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Discrete Tchebycheff
Orthonormal Polynomials
and Applications

Mission Planning and Analysis Division

April 1980

NASA
National Aeronautics and
Space Administration
Lyndon B. Johnson Space Center
Houston, Texas
SHUTTLE PROGRAM

DISCRETE TCHEBYCHEFF ORTHONORMAL POLYNOMIALS AND APPLICATIONS

By W. M. Lear, TRW

JSC Task Monitor: P. Pixley, Mathematical Physics Branch

Approved: 
Emil R. Schiesser, Chief
Mathematical Physics Branch

Approved:
Ronald L. Perry, Chief
Mission Planning and Analysis Division

Mission Planning and Analysis Division
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1.0 INTRODUCTION

Discrete Tchebycheff orthonormal polynomials offer a convenient way to make least-squares polynomial fits of uniformly spaced discrete data. Computer programs to do so are simple and fast, and appear to be less affected by computer roundoff error, for the higher order fits, than conventional least-squares programs.

P. Tchebycheff, a Russian mathematician, developed these polynomials about 100 years ago. Their application to modern computer technology is very germane, but few people know about them. They are useful for any application of polynomial least-squares fits: approximation of mathematical functions, noise analysis of radar data, and real time smoothing (filtering) of noisy data, to name a few.

Uniformly spaced, discrete data, as shown below

<table>
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<tr>
<td>x₁</td>
<td>1</td>
<td>y₁</td>
</tr>
<tr>
<td>x₂</td>
<td>2</td>
<td>y₂</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>xₙ</td>
<td>N</td>
<td>yₙ</td>
</tr>
</tbody>
</table>

has an Ith order least-squares polynomial fit given by

\[ y(n) = \hat{A}_0 \bar{T}_0(n) + \hat{A}_1 \bar{T}_1(n) + \hat{A}_2 \bar{T}_2(n) + \cdots + \hat{A}_I \bar{T}_I(n) \]  

(1.1)

where the \( \bar{T}_i(n) \) are the discrete Tchebycheff orthonormal polynomials of \( n \).

The \( \hat{A}_i \) coefficients are called the Fourier coefficients and are given by

\[ \hat{A}_i = \sum_{n=1}^{N} y_n \overline{T}_i(n) \]  

(1.2)

A simple recursion formula is used to generate the $T_i(n)$ polynomials. Note an important fact: the $\hat{A}_i$ coefficients are independent of the order of the polynomial fit. That is, an $(i+1)$ order fit uses the same values of $\hat{A}_i$ as an $i$th order fit, only the $\hat{A}_{i+1} T_{i+1}(n)$ term is added to increase the fit order. Let

$$y_n = y_n + \varepsilon_n \quad (1.3)$$

where $y_n$ is the error-free value of $y$, and $\varepsilon_n$ is the error (noise) adding to $y_n$. Equations will be developed to estimate the statistics of $\varepsilon$, and equations will be derived showing how accurately $y_n$ and its derivatives can be predicted.

When $\varepsilon_n$ is highly correlated, a surprising result is shown. The fit polynomial at the end points of the fit can be less accurate than the raw data. This newly discovered information bears reemphasizing. There are instances where the raw (noisy) data is more accurate than that of the polynomial fit (filtered) data.

2.0 THE DISCRETE TCHEBYCHEFF POLYNOMIALS

First shown will be a set of Tchebycheff orthogonal polynomials.

$$T_0 = 1$$
$$T_1 = 1 + a_1n$$
$$T_2 = 1 + a_1 n + a_2 n^2$$
$$T_3 = 1 + a_1 n + a_2 n^2 + a_3 n^3$$
$$\vdots$$
$$T_i = 1 + a_1 n + a_2 n^2 + a_3 n^3 + \cdots + a_i n^i$$

The scalar product of the two variables $f(n)$ and $g(n)$ is defined by

$$(f, g) = \sum_{n=1}^{N} f(n)g(n) \quad (2.1)$$
Since the polynomials are orthogonal, then by definition

\[(T_i, T_j) = 0 \text{ for } i \neq j \quad (2.2)\]

This equation may be used to determine the \(a_{ij}\) coefficients. That is

\[
\sum_{n=1}^{N} T_i(n)T_j(n) = 0 \quad i \neq j
\]

Note: the original discrete Tchebycheff polynomials used a summation of \(n = 0\) to \(n = N - 1\), op cit.

It has been determined that the norm of \(T_i\) is given by

\[
(T_i, T_i) = \sum_{n=1}^{N} T_i(n)^2 = \frac{N}{2i + 1} \frac{(N - 1)(N - 2) \cdots (N - i)}{(N + 1)(N + 2) \cdots (N + i)}
\]

The orthonormal polynomials are given by

\[
\overline{T}_i = \frac{T_i}{\sqrt{(T_i, T_i)}}
\]

where

\[
\overline{T}_i = \overline{a}_{i0} + \overline{a}_{i1}n + \overline{a}_{i2}n^2 + \overline{a}_{i3}n^3 + \cdots + \overline{a}_{in}n^i
\]

Note that

\[
(\overline{T}_i, \overline{T}_j) = 0 \quad \text{for } i \neq j
\]

\[
= 1 \quad \text{for } i = j
\]

The first eight sets of the \(a_{ij}\) coefficients are shown as follows.
\[ a_{11} = -\frac{2}{N + 1} \]

\[ a_{21} = -6 \frac{N + 1}{(N + 1)(N + 2)} \]

\[ a_{22} = \frac{6}{(N + 1)(N + 2)} \]

\[ a_{31} = -2 \frac{6N^2 + 15N + 11}{(N + 1)(N + 2)(N + 3)} \]

\[ a_{32} = 30 \frac{N + 1}{(N + 1)(N + 2)(N + 3)} \]

\[ a_{33} = -\frac{20}{(N + 1)(N + 2)(N + 3)} \]

\[ a_{41} = -10 \frac{2N^3 + 9N^2 + 17N + 10}{(N + 1)(N + 2)(N + 3)(N + 4)} \]

\[ a_{42} = 10 \frac{9N^2 + 21N + 17}{(N + 1)(N + 2)(N + 3)(N + 4)} \]

\[ a_{43} = -140 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)} \]

