Inversion and Approximation of Laplace Transforms

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INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

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1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programmable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let \( f(s) \) be a Laplace transform and \( F(t) \) its exact inverse. \( \hat{F}(t) \) will be the approximate inverse, given by

\[
\hat{F}(t) = A_1L_1(st) + A_2L_2(st) + \cdots + A_NL_N(st)
\]

(1)

where the \( L_n(st) \) are the new orthonormal functions (described below and in the appendix). The \( A_n \) values are the Fourier coefficients and are given by

\[
A_n(s) = \int_{0}^{\infty} F(t)L_n(st)dt
\]

(2)

\( s \) is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

\[
E(s) = \int_{0}^{\infty} (\hat{F}(t) - F(t))^2 dt = \int_{0}^{\infty} F(t)^2 dt - \sum_{n=1}^{N} A_n^2(s) \geq 0.
\]

(3)

To minimize the integral square error, \( s \) is chosen such that

\[
C = \sum_{n=1}^{N} A_n^2(s) \text{ is maximum}
\]

(4)
The new orthonormal functions are shown below.

\[ L_n = n_1 e^{-st} + n_2 e^{-2st} + n_3 e^{-3st} + \ldots + n_n e^{-nst} \]  

(5)

The values of \( n_i \) are chosen such that

\[ \int_{0}^{\infty} L_n L_m dt = 0 \quad \text{for} \ n \neq m \]

\[ = 1 \quad \text{for} \ n = m \]

(6)

The first 10 orthonormal functions are listed below.

\[ L_1 = \sqrt{2} e^{-st}, \quad s > 0 \]

\[ L_2 = \sqrt{4} e^{(-e^{st} + 3e^{-2st})} \]

\[ L_3 = \sqrt{6} (3e^{-st} - 12e^{-2st} + 10e^{-3st}) \]

\[ L_4 = \sqrt{8} (-3e^{-st} + 30e^{-2st} - 60e^{-3st} + 35e^{-4st}) \]

\[ L_5 = \sqrt{10} (5e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st}) \]

\[ L_6 = \sqrt{12} (-5e^{-st} + 10e^{-2st} - 560e^{-3st} + 1260e^{-4st} - 120e^{-5st} + 460e^{-6st}) \]

\[ L_7 = \sqrt{14} (7e^{-st} - 16e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st} - 5544e^{-6st} + 1716e^{-7st}) \]

\[ L_8 = \sqrt{16} (-3e^{-st} + 252e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st} + 36036e^{-6st} - 24024e^{-7st} + 6435e^{-8st}) \]

\[ L_9 = \sqrt{18} (-e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st} - 168168e^{-6st} + 181530e^{-7st} - 102960e^{-8st} + 24310e^{-5st}) \]
$$L_{10} = \sqrt{209}(-10e^{-st} + 495e^{-2st} - 7920e^{-3st} + 60600e^{-4st} - 252252e^{-5st}$$
$$+ 630630e^{-6st} - 960960e^{-7st} + 875160e^{-8st} - 437580e^{-9st}$$
$$+ 923780e^{-10st})$$

Figure 1 shows plots of the first four, $L_n$.

The values of the $n^{a_1}$ coefficients are given by

$$n^{a_1} = (-1)^{n+1} \frac{\sqrt{2}\sin n}{\pi} \frac{(n+1-1)!}{1!(1-1)!} \frac{1}{(n-1)!}$$

(7)

or

$$n^{a_1} = (-1)^{n+1} \frac{\sqrt{2}\sin n}{\pi} \frac{n}{1!(1-1)!} \prod_{j=1}^{i-1} (n^2 - j^2)$$

(8)

where

$$n^{a_1} = (-1)^{n+1} \sqrt{2}\sin n$$

(9)

