ANALYSIS OF THE RIM INERTIAL MEASURING SYSTEM (RIMS)

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Nonlinear equations of motion are derived for the Rim Inertial Measuring System (RIMS), which is mounted in a strapped-down configuration on a carrier vehicle. The RIMS can be used for measuring angular rates and linear accelerations and the equations derived can be easily interfaced with the dynamic model of a user-defined carrier vehicle. Two methods are presented for rate measurement, and the results of the nonlinear simulation are presented.
assumed in the derivations, actuator models can be easily inserted for a more realistic simulation. Results of the computer simulation of the RIMS are presented.

**SYMBOLS**

\[ A, \mathbf{B}_f, \mathbf{B}_{n_1}, \mathbf{B}_{n_2} \]  \hspace{1cm} \text{coefficient matrices}

\[ \mathbf{C} \]  \hspace{1cm} \text{measurement matrix}

\[ \mathbf{C}_p(\cdot) \]  \hspace{1cm} \text{cross-product matrix of a vector}

\[ \mathbf{e}_k \]  \hspace{1cm} \text{unit vector in } k \text{-direction (} k = 1, 2, 3 \text{)}

\[ \mathbf{f}_j \]  \hspace{1cm} \text{actuator force vector at station } j

\[ \mathbf{f}_r^j, \mathbf{f}_z^j \]  \hspace{1cm} \text{radial and axial actuator forces at station } j

\[ \mathbf{f}_{r_cj}^j, \mathbf{f}_{z_cj}^j \]  \hspace{1cm} \text{radial and axial actuator command forces at station } j

\[ \mathbf{f}_{vz} \]  \hspace{1cm} \text{z-direction force acting on vehicle}

\[ \mathbf{f}_z \]  \hspace{1cm} \text{axial actuator force vector}

\[ \mathbf{G} \]  \hspace{1cm} \text{feedback gain matrix}

\[ \mathbf{g} \]  \hspace{1cm} \text{gravitational acceleration vector}

\[ \mathbf{g}_m \]  \hspace{1cm} \text{measured axial centering errors (3 \times 1 vector)}

\[ \mathbf{g}_m^r \]  \hspace{1cm} \text{measured axial centering error rates (3 \times 1 vector)}

\[ \mathbf{g}_z \]  \hspace{1cm} \text{z-axis gravitational acceleration}

\[ \mathbf{H} \]  \hspace{1cm} \text{rim angular momentum about spin axis}

\[ \mathbf{H}_r \]  \hspace{1cm} \text{rim angular momentum vector}

\[ \mathbf{I}_a \]  \hspace{1cm} \text{transverse-axis rim inertia}

\[ \mathbf{I}_r \]  \hspace{1cm} \text{rim inertia matrix}

\[ \mathbf{I}_v \]  \hspace{1cm} \text{vehicle inertia matrix}

\[ \mathbf{K} \]  \hspace{1cm} \text{Kalman gain matrix}
rate gains
proportional and rate gain matrices
proportional gains
moment acting on rim
kth component of moment acting on rim of unit A
kth component of moment acting on rim of unit B
rim mass
vehicle mass
white noise torque acting on vehicle
rim radius
coefficient matrix defined in equation (29)
matrix consisting of the first two columns of T
transformation matrix ($j = 1,2,3$)
spin motor torque
torque acting on vehicle
jth column of T
covariance intensity matrix of $n_v$
actuator noise
inertial velocity of rim center of mass
proximity sensor noise
denotes coordinate frames; subscripts are described in the text
state vector
vector defined in equation (52)
$X$ and $Y$ components of $\rho_{rv}$
rim center-of-mass position in the inertial system
$z_v$  
vehicle center-of-mass position in the inertial system

$\alpha_a, \alpha_b$  
coefficient matrices

$\delta_{a_j}, \delta_{r_j}$  
axial and radial centering errors at station $j$

$\delta_{a_{cj}}, \delta_{r_{cj}}$  
axial and radial centering error commands at station $j$

$\delta_{a_{rcj}}, \delta_{r_{rcj}}$  
axial and radial centering error rate commands at station $j$

defined in equation (36)

$\rho_{a_j}$  
position of the point on the rim closest to actuator station $j$ (in $a$-system)

$\rho_j$  
position of actuator station $j$ in $v$-system

$\rho_T$  
position of rim center-of-mass in inertial system

$\rho_{r_v}$  
position of nominal rim center location in $v$-system

$\rho_V$  
position of vehicle center-of-mass in inertial system

$\tau_s$  
spin motor torque vector

$\phi_{e_1}, \phi_{e_2}$  
rotations (in order 1-2) for obtaining $e$-system from $v$-system