\[ a_{44} = \frac{70}{(N + 1)(N + 2)(N + 3)(N + 4)} \]
\[
\begin{align*}
a_{51} &= -2 \frac{15N^4 + 105N^3 + 365N^2 + 525N + 274}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)} \\
a_{52} &= 210 \frac{N^3 + 4N^2 + 8N + 5}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)} \\
a_{53} &= -140 \frac{4N^2 + 9N + 8}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)} \\
a_{54} &= 630 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)} \\
a_{55} &= -252 \frac{252}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)} \\
a_{61} &= -42 \frac{N^5 + 10N^4 + 55N^3 + 140N^2 + 178N + 84}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)} \\
a_{62} &= 42 \frac{10N^4 + 60N^3 + 215N^2 + 315N + 178}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)} \\
a_{63} &= -420 \frac{4N^3 + 15N^2 + 32N + 21}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)} \\
a_{64} &= 210 \frac{15N^2 + 33N + 32}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)} \\
a_{65} &= -2772 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)} 
\end{align*}
\]
\[
\begin{align*}
\mathbf{a}_{66} &= \frac{924}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)} \\
\mathbf{a}_{71} &= -4 \frac{14N^6 + 189N^5 + 1505N^4 + 5880N^3 + 13223N^2 + 14847N + 6534}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{72} &= 42 \frac{18N^5 + 150N^4 + 835N^3 + 2100N^2 + 2811N + 1414}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{73} &= -84 \frac{50N^4 + 275N^3 + 1030N^2 + 1540N + 937}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{74} &= 2310 \frac{5N^3 + 18N^2 + 41N + 26}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{75} &= -924 \frac{18N^2 + 39N + 41}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{76} &= 12012 \frac{N+1}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)} \\
\mathbf{a}_{77} &= -3432 \frac{6N^7 + 105N^6 + 1141N^5 + 6300N^4 + 21763N^3 + 42399N^2 + 44158N + 18264}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} \\
\mathbf{a}_{81} &= -12 \frac{6N^7 + 105N^6 + 1141N^5 + 6300N^4 + 21763N^3 + 42399N^2 + 44158N + 18264}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} \\
\mathbf{a}_{82} &= 6 \frac{210N^6 + 2310N^5 + 18375N^4 + 69300N^3 + 160951N^2 + 187110N + 88316}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} 
\end{align*}
\]
\[ a_{83} = -4620 \frac{2N^5 + 15N^4 + 66N^3 + 216N^2 + 305N + 162}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

\[ a_{84} = 2310 \frac{15N^4 + 78N^3 + 307N^2 + 408N + 305}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

\[ a_{85} = -3603i \frac{2N^3 + 7N^2 + 17N + 12}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

\[ a_{86} = 12012 \frac{7N^2 + 15N + 17}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

\[ a_{87} = -51480 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

\[ a_{88} = \frac{12870}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)} \]

Note that all the odd order polynomials in the numerators of the above coefficients have a factor of \( N + 1 \).

Several interesting relationships concerning the \( a_{ij} \) coefficients have been observed. Some of them are

\[ T_i(1) = 1 + a_{i1} + a_{i2} + \cdots + a_{ii} = \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)} \quad (2.7) \]

\[ T_i(N) = 1 + a_{i1}N + a_{i2}N^2 + \cdots + a_{ii}N^i = (-1)^i \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)} \quad (2.8) \]
For $i$ odd, the midpoint value of $T_i$ is

$$T_i \left( \frac{N + 1}{2} \right) = 1 + a_{i1} \frac{N + 1}{2} + a_{i2} \left( \frac{N + 1}{2} \right)^2 + \cdots + a_{iN} \left( \frac{N + 1}{2} \right)^1 = 0$$

(2.9)

For $i$ even:

$$T_i \left( \frac{N + 1}{2} \right) = \frac{1 \times 3 \times 5 \times \cdots \times (i - 1)}{2 \times 4 \times 6 \times \cdots \times i} \frac{(N - 1)(N - 3)\cdots(N - (i - 1))}{(N + 2)(N + 4)\cdots(N + i)}$$

(2.10)

Other, more complex relations are shown below. Let

$$a_{i0} = \frac{(N + 1)(N + 2)\cdots(N + i)}{(N + 1)(N + 2)\cdots(N + i)} = 1$$

(2.11)

$$a_{ij} = a_{ij} \text{ with } N = -N \text{ in the numerator.}$$

(2.12)

The for $k = 0, 1, 2, \cdots, i$

$$\sum_{j=k}^{i} \frac{j!}{(j - k)!} a_{ij} = k! a_{ik}$$

(2.13)

$$\sum_{j=k}^{i} \frac{j!}{(j - k)!} a_{ij} = (-1)^{i-k} k! a_{ik}$$

(2.14)

$^a_T$ $T_0 \left( \frac{N + 1}{2} \right) = 1$. 

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From these two equations, it is easily seen that

$$\sum_{j=k}^{i} \frac{j!}{(j-k)!} (Nj-k - (-1)^{i+k}) a_{ij} = 0 \quad (2.15)$$

Also observed were the two important relationships

$$a_{ii} = \frac{(-1)^i}{(2i - 1)} a_{i-1, i-1} \quad (2.16)$$

$$a_{i,i-1} = -\frac{i}{2} (N+1) a_{ii} \quad (2.17)$$

The $T_i(n)$ polynomials have the following symmetry properties about $n = (N+1)/2$.

$$T_i \left( \frac{N + 1}{2} + m \right) = (-1)^i T_i \left( \frac{N + 1}{2} - m \right) \quad (2.18)$$

Other interesting relationships are

$$(nJ, T_i) = \sum_{n=1}^{N} nJ^T_i = 0 \quad \text{for} \quad 0 \leq j < i \quad (2.19)$$

$$\sum_{n=1}^{N} n^i T_i = \frac{1}{a_{i+1,i}} \frac{(N+1)N(N-1)(N-2)\cdots(N-i)}{(N+1)(N+2)\cdots(N+i+1)} \quad (2.20)$$

$$\sum_{n=1}^{N} n^{i+1} T_i = \frac{1}{a_{i+1,i+1}} \frac{(N+1)N(N-1)(N-2)\cdots(N-i)}{(N+1)(N+2)\cdots(N+i+1)} \quad (2.21)$$

Note from equation 2.19 that $T_i (i > 0)$ has an average value of zero.

The relationship to Legendre polynomials is shown below. Let
\[ n = \frac{N}{2}(x + 1) \] (2.22)

Then for \( N \to \infty \)

\[ T_1(n) \to (-1)^nP_1(x) \] (2.23)

where the \( P_1(x) \) are the well known Legendre polynomials.

