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SUDDEN STRETCHING OF A
FOUR LAYERED-COMPOSITE PLATE

BY
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An approximate theory of laminated plates is developed by assuming that the extensional and thickness mode of vibration are coupled. The mixed boundary value crack problem of a four-layered composite plate is solved. Dynamic stress intensity factors for a crack subjected to suddenly applied stress are found to vary as a function of time and depend on the material properties of the laminate. Stress intensification in the region near the crack front can be reduced by having the shear modulus of the inner layers to be larger than that of the outer layers.
FOREWORD

The results in this report on the sudden stretching of a four-layered composite plate were obtained during the course of research supported by the NASA-Lewis Research Center in Cleveland, Ohio, for the period February 13, 1979 through February 12, 1980 under Grant NSG 3179 with the Institute of Fracture and Solid Mechanics at Lehigh University. The Principal Investigator of the Project is Professor George C. Sih who wishes to acknowledge Dr. Christos C. Chamis, the NASA Project Manager, for the encouragement he provided during this Project.
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LIST OF SYMBOLS

a
- half crack length

A, B, ..., D
- unknowns in integrals, functions of (s, p)

B_r
- Bromwich contour in complex p-plane

c_{21}
- shear wave speed for material 1

F, G
- known functions of (s, p)

h
- laminate thickness

H
- potential function

H^*
- Laplace transform of H

H(t)
- Heaviside unit step function

J_0
- Bessel function of order zero

k_1(t)
- dynamic stress intensity factor

k_1^*(p)
- Laplace transform of k_1(t)

L(ξ, n, p)
- kernel in Fredholm integral equation

N_0
- constant stress resultant

N_x, N_y, ..., N_{xy}
- stress resultants

p
- Laplace transform variable

R_x, R_y
- transverse shear forces

r, θ
- crack tip polar coordinates

s
- variable of integration

s_j
- parameter defined in equation (26) with j = 1, 2

t
- time

u_x, u_y, u_z
- displacement components in the (x, y, z) coordinate system

v_x, v_y, v_z
- displacement functions of x and y

x, y, z
- rectangular coordinates
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SUDDEN STRETCHING OF A FOUR-LAYERED COMPOSITE PLATE

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ABSTRACT

A research effort primarily concerned with the understanding of laminated composite plates with cracks subjected to time-dependent extensional loads is reported here. When loads are applied suddenly to a laminate, waves are reflected and refracted through the laminae and give rise to stresses and strains throughout the composite system. The process is three-dimensional in character and presents a formidable problem in the theory of elastodynamics, particularly in the presence of crack-like imperfections.

An approximate theory of laminated plates is developed by assuming that the extensional and thickness mode of vibration are coupled. The mixed boundary value crack problem of a four-layered composite plate is solved. Dynamic stress intensity factors for a crack subjected to suddenly applied stress are found to vary as a function of time and depend on the material properties of the laminate. Stress intensification in the region near the crack front can be reduced by having the shear modulus of the inner layers to be larger than that of the outer layers.
INTRODUCTION

The current interest in laminates for structural application is associated with the high strength-to-weight ratio which can be developed in laminates. These laminates are generally composed of layers which have been reinforced by embedding unidirectional fibers. The layers are adhered to each other such that the fiber direction varies from one layer to the next in a previously determined manner. The freedom of choice for fiber orientation in the layers of the composite system enables the development of laminates with special preferential directional properties for particular applications. Because of this characteristic of fibrous composites, the employment of these systems rather than equivalent homogeneous members will be clearly advantageous in many applications.

Because of the complicated internal structure of composite systems, stress analysis is much more difficult than for equivalent single-phase material. One fact which emerges very clearly from laminate studies is that the stress field in composite systems is truly three-dimensional in character. Thus, even the stress field in a symmetric laminate subjected to in-plane loading cannot be accurately modeled by standard two-dimensional methods of analysis. The previous work in this area further indicates that relatively little effort has been made to formulate laminate plate theories that can effectively solve for the redistribution of stresses and strains due to the presence of mechanical imperfections such as cracks.

One possible means of simplifying the three-dimensional equations of elasticity is to invoke the concept adopted in the formulation of plate theory. Approximate stress and strain dependence on the plate thickness coordinate are assumed such that the governing differential equations possess only two independent
space variables. In addition, special attention must be given to the state of affairs near the crack when formulating plate theories for analyzing crack problems. With this in mind, Hartranft and Sih [1] developed an approximate three-dimensional theory for a single material plate containing a through crack. The condition of plane strain was preserved ahead of the crack as suggested by Sih [2]. This theory was later extended to laminates by Badaliance, Sih and Chen [3] to solve the problem of a through crack in a laminar plate subjected to in-plane loading. The through crack configuration represents a preliminary effort to model the damage of composite plates. Additional complications arise when the load is time dependent. These considerations will be taken into account in the development of a new dynamic theory of laminated composite plates subjected to extensional loads.

This work is concerned with the formulation of a dynamic theory of laminated plates and reduces to that of Kane and Mindlin [4] for the single material plate. The idealized condition of stress and displacement continuity across the interface is replaced by assigning certain conditions of material nonhomogeneity in the thickness direction of the laminated plate as if it were a single layered nonhomogeneous plate. The nonhomogeneity is made equivalent to a symmetric laminate balanced with reference to its mid-plane. A through crack is assumed to exist in a four-layered laminate. Dynamic stress intensity factors are obtained for the case of a suddenly applied uniform in-plane loading and shown to vary as a function of time. Discussed are also the influence of material properties of the layers on the local stresses.
BASIC FORMULATION

The elastodynamic equations of generalized plane stress are adequate only if the frequency of vibration is lower than that of the first thickness mode and the wave length is large in comparison with the plate thickness. In other words, the coupling between extensional and thickness mode of vibration can be neglected. When laminated composite plates are stressed dynamically, loads are transmitted through the laminae by the reflection of thickness refraction of stress waves. The mode of vibration cannot be ignored, particularly in the vicinity of a crack-like imperfection where the stress state acquires a three-dimensional character.

A dynamic laminate plate theory will be developed to solve the problem of a four-layered composite plate with a through crack subjected to a suddenly applied uniform extensional load. The theory is a generalization to that given by Kane and Mindlin [4] for a single layer plate in which the extensional and thickness mode of vibration are assumed to be coupled. Accounted for is the lowest thickness-stretch mode such that the displacement is normal to the plate surface. Mindlin and Medick [5] have also considered a formulation in which the thickness-shear mode of vibration with displacement parallel to the plate surface is also included. The mid-plane of the plate is taken as the nodal plane of vibration.

