Inversion and Approximation of Laplace Transforms

Mission Planning and Analysis Division

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INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

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1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programmable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let \( f(s) \) be a Laplace transform and \( F(t) \) its exact inverse. \( \hat{F}(t) \) will be the approximate inverse, given by

\[
\hat{F}(t) = A_1 L_1(st) + A_2 L_2(st) + \cdots + A_N L_N(st)
\]

where the \( L_n(st) \) are the new orthonormal functions (described below and in the appendix). The \( A_n \) values are the Fourier coefficients and are given by

\[
A_n(s) = \int_0^\infty F(t) L_n(st) \, dt
\]

\( s \) is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

\[
E(s) = \int_0^\infty (\hat{F}(t) - F(t))^2 \, dt = \int_0^\infty F(t)^2 \, dt - \sum_{n=1}^N A_n^2(s) \geq 0.
\]

To minimize the integral square error, \( s \) is chosen such that

\[
C = \sum_{n=1}^N A_n^2(s) \text{ is maximum}
\]
The new orthonormal functions are shown below.

\[ L_n = n^ae^{-st} + n^2e^{-2st} + n^3e^{-3st} + \ldots + n^ne^{-n^2st} \]  \hspace{1cm} (5)

The values of \( n^a \) are chosen such that

\[ \int_{0}^{\infty} L_nL_mdt = 0 \quad \text{for} \ n \neq m \]
\[ = 1 \quad \text{for} \ n = m \]  \hspace{1cm} (6)

The first 10 orthonormal functions are listed below.

\[ L_1 = \sqrt{2}se^{-st} \quad s > 0 \]
\[ L_2 = \sqrt{3}e^{-st} + 3e^{-2st} \]
\[ L_3 = \sqrt{6}\left(30e^{-st} - 12e^{-2st} + 10e^{-3st}\right) \]
\[ L_4 = \sqrt{8}\left(-50e^{-st} + 30e^{-2st} - 60e^{-3st} + 36e^{-4st}\right) \]
\[ L_5 = \sqrt{10}\left(e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st}\right) \]
\[ L_6 = \sqrt{12}\left(-50e^{-st} + 106e^{-2st} - 560e^{-3st} + 1260e^{-4st} - 120e^{-5st} + 46e^{-6st}\right) \]
\[ L_7 = \sqrt{16}\left(70e^{-st} - 168e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st} - 5544e^{-6st} + 1716e^{-7st}\right) \]
\[ L_8 = \sqrt{18}\left(-8e^{-st} + 252e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st} + 36036e^{-6st} - 24024e^{-7st} + 6435e^{-8st}\right) \]
\[ L_9 = \sqrt{20}\left(20e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st} - 168168e^{-6st} + 185640e^{-7st} - 102960e^{-8st} + 24310e^{-9st}\right) \]
Figure 1 shows plots of the first four \( L_n \).

The values of the \( n^{a1} \) coefficients are given by

\[
\begin{align*}
  n^{a1} &= (-1)^{n+1} \sqrt{2\pi n} \frac{(n+1-1)!}{1!1!1!(n-1)!} \\
  &= (-1)^{n+1} \sqrt{2\pi n} \frac{n}{1!1!1!} \prod_{j=1}^{n-2} (n^2 - j^2)
\end{align*}
\]

where \( n^{a1} = (-1)^{n+1} \sqrt{2\pi} n \) (7)

The recursion relationship for the \( n^{a1} \) is given by

\[
\begin{align*}
  n^{a1} &= (-1)^{n+1} \sqrt{2\pi n} \\
  n^{a1} &= \frac{n^2 - (n-1)^2}{1!1!} n^{a1-1} \\
  &= \frac{n(n-1)}{2n-1} n^{a1-1} \\
  &= \frac{n(n-1)}{2n-1} n^{a1-1} \\
  &= \frac{n(n-1)}{2n-1} n^{a1-1}
\end{align*}
\]