The equations for \( \bar{T}_0 \) through \( \bar{T}_n(n) \) are shown below.

\[ \bar{T}_0 = \frac{1}{\sqrt{N}} \] (2.24)

\[ \bar{T}_1(n) = \frac{\sqrt{3}}{\sqrt{N}} \frac{1}{\sqrt{N^2 - 1^2}}(N + 1 - 2n) \] (2.25)

\[ \bar{T}_2(n) = \frac{\sqrt{5}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)}}[(N + 1)(N + 2) - 6(N + 1)n + 6n^2] \] (2.26)

\[ \bar{T}_3(n) = \frac{\sqrt{7}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)(N^2 - 3^2)}}[(N + 1)(N + 2)(N + 3) \\
- 2(6n^3 + 15n + 11)n + 30(N + 1)n^2 - 20n^3] \] (2.27)

\[ \bar{T}_4(n) = \frac{\sqrt{9}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)(N^2 - 3^2)(N^2 - 4^2)}}[(N + 1)(N + 2)(N + 3)(N + 4) \\
- 10(2n^3 + 9n^2 + 15n + 10)n + 10(9n^2 + 21n + 17)n^2 \\
- 140(N + 1)n^3 + 70n^4] \] (2.28)

Plots of \( \bar{T}_0 \) through \( \bar{T}_4(n) \) are shown in figure 1. The value of \( N \) was 119.

Note the symmetry about the midpoint of \( n = (N + 1)/2 = 60 \). Also note, from equation 2.19, that for \( i > 0 \) the mean value of \( \bar{T}_1(n) \) is zero.
Figure 1.- Plots of the orthonormal functions, $\bar{T}_n(n)$, ($N = 119$ points).
3.0 RECURSION EQUATIONS

The recursion equations are the best way to generate the Tchebycheff polynomials and the \( a_{ij} \) and \( \tilde{a}_{ij} \) coefficients.

\[
T_i(n) = \frac{1}{i(N + i)} \left( (2i - 1)(N + 1 - 2n)T_{i-1}(n) - (i - 1)(N + 1 - i)T_{i-2}(n) \right) \tag{3.1}
\]

where

\[
T_0(n) = 1 \tag{3.2}
\]

\[
T_1(n) = 1 - \frac{2n}{N + 1} \tag{3.3}
\]

The recursion relationship for the Tchebycheff orthonormal polynomials, \( \overline{T}_i(n) \), is given by

\[
\overline{T}_0 = 1/\sqrt{N} \tag{3.4}
\]

\[
\overline{T}_1 = \left( 1 - \frac{2n}{N + 1} \right) \sqrt{\frac{3(N + 1)}{N(N - 1)}} \tag{3.5}
\]

\[
S_i = \frac{1}{i} \sqrt{\frac{n^2 - 1}{N^2 - i^2}} \tag{3.6}
\]

\[
\overline{T}_i(n) = S_i \left[ (N + 1 - 2n)\overline{T}_{i-1}(n) - \frac{1}{S_{i-1}}\overline{T}_{i-2}(n) \right] \tag{3.7}
\]
To generate the $a_{ij}$ coefficients in

$$T_i(n) = a_{i0} + a_{i1}n + a_{i2}n^2 + \cdots + a_{i}n^i$$  \hspace{1cm} (3.8)

use

$$a_{i0} = 1 \quad i = 0, 1, \cdots, I$$  \hspace{1cm} (3.9)

$$a_{ii} = -\frac{2(2i - 1)}{i(N + i)}a_{i-1,i-1} \quad i = 1, 2, \cdots, I$$  \hspace{1cm} (3.10)

$$a_{i,i-1} = -\frac{i}{2(N + 1)}a_{ii} \quad i = 2, 3, \cdots, I$$  \hspace{1cm} (3.11)

$$a_{ij} = \frac{1}{i(N + i)}((2i - 1)((N + 1)a_{i-1,j} - 2a_{i-1,j-1})$$

$$- (i - 1)(N + 1 - i)a_{i-2,j})$$  \hspace{1cm} (3.12)

where $i = 3, 4, \cdots, I$ and $j = 1, 2, \cdots, i - 2$.

To generate the $\tilde{a}_{ij}$ coefficients in

$$\overline{T}_i(n) = \tilde{a}_{i0} + \tilde{a}_{i1}n + \tilde{a}_{i2}n^2 + \cdots + \tilde{a}_{i}n^i$$  \hspace{1cm} (3.13)

use

$$\tilde{S}_i = \frac{1}{i\sqrt{N^2 - i^2}}$$  \hspace{1cm} (3.14)

$$\tilde{a}_{00} = \frac{1}{\sqrt{N}}$$  \hspace{1cm} (3.15)
\[
\bar{a}_{10} = \frac{3(N + 1)}{N(N - 1)}
\] (3.16)

\[
\bar{a}_{11} = -\frac{2}{N + 1} \bar{a}_{10}
\] (3.17)

\[
\bar{a}_{i0} = \sqrt{\frac{2i + 1}{N} \frac{(N + 1)(N + 2)\cdots(N + i)}{(N - 1)(N - 2)\cdots(N - i)}}
\] \(i = 2, 3, \ldots, I\) (3.18)

\[
\bar{a}_{ii} = -2S_i \bar{a}_{i-1,i-1}
\] \(i = 2, 3, \ldots, I\) (3.19)

\[
\bar{a}_{i,i-1} = -\frac{i}{2(N + 1)} \bar{a}_{ii}
\] \(i = 2, 3, \ldots, I\) (3.20)

\[
\bar{a}_{ij} = S_i \left[(N + 1)\bar{a}_{i-1,j} - 2\bar{a}_{i-1,j-1} - \frac{1}{S_{i-1}} \bar{a}_{i-2,j}\right]
\] (3.21)

for \(i = 3, 4, \ldots, I\) and \(j = 1, 2, \ldots, i - 2\).

Let

\[
\bar{T}_i'(n) = \frac{dT_i}{dn}
\] (3.22)

The recursion relationship for the first derivative is obtained by differentiating equation 3.7.

\[
\bar{T}_i'(n) = S_i \left[-2\bar{T}_{i-1}' + (N + 1 - 2n)\bar{T}_{i-1}'(n) - \frac{1}{S_{i-1}} \bar{T}_{i-2}'(n)\right]
\] (3.23)

The following interesting relationships can be obtained for derivatives.
For $i$ even,
\begin{equation}
\sum_{n=1}^{N} \frac{d^{k}T_{i}}{d n^{k}} = 0 \quad \text{for } k = 1, 3, 5, 7, \ldots
\end{equation}

(3.24)

For $i$ odd,
\begin{equation}
\sum_{n=1}^{N} \frac{d^{k}T_{i}}{d n^{k}} = 0 \quad \text{for } k = 0, 2, 4, 6, \ldots
\end{equation}

(3.25)

4.0 LEAST-SQUARES FITS AND ACCURACY

Consider a uniformly spaced set of data to be fit, $y_{n}$, as shown below.