Consider a four-layered composite plate of thickness h as shown in Figure 1 where each layer has the same thickness h/4. The two outer layers have material properties \((\mu_2, \nu_2)\) or \((\lambda_2, \mu_2)\) while the two inner layers have material properties \((\mu_1, \nu_1)\) or \((\lambda_1, \mu_1)\). The Lamé coefficients are denoted by \(\lambda_j\) and \(\mu_j\) \((j = 1, 2)\). The layers are stacked such that symmetry prevails across the mid-plane of the laminate composite. The crack of width 2a cuts through the entire thickness of the laminate.
Figure 1 - A four-layered composite plate with a crack
The time-dependent displacement field is assumed to be

\[ u_x = v_x(x, y, t) \]
\[ v_y = v_y(x, y, t) \]  \hspace{1cm} (1)
\[ w_z = \frac{2z}{h} v_z(x, y, t) \]

It follows that the strain components can be written as

\[ \varepsilon_x(x, y, t) = \frac{\partial v_x}{\partial x} \]
\[ \varepsilon_y(x, y, t) = \frac{\partial v_y}{\partial y} \]
\[ \varepsilon_z(x, y, t) = \frac{2z}{h} v_z \]  \hspace{1cm} (2)
\[ \gamma_{xy}(x, y, t) = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \]
\[ \gamma_{yz}(x, y, t) = \frac{2z}{h} \frac{\partial v_z}{\partial y} \]
\[ \gamma_{xz}(x, y, t) = \frac{2z}{h} \frac{\partial v_z}{\partial x} \]

in which the transverse normal and shear strains are assumed to be linear in the thickness coordinate \( z \). If each layer of the laminated composite plate is isotropic, then the following stress-strain relationships may be used:

\[ \sigma_x = (\lambda + 2\mu) \varepsilon_x + \lambda (\varepsilon_y + \kappa \varepsilon_z) \]
\[ \sigma_y = (\lambda + 2\mu) \varepsilon_y + \lambda (\varepsilon_x + \kappa \varepsilon_z) \]
\[ \sigma_z = (\lambda + 2\mu) \kappa^2 \varepsilon_z + \lambda \kappa (\varepsilon_x + \varepsilon_y) \]
\[ \tau_{yz} = \mu Y_{yz} \]
\[ \tau_{zx} = \mu Y_{zx} \]
\[ \tau_{xy} = \mu Y_{xy} \]

The constant
\[ \kappa = \pi / \sqrt{12} \]

accounts for the coupling between the extensional and thickness mode of vibration. It is determined from the three-dimensional equations of elasticity. As in the development of plate theories, the resultant strain quantities \((\Lambda_x)_j, (\Lambda_y)_j, \ldots, (\Lambda_{xy})_j \) (j = 1, 2) will be defined:

\[
[(\Lambda_x)_1, (\Lambda_y)_1, (\Lambda_z)_1, (\Lambda_{xy})_1] = \frac{2}{h} \int_{-h/4}^{h/4} \left[ e_{x}^2 e_y e_z + \gamma_{xy} \right] dz
\]

\[
[(\Lambda_x)_2, (\Lambda_y)_2, (\Lambda_z)_2, (\Lambda_{xy})_2] = \frac{2}{h} \left\{ \frac{h/2}{h/4} \left[ e_{x} e_y e_z + \gamma_{xy} \right] dz + \frac{h/2}{-h/2} \left[ e_{x}^2 e_y e_z + \gamma_{xy} \right] dz \right\}
\]

\[
[(\Lambda_{xz})_1, (\Lambda_{yz})_1] = \frac{96}{h^3} \int_{-h/4}^{h/4} [\gamma_{xz} + \gamma_{yz}] dz
\]

\[
[(\Lambda_{xz})_2, (\Lambda_{yz})_2] = \frac{96}{7h^3} \int_{-h/2}^{h/2} [\gamma_{xz} + \gamma_{yz}] dz + \frac{h/2}{h/4} [\gamma_{xz} + \gamma_{yz}] dz
\]

Substituting equations (2) into (5), it is found that

\[
(\Lambda_{x})_1 = (\Lambda_{x})_2 = \frac{\partial v_x}{\partial x}
\]
\[
\begin{align*}
(\lambda_y)_1 &= (\lambda_y)_2 = \frac{\partial v_x}{\partial y} \\
(\lambda_z)_1 &= (\lambda_z)_2 = \frac{2}{h} v_z \\
(\lambda_{xy})_1 &= (\lambda_{xy})_2 = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \\
(\lambda_{xz})_1 &= (\lambda_{xz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial x} \\
(\lambda_{yz})_1 &= (\lambda_{yz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial y}
\end{align*}
\]

The laminate plate theory can be most conveniently formulated in terms of the stress resultants

\[
(N_x, N_y, N_z, N_{xy}) = \frac{h}{2} \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}) dz
\]

and the transverse shears

\[
(R_x, R_y) = \frac{h}{2} \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz
\]

The stress-strain relations in equations (3) when enforced yield

\[
\begin{align*}
N_x(x,y,z) &= \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_x}{\partial x} + \beta \frac{\partial v_y}{\partial y}] + \beta \kappa v_z \\
N_y(x,y,z) &= \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_y}{\partial y} + \beta \frac{\partial v_x}{\partial x}] + \beta \kappa v_z \\
N_z(x,y,z) &= (\beta+2\gamma) \kappa^2 v_z + \frac{1}{2} \beta \kappa h \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \\
N_{xy}(x,y,z) &= \frac{1}{2} \gamma h \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)
\end{align*}
\]
and

\[ R_x(x,y,t) = \frac{1}{48} \delta h^2 \frac{\partial v_z}{\partial x} \]

\[ R_y(x,y,t) = \frac{1}{48} \delta h^2 \frac{\partial v_z}{\partial y} \]

In equations (9) and (10), \( \beta, \gamma \) and \( \delta \) stand for

\[ \beta = \lambda_1 + \lambda_2, \quad \gamma = \mu_1 + \mu_2, \quad \delta = \nu_1 + 7\nu_2 \]

Denoting \( \rho_1 \) and \( \rho_2 \) as the mass density of the inner and outer layers of the laminate, the three equations of motion governing \( N_x, N_y, \ldots, R_y \) become

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = \frac{1}{2} h(\rho_1 + \rho_2) \frac{\partial^2 v_x}{\partial t^2} \]

\[ \frac{\partial N_y}{\partial x} + \frac{\partial N_z}{\partial y} = \frac{1}{2} h(\rho_1 + \rho_2) \frac{\partial^2 v_y}{\partial t^2} \]

\[ \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} - N_z = \frac{1}{48} h^2(\rho_1 + 7\rho_2) \frac{\partial^2 v_z}{\partial t^2} \]

The result of substituting equations (9) and (10) into (12) renders

\[ \gamma \nu^2 v_x + (\beta + \gamma) \frac{\partial}{\partial x} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) + \frac{2\beta \nu}{h} \frac{\partial v_z}{\partial x} = (\rho_1 + \rho_2) \frac{\partial^2 v_x}{\partial t^2} \]

\[ \gamma \nu^2 v_y + (\beta + \gamma) \frac{\partial}{\partial y} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) + \frac{2\beta \nu}{h} \frac{\partial v_z}{\partial y} = (\rho_1 + \rho_2) \frac{\partial^2 v_y}{\partial t^2} \]

\[ \delta \nu^2 v_z - \frac{48}{h^2} (\beta + 2\gamma) \kappa^2 v_z - \frac{24\beta \nu}{h} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = (\rho_1 + 7\rho_2) \frac{\partial^2 v_z}{\partial t^2} \]
where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator in two dimensions.