The recursion relationship for the \( L_n \) is given by

\[
\begin{align*}
  V_n &= 2 \frac{2n-1}{\sqrt{n(n-1)}} \\
  &= 2 \frac{2n-1}{\sqrt{n(n-1)}} \\
  &= 2 \frac{2n-1}{\sqrt{n(n-1)}} \\
  &= 2 \frac{2n-1}{\sqrt{n(n-1)}} \\
  &= 2 \frac{2n-1}{\sqrt{n(n-1)}}
\end{align*}
\]
This is the relationship that should be used to compute the $L_n$ in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of $F(t)$. In terms of the Laplace transform, $F(s)$, they are given by

$$A_n = \sum_{i=1}^{n} \frac{\alpha_i f(is)}{i!}$$

(Note that as $n$ increases, so does the magnitude of the $\alpha_i$, which has an oscillating sign. This can cause serious roundoff error problems in computing the $A_n$. It is speculated that the maximum value of $n = N$ be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set $s = 1$ and compute the $A_n$. Theoretically

$$A_1 = \frac{1}{\sqrt{2}}$$

$$A_n = 0 \quad \text{for} \quad n > 1$$

\(^3\)Also see theorem 15 in the appendix.
Due to roundoff error, the theoretical values will not be achieved for $N$ large.

Perhaps a better way of computing the $A_n$ (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes $C$.

\[
C = \sum_{n=1}^{N} A_n^2 = 0.5
\]

\[
C = 0
\]

DO c $n = 1, N$

$A_n = f(n,a)$

IF (n.EQ.1) GOTO b

$\delta = 1$

DO a $i = 1, n - 1$

$A_n = \frac{((n-i))}{((n+1-i):n-i)} A_n - \delta f((n-i)a)$

a $\delta = -\delta$

b $A_n = \sqrt{2\pi n} A_n$

$C = C + A_n^2$

c PRINT C
Note that \( a \) should be chosen such that \( C \) is maximum.

All the \( L_n \) approach zero as \( t \) approaches infinity. Therefore, the approximations work well only when \( F(t) < 0 \) as \( t \) approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when \( F(t) \) does not decay to zero is shown below. Let

\[
g(s) = \frac{1 - e^{-2s}}{s^2}
\]

Apply the final value theorem.

\[
G(\infty) = \lim_{s \to 0} s g(s) = 2
\]

So instead of inverting \( g(s) \), invert

\[
f(s) = g(s) - \frac{2}{s}
\]

Now \( F(t) < 0 \) as \( t \) approaches infinity and \( G(t) = F(t) + 2 \). Thus

\[
G(t) = 2 + \frac{e^{-2t}}{s^2}
\]

3.0 EXAMPLES

As the first example, let

\[
f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2}
\]

(18)

The exact inverse is

\[
F(t) = e^{-t} \cos(\pi t)
\]

(19)
Figures 2 through 9 show the values of

$$C = \sum_{n=1}^{N} A_n^2$$

versus \( s \) for values of \( N \) from 1 to 14. The maximum value that \( C \) can obtain (neglecting roundoff errors) is 0.27300 since

$$\int_{0}^{\infty} F(t)^2 dt = 0.27300$$

(21)

It is seen that each value of \( N \) has its own optimum value of \( s \), and the choice of \( s \) can greatly influence the accuracy of the fit.

Figure 10 shows plots of \( F(t), \; zF(t), \; \tau, \; d^2F(t) \). For \( N = 3 \) the optimum value of \( s \) was \( s = 2.2 \). In this case

$$A_1 = 0.33378 \; 95910$$

$$A_2 = 0.28719 \; 57099$$

$$A_3 = -0.21441 \; 58487$$

$$zF(t) = -3.545e^{-2.2t} + 11.921e^{-4.4t} - 7.805e^{-6.6t}$$

(22)

The approximate Laplace transform is thus seen to be

$$3f(s) = \frac{-3.545}{s + 2.2} + \frac{11.921}{s + 4.4} - \frac{7.805}{s + 6.6}$$

(22)