$\theta_{a_1}, \theta_{a_2}$  
rotations (in order 1-2) for obtaining $a$-system from inertial system

$\theta_e$  
vector defined in equation (33)

$\theta_v, \theta_{v_1}, \theta_{v_2}, \theta_{v_3}$  
vehicle attitude angles

$\psi_T$  
rim rotation angle relative to $a$-system

$\Omega$  
matrix defined in equation (35)

$\Omega_v = (\theta_{v_1}, \theta_{v_2})^T$  

$\omega_o$  
rim spin speed relative to $v$-system

$\omega_a$  
absolute angular velocity of $a$-system

$\omega_{a_j}$  
components of $\omega_a$ ($j = 1, 2, 3$)

$\omega_r$  
absolute angular velocity of rim
\[ \omega_r \] components of \( \omega_r \)

\[ \omega_{sp} = (0,0,\omega_o) \]

\[ \omega_{spin} = (0,0,\dot{\psi}_r) \]

\( \omega_v \) absolute angular velocity of \( v \)-system

\( \omega'_v \) absolute angular velocity of \( v \)-system, expressed in e-system

\[ \zeta = (Y_{rv}' - X_{rv}') \]

**Superscripts**

\( T \) transpose of a matrix

\( \cdot \) first derivative

\( \ldots \) second derivative

\( \wedge \) estimated value of a variable

**MATHEMATICAL MODEL DEVELOPMENT**

**Coordinate Systems**

Let \( (X_i, Y_i, Z_i) \) be the inertial coordinate frame and let \( (X_a, Y_a, Z_a) \) denote a coordinate frame obtained by a rotation \( \theta_{a_1} \) about the \( X_i \)-axis and \( \theta_{a_2} \) about the new \( Y_i \)-axis. The origin of the \( a \)-system is the rim center of mass which is also the rim center. Let \( (X_{rb}, Y_{rb}, Z_{rb}) \) denote a coordinate system fixed to the rim. This system is obtained from the \( (X_a', Y_a', Z_a') \) system by a rotation \( \psi_r \) (rim roll angle) about the \( a \)-axis. Let \( (X_v, Y_v, Z_v) \) denote a coordinate system which is fixed to the vehicle and is centered at the vehicle center of mass. The \( v \)-system is obtained from the \( i \)-system by rotations \( \theta_{v_1}, \theta_{v_2}, \theta_{v_3} \) in the order 1-2-3. Let \( (X_v', Y_v', Z_v') \) denote a vehicle-fixed coordinate system parallel to the \( v \)-system, centered at the nominal location of the rim c.m.

**Equations of Motion of the Rim**

The equations of motion are similar to those derived in reference 3 for the AMCD.
Rotational equations. - Let \( \omega_a \) denote the absolute angular velocity of the a-system, and \( \omega_{\text{spin}} \) the angular velocity of the rim with respect to the a-system. Then the absolute angular velocity of the rim expressed in the a-system is given by

\[
\omega_r = \omega_a + \omega_{\text{spin}}
\]  

(1)

where

\[
\omega_{\text{spin}} = (0, 0, \psi_r)
\]  

(2)

Rim angular momentum is given by

\[
H_r = I_r \omega_r = I_r (\omega_a + \omega_{\text{spin}})
\]  

(3)

\[
\dot{H}_r + \omega_a \times H_r = M_r
\]  

(4)

where \( M_r \) is the total momentum acting on the rim. Therefore,

\[
\dot{\omega}_r = -I_r^{-1} \left[ C_p(\omega_a) I_r \omega_r - \sum_{j=1}^{3} \rho_{aj} \times f_j - \tau_s \right]
\]  

(5)

where \( C_p(\cdot) \) denotes the cross-product matrix of a vector, \( \rho_{aj} \) denotes the position of the point on the rim closest to actuator \( j \), \( f_j \) is the force at actuator \( j \) in the a-system and \( \tau_s \) denotes the spin motor torque vector.