**METHOD OF SOLUTION**

Equations (13) will be solved by introducing two potential functions $\phi(x,y,t)$ and $H(x,y,t)$ as

\[
\begin{align*}
\nu_x(x,y,t) &= \frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial y} \\
\nu_y(x,y,t) &= \frac{\partial \phi}{\partial y} - \frac{\partial H}{\partial x}
\end{align*}
\]

Making the appropriate algebraic manipulations, the governing equations for the potential functions can be derived by enforcing equations (13):

\[
\begin{align*}
\gamma \nabla^2 H &= (\rho_1 + \rho_2) \frac{\partial^2 H}{\partial t^2} - \delta(\beta + 2\gamma) \nu^4 \phi + \frac{192}{h^2} \frac{\nu^4}{h^2} (\beta + 2\gamma) \nu^2 \phi \\
&= ((\rho_1 + \rho_2)[(\rho_1 + 7\rho_2) \frac{\partial^4 \phi}{\partial t^4} + \frac{48}{h^2} (\beta + 2\gamma) \frac{\partial^2 \phi}{\partial t^2}]
- [\delta(\rho_1 + \rho_2) + (\beta + 2\gamma)(\rho_1 + 7\rho_2) \frac{\partial^2 \phi}{\partial t^2} (\nu^2 \phi)])
\end{align*}
\]

Once $\phi(x,y,t)$ and $H(x,y,t)$ are known, $\nu_x$ and $\nu_y$ can be obtained from equations (14) and

\[
\begin{align*}
\nu_z(x,y,t) &= \frac{h}{2\beta \kappa} [(\rho_1 + \rho_2) \frac{\partial^2 \phi}{\partial t^2} - (\beta + 2\kappa) \nu^2 \phi]
\end{align*}
\]

Suppose that a uniform stress resultant $N_0$ is applied suddenly to the crack surfaces and the resulting deformation is symmetrical with respect to the $x$-axis, then the following conditions are to be specified:
\[ N_y(x,0,t) = -N_0 H(t), \quad x < a \] (17)

\[ v_y(x,0,t) = 0, \quad x \geq a \]

where \( H(t) \) is the Heaviside unit step function. The condition of symmetry further requires that

\[ N_{xy}(x,0,t) = R_y(x,0,t) = 0, \quad \text{for all } x \] (18)

Use will now be made of the Laplace transform. Let \( \phi^*(x,y,p) \), \( H^*(x,y,p) \), etc., denote the Laplace transforms of the functions \( \phi(x,y,t) \), \( H(x,y,t) \), etc. Equation (15) when expressed in the Laplace transform domain become

\[ (v^2 - \omega_1^2) \phi_1^*(x,y,p) = 0 \]

\[ (v^2 - \omega_2^2) \phi_2^*(x,y,p) = 0 \] (19)

\[ (v^2 - \omega_3^2) H^*(x,y,p) = 0 \]

where the potential \( \phi(x,y,t) \) has been separated into two parts:

\[ \phi(x,y,t) = \phi_1(x,y,t) + \phi_2(x,y,t) \] (20)

in terms of time \( t \) or

\[ \phi^*(x,y,p) = \phi_1^*(x,y,p) + \phi_2^*(x,y,p) \] (21)

in terms of the Laplace transform variable \( p \). The parameters \( \omega_j \) \((j = 1,2,3)\) in equations (19) are defined as
\[ \omega_{1,2}^2 = \frac{6\pi^2}{n\delta_0} \left[(\alpha_0 + \delta_0)(\frac{\rho}{\omega})^2 + \rho_o \pm \{[(\alpha_0 + \delta_0)(\frac{\rho}{\omega})^2 + \rho_o]^2 \right. \\
\left. - 4\alpha_0\delta_0 \left(\frac{\rho}{\omega_o}\right)^2 \left(\frac{\rho_o}{\omega_o}\right)^1 \right] \]

\[ \omega_3^2 = (\rho_1 + \rho_2)p^2\gamma^{-1} \]

in which the newly defined quantities are

\[ \alpha_0 = \frac{\beta+2\gamma}{(\rho_1 + \rho_2)\gamma_0}, \quad \delta_0 = \frac{\delta}{(\rho_1 + \rho_2)\gamma_0} \]

\[ \rho_0 = \frac{4(\rho_1 + \rho_2)}{\rho_1 + \rho_2}, \quad \omega_0 = \frac{12\pi^2(\beta+2\gamma)}{h^2(\rho_1 + \rho_2)} \]

and \( \gamma_0 \) takes the form

\[ \gamma_0 = \frac{4\gamma(\beta+\gamma)}{(\rho_1 + \rho_2)(\beta+2\gamma)} \]

Equations (19) then give

\[ \Phi_1^*(x,y,p) = \frac{2}{\pi} \int_0^\infty A(s,p) \cos(sx) \exp(-s\gamma)ds \]

\[ \Phi_2^*(x,y,p) = \frac{2}{\pi} \int_0^\infty B(s,p) \cos(sx) \exp(-s\gamma)ds \]

\[ \Phi_3^*(x,y,p) = \frac{2}{\pi} \int_0^\infty C(s,p) \sin(sx) \exp(-s\gamma)ds \]

with \( s_j \) being given by

\[ s_j = \sqrt{s^2 + \omega_j^2}, \quad j = 1,2,3 \]
The dynamic problem has now been reduced to finding the three unknown functions $A(s,p)$, $B(s,p)$ and $C(s,p)$.