The recursion relationship for the $n^{a_1}$ is given by

$$n^{a_1} = (-1)^{n+1} \sqrt{2}\sin n$$

(10)

$$n^{a_1} = \frac{n^2 - (i-1)^2}{i(1-1)} n^{a_1-1}$$

(11)

for $i = 2, 3, \ldots, n$

$n = 2, 3, \ldots, N$

The recursion relationship for the $L_n$ is given by

$$U_n = 2 \frac{2n-1}{\sqrt{n(n-1)}}$$

(12)

for $n = 2, 3, \ldots, N$

$$V_n = \frac{n(n-1)}{2n-1}$$

(13)

for $n = 2, 3, \ldots, N$
This is the relationship that should be used to compute the $L_n$ in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of $F(t)$. In terms of the Laplace transform, $f(s)$, they are given by

$$A_n = \sum_{i=1}^{n} \alpha_i f(is)$$

(17)

Note that as $n$ increases, so does the magnitude of the $\alpha_1$, which has an oscillating sign. This can cause serious roundoff error problems in computing the $A_n$. It is speculated that the maximum value of $n = N$ be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set:

$$f(s) = \frac{1}{s + 1}$$

Let $s = 1$ and compute the $A_n$. Theoretically

$$A_1 = 1/\sqrt{2}$$

$$A_n = 0 \quad \text{for} \quad n > 1$$

(Also see theorem 15 in the appendix.)
and

$$C = \sum_{n=1}^{N} a_n^2 = 0.5$$

Due to roundoff error, the theoretical values will not be achieved for $N$ large.

Perhaps a better way of computing the $A_n$ (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes $C$.

c = 0

DO c n = 1, N

$A_n = f(n,a)$

IF (n.EQ.1) GOTO b

$\delta = 1$

DO a i = 1, n - 1

$A_n = \frac{(\gamma_{n-i})}{(n+1-i)!(n-i)!} A_n - \delta f((n-i)a)$

$\delta = -\delta$

$A_n = \sqrt{2\pi n} nA_n$

$C = C + A_n^2$

PRINT C
Note that \( s \) should be chosen such that \( C \) is maximum.

All the \( L_n \) approach zero as \( t \) approaches infinity. Therefore, the approximations work well only when \( F(t) + 0 \) as \( t \) approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when \( F(t) \) does not decay to zero is shown below. Let

\[
g(s) = \frac{1 - e^{-2s}}{s^2}
\]

Apply the final value theorem.

\[
G(\infty) = \lim_{s \to 0} sg'(s) = 2
\]

So instead of inverting \( g(s) \), invert

\[
f(s) = g(s) - \frac{2}{s}
\]

Now \( F(t) + 0 \) as \( t \) approaches infinity and \( G(t) = F(t) + 2 \). Thus

\[
G(t) = 2 + \frac{F(t)}{s}
\]

3.0 EXAMPLES

As the first example, let

\[
f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2}
\]

(18)

The exact inverse is

\[
F(t) = e^{-t} \cos (\pi t)
\]

(19)
Figures 2 through 9 show the values of

\[ C = \sum_{n=1}^{N} A_n^2 \]  

versus \( s \) for values of \( N \) from 1 to 14. The maximum value that \( C \) can obtain (neglecting roundoff errors) is 0.27300 since

\[ \int_{0}^{\infty} F(t)^2 dt = 0.27300 \]  

(21)

It is seen that each value of \( N \) has its own optimum value of \( s \), and the choice of \( s \) can greatly influence the accuracy of the fit.

Figure 10 shows plots of \( F(t) \), \( \frac{dF(t)}{dt} \), and \( \frac{d^2F(t)}{dt^2} \). For \( N = 3 \) the optimum value of \( s \) was \( s = 2.2 \). In this case

\[ A_1 = 0.33372 \ 95910 \]
\[ A_2 = 0.28719 \ 57089 \]
\[ A_3 = -0.21431 \ 58487 \]

\[ 3F(t) = -5.345e^{-2.2t} + 11.921e^{-4.4t} - 7.805e^{-6.6t} \]  

(22)

The approximate Laplace transform is thus seen to be

\[ 3f(s) = \frac{3.345}{s + 2.2} \cdot \frac{11.921}{s + 4.4} \cdot \frac{7.805}{s + 6.6} \]  

(22)