Since the a-system is obtained by rotations \( \theta_{a1} \) and \( \theta_{a2} \) about the \( X_a \) and \( Y_a \) axes,

\[
\begin{bmatrix}
\omega_{a1} \\
\omega_{a2} \\
\omega_{a3}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_{a1} & 0 & 0 \\
0 & 1 & 0 \\
\sin \theta_{a2} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_{a1} \\
\dot{\theta}_{a2} \\
\dot{\theta}_{a3}
\end{bmatrix}
\]  

(6)
Therefore,
\[
\begin{bmatrix}
\dot{\theta}_{a1} \\
\dot{\theta}_{a2}
\end{bmatrix}
= \begin{bmatrix}
\sec \theta_{a2} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega_{a1} \\
\omega_{a2}
\end{bmatrix}
\] (7)

Since
\[
\omega_{r3} = \omega_{a3} + \omega_{\text{spin}} = \omega_{a3} + \dot{\psi}_r
\] (8)

substituting for \( \omega_a \) from equation (6) gives
\[
\dot{\psi}_r = -\sin \theta_{a2} \dot{\theta}_{a1} + \omega_{r3}
\] (9)

Assuming that the magnetic actuator forces act only axially and radially, there is no torque generated by the actuator forces about the \( Z_a \)-axis. Assuming further that the spin motor generates a torque \( T_s \) only about the \( Z_a \)-axis, \( \tau_s \) in equation (5) is given by
\[
\tau_s = (0, 0, T_s)^T
\] (10)

(The spin-motor torque is included here for completeness, and is not used in the simulation described later.) Thus, \( \omega_r \) is obtained by solving the differential equation (5); the rim spin speed \( \dot{\psi}_r \) is then given by equation (9).

Translation equations. - Let \( v_r \) denote the inertial velocity of the rim center of mass expressed in the \( a \)-system. Then
\[
m_r \left( v_r + C_p(\omega_a) v_r \right) = \sum_{j=1}^{3} f_j + m_r g
\] (11)

where \( m_r \) is the rim mass and \( m_r g \) is the gravitational force (in \( a \)-system).
Let $\rho_r$ denote the position of the rim c.m. in the i-system. Then

$$\dot{\rho}_r = T_1^T(\theta_{a_1}) T_2^T(\theta_{a_2}) v_r$$

(12)

where $T_k$ denotes the transformation matrix corresponding to rotation about axis $k$ ($k = X,Y,Z$).

Expressions for the Actuator Gaps

The vehicle on which the RIMS would be installed could be a spacecraft, aircraft, etc. Let $\rho_v$ denote the position of the vehicle c.m. in the i-system, and $\omega_v$ its absolute angular velocity in the v-system.

It is necessary to obtain expressions for the magnetic actuator gaps. An actuator gap or centering error is defined as the distance between the point midway between the actuator pole faces and the point on the rim (assumed to be a perfect circle of zero thickness for gap calculation) which is closest to it. Thus the exact expression for an actuator gap will involve (1) deriving an expression for the rim plane, (2) obtaining the point $P_r$ on the plane through which the vector normal to the plane passes, such that this normal vector also passes through the point $S$ located midway between the actuator pole faces, and (3) obtaining the point $Q$ on the rim which lies on the line joining the rim center to $P$. The distance between the points $S$ and $Q$ then gives the actuator gap. Although the expression obtained in this manner will be exact, it is also quite cumbersome to evaluate. However, since the rim motion within the actuator pole faces is small, it is advisable to obtain an approximate expression for the actuator gaps. An approximate expression (in inertial coordinates) for the gap at actuator $j$ is

$$\delta_j = \rho_r - \rho_v - T_1^T(\theta_{v_1}) T_2^T(\theta_{v_2}) T_3^T(\theta_{v_3}) \rho_v$$

$$+ \left\{ T_1^T(\theta_{a_1}) T_2^T(\theta_{a_2}) - T_1^T(\theta_{v_1}) T_2^T(\theta_{v_2}) \right\} T_3^T(\theta_{v_3}) \rho_j$$

(13)
where $\rho_j$ denotes the location of actuator $j$ in the $v'$-system, and $\rho_{r_{v'}}$ denotes the nominal position of the rim c.m. in the $v$-system. In the vehicle coordinates, the gap at actuator $j$ is given by

$$\delta_{v_j} = T_3(\theta_{v_3})T_2(\theta_{v_2})T_1(\theta_{v_1})\delta_j$$

(14)