**DUAL COUPLED INTEGRAL EQUATIONS**

Before the boundary and symmetry conditions can be enforced, it is necessary to obtain $v_x^*(s,y,p)$, $v_y^*(x,y,p)$, etc., in terms of the unknowns in equations (25). With the help of equations (14) and (16), it can be shown that

$$v_x^*(x,y,p) = -\frac{2}{\pi} \int_0^\infty [sA(s,p) \exp(-s_1y) + sB(s,p) \exp(-s_2y)$$
$$+ sC(s,p) \exp(-s_3y)] \sin(sx)ds$$

$$v_y^*(x,y,p) = -\frac{2}{\pi} \int_0^\infty [s_1A(s,p) \exp(-s_1y) + s_2B(s,p) \exp(-s_2y)$$
$$+ sC(s,p) \exp(-s_3y)] \cos(sx)ds$$

$$v_z^*(x,y,p) = \frac{2}{\pi} \int_0^\infty [\Delta_1A(s,p) \exp(-s_1y) + \Delta_2B(s,p) \exp(-s_2y)] \cos(sx)ds$$

The quantities $\Delta_j$ ($j = 1, 2$) are given by

$$\Delta_j = \frac{h(\beta + 2\gamma)}{2\kappa} \left[ \frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_j^2 \right], \quad j = 1, 2$$

Similarly, the Laplace transform of $N_x^*(x,y,p)$, $N_y^*(x,y,p)$ become

$$N_x^*(x,y,p) = \frac{2}{\pi} \gamma h \int_0^\infty \left[ \frac{(\rho_1 + \rho_2)p^2}{2\gamma} - \omega_1^2 \right] A(s,p) \exp(-s_1y)$$
$$+ \frac{(\rho_1 + \rho_2)p^2}{2\gamma} - \omega_2^2 \right] B(s,p) \exp(-s_2y)$$
$$- ss_3C(s,p) \exp(-s_3y) \cos(sx)ds$$

-13-
\[ N_y^*(x,y,p) = \frac{2}{\pi} \gamma h \int_0^\infty \left[ s^2 + \frac{(\rho_1 + \rho_2)p^2}{2\gamma} \right] \left[ A(s,p) \exp(-s_1 y) + B(s,p) \exp(-s_2 y) \right] \]

\[ + \frac{ss_3 C(s,p)}{\pi} \exp(-s_3 y) \cos(sx)ds \]

\[ N_{xy}^*(x,y,p) = \frac{2}{\pi} \gamma h \int_0^\infty \left( s_1 A(s,p) \exp(-s_1 y) + s_2 B(s,p) \exp(-s_2 y) \right) \]

\[ + \frac{1}{2} \left( s^2 + s_3^2 \right) C(s,p) \exp(-s_3 y) \sin(sx)ds \]

while \( R_x^*(x,y,p) \) and \( R_y^*(x,y,p) \) take the forms

\[ R_x^*(x,y,p) = -\frac{\Delta^2}{2\pi} \int_0^\infty \left[ s_1 \Delta A(s,p) \exp(-s_1 y) \right. \]

\[ + \left. s_2 \Delta B(s,p) \exp(-s_2 y) \right] \sin(sx)ds \]

\[ R_y^*(x,y,p) = -\frac{\Delta^2}{2\pi} \int_0^\infty \left[ s_1 \Delta A(s,p) \exp(-s_1 y) \right. \]

\[ + \left. s_2 \Delta B(s,p) \exp(-s_2 y) \right] \cos(sx)ds \]

The symmetry conditions in equations (18) when applied show that \( A(s,p), B(s,p) \)

and \( C(s,p) \) can be expressed in terms of a single unknown \( D(s,p): \)

\[ A(s,p) = \frac{s^2 + s_3^2}{s_1} D(s,p) \]

\[ B(s,p) = -s_1 \left[ \frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_1^2 \right] \left( \frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_2^2 \right) A(s,p) \]

\[ C(s,p) = \frac{-2ss_1(\omega_1^2 - \omega_2^2)}{(s^2 + s_3^2)[s_1(\frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_2^2)]} A(s,p) \]

Application of the mixed boundary conditions in equations (17) leads to a system of dual integral equations
\[
\int_0^\infty D(s,p) \cos(sx)ds = 0, \ x > a \tag{32}
\]

\[
\int_0^a sF(s,p) \ D(s,p) \cos(sx)ds = -\frac{\pi N_0}{2\gamma hp}, \ x < a
\]

The function \( F(s,p) \) is known:

\[
F(s,p) = \frac{s^2 + s^2_3}{ss_1} \left[ \frac{s^2 + (p_1 + p_2)p^2}{2\gamma} \right] \left( 1 - \frac{s_1[p_1 + p_2 p^2]}{s_2[\frac{p_1 + p_2}{\beta + 2\gamma} - \omega^2]} \right) - \frac{2s^2s_1s_3}{s^2 + s^2_3} \left[ \frac{\omega_1^2 - \omega_2^2}{(p_1 + p_2)p^2/(\beta + 2\gamma) - \omega_2^2} \right] \tag{33}
\]

The standard procedure by Copson [6] may be applied to solve equations (32) and the result is

\[
D(s,p) = -\frac{\pi N_0 a^2}{2\gamma hp} \left\{ \frac{(p_1 + p_2)p^2/(\beta + 2\gamma) - \omega_2^2}{(p_1 + p_2)p^2} \left( 1 - \frac{\gamma}{\beta + 2\gamma} (\omega_1^2 - \omega_2^2) \right) \right\} \\
\times \int_0^1 \sqrt{\xi} \ \phi^*(\xi, p) \ J_0(sa\xi) d\xi \tag{34}
\]

in which \( \phi^*(\xi, p) \) can be computed from a Fredholm integral equation of the second kind:

\[
\phi^*(\xi, p) + \int_0^1 \phi^*(n, p) \ L(\xi, n, p)dn = \sqrt{\xi} \tag{35}
\]

The kernel \( L(\xi, n, p) \) is

\[
L(\xi, n, p) = \sqrt{\xi n} \int_0^\infty s[G(s, p) - 1] J_0(s\xi) J_0(sn)ds \tag{36}
\]
while the function $G(s, p)$ is related to $F(s, p)$ in equation (33):

$$G(s, p) = \frac{\{[(\rho_1 + \rho_2)p^2/[(\beta+2\gamma)] - \omega_2\}^\gamma}{p^2[1 - \gamma/[(\beta+2\gamma)](\omega_1^2 - \omega_2^2)]} F(s, p)$$  \hspace{1cm} (37)

**DYNAMIC STRESS INTENSITY FACTORS**

Of interest is the intensification of the dynamic stresses ahead of the crack. Hence, the integrals in equations (29) and (30) must be evaluated for large values of $s$ which corresponds to distances near the crack edge $x = \pm a$ and $y = 0$. In terms of the polar coordinates $r$ and $\theta$ in Figure 1, the Laplace transform of the stress resultants for small $r$ are found:

$$N_x^*(r, \theta, p) = \frac{k_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) + O(r^0)$$

$$N_y^*(r, \theta, p) = \frac{k_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) + O(r^0)$$

$$N_{xy}^*(r, \theta, p) = \frac{k_1^*(p)}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0)$$

$$R_x^*(r, \theta, p) = R_y^*(r, \theta, p) = O(r^0)$$

in which $k_1^*(p)$ is the Laplace transform of $k_1(t)$:

$$k^*(p) = \frac{\delta^*(1+p)}{p} N_0 \sqrt{a}$$  \hspace{1cm} (39)