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<tr>
<th>$x$</th>
<th>$n$</th>
<th>$y_{n}$</th>
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<tr>
<td>$x_{1}$</td>
<td>1</td>
<td>$y_{1}$</td>
</tr>
<tr>
<td>$x_{2}$</td>
<td>2</td>
<td>$y_{2}$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$x_{N}$</td>
<td>$N$</td>
<td>$y_{N}$</td>
</tr>
</tbody>
</table>

where $\Delta x = x_{i} - x_{i-1}$ is constant, and

\begin{equation}
n = \frac{x - x_{1}}{\Delta x} + 1
\end{equation}

(4.1)

For an $I$th order polynomial fit, let

\begin{equation}
y_{n} = \hat{A}_{0}T_{0} + \hat{A}_{1}T_{1} + \hat{A}_{2}T_{2} + \cdots + \hat{A}_{I}T_{I}
\end{equation}

(4.2)
where

\[ \hat{A}_i = \sum_{n=1}^{N} y_n \hat{T}_i(n) \]  

(4.3)

are called the Fourier coefficients. As will be seen, this will result in a conventional least-squares fit.

The residual is defined by

\[ r_n = y_n - \hat{y}_n \]  

(4.4)

For a least-squares fit, the \( \hat{A}_i \) are chosen such that

\[ \sum_{n=1}^{N} r_n^2 = 0 \]  

is minimized

That is,

\[ \frac{\partial}{\partial \hat{A}_i} \sum_{n=1}^{N} r_n^2 = 0 \quad \text{for } i = 0, 1, 2, \ldots, I \]  

(4.5)
It is easily seen that

\[
\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} (y_n^*)^2 + \hat{\gamma}_0^2 + \hat{\gamma}_1^2 + \hat{\gamma}_2^2 + \cdots + \hat{\gamma}_I^2
\]

\[= 2\gamma_0 \sum_{n=1}^{N} \frac{y_n^*}{\bar{r}_0} - 2\gamma_1 \sum_{n=1}^{N} \frac{y_n^*}{\bar{r}_1} - \cdots \]

\[- 2\gamma_I \sum_{n=1}^{N} \frac{y_n^*}{\bar{r}_I} \quad (4.6)\]

Differentiating with respect to \(\hat{\gamma}_I\) gives

\[2\hat{\gamma}_I - 2 \sum_{n=1}^{N} \frac{y_n}{\bar{r}_I} = 0 \quad (4.7)\]

which yields equation 4.3 for \(\hat{\gamma}_I\).

An interesting aspect of a least-squares fit is that

\[\sum_{n=1}^{N} r_n = 0 \quad (4.8)\]

This is seen from

\[\frac{\partial}{\partial \hat{\gamma}_I} \sum_{n=1}^{N} r_n^2 = 2 \sum_{n=1}^{N} \frac{\partial r_n}{\partial \hat{\gamma}_I} = - 2 \sum_{n=1}^{N} \frac{\partial y_n}{\partial \hat{\gamma}_I} = 0 \quad (4.9)\]
But $\bar{T}_0$ is the constant $1/\sqrt{N}$, which will factor out. Hence, the sum of the residuals will be zero. Equation 4.10 indicates that the Fourier coefficients of the residuals will all be zero. That is, an $I$th order polynomial fit of the residuals will be zero, and so will an $I - 1$ order fit, an $I - 2$ order fit, etc.

The $y_n$ values can be represented as

\[ y_n = y_n + \epsilon_n \quad (4.11) \]

where $y_n$ is the error-free value and $\epsilon_n$ is the error (noise) adding to $y_n$. It will be assumed that $\epsilon_n$ is zero mean. That is, the expected value of $\epsilon_n$ is

\[ E(\epsilon_n) = 0 \quad (4.12) \]

The autocorrelation function of $\epsilon_n$ is defined by

\[ \phi_i = E(\epsilon_n \epsilon_{n+i}) \quad (4.13) \]

$\phi_i$ is assumed independent of $n$, stationary process, and $\phi_i$ will be symmetric.

\[ \phi_i = \phi_{-i} \quad (4.14) \]

The standard deviation of $\epsilon_n$ is defined by

\[ \sigma_\epsilon = \sqrt{\phi_0} \quad (4.15) \]
If the error is "uncorrelated" in time, then

\[ \phi_i = 0 \quad \text{for } i \neq 0 \] (4.16)

The Fourier coefficients of \( \epsilon \) are given by

\[ E_i = \sum_{n=1}^{N} \epsilon_n \overline{T}_i(n) \] (4.17)

With a little effort, it can be shown that

\[ E(\epsilon_i \epsilon_j) = \sigma_\epsilon^2 \delta_{ij} + \sum_{k=1}^{N-1} \phi_k \sum_{n=1}^{N-k} \overline{T}_i(n) \overline{T}_j(n + k) + \overline{T}_j(n) \overline{T}_i(n + k) \] (4.18)

where

\[ \delta_{ij} = 1 \quad \text{for } i = j \]
\[ = 0 \quad \text{for } i \neq j \]

Note that for uncorrelated noise

\[ E(\epsilon_i^2) = \sigma_\epsilon^2 \] (4.19)
\[ E(\epsilon_i \epsilon_j) = 0 \quad \text{for } i \neq j \] (4.20)

It will be assumed that \( y_n \) is adequately represented by the \( I \)th order polynomial

\[ y_n = A_0 \overline{T}_0 + A_1 \overline{T}_1 + A_2 \overline{T}_2 + \cdots + A_I \overline{T}_I \] (4.21)
where

$$A_i = \sum_{n=1}^{N} y_n T_1(n)$$  \hspace{1cm} (4.22)

"Adequately" means that the error in equation 4.21 is small compared to $\sigma_c$.