For \( N = 6 \) the optimum value of \( s \) is 0.3 and

\[ A_1 = 0.18050 \ 67715 \]
\[ A_2 = 0.36507 \ 61717 \]
\[ A_3 = 0.22334 \quad 71.754 \quad A_4 = -0.14742 \quad 23756 \]
\[ A_5 = -0.11047 \quad 63746 \quad A_6 = 0.11492 \quad 41046 \]
\[
\delta F(t) = -1.916e^{-0.9t} + 43.527e^{-1.8t} - 252.178e^{-2.7t} + 554.831e^{-3.6t} - 517.636e^{-4.5t} - 174.488e^{-5.4t}
\]  

(28)

From figure 8 it is seen that \( N = 10 \) and \( s = 0.65 \) will give an excellent fit. For this case

\[
A_1 = 0.14940 \quad 23073 \quad A_2 = 0.31134 \quad 50651
\]
\[
A_3 = 0.31771 \quad 10153 \quad A_4 = 0.03684 \quad 84058
\]
\[
A_5 = -0.18661 \quad 71768 \quad A_6 = -0.03590 \quad 26947
\]
\[
A_7 = 0.10364 \quad 31938 \quad A_8 = -0.02596 \quad 24761
\]
\[
A_9 = -0.03656 \quad 09318 \quad A_{10} = 0.04125 \quad 86606
\]

and

\[
10 F(t) = -0.822e^{-0.65t} + 69.854e^{-1.3t} - 1195.825e^{-1.95t} + 10 \; 133.374e^{-2.6t} - 44 \; 260.068e^{-3.25t} + 110.050.528e^{-3.9t} - 169 \; 040.633e^{-4.55t} + 142.050.134e^{-5.2t} - 68 \; 134.618e^{-5.85t} + 12 \; 742.171e^{-6.5t}
\]

(29)

For the next example

\[
f(s) = \ln\left(\frac{s + 2}{s + 1}\right) - \frac{1}{s + 2}
\]

(29)

The exact inverse is
\[ F(t) = \frac{1}{t} (e^{-t} - e^{-2t}) - e^{-2t} \]  

(27)

Figure 11 shows plots of \( C = \sum_{n=1}^{N} a_n \) versus \( s \) for \( N = 4, 8, \) and 12. It is clear that for \( s = 0.5 \), only four terms are needed to give an excellent fit. In this case

\[ qF(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \]  

(28)

Figure 11 shows plots of \( F(t) \) and \( qF(t) \). There is no visible difference between \( F(t) \) and \( qF(t) \).

Let

\[ g(s) = \ln \left( \frac{s + 2}{s + 1} \right) \]  

(29)

Then from equations 26 and 28, \( g(s) \) is approximated by

\[ qg(s) = \frac{0.00261}{s + 0.5} + \frac{0.17291}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \]  

(30)

For \( s > 0 \), \( qg(s) \) is an excellent approximation of \( g(s) \), as seen below.
\[ s \quad g(s) \quad \eta g(s) \]

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<td>0</td>
<td>0.69315</td>
<td>0.69304</td>
</tr>
<tr>
<td>0.1</td>
<td>0.64663</td>
<td>0.64660</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>0.28769</td>
</tr>
<tr>
<td>5</td>
<td>0.15415</td>
<td>0.15415</td>
</tr>
<tr>
<td>10</td>
<td>0.08701</td>
<td>0.08701</td>
</tr>
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</table>

Note

\[ \eta G(t) = -0.00261 e^{-0.5t} + 0.17392 e^{-t} + 0.66316 e^{-1.5t} + 0.1664 e^{-2t} \] (31)

where

\[ G(t) = \frac{1}{t} (e^{-t} - e^{-2t}) \] (32)

Note \( G(0) = 1 \) and \( \eta G(0) = 0.99994 \).

For the final example,

\[ f(s) = e^{-\sqrt{s}} \] (33)

which has an exact inverse of

\[ F(t) = \frac{1}{\sqrt{t}} \exp \left( -\frac{1}{4t} \right) \] (34)

Figure 13 shows plots of \( C = \sum_{n=1}^{N} \frac{2}{A_n} \) versus \( s \) for values of \( N = 6, 10, \) and \( 14 \).