The Laplace inversion theorem may now be applied to give
\[ N_x(r, \theta, t) = \frac{k_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + O(r^0) \]

\[ N_y(r, \theta, t) = \frac{k_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + O(r^0) \] (40)

\[ N_{xy}(r, \theta, t) = \frac{k_1(t)}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \]

\[ R_x(r, \theta, t) = R_y(r, \theta, t) = O(r^0) \]

Equations (40) reveal that dynamic loading does not affect the functional relationship of \( r \) and \( \theta \). The stress intensity factor, however, is a function of time:

\[ k_1(t) = \frac{N_0 \sqrt{a}}{2\pi} \int_{Br} \frac{\phi^*(1,p)}{p} \exp(pt)dp \] (41)

where \( Br \) denotes the Bromwich path of integration. Once \( \phi^*(\xi, p) \) is calculated from equation (35) and evaluated at \( \xi = 1 \), equation (41) may be solved numerically.

Figure 2 gives a plot of \( \phi^*(1,p) \) as a function of \( c_{21}/pa \) where \( c_{21} = (\nu_1/\rho_1)^{1/2} \) is the shear wave velocity referred to the material in the inner layers. For \( \rho_1 = \rho_2, \nu_1 = \nu_2 = 0.3 \) and \( \rho_1 = \rho_2, \phi^*(1,p) \) is seen to increase monotonically with \( c_{21}/pa \). Three different ratios of \( \nu_2/\nu_1 \) = 0.2, 1.0 and 5.0 are considered. Making use of the results in Figure 2, \( k_1(t) \) in equation (41) may be computed. Refer to Figure 3 for a display of \( k_1(t)/N_0 \sqrt{a} \) versus \( c_{21} t/a \). The resultant stress intensity factors are observed to vary as a function of time. Their amplitude rise quickly reaching a peak and then declines. The solution for a homogeneous plate corresponds to \( \nu_2/\nu_1 = 1.0 \) as the Poisson's ratio and mass density.
Figure 2 - Numerical values of $\phi^*(l,p)$ as a function of $c_{21}/pa$ for $a/h = 1.0$
Figure 3 - Normalized resultant stress intensity factor versus $c_{21}t/a$ for $a/h = 1.0$

$\nu_1 = \nu_2 = 0.3$

$\rho_1 = \rho_2$

$\mu_2/\mu_1 = 5.0$
for the inner and outer layers are assumed to be equal. The peak value of \( k_1(t) \) is greater than that of the homogeneous plate solution for \( \mu_2 > \mu_1 \) while the opposite is found for \( \mu_2 < \mu_1 \). Hence, the intensity of the crack border stress field can be reduced by having the shear modulus of the outer layers to be smaller than that of the inner layers.

**CONCLUDING REMARKS**

A dynamic laminate plate theory has been developed for solving crack boundary value problems. The complexity of the problem owing to material nonhomogeneity and dynamic stress analysis necessitates certain simplifying assumptions so that effective analytical solutions can be obtained. It is shown that the dynamic stresses near a mechanical imperfection such as a crack are intensified depending on the stacking sequence of the laminae. In general, this intensity tends to increase quickly for small time reaching a peak and then decreases to the static solution for sufficiently long time. When the modulus of the outer layers are smaller than that of the inner layers, the crack border stress intensity reaches a maximum quicker than the homogeneous solution but with a smaller magnitude. The opposite holds for the case when the outer layers are stiffer than the inner layers. Information of this type is useful for evaluating the resistance of laminate plates to impact loadings.
REFERENCES


COMPUTER PROGRAM: DYNAMIC LAMINATE PLATE THEORY WITH A CRACK

1
PROGRAM FLAP(INPUT, OUTPUT)
REAL NON(4), F(4, 4, 2), G(4, 4), D(4), PT(4)
REAL B(4, C(4)
REAL LP(I9), DTA(I9)
EQUIVALENCE (NON, R)
COMMON K1, K2, K3, K4
COMMON/AUX/H, P, PK1, PK2, BMU, X, Y
LP(I) = 0.0
DTA(I) = 0.0
10 READ 2 * K1, K2, K3, K4
2 FORMAT(I2)

* K1 = ORDER OF SYSTEM OF EQUATIONS
* K2 = NO. OF DISTINCT KERNELS
* K3 = NO. OF DATA POINTS
* K4 = NO. OF DATA SETS TO BE EVALUATED

5 SET UP DATA POINTS
AK = K3
DO 5 N = 1, K3
AN = N
5 PT(N) = AN/AK

* SET UP INTEGRATION MATRIX
M = K3 - 2
N = K3 - 1
A = K3
D = 1/(3 * A)
DO 10 K = 2, M + 2
10 O(K) = 2 * A
D0 15 K = I, N + 2
15 O(K) = 4 * A
35 P(K) = A
30

* CALCULATE NONHOMOGENEOUS TERMS
PHS = 1.0
DO 22 I = 1, K2
PRINT 9
9 FORMAT(1H1)
DO 22 I = 1, K4
DO 35 N = 1, K3
35 NON(N) = RMS * SORT(PT(N))
CALL CONST(I)
40

* CALCULATE KERNEL MATRICFS
DO 20 N = 1, K3
DO 20 M = 1, K3
F(M, N, I) = F(U(I, PT(M), PT(N))
20 CONTINUE

CALL CHANGE(F, G, D, 1)
CALL LINEO(G, R, C, K3)
DO 40 L = 1, K3
PRINT A * PT(L), NON(L)
6 FORMAT(5X, F8.4, F15.6)
40 CONTINUE

LP(I + 1) = NON(K3)
DTA(I + 1) = P
999 CONTINUE
CALL LAPINV(DTA, LP)
55 CONTINUE
FND
FUNCTION SIMP(I,A,P)
COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
MXYZ=2**15
DEL=0.25*(B-A)
IF (DEL) 40,45,50
5  SIMP=0.0
  RETURN
45  CONTINUE
  SA=Z(I,A)*Z(I,B)
  SB=Z(I,A+2,DEL)
  SC=Z(I,A+DEL)*Z(I,A+3,DEL)
  S1=(DEL/3.)*(SA+2,*SR+4,*SC)
  IF (S1.EQ.0.0) GO TO 45
  K=8
10  35  SB=SB+SC
    DEL=0.5+DEL
    SC=Z(I,A+DEL)
    J=K-1
20  5  DO 5 N=3,J,2
    AN=N
25  5  SC=SC*Z(I,A+AN*DEL)
    S2=(DEL/3.)*(SA+2,*SR+4,*SC)
    DIF=ABS((S2-S1)/S1)
    ER=0.01
30  IF (DIF-ER) 30,25,25
    SIMP=S2
    RETURN
25  K=2*K
30  S1=S2
35  IF (K-MXYZ) 35,35,40
40  PRINT 42,1,AB
    42  FORMAT(SX9.0,INT*,DONES NOT CONVERGE *,I3,2F9.4)
    PRINT 60,X,Y
60  FORMAT(2F10.5)
35  DO 70 J=1,10
    DIP=J
    DIP=DIP/10.
    W=Z(I,DIP)
70  PRINT 60,W
40  CONTINUE
70  CALL EXIT
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
  4  SIMP