The Fourier coefficients for $y_n$ were

$$A_i = \sum_{n=1}^{N} y_n T_1(n)$$

But $y_n = y_f + \epsilon_n$, so

$$\hat{A}_i = A_i + E_i$$  \hspace{1cm} (4.23)

And

$$\hat{y}_n = \hat{A}_0 T_0 + \hat{A}_1 T_1 + \ldots + \hat{A}_I T_I$$

becomes

$$\hat{y}_n = y_n + E_0 T_0 + E_1 T_1 + E_2 T_2 + \ldots + E_I T_I$$  \hspace{1cm} (4.24)

The mean value of the error in $\hat{y}_n$ is easily seen to be

$$E(\hat{y}_n - y_n) = 0$$  \hspace{1cm} (4.25)

The variance of the error for an $I$th order fit is

$$E((\hat{y}_n - y_n)^2) = \sum_{i=0}^{I} T_i^2 E(E_i^2) + 2 \sum_{j=1}^{I-1} \sum_{i=0}^{I-j} T_i T_{i+j} E(E_i E_{i+j})$$  \hspace{1cm} (4.26)
For uncorrelated noise, equations 4.19 and 4.20,

\[
\begin{align*}
E((\hat{y}_n - y_n)^2) &= \sigma_e^2 \sum_{i=0}^{I} \overline{T}_{i}^2
\end{align*}
\]  

(4.27)

From equations 2.4, 2.7 and 2.8, the endpoint values of \( \overline{T}_{i}^2 \) are

\[
\overline{T}_{1}^2(1) = \overline{T}_{1}^2(N) = \frac{2I + 1}{N} \frac{(N-1)(N-2)\cdots(N-I)}{(N+1)(N+2)\cdots(N+I)}
\]  

(4.28)

For \( N \) large and \( n = 1 \) or \( N \)

\[
E((\hat{y}_n - y_n)^2) = \frac{1}{n} \sigma_e^2 (1 + 3 + 5 + 7 + \cdots + (2I + 1))
\]  

(4.29)

or

\[
E((\hat{y}_1 - y_1)^2) = E((\hat{y}_N - y_N)^2) = (I + 1)^2 \sigma_e^2 / N
\]  

(4.30)

where \( I \) is the order of the fit.

From equations 2.4, 2.9, and 2.10, for the midpoint value of \( n = (N + 1)/2 \)

\[
\overline{T}_{i}^2 \left( \frac{N+1}{2} \right) = 0 \quad \text{for } i \text{ odd}
\]  

(4.31)

\[
\overline{T}_{i}^2 \left( \frac{N+1}{2} \right) = \frac{1}{N} \quad \text{for } i = 0
\]  

(4.32)
And for $i$ even

$$
\bar{T}_1^2 \left( \frac{N + 1}{2} \right) = \frac{2i + 1}{N} \left( \frac{1 \times 3 \times 5 \times \cdots \times (1 - 1)}{2 \times 4 \times 6 \times \cdots \times 1} \right)^2 \\
\frac{(N^2 - 1)(N^2 - 3^2) \cdots (N^2 - (1 - 1)^2)}{(N^2 - 2^2)(N^2 - 4^2) \cdots (N^2 - 1^2)}
$$

(4.33)

For $N$ large and $n = (N + 1)/2$

$$
E((\hat{y}_n - y_n)^2) = \frac{1}{N^2} e^2 \left[ 1 + 0 + \frac{5}{4} + 0 + \frac{81}{64} + 0 + \frac{2925}{2304} + \cdots \right]
$$

$I = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6$

(4.34)

The residual is $r_n = y_n - \hat{y}_n$. Substituting equations 4.11 and 4.24 into this yields

$$
r_n = \epsilon_n - E_0\bar{T}_0 - E_1\bar{T}_1 - E_2\bar{T}_2 - \cdots - E_I\bar{T}_I
$$

(4.35)

The sum of the residuals squared is

$$
\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \epsilon_n^2 - 2 \sum_{n=1}^{N} (E_0\epsilon_n\bar{T}_0 \pm E_1\epsilon_n\bar{T}_1 \pm \cdots \pm E_I\epsilon_n\bar{T}_I)
$$

$$
+ \sum_{n=1}^{N} (E_0\bar{T}_0 \pm E_1\bar{T}_1 \pm E_2\bar{T}_2 \pm \cdots \pm E_I\bar{T}_I)^2
$$

22
which becomes

\[ \sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \varepsilon_n^2 - 2(E_0^2 + E_1^2 + \cdots + E_I^2) + E_0^2 + E_1^2 + \cdots + E_I^2 \]

or

\[ \sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \varepsilon_n^2 - E_0^2 - E_1^2 - E_2^2 - \cdots - E_I^2 \]  \hspace{1cm} (4.36)

Taking the expected value gives

\[ E \left[ \sum_{n=1}^{N} r_n^2 \right] = N \sigma^2 - E(E_0^2) - E(E_1^2) - \cdots - E(E_I^2) \]  \hspace{1cm} (4.37)

Assuming the noise is uncorrelated yields

\[ E \left[ \sum_{n=1}^{N} r_n^2 \right] = N \sigma^2 - (I + 1) \sigma^2 \]

or

\[ \sigma^2 = \frac{1}{N - I - 1} E \left[ \sum_{n=1}^{N} r_n^2 \right] \]  \hspace{1cm} (4.38)

From equation 4.24, it is seen that
\[
\frac{d\hat{y}_n}{dx} - \frac{dy_n}{dx} = \left( E_0 \frac{dT_0}{dn} + E_1 \frac{dT_1}{dn} + E_2 \frac{dT_2}{dn} + \cdots + E_I \frac{dT_I}{dn} \right) \frac{dn}{dx} \tag{4.39}
\]

where, from equation 4.1

\[
\frac{dn}{dx} = \frac{1}{\Delta x} \tag{4.40}
\]

Assuming uncorrelated noise

\[
E \left[ \left( \frac{d\hat{y}_n}{dx} - \frac{dy_n}{dx} \right)^2 \right] = \sigma_e^2 \left[ \frac{(dT_0)^2}{(dn)^2} + \frac{(dT_1)^2}{(dn)^2} + \cdots + \frac{(dT_I)^2}{(dn)^2} \right] \tag{4.41}
\]

where \( d\overline{T}_0/dn = 0 \). In general, for uncorrelated noise

\[
E \left[ \left( \frac{d^k\hat{y}_n}{dx^k} - \frac{d^k y_n}{dx^k} \right)^2 \right] = \sigma_e^2 \left[ \frac{(d^k \overline{T}_k)^2}{(dn^k)^2} + \frac{(d^k \overline{T}_{k+1})^2}{(dn^k)^2} + \cdots + \frac{(d^k \overline{T}_I)^2}{(dn^k)^2} \right] \tag{4.42}
\]

Figures 1 through 5 show the accuracies of \( \hat{y}_n \) and \( d\hat{y}_n/dx \) versus \( N \) for uncorrelated noise for \( n = 1, N \) and \((N + 1)/2\). Figures 6 and 7 show the accuracies of \( \hat{y}_n \) and \( d\hat{y}_n/dx \) versus \( n \) for uncorrelated noise for \( N = 99 \). The figures are graphically informative. Clearly the midpoint estimates of