For \( N = 6 \) the optimum value of \( s \) is 0.3 and

$$A_4 = 0.18609 \; 67615$$

$$A_6 = 0.06507 \; 6172$$
From figure 8 it is seen that $N = 10$ and $s = 0.65$ will give an excellent fit. For this case

\[ A_1 = 0.14940 \quad 23073 \quad A_2 = 0.31134 \quad 50651 \]

\[ A_3 = 0.31771 \quad 10153 \quad A_4 = 0.03684 \quad 84058 \]

\[ A_5 = -0.18661 \quad 71768 \quad A_6 = -0.03590 \quad 26947 \]

\[ A_7 = 0.10504 \quad 31538 \quad A_8 = -0.02596 \quad 24761 \]

\[ A_9 = -0.03656 \quad 09918 \quad A_{10} = 0.04125 \quad 86606 \]

and

\[ 10F(t) = -0.822e-0.65t + 69.854e^{-1.3t} - 1195.825e^{-1.95t} \]

\[ + 10 \quad 138.374e-2.6t - 44 \quad 260.068e-3.25t + 110.056 \quad 528e-3.9t \]

\[ - 169.040 \quad 633e-4.55t + 142.528 \quad 134e-5.2t \]

\[ - 68.134 \quad 680e-5.85t + 13 \quad 742.171e-6.5t \] (25)

For the next example

\[ f(a) = \ln\left(\frac{a + 2}{a + 1}\right) - \frac{1}{a + 2} \] (26)

The exact inverse is
\[ F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t} \]  

(27)

Figure 11 shows plots of \( C = \sum_{n=1}^{N} A_n \) versus \( s \) for \( N = 4, 8, \) and 12. It is clear that for \( s = 0.5 \), only four terms are needed to give an excellent fit. In this case

\[ 4F(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \]  

(28)

Figure 11 shows plots of \( F(t) \) and \( 4F(t) \). There is no visible difference between \( F(t) \) and \( 4F(t) \).

Let

\[ g(s) = \ln \left( \frac{s + 2}{s + 1} \right) \]  

(29)

Then from equations 26 and 28, \( g(s) \) is approximated by

\[ 4g(s) = -\frac{0.00261}{s + 0.5} + \frac{0.17292}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \]  

(30)

For \( s > 0 \), \( 4g(s) \) is an excellent approximation of \( g(s) \), as seen below.
Note

\[ g(s) = e^{-0.00261t - 0.5t + 0.17392e^{-t} + 0.66316e^{-1.5t} + 0.1664e^{-2t}} \]  

(31)

where

\[ G(t) = \frac{1}{t} (e^{-t} - e^{-2t}) \]  

(32)

Note \( G(0) = 1 \) and \( \eta G(0) = 0.99994 \).

For the final example

\[ f(s) = e^{-\sqrt{s}} \]  

(33)

which has an exact inverse of

\[ F(t) = \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{1}{4t} \right) \]  

(34)

Figure 13 shows plots of \( C = \sum_{n=1}^{N} \frac{A_n^2}{n} \) versus \( s \) for values of \( N = 6, 10, \) and \( 14 \).

For \( N = 6 \) the optimum value of \( s = 0.8 \), and
\[ 6F(t) = \text{1.4951e-0.8t - 15.6761e-1.6t + 83.8937e-2.4t} \]
\[ - 204.8870e-3.2t + 232.570e-4.4t - 97.7713e-5.4t \]
(35)

As seen from figure 14, \( 6F(t) \) is a very good approximation of \( F(t) \), which is remarkable since \( F(t) \) is a complicated function of time that is very dissimilar to a power series in \( e^{-0.8t} \).
Figure 1 - Plot of first four orthogonal functions.
Figure 7. $\Sigma_{n=1}^{\infty} \frac{1}{n^s}$ for $F(t) = e^{-t} \cos(\pi t)$

Figure 8. $\Sigma_{n=1}^{\infty} \frac{1}{n^{s/2}}$ for $F(t)$ as in Figure 7.
Figure 4. $\sum_{n=1}^{\infty} \delta_n$ for $F(t) = e^{-t} \cos(\pi t)$. 