VARIABLES      SN   TYPE     RELOCATION
  0  A   REAL     F.P.     260  AN   REAL
  0  B   REAL     F.P.     4   BMU REAL
  250  DEL  REAL     262  DIF REAL
  264  DIP  REAL     263  ER  REAL

-23-
SUBROUTINE CHANGE(F,G,D,I)
COMMON K1,K2,K3,K4
REAL F(4,4),G(4,4),D(4)
DO 10 N=1*K3
DO 10 M=1*K3
G(M,N) = F(M,N)*I*D(N)
CONTINUE
DO 20 N=1*K3
G(N,N)=G(N,N)+1.0
10 CONTINUE
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 CHANGE

VARIABLES SN TYPE RELLOCATION
0 D REAL ARRAY F.P. 0 F REAL
0 G REAL ARRAY F.P. 0 I INTEGER
0 K1 INTEGER / / 1 K2 INTEGER
2 K3 INTEGER / / 3 K4 INTEGER
53 M INTEGER 52 N INTEGER

STATEMENT LABELS
0 10 0 20

LOOPS LABEL INDEX FROM-TO LENGTH PROPERTIES
17 10 N 4 7 17R NOT INNER
30 10 M 5 7 1R INSTACK
43 20 N 8 9 48 INSTACK

COMMON BLOCKS LENGTH
/ / 4

STATISTICS
PROGRAM LENGTH 658 53
SCM BLANK COMMON LENGTH 48 4
47000R SCM USED
SURROUNTE LINEQ(A,B,T,N)
REAL A(N,N),R(N),T(N)
DO 5 I=2,N
5 A(I,1)=A(I,1)/A(1,1)
DO 10 K=2,N
M=K-1
DO 15 I=1,N
15 T(I)=A(I,K)
DO 20 J=1,M
A(J,K)=T(J)
J1=J+1
DO 20 I=J1,N
T(I)=T(I)-A(I,J)*T(J)
20 CONTINUE
A(K,K)=T(K)
IF(K.EQ.N) GO TO 10
M=K+1
DO 25 I=M,N
A(I,K)=T(I)/A(K,K)
20 CONTINUE
BACK SUBSTITUTE
DO 31 I=1,N
T(I)=B(I)
M=I+1
IF(M.GT.N) GO TO 31
DO 30 J=M,N
B(J)=B(J)-A(J,I)*T(I)
30 CONTINUE
31 CONTINUE
30 CONTINUE
K=N+1-I
P(K)=T(K)/A(K,K)
K1=K-1
IF(K1.EQ.0) GO TO 75
35 CONTINUE
36 CONTINUE
35 CONTINUE
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LINEQ

VARIABLES    SN TYPE       RELOCATION
0   A       REAL     ARRAY     F,P.     0   R       REAL
172  I       INTEGER  175  J       INTEGER
176  J1      INTEGER  173  K       INTEGER
177  K1      INTEGER  174  M       INTEGER
0   N       INTEGER  0   T       REAL

-25-
FUNCTION FU(I,A,B)
COMMON/AUX/H,P,PK1,PK2,AMUX,Y
X=A
Y=R
5 IF(A*B)5 10.5
10 FU=0.0
RETURN
5 SUM=SIMP(I,0,0,5,0)
ER=0.01
10 DEL=5.0
20 UP=DEL+5.0
ADDL=SIMP(I,DEL,UP)
DEL=UP
TEST=ABS(ADDL/SUM)
15 SUM=SUM+ADDL
IF(TEST-ER)15 20,20
15 FU=SORT(X*Y)*SUM
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 FU

VARIABLES SN TYPE RELOCATION
0 A REAL F.P. 62 ADDL REAL
0 B REAL F.P. 4 BMU REAL
60 DEL REAL 57 ER REAL
55 FU REAL 0 H REAL
0 I INTEGER F.P. 1 P REAL
2 PK1 REAL AUX 3 PK2 REAL
56 SUM REAL 63 TEST REAL
61 UP REAL 5 X REAL
6 Y REAL AUX

EXTERNALS TYPE ARGS
SIMP REAL 3 SORT REAL

INLINED FUNCTIONS TYPE ARGS
ABS REAL 1 INTRIN

STATEMENT LABELS
14 5 0 10 INACTIVE
22 20

COMMON BLOCKS LENGTH
AUX 7

STATISTICS
PROGRAM LENGTH 64B 52
SCM LABELED COMMON LENGTH 7R 7
47000B SCM USED
FUNCTION BESJO(A)

IF(A=3.)5*5+10

5  B=A*A/9.
    W=1.-2.2499997*B
    Z=B*B
    W=W+1.2656208*Z
    Z=Z*B
    W=W-.3163866*Z
    Z=Z*B
    W=W+.044479*Z
    Z=Z*B
    W=W-.0039444*Z
    Z=Z*B
    BESJO=W*.00021*Z
    RETURN

10  B=3./A
    W=.79788456-.00000077*R
    V=A-.78539816-.04163978*B
    Z=Z*B
    W=W-.0055274*Z
    V=V-.00003954*Z
    Z=Z*B
    W=W-.00009512*Z
    V=V+.00262573*Z
    Z=Z*B
    W=W+.00137237*Z
    V=V-.00054125*Z
    Z=Z*B
    W=W-.00072805*Z
    V=V+.00293333*Z
    Z=Z*B
    W=W+.00014476*Z
    V=V-.00052743*Z
    Z=Z*B
    BESJO=W/SQRT(A)*COS(V)

35  RETURN

END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4  BESJO

VARIABLES  SN  TYPE  RELOCATION
.  A  REAL  F.P.  114  B  REAL
113  BESJO  REAL
115  W  REAL

EXTERNALS  TYPE  ARGS
COS  REAL  1 LIBRARY  SORT  REAL

STATEMENT LABELS
0  5  INACTIVE  26  10
SUBROUTINE CONST(I)
COMMON/AUX/H,P,PK1,PK2,RMU,X,Y
PK1=0.3
PK2=0.3
BMU=50.0
H=1.0
READ 2,P
2 FORMAT(F10.5)
HH=1./H
PRINT 1,BMU,PK1,PK2,HH,P
1 FORMAT(////SMU?/SU1 =*F6.2*,SU1 =*F4.2*,SU1 =
A/H =*F4.2*,C21/P1 =*F4.2//)
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 Const

