text{for } n \neq m$$
$$= 1 \quad \text{for } n = m$$

(11)

Definition 4:
The generating function $g(z,t)$ is defined as

$$g(z,t) = 1 - \frac{1}{\sqrt{1 + \frac{e^{-st}}{(1-z)^2}}}. \quad (12)$$

Theorem 2:
Expansion of $g(z,t)$ into Maclaurin's series gives

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n^3}} \text{L}_n(st) \quad z^2 \leq 1 \quad (13)$$

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n^3}} \frac{1}{zn} \text{L}_n(st) \quad z^2 \geq 1. \quad (14)$$

Theorem 3:
The difference equation satisfied by $L_n(st)$ is
\[ L_n = 2 \frac{2n - 1}{\sqrt{n(n-1)}} \left\{ e^{-st} - \frac{n(n-1)}{2n-1} + \frac{(n-1)(n-2)}{2n-3} \right\} L_{n-1} \]
\[ - \frac{1}{2} \frac{(n-1)(n-2)}{2n-3} L_{n-2} \]  

(15)

Theorem 4:

The differential equation satisfied by \( L_n \) is
\[ (e^{st} - 1)L_n + se^{st} L_n + s^2 n^2 L_n = 0 \]  

(16)

Also of interest is
\[ L_n = (-1)^{n+1} \frac{\sqrt{2n^3}}{(n-1)!} \frac{d^{n-1}}{d(e^{st})^{n-1}} [e^{-nst} - e^{-st}]^{n-1} \]  

(17)

Theorem 5:

\[ e^{-nst} = \sum_{i=1}^{\infty} \frac{\sqrt{1}}{(n+i)! (n-1)!} L_i(st) \]  

(18)

Definition 5:

Let
\[ \int_{0}^{\infty} F(t)^2 dt \]

be finite.

Let \( \tilde{F}(t) \) be an approximation of \( F(t) \). The integral square error is defined by
Theorem 6:

The best approximation of \( F(t) \) in the integral square error sense (\( E \) minimized) is given by

\[
E = \int_0^\infty (Nf(t) - F(t))^2 \, dt
\]

(19)

\[
NF(t) = \sum_{n=1}^N A_n \phi_n(t)
\]

(20)

where

\[
A_n(s) = \int_0^\infty F(t) \phi_n(st) \, dt
\]

(21)

The integral square error is now given by

\[
E = \int_0^\infty F(t)^2 \, dt - \sum_{n=1}^N A_n^2 \geq 0
\]

(22)

\( E \) is minimized by choosing \( s \) such that \( \sum A_n^2 \) is maximum.

Theorem 7, completeness theorem:

If
\[ \int_{0}^{\infty} F(t)^2 dt \]
is finite, and the Laplace transform of \( F(t) \), \( f(s) \) exists, then

\[ \frac{E}{N} \to 0 \quad \text{as} \quad N \to \infty. \]

**Theorem 8:**

Let the Laplace transform of \( F(t) \) be

\[ f(s) = \int_{0}^{\infty} F(t)e^{-st} dt \quad (23) \]

Then

\[ A_n(s) = \sum_{i=1}^{n} n_1 f(is) \quad (24) \]

**Theorem 9:**

\( N^F(t) \) can be written as

\[ N^F(t) = \sum_{n=1}^{N} N^b_n e^{-n at} \quad (25) \]

where

\[ N^b_n = 2^n \left[ \sum_{i=1}^{n} N^b_i f(is) + N^b_{n+1} f((n+1)s) + \cdots + N^b_{2n} f((2n)s) \right] \quad (26) \]
and where

\[ N^{b_{ij}} = n^{b_{ji}} = \frac{1}{2s} \sum_{k=1}^{N} k^{b_{ij}} k^{b_{ij}} \quad (k_{am} = 0 \text{ for } m > k) \]  \hspace{1cm} (27)

or

\[ N^{b_{ij}} = n^{b_{ji}} = \frac{(-1)^{i+j}}{2(i+j)} \cdot \frac{1}{1(i-1)!} \cdot \frac{1}{j(j-1)!} \cdot \frac{(N+i)!}{(N-i)!} \cdot \frac{(N+j)!}{(N-j)!} \]  \hspace{1cm} (28)

Lemma 2:

\[ \sum_{i=1}^{N} \frac{N^{b_{ij}}}{x+i} = \frac{(-1)^{N-j}}{2} \cdot \frac{(N+j)!}{(N-j)! j! (j-1)!} \]  \hspace{1cm} \[ \frac{(x-1)(x-2) \cdots (x-N)}{(x+1)(x+2) \cdots (x+N)} \cdot \frac{1}{x-j} \]  \hspace{1cm} (29)