Figure 5. $\sum_{n=1}^{\infty} \delta_n$ for $F(t) = e^{-t} \cos(\pi t)$. 
Figure 1. \( \sum_{n=1}^{n} f(t) = e^{-t} \cos(nt) \).

Figure 2. \( \sum_{n=1}^{n} f(t) = e^{-t} \sin(nt) \).
Figure 2: $\sum_{n=1}^{10} a_n \cos n\pi$ for $F(t) = e^{-t}$ 

$S = 0.05$ (MAX 0.2720)
Figure 12: A comparison of $f(t)$ and $e^{-t} \cos(t)$. The equation $f(S) = \frac{S + 1}{(S + 1)^2 + 1}$ is also shown.
Figure 11: \[ f(s) = \ln\left(\frac{S + \frac{2}{s}}{1 + \frac{2}{s}}\right) - \left(\frac{2}{s} - 2\right) \]
Figure 12. Approximation of $F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t}$. 
Figure 14.- Approximation of \( f(t) = \frac{1}{2\sqrt{\pi} t^3} \exp\left(-\frac{1}{2t^2}\right) \).
APPENDIX

THE \( L_n \) FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:

The scalar product of \( f(t) \) and \( g(t) \) will be defined by

\[
(f, g) = \int_0^\infty f(t)g(t)dt
\]  

(1)

Definition 2:

Define \( L_n(st) \) by

\[
L_n(st) = \sum_{i=1}^n \rho_i e^{-ist}
\]  

(2)

where

\[
\rho_i = (-1)^{i+1} \sqrt{\frac{(n+i-1)!}{i!(i+1)!}}
\]  

(3)

Alternately

\[
\rho_i = (-1)^{i+1} \sqrt{\frac{n}{(i+1)!}} \frac{i-1}{j_{i-1}^2(j_i^2 - j_{i-1}^2)}
\]  

(4)

where

\[
\rho_i = (-1)^{i+1} \sqrt{n}
\]  

(5)
Lemma 1:

For $n > 1$

$$\sum_{i=1}^{n} \frac{n^{a_i}}{x+i} = \sqrt{2^{sn}} \frac{(x-1)(x-2)\cdots(x-(n-1))}{(x+1)(x+2)\cdots(x+(n-1))(x+n)}$$  \hspace{1cm} (6)

Corollary A:

$$\int_{0}^{\infty} l_n(st)dt = (-1)^{n+1} \frac{2^{sn+1}}{\sqrt{2^{sn}}}$$  \hspace{1cm} (7)

or

$$\sum_{i=1}^{n} \frac{n^{a_i}}{i} = (-1)^{n+1} \sqrt{2^{sn}} \frac{1}{n}$$  \hspace{1cm} (8)

Corollary B:

$$L_n(0) = \sqrt{2^{ns}}$$  \hspace{1cm} (9)

or

$$\sum_{i=1}^{n} n^{a_i} = \sqrt{2^{ns}}$$  \hspace{1cm} (10)
Theorem 1:
The system of functions $L_n(st)$ are orthonormal. That is

$$ (L_n, L_m) = \begin{cases} 
0 & \text{for } n \neq m \\
1 & \text{for } n = m 
\end{cases} $$

(11)

Definition 4:
The generating function $g(z,t)$ is defined as

$$ g(z,t) = 1 - \frac{1}{\sqrt{1 + \frac{e^{-st}}{(1 - z)^2}}} = g(1/z,t) $$

(12)

Theorem 2:
Expansion of $g(z,t)$ into Maclaurin's series gives

$$ g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n^3}} n^2 L_n(st) z^{2n} \leq 1 $$

(13)

$$ g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n^3}} n^2 L_n(st) z^{2n} \geq 1 $$

(14)

Theorem 3:
The difference equation satisfied by $L_n(st)$ is
Theorem 4:

The differential equation satisfied by \( L_n \) is

\[
(\text{est} - 1)L_n + \text{est} L_n + s^2 n^2 L_n = 0
\]  

(16)

Also of interest is

\[
L_n = (-1)^{n+1} \frac{\sqrt{2n} e^{-st}}{(n-1)!} \frac{d^{n-1}}{d(e-st)^{n-1}} [e^{-nst}(1 - e^{-st})^{n-1}] 
\]  

(17)

Theorem 5:

\[
e^{-nst} = \frac{2}{s} \frac{n!(n-1)!}{e^{-st}} \sum_{i=1}^{n} \frac{\sqrt{1}}{(n+i)!(n-i)!} L_i(st) 
\]  

(18)

Definition 5:

Let

\[
\int_{0}^{\infty} F(t)^2 dt
\]

be finite.

Let \( F(t) \) be an approximation of \( F(t) \). The integral square error is defined by

\[
\int_{0}^{\infty} [F(t) - \hat{F}(t)]^2 dt
\]
Theorem 6:

The best approximation of $F(t)$ in the integral square error sense (E minimized) is given by

\[
NF(t) = \sum_{n=1}^{N} A_n L_n(st)
\]  

where

\[
A_n(s) = \int_{0}^{\infty} F(t)L_n(st)dt
\]  

The integral square error is now given by

\[
E = \int_{0}^{\infty} (NF(t) - F(t))^2 dt
\]  

E is minimized by choosing $s$ such that $\sum_{n=1}^{N} A_n^2$ is maximum.

Theorem 7, completeness theorem:

If
\[ \int_{0}^{\infty} F(t)^2 dt \]

is finite, and the Laplace transform of \( F(t) \), \( f(s) \) exists, then

\[ E \to 0 \quad \text{as} \quad N \to \infty \]

Theorem 8:

Let the Laplace transform of \( F(t) \) be

\[ f(s) = \int_{0}^{\infty} F(t)e^{-st} dt \quad (23) \]

Then

\[ A_n(s) = \sum_{i=1}^{n} n^a_i f(is) \quad (24) \]

Theorem 9:

\( N^p(t) \) can be written as

\[ N^p(t) = \sum_{n=1}^{N} N^p_n e^{-nt} \quad (25) \]

where

\[ N^p_n = 2^p \left[ c_1^{n_1} f(1) + c_2^{n_2} f(2) + c_3^{n_3} f(3) + \cdots + c_m^{n_m} f(m) \right] \quad (26) \]
and where

\[ N^{bij} = \frac{1}{2^s} \sum_{k=1}^{\sqrt{2}} k^a_i k^a_j \quad (k^m_m = 0 \text{ for } m > k) \quad (27) \]

or

\[ N^{bij} = \frac{(-1)^{i+j}}{2^{i+j}} \frac{1}{i!(i-1)!} \frac{1}{j!(j-1)!} \frac{(N+i)!}{(N-i)!} \frac{(N+j)!}{(N-j)!} \quad (28) \]

Lemma 2:

\[ \sum_{i=1}^{N} \frac{N^{bij}}{x+i} = \frac{(-1)^{N-j}}{2} \frac{(N+j)!}{(N-j)!j!(j-1)!} \frac{1}{(x-1)(x-2)\cdots(x-N)} \frac{1}{(x+1)(x+2)\cdots(x+N)} x-j \quad (29) \]

Theorem 10:

\[ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{N^n}{i+n} \quad (30) \]

Theorem 11:

\[ \sum_{n=1}^{N} {a_n}^2 = \sum_{n=1}^{N} N^n f(ns) \quad (31) \]

where \( N^n \) was given by equation 26.
Theorem 12:

\[ N \mathcal{F}(0) = \sum_{n=1}^{N} \sqrt{2\pi n} A_n \]  

(32)

Theorem 13:

\[ \int_{0}^{\infty} G(t)L_n(st)L_m(st)dt = \sum_{j=1}^{m} m^a_j n^s_j \]  

(33)

where

\[ n^s_j = \sum_{i=1}^{n} n^a_i g((1+j)s) \]  

(34)

where \( g(s) \) is the Laplace transform of \( G(t) \).