For \( N = 6 \) the optimum value of \( s = 0.8 \), and
\[ 6F(t) = 1.4551e^{-0.8t} - 15.6761e^{-1.6t} + 83.8937e^{-2.4t} \]
\[ - 204.8879e^{-3.2t} + 232.5763e^{-4.0t} - 97.7713e^{-4.8t} \]  

(35)

As seen from Figure 14, \( 6F(t) \) is a very good approximation of \( F(t) \), which is remarkable since \( F(t) \) is a complicated function of time that is very dissimilar to a power series in \( e^{-0.8t} \).
Figure 1. Plot of first four orthonormal functions.
\[ f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2} \]

\[ s = \text{F.I.} \ (\text{MAX. 171}) \]

Figure 1. \( \sum_{n=1}^{\infty} c_n \) for \( f(t) = e^{-t} \cos(\pi t) \)

\[ f(s) = \frac{s}{(s + 1)^2 + \pi^2} \]

\[ s = 1.5 \ (\text{MAX. 224}) \]

Figure 2. \( \sum_{n=1}^{\infty} c_n \) for \( f(t) = e^{-t} \cos(\pi t) \)
Figure 4. \( \sum_{n=1}^{\infty} \gamma_n \) for \( F(t) = e^{-t} \cos(\pi t) \).

Figure 5. \( \sum_{n=1}^{\infty} \gamma_n \) for \( F(t) = e^{-t} \cos(\pi t) \).
Figure 1: $\sum_{n=1}^{\infty} f(t)$ for $f(t) = e^{-t} \cos(nt)$.

Figure 2: $\sum_{n=1}^{\infty} g(t)$ for $g(t) = e^{-t} \sin(nt)$. 

$S = 1.2$ (MAX: 261) 

$S = 0.5$ (MIN: 100)
Figure 2: \( \sum_{k=0}^{10} \beta_k \) for \( F(t) = e^{-t} \cos(nt) \).
Figure 10. An expression of $f(t) = e^{-t} \cos(\pi t)$.
\[ f(s) = \ln\left(\frac{s + \frac{1}{2}}{\frac{1}{2}}\right) - \left(s - \frac{1}{2}\right) \]
Figure 12.- Approximation of $F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t}$. 
Figure 14. Approximation of \( f(t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{1}{4t}\right) \).
APPENDIX

THE $L_n$ FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:

The scalar product of $f(t)$ and $g(t)$ will be defined by

$$ (f,g) = \int_0^\infty f(t)g(t)\,dt \quad (1) $$

Definition 2:

Define $L_n(s,t)$ by

$$ L_n(s,t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!(i-1)!} s^{i-1} \quad (2) $$

where

$$ n_1 = (-1)^{i-1} \frac{(n-1)!}{i!(i-1)!} \frac{(n+i-1)!}{i!(i-1)!} \quad (3) $$

Alternately

$$ n_1 = \frac{(-1)^{i-1} \frac{(n-1)!}{i!(i-1)!}}{i!} \frac{\left(\frac{1}{2}(i-1)!\right)^{i-1}}{j^{i-1}} \quad (4) $$

where

$$ n_1 = (-1)^{i-1} \frac{1}{i!} \cdot n \quad (5) $$
Lemma 1:

For $n > 1$

$$\sum_{i=1}^{n} \frac{n^a_i}{x + i} = \sqrt{2\pi n} \frac{(x - 1)(x - 2)\cdots(x - (n - 1))}{(x + 1)(x + 2)\cdots(x + n - 1)(x + n)}.$$  \hspace{1cm} \text{(6)}

Corollary A:

$$\int_{0}^{\infty} l_{n}(st)dt = (-1)^{n+1} \frac{\sqrt{2\pi n}}{2^{n+1}} \frac{2}{\sqrt{2\pi n}}.$$  \hspace{1cm} \text{(7)}

or

$$\sum_{i=1}^{n} \frac{n^a_i}{i} = (-1)^{n+1} \frac{1}{\sqrt{2\pi n}} \frac{1}{n}.$$  \hspace{1cm} \text{(8)}