\( \hat{y}_n \) and \( d\hat{y}_n/dx \) are much more accurate than the estimates at the end points. Also at the midpoint value of \( n = (N + 1)/2 \), it is seen that the accuracy of

\( \hat{y}_n \) is the same for \( I = 0 \) order fit and \( I = 1, I = 2 \) and \( I = 3, I = 4 \) and \( I = 5 \), etc. The reason for this is interesting. Let

\( \hat{y}_n = \text{estimate using Ith order fit} \)
Then for $n = (N + 1)/2$

\[ I^y(N+1)/2 = I+1^y(N+1)/2 \quad \text{for} \quad I = 0, 2, 4, \ldots \]  \hfill (4.43)

Likewise for the midpoint derivative $\hat{y}_n = dy_n/dx$

\[ I^y(N+1)/2 = I+1^y(N+1)/2 \quad \text{for} \quad I = 1, 3, 5, \ldots \]  \hfill (4.44)

The extension to higher-order derivatives is easily seen from the above.
Figure 3. Accuracy of $Y_n$ at midpoint of fit versus $N$.

\[
\frac{2}{(1+N)} = u \left[ \frac{2(u - u)}{2(u - u)} \right]^3
\]
Figure 5: Accuracy of $dy/dx$ at fit endpoint versus $N$.
Figure 6. - Accuracy of $\tilde{y}_n$ versus $n$ ($N = 99$ points).
Figure 7. - Accuracy of $\hat{y}_n'/x$ versus $n$ ($N = 99$ points).
Correlated noise vastly complicates the accuracy equations. In this section, exponentially correlated noise will be considered when using a second order fit with \( N = 101 \) points. A comparison of the fit accuracy will be made with the uncorrelated noise case. The autocorrelation function of exponentially correlated noise is given by

\[
\phi_k = \sigma_e^2 a^k = \mathbb{E}(\varepsilon_n \varepsilon_{n+k})
\]  

(5.1)

where

\[ 0 < a < 1 \]

Let

\[
k_{ij} = \sum_{n=1}^{N-k} (\overline{T_i(n)} \overline{T_j(n+k)} + \overline{T_j(n)} \overline{T_i(n+k)})
\]  

(5.2)

From equation 4.18

\[
\mathbb{E}(E_i E_j) = \sigma_e^2 \delta_{ij} + \sigma_e^2 \sum_{k=1}^{N-1} a^k k_{ij}
\]  

(5.3)

For a second-order polynomial fit, the following can be shown after some effort.

\[
k_{00} = \frac{2}{N}(N - k)
\]  

(5.4)

\[
k_{01} = k_{10} = 0
\]  

(5.5)

\[
k_{02} = k_{20} = -\frac{2 \sqrt{5}}{N \sqrt{(N^2 - 1)(N^2 - 4)}} \frac{(N - k)k}{(N - 2k)}
\]  

(5.6)
\[ k_{T11} = \frac{2}{N} \frac{N - k}{N^2 - 1} (N^2 - 1 - 2(N + k)k) \] (5.7)

\[ k_{T12} = k_{T21} = 0 \] (5.8)

\[ k_{T22} = \frac{2(N - k)}{N(N^2 - 1)(N^2 - 4)} ((N^2 - 1)(N^2 - 4) \]
\[- (4N^2 - 10)(N + k)k + 6(N + k)k^3) \] (5.9)

For \( a = 0 \) (uncorrelated noise)

\[ E(E_0^2) = \sigma_e^2 \] (5.10)

\[ E(E_1^2) = \sigma_e^2 \] (5.11)

\[ E(E_2^2) = \sigma_e^2 \] (5.12)

\[ E(E_0E_2) = E(E_2E_0) = 0 \] (5.13)

\[ E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \] (5.14)

The following results use \( N = 101 \).

For \( a = 0.4 \)

\[ E(E_0^2) = 2.3113 31133 \sigma_e^2 \] (5.15)

\[ E(E_1^2) = 2.2673 55494 \sigma_e^2 \] (5.16)

\[ E(E_2^2) = 2.2234 65949 \sigma_e^2 \] (5.17)

\[ E(E_0E_2) = E(E_2E_0) = -0.045873 81831 \sigma_e^2 \] (5.18)

\[ E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \] (5.19)
For \( a = 0.8 \)

\[
\begin{align*}
E(E_0^2) & = 8.6039 \ 60396 \ \sigma_e^2 & (5.20) \\
E(E_1^2) & = 7.8211 \ 99767 \ \sigma_e^2 & (5.21) \\
E(E_2^2) & = 7.0652 \ 98249 \ \sigma_e^2 & (5.22) \\
E(E_0E_2) = E(E_2E_0) & = -0.67000 \ 73131 \ \sigma_e^2 & (5.23) \\
E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) & = 0 & (5.24)
\end{align*}
\]

For \( a = 0.9 \)

\[
\begin{align*}
E(E_0^2) & = 17.217 \ 86439 \ \sigma_e^2 & (5.25) \\
E(E_1^2) & = 13.841 \ 98613 \ \sigma_e^2 & (5.26) \\
E(E_2^2) & = 10.933 \ 27966 \ \sigma_e^2 & (5.27) \\
E(E_0E_2) = E(E_2E_0) & = -2.1591 \ 29562 \ \sigma_e^2 & (5.28) \\
E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) & = 0 & (5.29)
\end{align*}
\]

For \( a = 0.99 \)

\[
\begin{align*}
E(E_0^2) & = 73.999 \ 66291 \ \sigma_e^2 & (5.30) \\
E(E_1^2) & = 13.714 \ 80539 \ \sigma_e^2 & (5.31) \\
E(E_2^2) & = 4.5863 \ 43888 \ \sigma_e^2 & (5.32) \\
E(E_0E_2) = E(E_2E_0) & = -4.6841 \ 13901 \ \sigma_e^2 & (5.33) \\
E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) & = 0 & (5.34)
\end{align*}
\]
For \( a = 0.999 \)

\[
E(E_0^2) = 97.68252696 \sigma^2_e \quad (5.35)
\]

\[
E(E_1^2) = 1.957674533 \sigma^2_e \quad (5.36)
\]

\[
E(E_2^2) = 0.4858440305 \sigma^2_e \quad (5.37)
\]

\[
E(E_0E_2) = E(E_2E_0) = -0.7231919335 \sigma^2_e \quad (5.38)
\]

\[
E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \quad (5.39)
\]