Theorem 14:

The best approximation to the \( j \)th derivative of \( F(t) \) is

\[ N \mathcal{F}(j)(t) = \sum_{n=1}^{N} j A_n L_{n}(st) \]  

(35)

where
\[ jA_n = \sum_{i=1}^{n} n^a_i(is) f(is) - F(\pm 0) \sum_{i=1}^{n} n^a_i(is) j^{-1} \]  
\[ - \frac{dF}{dt} \bigg|_{t=\pm 0} \sum_{i=1}^{n} n^a_i(is) j^{-2} - \frac{d^2F}{dt^2} \bigg|_{t=\pm 0} \sum_{i=1}^{n} n^a_i(is) j^{-3} \]
\[ \cdots - \frac{d^{j-1}F}{dt^{j-1}} \bigg|_{t=\pm 0} \sum_{i=1}^{n} n^a_i \]

Note

\[ n^p(j)(t) \neq \frac{d^jF(t)}{dt^j} \]  

For example, if \( j = 1 \), the first derivative, then

\[ 1A_n = \sum_{i=1}^{n} n^a_i(is) f(is) - \sqrt{2\pi n} F(\pm 0) \]  

Note equation 10, corollary B,

\[ \sum_{i=1}^{n} n^a_i = \sqrt{2\pi n} \]

was used to obtain equation 38. The value of \( F(\pm 0) \) can be obtained from the initial value theorem:

\[ F(\pm 0) = \lim_{s \to \infty} s f(s) \]
If \( j = 2 \), the second derivative, then

\[
2A_n = \sum_{i=1}^{n} n^3 i (is)^2 f(is) - \sqrt{2} \sin \frac{n^2}{2} sF(0) - \sqrt{2} \sin \frac{dt}{dt} \bigg|_{t=0} (40)
\]

If \( j = 3 \)

\[
3A_n = \sum_{i=1}^{n} n^3 i (is)^3 f(is) - \sqrt{2} \sin \frac{n^2}{2} (n^2 + 1) s^2 F(0)
\]

\[
- \sqrt{2} \sin \frac{n^2}{2} s \frac{dF}{dt} \bigg|_{t=0} - \sqrt{2} \sin \frac{d^2F}{dt^2} \bigg|_{t=0} (41)
\]

For \( j = 4 \)

\[
4A_n = \sum_{i=1}^{n} n^3 i (is)^4 f(is) - \sqrt{2} \sin \frac{n^2}{6} (n^4 + 4n^2 + 1) s^3 F(0)
\]

\[
- \frac{1}{2} \sqrt{2} \sin \frac{n^2}{2} (n^2 + 1) s^2 \frac{dF}{dT} \bigg|_{t=0} - \sqrt{2} \sin \frac{n^2}{2} s \frac{d^2F}{dt^2} \bigg|_{t=0}
\]

\[
- \sqrt{2} \sin \frac{d^3F}{dt^3} \bigg|_{t=0} (42)
\]

Theorem 15: If

\[
f(s) = \frac{A}{s + a} (43)
\]
ther

\[ A_n = (-1)^n \sqrt{2s} \frac{(s-a)(2s-a)\cdots((n-1)s-a)}{(s+a)(2s+a)\cdots(ns+a)} \]  \hspace{1cm} (44)

\[ A_1 = A \sqrt{2s} \frac{1}{s+a} \]  \hspace{1cm} (45)

Note the results for \( A = 1 \) and \( a = 0 \), \( F(t) \) a unit step function. In this case

\[ A_n = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \]  \hspace{1cm} (46)

hence

\[ nF(t) = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{\sqrt{2ns}} L_n(st) \]  \hspace{1cm} (47)

From corollary B, \( L_n(0) = \sqrt{2ns} \), hence

\[ nF(0) = 0 \quad N \text{ even} \]  \hspace{1cm} (48)

\[ = 2 \quad N \text{ odd} \]

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.