The axial centering error at actuator $j$ is

$$\delta_{a_j} = e_3^T \delta_{v_j}, \ j = 1, 2, 3$$

(15)

where $e_k^T$ denotes a $3 \times 1$ vector with all zeros except a "1" in the $k$th position. The radial centering errors are obtained by appropriate rotations about the Z-axis:

$$\delta_{r_1} = e_1^T T_3(60^\circ) \delta_{v_1}$$

(16)

$$\delta_{r_2} = e_1^T T_3(-60^\circ) \delta_{v_2}$$

(17)

$$\delta_{r_3} = -e_1^T \delta_{v_3}$$

(18)

Radial centering errors are defined as being positive outwards and

$$T_3(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(19)

The position of actuator $j$ in the $a$-system

$$\rho_{{a_j}} = T_3^T(\theta_{v_3})\rho_j$$

(20)
Let $f_{r_i}$ and $f_{z_i}$ ($i = 1, 2, 3$) denote the radial and axial actuator forces generated at station $i$. Forces $f_i$ at station $i$ (in the $v$-system) are given by

$$
\begin{align*}
\begin{bmatrix}
  f_{r_1} \\
  f_{r_1} \cos (60^\circ) \\
  f_{r_1} \sin (60^\circ) \\
  f_{z_1} \\
  f_{r_2} \cos (60^\circ) \\
  f_{r_2} \sin (60^\circ) \\
  f_{z_2} \\
  -f_{r_3} \\
  0 \\
  f_{z_3}
\end{bmatrix}
\end{align*}
$$

(21)

(22)

(23)

Since the forces $f_i$ are in the $v$-system, they must be transformed into the $a$-system before their use in equation (5).

The control laws for the magnetic actuators are assumed to be proportional plus derivative type. The radial and axial command forces at actuator $i$ are given by

$$
\begin{align*}
  f_{r_{ci}} &= K_p \left( \delta_{r_i} - \delta_{r_{ci}} \right) + K_D \left( \dot{\delta}_{r_i} - \dot{\delta}_{r_{ci}} \right) \\
  f_{z_{ci}} &= K_p \left( \delta_{a_i} - \delta_{a_{ci}} \right) + K_D \left( \dot{\delta}_{a_i} - \dot{\delta}_{a_{rci}} \right)
\end{align*}
$$

(24)

(25)
where \( \delta_{\text{rci}} \) and \( \delta_{\text{aci}} \) denote the radial and axial centering error commands, and \( \delta_{\text{rci}} \), \( \delta_{\text{aci}} \) denote the centering error rate commands at the \( i \)th actuator. These commands are normally zero. \( K_{pj} \), \( K_{Dj} \) \((j = a, r \text{ for axial and radial})\) are the proportional and derivative gains.

**ANGULAR RATE MEASUREMENT SCHEMES**

In order to gain insight into the angular rate measurements, it is convenient to examine the linearized equations of motion of the carrier vehicle/RIMS. The linearization is performed about nonrotating carrier axes.

**Linearized Equations of Motion**

The linearized equations of motion of the rim are given by:

\[
I_a \ddot{\theta}_a + H \dot{\theta}_a = t_1^T f_z \tag{26}
\]

\[
I_a \ddot{\theta}_a - H \dot{\theta}_a = t_2^T f_z \tag{27}
\]

\[
m_{rz} = t_3^T f_z + m_{rg} \tag{28}
\]

where \( I_a \) is the transverse rim inertia, \( z_r \) the Z-axis rim c.m. position, \( m_{rg} \) the Z-axis gravitational force, \( f_z \) the \( 3 \times 1 \) axial force vector, \( H \) is the rim angular momentum (about spin axis) and \( t_i \) is the \( i \)th column of matrix \( T \):

\[
T = \begin{bmatrix}
\sqrt{5}r/2 & -r/2 & 1 \\
-\sqrt{5}r/2 & -r/2 & 1 \\
0 & r & 1
\end{bmatrix} \tag{29}
\]

where \( r \) represents the rim radius.
Linearized two-axis vehicle equations are

\[
\begin{bmatrix}
\dot{\theta}_v^1 \\
\dot{\theta}_v^2
\end{bmatrix} = I_v^{-1}(T_v + n_v)
\]  \hspace{1cm} (30)

where $I_v$ and $T_v$ denote the vehicle inertia matrix and the torque acting on the vehicle and $n_v$ is a disturbance input torque which is assumed to be a zero-mean white noise with covariance intensity matrix $V$. It is assumed that RIMS actuator forces have a negligible effect on the vehicle. The vehicle c.m. position $z_v$ (Z-axis) is given by

\[
m_v \ddot{z}_v = f_{v_z} + m_v g_v
\]  \hspace{1cm} (31)

where $f_{v_z}$ is the (Z-axis) force acting on the vehicle. Let $X_{r_v}$ and $Y_{r_v}$ denote the X and Y components of $\rho_{r_v}$, the nominal position of the rim c.m. in the v-system, and let $\xi = (Y_{r_v} - X_{r_v})^T$.