For \( a = 1 \)

\[
E(E_0^2) = 101 \sigma^2_e \quad (5.40)
\]

\[
E(E_1^2) = 0 \quad (5.41)
\]

\[
E(E_2^2) = 0 \quad (5.42)
\]

\[
E(E_0E_2) = E(E_2E_0) = 0 \quad (5.43)
\]

\[
E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \quad (5.44)
\]

From equation 4.26, for a second order fit, the accuracy of \( \hat{y}_n \) is given by

\[
E((\hat{y}_n - y_n)^2) = T_0^2 E(E_0^2) + T_1^2 E(E_1^2) + T_2^2 E(E_2^2) + 2T_0T_1 E(E_0E_1) + 2T_1T_2 E(E_1E_2) + 2T_0T_2 E(E_0E_2)
\]

But

\[
E(E_0E_1) = E(E_1E_2) = 0
\]
\[ E((\hat{y}_n - y_n)^2) = T_0^2E(E_0^2) + T_1^2E(E_1^2) + T_2^2E(E_2^2) \]
\[ + 2T_0T_2E(E_0E_2) \]  
(5.45)

where

\[ T_0 = \frac{1}{\sqrt{N}} \]  
(5.46)

\[ T_1 = \frac{\sqrt{3}}{\sqrt{N}} \frac{1}{\sqrt{N^2 - 1}}(N + 1 - 2n) \]  
(5.47)

\[ T_2 = \frac{\sqrt{5}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1)(N^2 - 4)}} ((N + 1)(N + 2) - 6(N + 1)n + 6n^2) \]  
(5.48)

Table 5-I shows the accuracy of \( \hat{y}_n \) at the end points of \( n = 1 \) and \( n = N \) for various values of \( a \). Note that \( N = 101 \) and a second-order fit is used. Table 5-II shows the accuracy of \( \hat{y}_n \) at the midpoint of \( n = (N + 1)/2 = 51 \) for various values of \( a \). Note that \( a = 1 \) is for a constant-bias error.
### TABLE 5-I. - ACCURACY OF \( \hat{y}_1 \) AND \( \hat{y}_{101} \) FOR A SECOND-ORDER FIT AND \( N = 101 \)

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \sqrt{E((\hat{y}<em>1 - y_1)^2)} = \sqrt{E((\hat{y}</em>{101} - y_{101})^2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.29270 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.4</td>
<td>0.43665 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.8</td>
<td>0.78341 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.9</td>
<td>0.99538 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.99</td>
<td>1.06989 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.999</td>
<td>1.00784 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>1 ( \sigma_\varepsilon )</td>
</tr>
</tbody>
</table>

### TABLE 5-II. - ACCURACY OF \( \hat{y}_{51} \) FOR A SECOND-ORDER FIT AND \( N = 101 \)

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \sqrt{E((\hat{y}<em>{51} - y</em>{51})^2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.14927 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.4</td>
<td>0.22677 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.8</td>
<td>0.43300 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.9</td>
<td>0.59467 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.99</td>
<td>0.94507 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>0.999</td>
<td>0.99458 ( \sigma_\varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>1 ( \sigma_\varepsilon )</td>
</tr>
</tbody>
</table>
The results shown in tables 5-I and 5-II indicate that increasingly correlated noise decreases the accuracy of the fit. This is not unexpected. However, the large amount of degradation is surprising. From table 5-I, it is seen that if \( a > 0.9 \), the fit error is greater than that using the raw data. This is so surprising that the equations leading to this result were double checked. No error was found. The calculator used (a Texas Instruments 59) had 13 decimal digits of internal accuracy, so it is believed that roundoff error was negligible.

Correlated error also affects the equation used to compute the standard deviation of the noise. From equation 4.37, for a second order fit,

\[
E \left[ \sum_{n=1}^{N} r_n^2 \right] = N\sigma_e^2 - E(E_0^2) - E(E_1^2) - E(E_2^2)
\]  

(5.49)

If \( a = 0 \), uncorrelated noise,

\[
E(E_0^2) = E(E_1^2) = E(E_2^2) = \sigma_e^2
\]

and

\[
\sigma_e^2 = E \left[ \sum_{n=1}^{N} r_n^2 \right] / (N - 3)
\]  

(5.50)

The equations for other values of \( a \), for \( N = 101 \), are shown in table 5-III. Note that for \( a = 1 \), theoretically

\[
\sum_{n=1}^{N} r_n^2 = 0,
\]

thus \( \sigma_e^2 = 0/0 \), as shown in the table.
### TABLE 5-III. - EQUATIONS FOR $\sigma^2$ FOR A SECOND-ORDER FIT AND $N = 101$

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<th>$\sigma^2$</th>
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<tr>
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<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 94.2$</td>
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<tr>
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<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 77.5$</td>
</tr>
<tr>
<td>.9</td>
<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 59.0$</td>
</tr>
<tr>
<td>.99</td>
<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 8.70$</td>
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<tr>
<td>.999</td>
<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 0.873$</td>
</tr>
<tr>
<td>1</td>
<td>$E \left[ \sum_{n=1}^{N} r_n^2 \right] / 0$</td>
</tr>
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</table>
6.0 NUMERICAL EXAMPLES

A numerical example with correlated noise will be considered first. A second-order fit will be made with \( N = 101 \) total fit points. The true value, \( y_n \), will be chosen to be zero.

\[
y_n = 0
\]  

(6.1)

The noisy measurements, \( y_n^* \), will be given by

\[
y_n^* = y_n + \varepsilon_n
\]  

(6.2)

Table 6-I shows \( n, y_n, y_n^* = \varepsilon_n, \hat{y}_n - y_n = y_n \) and the residual \( r_n = y_n - y_n^* \). The actual statistics of \( \varepsilon_n \) are (the bar signifies average value)

\[
\bar{\varepsilon}_n = 0.027
\]  

(6.3)

\[
\sqrt{\bar{\varepsilon}_n^2} = \sigma_\varepsilon = 0.99
\]  

(6.4)

\[
a = \frac{\bar{\varepsilon}_n\varepsilon_{n-1}}{\sigma_\varepsilon^2} = 0.89
\]  

(6.5)

The Fourier coefficients were

\[
A_0 = E_0 = 0.27273 \, 96938
\]

\[
A_1 = E_1 = -6.1441 \, 75478
\]

\[
A_2 = E_2 = -2.4988 \, 18869
\]

Higher-order fits were also made by the computer program and may be of interest.

\[
A_3 = E_3 = 4.3008 \, 65931
\]

\[
A_4 = E_4 = -0.47203 \, 26435
\]

\[
A_5 = E_5 = 1.7513 \, 69813
\]
\[ A_6 = E_6 = 1.19058688 \]
\[ A_7 = E_7 = -0.4954058757 \]
\[ A_8 = E_8 = -0.1822110594 \]