<table>
<thead>
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<th>VARIABLES</th>
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<th>TYPE</th>
<th>RELOCATION</th>
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</tr>
<tr>
<td>Y</td>
<td>6</td>
<td>REAL</td>
<td>AUX</td>
</tr>
</tbody>
</table>

FILE NAMES MODE
INPUT FMT OUTPUT FMT

STATEMENT LABELS
37 1 FMT 25 2 FMT

COMMON BLOCKS LENGTH
AUX 7

STATISTICS
PROGRAM LENGTH 56B 46
SCM LABELED COMMON LENGTH 7R 7
47000B SCM USED
FUNCTION Z(I,S)
COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
COMPLEX DA,DL1,DL2,SA,SB,SC,SD
COMPLEX GA,GB,CA,CR,CC,F,G
PI=3.1415926
PP=P*P
PG=2./PP/(1.+BMU)
AA=2.*((1.-PK1)/(1.-2.*PK1))
AB=2.*((1.-PK2)/(1.-2.*PK2))
PA=2./PP/(AA+BMU*AA)
PO=2.*H/H/PI/PI/(AA+BMU*AB)/PP
RA=(1.+BMU)/(AA+BMU*AB)
BB=1.-BA
BC=(1.+7.*BMU)/4./(1.+BMU)
BD=PI*PI/2.*H/H
ALP=1.+4./RA/BB
DLP=BC/4.*PB
DD=((ALP+DLP)*PO+1.)*2.-4.*ALP*DLP*PO*(PO+1.)
G=CMPLX(DD,0.0)
DA=CSORT(G)
DL1=BD/DLP*((ALP+DLP)*PO+1.+DA)
DL2=BD/DLP*((ALP+DLP)*PO+1.-DA)
SC=S*S+DL1
SD=S*S+DL2
GA=CSORT(SC)
GB=CSORT(SD)
GC=CSORT(S*S+PG)
SA=(PA-DL2)/(DL1-DL2)
SB=(PA-DL1)/(PA-DL2)
CA=SA/PG/BB
CB=2.*((S*S+PG/2.)*2./GA*(1.-GA/GB*S9)
CC=2.*S*S*GC/SA
F=CA*(CB-CC)
O=REAL(F)
OA=AIMAG(F)
IF(OA<0.0)5,10.5
RETURN
9 PRINT 9,P,S,F
40 FORMAT(4F10.5)
CALL EXIT
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 Z

VARIABLES SN TYPE RELOCATION
276 AA REAL 277 AB REAL
304 ALP REAL 302 BA REAL
303 BB REAL 304 BC REAL
305 BD REAL 4 BMU REAL
SUBROUTINE LAPINV(GLAM, PHI)
C THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C INVERSION INTEGRAL

REAL MUL
DIMENSION A(50), GLAM(50), PHI(50), C(4, 50)
DIMENSION BK(101), TT(101)
COMMON/TI, TF, DT, MN, BK, TT
READ 1*NN, MN, MM
1 FORMAT(3I2)
READ 2*TI, TF, DT
2 FORMAT(3F10.5)
PRINT 99
PRINT 101
CALL SPLICE(GLAM, PHI, MM, C)
PRINT 102
101 FORMAT(///5X, GLAM, PHI *)
PRINT 102*(GLAM(I), PHI(I), I=1, MM)
102 FORMAT(5X, F10.5, 5X, F10.5)
M1 = MM-1
PRINT 99
DO 10 I=1, NN
READ 3*BET, DEL
3 FORMAT(2F10.5)
PRINT 98
PRINT 98 FORMAT(////5X, DETA = F5.3, DELTA = F5.3)
DO 11 L=1, MN
AL = L
S = 1./(AL*BET)/DEL
CALL SPLINE(GLAM, PHI, MM, C, S, G)
F = G*S
IF(AL = 2.) 81, 82, 83
81 A(1) = (1.*BET)*DEL*F
GO TO 11
82 A(2) = (((2.*BET)*DEL*F-A(1))*3.*BET)
GO TO 11
83 CONTINUE
TOP = 1.
L1 = L - 1
AL1 = L1
DO 12 J=1, L1
AJ = J
TOP = AJ*TOP
12 CONTINUE
L2 = 2*L - 1
BOT = 1.
DO 13 J=L, L2
AJ = J
BOT = (AJ*DEL)*BOT
13 CONTINUE
MUL = BOT/TOP
SUM = 0.0
DO 14 N=1, L1
AN = N
14 IF(AN = 2.) 85, 86, 87
85 TOD = 1.
GO TO AR
86 TOD=AL1
GO TO 88
60
87 CONTINUE
TOD=1.
ICH=L1-(N-2)
DO 15 J=ICH+L1
AJ=J
65
TOD=AJ*TOD
15 CONTINUE
88 CONTINUE
BOD=1.
JA=L1+N
DO 16 J=L1*JA
AJ=J
BOD=BOD*(AJ*BET)
16 CONTINUE
CO=TOD/BOD
SUM=SUM*CO*A(N)
75
14 CONTINUE
A(L)=MUL*(DEL*F-SUM)
11 CONTINUE
CALL JACSER(DEL,A,FET)
80
10 CONTINUE
999 CONTINUE
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LAPINV

VARIABLES SN TYPE PELOCATION
377 A REAL ARRAY 364 AJ REAL
354 AL REAL 362 AL1 REAL
371 AN REAL 351 BET REAL
4 BK REAL ARRAY 2 374 BOD REAL
366 B0T REAL 461 C REAL
376 CO REAL 352 DEL REAL
2 DT REAL 2 357 F REAL
354 G REAL 0 GLAM REAL
347 I INTEGER 373 ICH INTEGER
363 J INTEGER 375 JA INTEGER
353 L INTEGER 361 L1 INTEGER
365 L2 INTEGER 346 MM INTEGER
3 MN INTEGER 2 344 MUL REAL
350 M11 INTEGER 370 N INTEGER
348 NN INTEGER 0 PHI REAL
355 S REAL 367 SUM REAL
1 TF REAL 2 0 TI REAL
372 TOD REAL 360 TOP REAL
151 TT REAL ARRAY 2
SUBROUTINE JACSER(n,C,P)
DIMENSION C(50),SF(50),P(50)
DIMENSION BK(101),TT(101)
COMMON/TI,TF,DT,MN,BK,TT
TT(1)=0.0
BK(1)=0.0
LM=1
T=TI
10 T=T+DT
X=2.*EXP(-DT)-1.
CALL JACOBI(MN*X,P)
SF(1)=C(1)*P(1)
DO 10 L=2,MN
L1=L-1
AL=L
SF(L)=SF(L1)+C(L)*D(L)
10 CONTINUE
LM=LM+1
BK(LM)=SF(1)
15 TT(LM)=T
IF(T.LE.TF) GO TO 12
PRINT 97
97 FORMAT(////5X*T,K*T,K)
12 CONTINUE
20 RETURN
35 END