Dividing equations (26) and (27) by $I_a$ and subtracting equation (30) from the result, we have

\[
\dot{\theta}_e + \Omega \dot{\theta}_e + \Omega^2 \dot{\theta}_v = \frac{1}{I_a} T_a \ddot{f}_z - I_v^{-1} (T_v + n_v)
\]  \hspace{1cm} (32)

where $\theta_v$ denotes $(\theta_v^1, \theta_v^2)^T$ and

\[
\theta_e = \begin{bmatrix} \theta_{e1} \\ \theta_{e2} \end{bmatrix} = \begin{bmatrix} \theta_{a1} - \theta_v^1 \\ \theta_{a2} - \theta_v^2 \end{bmatrix}
\]  \hspace{1cm} (33)
Let

\[ \varepsilon = z_r - (z_v + \zeta_T^T \theta_v) \]  

(36)

Dividing equation (28) by \( m_a \), equation (31) by \( m_v \), subtracting the latter from the former and using equations (36) and (30) yield

\[ \varepsilon = \frac{1}{m_r} t_3^T f - \frac{1}{m_v} f_v - \zeta_T^T I_v^{-1} (T_v + n_v) \]  

(37)

The measured axial centering errors at the three actuator stations are given by:

\[ g_m = T_a \theta e + t_3 \varepsilon + w_g \]  

(38)

where \( w_g \) represents the position sensor noise. The axial actuator forces are given by:

\[ f_z = K_p g_m + K_D g_m_r \]  

(39)

where \( K_p \) and \( K_D \) denote the \( 3 \times 3 \) position and rate gain matrices and \( g_m_r \) denotes the measured (or derived) axial actuator centering error rate vector.

**Estimation Schemes**

**Scheme I - direct inversion.** - The actuator gains \( K_p \) and \( K_D \) are selected in such a manner that the centering errors are quickly corrected
and returned to zero in steady state. The closed-loop bandwidth of the centering error loop is usually much higher than that of the vehicle motion: that is, the actuator centering errors quickly attain steady state. Thus, substituting $\ddot{\theta}_e = \dot{\theta}_e = 0$ in equation (32) and solving for $\dot{\theta}_v$ yield an estimate of the two-axis vehicle angular rates:

$$\dot{\theta}_v = \Omega^{-1} \left[ \frac{1}{T_a} T_s T_{fz} - I_{v}^{-1} (T_{v} + n_{v}) \right]$$

(40)

The second term in the brackets denotes an error term which is a characteristic of rate gyros. This error term will be assumed negligible in this estimation scheme. The first term can be simplified by using equation (39) and making the centering error rate $g_{m_{r}}$ equal to 0. Thus an estimate of vehicle angular rates based on position sensor outputs is given by

$$\dot{\theta}_v = \frac{1}{T_a} \Omega^{-1} T_a T_{p} g_{m}$$

(41a)

The same analysis was presented in reference 1 using the frequency-domain approach. From equations (31) and (37), the linear acceleration (Z-direction) is given by

$$z_v \approx \frac{1}{m_r} t_3 T_{fz} + g_z$$

or

$$z_v \approx \frac{1}{m_r} T_{p} g_{m} + g_z$$

(41b)

Although these rate estimates are based on the linear analysis, it is shown below that they are also valid for the general nonlinear equations when the vehicle is rotating about all three axes.
Nonlinear analysis. - Let \((X_e, Y_e, Z_e)\) denote a coordinate system obtained from the \(v\)-system by Euler angle rotations \(\phi_{e1}\) and \(\phi_{e2}\) in the order 1, 2. The origin of this coordinate system is fixed to the rim center and translates with it. Thus the \((X_e, Y_e)\) plane defines the rim plane. Let \(\omega_e\) denote the angular velocity of the \(e\)-system relative to the vehicle, expressed in the \(e\)-system. Let \(\omega_{sp}\) be the angular velocity of the rim relative to the \(e\)-system. Then \(\omega_{sp} = (0, 0, \omega_o)^T\) where \(\omega_o = \) rim spin speed (relative to the vehicle). The absolute angular velocity of the rim, expressed in the \(e\)-system, is

\[
\omega_r = \omega_v + \omega_e + \omega_{sp}
\]

where \(\omega_v\) = absolute angular velocity of the vehicle expressed in the \(e\)-system. The rim rotational equation is given by [from eq. (4)]:

\[
I_r \left( \dot{\omega}_v + \dot{\omega}_e + \dot{\omega}_{sp} \right) + C_p \left( \omega_v + \omega_e \right) I_r \left( \omega_v + \omega_e + \omega_{sp} \right) = M_r
\]