Theorem 10:

\[ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{N^{b_{in}}}{i+n} \]  \hspace{1cm} (30)

Theorem 11:

\[ \sum_{n=1}^{N} a_{n}^{2} = \sum_{n=1}^{N} N^{b_{in}} f(ns) \]  \hspace{1cm} (31)

where \( N^{b_{in}} \) was given by equation 26.
Theorem 12:

\[ N F(0) = \sum_{n=1}^{N} \sqrt{2 \pi n} \, A_n \]  
(32)

Theorem 13:

\[ \int_{0}^{\infty} G(t) L_m(st) L_n(st) dt = \sum_{j=1}^{m} m^a_j \, n^b_j \]  
(33)

where

\[ n^b_j = \sum_{i=1}^{n} i^a_i \, \delta(i + j) \]  
(34)

where \( g(s) \) is the Laplace transform of \( G(t) \).

Theorem 14:

The best approximation to the \( j \)th derivative of \( F(t) \) is

\[ N F(j)(t) = \sum_{n=1}^{N} j A_n L_n(st) \]  
(35)

where
\[ jA_n = \sum_{i=1}^{n} n^{a_i}_1(is)f(is) - F(+0) \sum_{i=1}^{n} n^{a_i}_3(is)j^{3-1} \]  

\[ \left. - \frac{dF}{dt} \right|_{t=+0} \sum_{i=1}^{n} n^{a_i}_1(is)j^{j-2} - \frac{d^2F}{dt^2} \left. \right|_{t=+0} \sum_{i=1}^{n} n^{a_i}_1(is)j^{j-3} \]

\[ \vdots - \frac{d^{j-1}F}{dt^{j-1}} \left. \right|_{t=+0} \sum_{i=1}^{n} n^{a_i}_1 \]

Note

\[ N^F(j)(t) \neq \frac{d^jF(t)}{dt^j} \]  

For example, if \( j = 1 \), the first derivative, then

\[ 1A_n = \sum_{i=1}^{n} n^{a_i}_1(is)f(is) - \sqrt{2\pi n} F(+0) \]  

Note equation 10, corollary B,

\[ \sum_{i=1}^{n} n^{a_i}_1 = \sqrt{2\pi n} \]

was used to obtain equation 38. The value of \( F(+0) \) can be obtained from the initial value theorem:

\[ F(+0) = \lim_{s \to \infty} sf(s) \]  

(39)
If \( j = 2 \), the second derivative, then

\[
2A_n = \sum_{i=1}^{n} n^2 (i^2 f(i^2)) - \sqrt{2n^4 n^2} \frac{dF}{dt}
\]

If \( j = 3 \)

\[
3A_n = \sum_{i=1}^{n} n^3 (i^3 f(i^3)) - \sqrt{2n^6} \frac{n^2}{2} \left( n^2 + 1 \right) g^2 F(+0)
\]

\[
- \sqrt{2n^4 n^2} \frac{dF}{dt} - \sqrt{2n^2} \frac{d^2F}{dt^2}
\]

For \( j = 4 \)

\[
4A_n = \sum_{i=1}^{n} n^4 (i^4 f(i^4)) - \sqrt{2n^8} \frac{n^2}{6} \left( n^4 + 4n^2 + 1 \right) a^3 F(+0)
\]

\[
- \frac{1}{2} \sqrt{2n^6 n^2} \left( n^2 + 1 \right) g^2 \frac{dF}{dT} - \sqrt{2n^4 n^2} \frac{d^2F}{dt^2}
\]

\[
- \sqrt{2n^2} \frac{d^3F}{dt^3}
\]

Theorem 15:

If

\[
f(a) = \frac{A}{s + a}
\]
there

\[ A_n = (-1)^{n+1} A \sqrt{2n} \frac{(s-a)(2s-a) \cdots ((n-1)s-a)}{(s+a)(2s+a) \cdots (ns+a)} \quad (44) \]

\[ A_1 = A \sqrt{2} \frac{1}{n+a} \quad (45) \]

Note the results for \( A = 1 \) and \( a = 0 \), \( F(t) \) a unit step function. In this case

\[ A_n = (-1)^{n+1} \frac{2}{\sqrt{2n}} \quad (46) \]

hence

\[ N F(t) = 2 \sum_{n=1}^{N} (-1)^{n+1} \frac{L_n(st)}{\sqrt{2n}} \quad (47) \]

From corollary B, \( L_n(0) = \sqrt{2n} \), hence

\[ NF(0) = 0 \quad N \text{ even} \quad (48) \]

\[ = 2 \quad N \text{ odd} \]

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.