SYMBOLIC REFERENCE MAP (P=1)

ENTRY POINTS
3  JACSER

VARIABLES  SN  TYPE  RELOCATION
151  AL  REAL  0  B  REAL
4  BK  REAL  ARRAY  2  C  REAL
0  D  REAL  F*P.  2  DT  REAL
147  L  INTEGER  144  LM  INTEGER
150  L1  INTEGER  153  MA  INTEGER
154  MB  INTEGER  155  MC  INTEGER
156  MD  INTEGER  3  MN  INTEGER
152  MY  INTEGER  241  P  REAL
157  SF  REAL  ARRAY  2  T  REAL
1  TF  REAL  0  TI  REAL
151  TT  REAL  ARRAY  2  146  X  REAL
SUBROUTINE JACOBI(N,X,B,PB)

C THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER K=1 WITH ARG X AND PARAMETER B GT -1

DIMENSION PB(N)

AN=N

IF(AN-2.)1,2,3

1 PB(1)=1.

RETURN

2 PB(1)=1.

PB(2)=X-B*(1.-X)/2.

RETURN

3 BSO=B*B

BONE=B+1.

PB(1)=1.

PB(2)=X-B*(1.-X)/2.

DO 4 K=3,N

AK=K

AK1=AK-1.

AK2=AK-2.

K1=K-1

K2=K-2

C01=((2.*AK1)+8)*X

C01=((2.*AK2)+B)*C01

C01=((2.*AK2)+BONE)*0(C01-BSO)

C02=2.*AK2*(AK2+B)+((2.*AK1)+B)

C0=2.*AK1*(AK1+B)*((2.*AK2)+B)

4 PB(K)=(C01*PB(K1)-C02*PB(K2))/C0

RETURN

END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS

7 JACOBI

VARIABLES SN TYPE RELOCATION

104 AK REAL 107 AK1 REAL

110 AK2 REAL 102 AN REAL

0 B REAL F.P. 104 BONE REAL

103 BSO REAL 115 CO REAL

113 CO1 REAL 114 CO2 REAL

105 K INTEGER 111 K1 INTEGER

112 K2 INTEGER 0 N INTEGER

0 PB REAL ARRAY F.P. 0 X REAL

STATEMENT LABELS

0 1 INACTIVE 24 2

0 4

LOOPS LABEL INDEX FROM TO LENGTH PROPERTIES

47 4 K 16 27 2=B OPT

-33-
SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
DIMENSION X(50),Y(50),C(4,50)
IF(XINT-X(1))1,10,11
10 YINT=Y(1)
   RETURN
11 CONTINUE
   IF(X(M)=XINT)1,12,13
12 YINT=Y(M)
   RETURN
13 CONTINUE
   K=M/2
   N=M
2 CONTINUE
   IF(X(K)=XINT)3,14,=
14 YINT=Y(K)
   RETURN
3 CONTINUE
   IF(XINT-X(K+1))4,15,7
15 YINT=Y(K+1)
   RETURN
4 CONTINUE
   YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
   YINT=YINT*(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
   RETURN
5 CONTINUE
   IF(X(K-1)=XINT)6,14,17
6 K=K-1
   GO TO 4
16 YINT=Y(K-1)
   RETURN
17 N=K
   K=K/2
   GO TO 2
7 LL=K
35 K=(N+K)/2
8 CONTINUE
   IF(X(K)=XINT)3,14,18
18 CONTINUE
   IF(X(K-1)=XINT)6,16,19
40 N=K
   K=(LL+K)/2
   GO TO 8
1 PRINT 101
101 FORMAT(* OUT OF RANGE FOR INTERPOLATION *)
   STOP
END

SYMBOL REFERENCE MAP (R=1)
ENTRY POINTS
3 SPLINE
SUBROUTINE SPLICE(X,Y,M,C)
DIMENSION X(50), Y(50), P(50), E(50), C(4,50)
DIMENSION A(50,3), R(50), Z(50)
MM=M-1
DO 2 K=1,MM
  D(K)=X(K+1)-X(K)
  P(K)=D(K)/6.
  E(K)=(Y(K+1)-Y(K))/D(K)
2 E(K)=E(K)-E(K-1)
DO 3 K=2,MM
  B(K)=E(K)-E(K-1)
  A(1,2)=D(1)/D(2)
  A(2,2)=2*(P(1)+P(2))-P(1)*A(1,2)
  A(2,3)=A(2,2)/A(2,2)
  B(2)=B(2)/A(2,2)
3 B(K)=B(K)-B(K-1)
  A(K,2)=D(M-2)/D(M-1)
  A(M,1)=1.+Q*A(M-2,1)
  A(M,2)=0.-A(M,1)*A(M-1,3)
  R(M)=B(M-2)-A(M,1)*B(M-1)
  Z(M)=B(M)/A(M,2)
MN=M-2
DO 6 I=1,MN
  K=M-I
6 Z(K)=B(K)-A(K,3)*Z(K+1)
  Z(1)=A(1,2)+Z(2)-A(1,3)*Z(3)
DO 7 K=1,MM
  Q=1./D(K)
  C(1,K)=Z(K)*Q
7 C(2,K)=Z(K)+Q
  C(3,K)=Y(K)/D(K)-Z(K)*P(K)
  C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
RETURN
END

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 SPLICE

<table>
<thead>
<tr>
<th>VARIABLES</th>
<th>SN</th>
<th>TYPE</th>
<th>RELLOCATION</th>
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<tbody>
<tr>
<td>373 A</td>
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<td>ARRAY</td>
<td>621 R</td>
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<tr>
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<td>F.P.</td>
<td>145 D</td>
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<tr>
<td>311 E</td>
<td>REAL</td>
<td>ARRAY</td>
<td>144 I</td>
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<td>ARRAY</td>
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</tr>
<tr>
<td>140 MM</td>
<td>INTEGER</td>
<td>MN</td>
<td>143 MN</td>
</tr>
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<td>REAL</td>
<td>ARRAY</td>
<td>142 Q</td>
</tr>
<tr>
<td>0 X</td>
<td>REAL</td>
<td>F.P.</td>
<td>0 Y</td>
</tr>
</tbody>
</table>

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