Corollary B:

$$L_n(0) = \sqrt{2\pi n}.$$  \hspace{1cm} \text{(9)}

or

$$\sum_{i=1}^{n} n^a_i = \sqrt{2\pi n}.$$  \hspace{1cm} \text{(10)}
Theorem 1:
The system of functions $L_n(st)$ are orthonormal. That is

$$(L_n, L_m) = 0 \quad \text{for} \quad n \neq m$$

$$= 1 \quad \text{for} \quad n = m \quad (11)$$

Definition 4:
The generating function $g(z,t)$ is defined as

$$g(z,t) = 1 - \frac{1}{\sqrt{1 + \frac{e^{-st}}{(1-z)^2}}} = g(1/z,t)$$

$$(12)$$

Theorem 2:
Expansion of $g(z,t)$ into Maclaurin's series gives

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\sqrt{2\pi n}} L_n(st) \quad z^2 \leq 1$$

$$(13)$$

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\sqrt{2\pi n}} \frac{1}{z^n} L_n(st) \quad z^2 \geq 1$$

$$(14)$$

Theorem 3:
The difference equation satisfied by $L_n(st)$ is
Theorem 4:

The differential equation satisfied by \( L_n \) is

\[
(e^{-st} - 1)L_n + s e^{st} L_n + s^2 n^2 L_n = 0
\]  

(16)

Also of interest is

\[
L_n = (-1)^{n+1} \frac{\sqrt{2\pi n}}{(n-1)!} \frac{d^{n-1}}{d(e^{-st})^{n-1}} \left[ e^{-nst} (1 - e^{-st})^{n-1} \right]
\]  

(17)

Theorem 5:

\[
e^{-nst} = \frac{2}{s} n!(n-1)! \sum_{i=1}^{n} \frac{\sqrt{1}}{(n+1)! (n-i)! L_i(st)}
\]  

(18)

Definition 5:

let

\[
\int_{0}^{\infty} F(t)^2 dt
\]

be finite.

Let \( \hat{F}(t) \) be an approximation of \( F(t) \). The integral square error is defined by
Theorem 6:

The best approximation of $F(t)$ in the integral square error sense (E minimized) is given by

$$N \left( \sum_{n=1}^{N} a_n \gamma(n(st)) \right)$$

where

$$a_n(s) = \int_{0}^{\infty} F(t) \gamma(n(st)) dt$$

The integral square error is now given by

$$E = \int_{0}^{\infty} F(t)^2 dt - \sum_{n=1}^{N} a_n^2 \geq 0$$

E is minimized by choosing $s$ such that $\sum_{n=1}^{N} a_n^2$ is maximum.

Theorem 7, completeness theorem:

If
\[ \int_0^\infty F(t)^2 dt \]

is finite, and the Laplace transform of \( F(t) \), \( f(s) \) exists, then

\[ E \to 0 \quad \text{as} \quad N \to \infty. \]

**Theorem 8:**

Let the Laplace transform of \( F(t) \) be

\[ f(s) = \int_0^\infty F(t)e^{-st} dt \quad (23) \]

Then

\[ A_n(s) = \sum_{i=1}^n n^a_i f(is) \quad (24) \]

**Theorem 9:**

\( N^F(t) \) can be written as

\[ N^F(t) = \sum_{n=1}^N N^{\tilde{n}_n - n} \quad (25) \]

where

\[ N^{\tilde{n}_n} = 2^n \left[ n^{a_1 f(is)} + N^{a_2 f(2s)} + N^{a_3 f(3s)} + \cdots + N^{a_n f(ns)} \right] \quad (26) \]
and where

\[ N_{bij} = N_{bji} = \frac{1}{2^s} \sum_{k=1}^{N} k^a_i k^a_j \]  \hspace{1cm} (k_m = 0 \text{ for } m > k) \quad (27)

or

\[ N_{bij} = N_{bji} = \frac{(-1)^{i+j}}{2(i+j)} \frac{1}{1!(i-1)!} \frac{1}{j!(j-1)!} \frac{(N+i)!(N+j)!}{(N-i)!(N-j)!} \quad (28) \]

Lemma 2:

\[ \sum_{i=1}^{N} \frac{N_{bij}}{x+i} = \frac{(-1)^{N-j}}{2} \frac{(N+j)!}{(N-j)!(j-1)!} \frac{1}{(x-1)(x-2)\cdots(x-N)(x-j)} \quad (29) \]

Theorem 10:

\[ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{N_{b_n}}{1+n} \quad (30) \]