From \( A_0, A_1 \) and \( A_2 \) may be computed

\[ y_n = -1.615062680 + 0.0543334609n - 0.0003270951n^2 \] (6.6)

The residual statistics are

\[ \hat{a} = \frac{\hat{r}_n\hat{r}_{n-1}}{\hat{r}_n^2} = 0.77 \] (actual value 0.89) (6.7)

\[ \hat{\sigma}_\varepsilon = \sqrt{\frac{1}{98} \sum r_n^2} = 0.75 \] (for \( a = 0 \)) (6.8)

\[ \hat{\sigma}_\varepsilon = \sqrt{\frac{1}{78} \sum r_n^2} = 5.84 \] (for \( a = 0.77 \)) (6.9)

\[ \hat{\sigma}_\varepsilon = \sqrt{\frac{1}{58} \sum r_n^2} = 0.97 \] (for \( a = 0.9 \)) (6.10)

It is seen from table 6-I that the residuals are, in general, not a very accurate representation of \( \varepsilon_n \) when the noise is strongly correlated. However, \( \hat{\sigma}_\varepsilon = 0.97 \) (actual 0.99) is an excellent estimate when using \( a = 0.9 \). However, one has no way of knowing that \( a \) is this large and must use \( a = 0.77 \), which gives \( \hat{\sigma}_\varepsilon = 0.84 \), which is low but is still a "ball park" answer.
### TABLE 6-I. - SECOND-ORDER FIT RESULTS FOR $a = 0.9$

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<th>$\hat{y}_n = y_n + \epsilon_n$</th>
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TABLE 6-I.- Continued

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Table 6-II shows the results of a second-order fit for \( N = 101 \) when \( a = 0 \), uncorrelated noise. The actual statistics of \( \varepsilon_n \) were (the bar signifies average value)

\[
\bar{\varepsilon}_n = 0 \tag{6.11}
\]

\[
\sqrt{\langle \varepsilon_n^2 \rangle} = \sigma_\varepsilon = 1.01 \tag{6.12}
\]

\[
\Delta = \frac{\varepsilon_n \varepsilon_{n-1}}{\sigma_\varepsilon^2} = 0.02 \tag{6.13}
\]

The Fourier coefficients were

\[
\hat{A}_0 = E_0 = 0
\]

\[
\hat{A}_1 = E_1 = 0.7478929138
\]

\[
\hat{A}_2 = E_2 = 1.882400360
\]

\[
\hat{A}_3 = E_3 = 0.8417525686
\]

\[
\hat{A}_4 = E_4 = -0.6117760429
\]

\[
\hat{A}_5 = E_5 = 0.2278877925
\]

\[
\hat{A}_6 = E_6 = 0.3977181088
\]

\[
\hat{A}_7 = E_7 = 0.4524792538
\]

\[
\hat{A}_8 = E_8 = -0.1976251796
\]

Let the left subscript indicate the order of the fit. Then

\[
\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{N - I - 1} \sum I n^2} \quad \text{(for } a = 0) \tag{6.14}
\]
Based on the residuals from various order fits,

\[
\hat{\gamma}_c = 1.01 \quad \hat{\gamma}_a = -0.03
\]

\[
\hat{\gamma}_c = 1.00 \quad \hat{\gamma}_a = -0.06
\]

\[
\hat{\gamma}_c = 1.00 \quad \hat{\gamma}_a = -0.07
\]

\[
\hat{\gamma}_c = 1.01 \quad \hat{\gamma}_a = -0.08
\]

\[
\hat{\gamma}_c = 1.01 \quad \hat{\gamma}_a = -0.08
\]

\[
\hat{\gamma}_c = 1.01 \quad \hat{\gamma}_a = -0.08
\]

\[
\hat{\gamma}_c = 1.02 \quad \hat{\gamma}_a = -0.08
\]

\[
\hat{\gamma}_c = 1.02 \quad \hat{\gamma}_a = -0.08
\]

\[
\hat{\gamma}_c = 1.03 \quad \hat{\gamma}_a = -0.08
\]

There are very good estimates of the statistics of \( \varepsilon \). As can be seen from table 6-II, when \( a = 0 \), the values of the residuals are a better estimate of \( \varepsilon_n \) than when \( a = 0.9 \), as in table 6-I. Using figure 6 for a second-order fit, and using table 6-II, it is seen that

\[
\sqrt{E((\hat{y}_1 - y_1)^2)} = 0.295 \quad \text{(actual error} = -0.53)\]

\[
\sqrt{E((\hat{y}_{101} - y_{101})^2)} = 0.295 \quad \text{(actual error} = -0.28)\]

\[
\sqrt{E((\hat{y}_{51} - y_{51})^2)} = 0.15 \quad \text{(actual error} = 0.21)\]

\[
\sqrt{E((\hat{y}_{27} - y_{27})^2)} = 0.135 \quad \text{(actual error} = 0.01)\]

\[
\sqrt{E((\hat{y}_{74} - y_{74})^2)} = 0.135 \quad \text{(actual error} = 0.14)\]
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7.0 CONCLUSIONS

Discrete Tchebycheff orthonormal polynomials offer a convenient way to make least-squares polynomial fits of uniformly spaced discrete data. Computer programs to do so are simple and fast, and appear to be less affected by computer roundoff error, for the higher-order fits, than conventional least-squares programs.

When analyzing noisy data to obtain the error statistics, use the lowest-order fit that is "adequate." One way to ascertain this order is to examine $\hat{\sigma}_e$ to find the first two values of $I$ (the fit order) that give similar results. The lower value will be adequate. Probably the best way to obtain accurate residuals for generating error statistics is to use the midpoint residuals of a sliding polynomial fit of order 0, 2, 4, or 6, etc. Note that the results are identical to those with fits of 1, 3, 5, or 7, etc. For example, if a second-order midpoint fit is not adequate, then a fourth-order fit must be tried since a third-order fit will not alter the value of a midpoint residual.

For real-time polynomial smoothing (filtering), try to avoid using the endpoint ($n = N$) estimate. Estimates a short way from the end of the interval can double the accuracy, particularly if derivatives are needed. Again, the lowest-order adequate fit should be used.

In section 5.0, it was seen that exponentially correlated noise can greatly decrease the fit accuracy. In fact, a very surprising result was obtained. If the errors adding to the data are very strongly correlated, it is possible that the end point fit error standard deviation can be worse than that using the raw data.