Theorem 11:

\[ \sum_{n=1}^{N} \alpha_n^2 = \sum_{n=1}^{N} N_{b_n} f(ns) \quad (31) \]

where \( N_{b_n} \) was given by equation 26.
Theorem 12:

\[ N F(0) = \sum_{n=1}^{N} \sqrt{2 \pi n} A_n \]  

(32)

Theorem 13:

\[ \int_0^\infty G(t) L_n(st) L_n(st) dt = \sum_{j=1}^{m} m^{aj} n^{\xi_j} \]  

(33)

where

\[ n^{\xi_j} = \sum_{i=1}^{n} n^{ai} \xi(i + j)s \]  

(34)

where \( g(s) \) is the Laplace transform of \( G(t) \).

Theorem 14:

The best approximation to the \( j \)th derivative of \( F(t) \) is

\[ N F(j)(t) = \sum_{n=1}^{N} j A_n L_n(st) \]  

(35)

where
\[ jA_n = \sum_{i=1}^{n} n_{i}^{a} f(i) - F(\pm 0) \sum_{i=1}^{n} n_{i}^{a} j^{-1} \]  
\[ \frac{dF}{dt} \bigg|_{t=\pm 0} \sum_{i=1}^{n} n_{i}^{a} j^{-2} - \frac{d^{2}F}{dt^{2}} \sum_{i=1}^{n} n_{i}^{a} j^{-3} \]  
\[ \vdots - \frac{d^{j-1}F}{dt^{j-1}} \sum_{i=1}^{n} n_{i}^{a} \]

Note

\[ n^{P}(j)(t) \neq \frac{d^{j}F(t)}{dt^{j}} \]  
(37)

For example, if \( j = 1 \), the first derivative, then

\[ 1A_n = \sum_{i=1}^{n} n_{i}^{a} f(i) - \sqrt{2\pi} n F(\pm 0) \]  
(38)

Note equation 10, corollary D,

\[ \sum_{i=1}^{n} n_{i}^{a} = \sqrt{2\pi n} \]

was used to obtain equation 38. The value of \( F(\pm 0) \) can be obtained from
the initial value theorem.

\[ f(\pm 0) = \lim_{s \to \infty} sf(s) \]  
(39)
If \( j = 2 \), the second derivative, then

\[
p_A = \sum_{i=1}^{n} n^4 i^2 \int f(s) \frac{dF}{ds} - \sqrt{2 \pi n} n^2 sF(0) - \sqrt{2 \pi n} \left. \frac{dF}{ds} \right|_{t=0} (40)
\]

If \( j = 3 \)

\[
p_A = \sum_{i=1}^{n} n^4 i^3 \int f(s) \frac{dF}{ds} - \sqrt{2 \pi n} \frac{n^2}{2} (n^2 + 1) \left. \frac{dF}{ds} \right|_{t=0} (41)
\]

For \( j = 4 \)

\[
p_A = \sum_{i=1}^{n} n^4 i^4 \int f(s) \frac{dF}{ds} - \sqrt{2 \pi n} \frac{n^2}{6} (n^4 + 4n^2 + 1) \left. \frac{dF}{ds} \right|_{t=0} (42)
\]

Theorem 15:
If

\[
f(s) = \frac{A}{s + a} (43)
\]
\[ A_n = (-1)^{n+1} \sqrt{\frac{s-a}{2^n s}} \frac{(s-a)(2s-a) \cdots (ns-a)}{(s+a)(2s+a) \cdots (ns+a)} \] (44)

\[ A_1 = \sqrt{\frac{1}{s+a}} \] (45)

Note the results for \( A = 1 \) and \( a = 0 \), \( F(t) \) a unit step function. In this case

\[ A_n = (-1)^{n+1} \frac{2}{\sqrt{2^n s}} \] (46)

 Hence

\[ n F(t) = 2 \sum_{n=0}^{N} (-1)^{n+1} \sqrt{\frac{s-a}{2^n s}} L_n(st) \] (47)

From corollary B, \( L_n(0) = \sqrt{\frac{s-a}{2^n s}} \), hence

\[ n F(0) = \begin{cases} 0 & N \text{ even} \\ 2 & N \text{ odd} \end{cases} \] (48)

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.