In steady state (of the magnetic actuator loops), the \(e\)-system does not move relative to the vehicle, i.e., \(\omega_e = \dot{\omega}_e = 0\). Also, spin rate is assumed to be constant with respect to the vehicle. Therefore, in steady state,

\[
I_r \left( \dot{\omega}_v \right) + C_p \left( \omega_v \right) I_r \left( \omega_v + \omega_{sp} \right) = M_r
\]

The second term on the left-hand side is

\[
\begin{bmatrix}
0 & -\omega_{v3} & \omega_{v2} \\
\omega_{v3} & 0 & -\omega_{v1} \\
-\omega_{v2} & \omega_{v1} & 0
\end{bmatrix}
\begin{bmatrix}
I \omega_{v1} \\
I \omega_{v2} \\
2I_o \left( \omega_{v3} + \omega_o \right)
\end{bmatrix}
\]

\[45\]
where $I_a$ denotes the transverse-axis rim inertia. Therefore, in steady state,

$$\begin{bmatrix}
I_a\left(\omega'_v + 2\omega_v\right) \omega'_v \\
-I_a\left(\omega'_v + 2\omega_v\right) \omega'_v \\
0
\end{bmatrix} = M_r - I_r \omega'_v$$  \hspace{1cm} (46)

The acceleration term $I_r \omega'_v$ represents an error term which is present in all rate gyros. Ignoring this error term,

$$I_a \begin{bmatrix}
2\omega_o \omega'_v + \omega'_v \omega'_v \\
-2\omega_o \omega'_v - \omega'_v \omega'_v
\end{bmatrix} = \begin{bmatrix} M_{r_1} \\
M_{r_2} \end{bmatrix}$$  \hspace{1cm} (47)

where the right-hand side represents the steady-state torque exerted by the magnetic actuators. The torque can be expressed as a moment-arm matrix times axial and radial force vector, which, in steady state, can be expressed as a proportional gain matrix times the axial and radial centering error vector. Thus $M_{r_a}$ can be expressed as

$$M_{r_a} = \begin{bmatrix} a_a \\
\delta_a \\
\delta_r \end{bmatrix}$$  \hspace{1cm} (48)

where $a_a$ is a $2 \times 6$ matrix.

If the vehicle is not rotating about the yaw-axis, $\omega'_v = 0$ and equation (47) can be solved for $\omega'_v$ and $\omega'_v$, the solution being basically the same as equation (41a). However, if $\omega'_v \neq 0$, it is necessary to use at least one more unit for complete rate measurement. The equations for the second unit (unit B), which consists of a rim suspended in the $Y_v - Z_v$ plane, can be similarly derived:
In steady state,

\[ M_{rb} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \]

Equations (47) and (49) represent four algebraic equations in three unknowns: \( \omega'_{v1} \), \( \omega'_{v2} \), and \( \omega'_{v3} \). From these equations:

\[ 2\omega \left( \omega'_{v3} - \omega'_{v1} \right) = \frac{1}{I_a} \left( M_{rb1} + M_{ra2} \right) \]

or

\[ \omega'_{v3} = \omega'_{v1} + \frac{1}{2I_a \omega} \left( M_{rb1} + M_{ra2} \right) \]

Substituting for \( \omega'_{v3} \) from equation (51) into the top equation of the vector equation (49), a quadratic equation in \( \omega'_{v1} \) is obtained. The two roots of the equation can be computed. In order to determine the root which represents \( \omega'_{v1} \), the first root is substituted in equation (51), which gives \( \omega'_{v3} \). Using these values of \( \omega'_{v1} \) and \( \omega'_{v3} \), \( \omega'_{v2} \) is computed from the top part of equation (47), and also from the bottom part of equation (49). If these two estimates of \( \omega'_{v2} \) are equal (within some tolerance), the root chosen was the correct representation of \( \omega'_{v1} \). If not, the other root must represent \( \omega'_{v1} \). Three-axis rate information can be obtained from two orthogonal units in this manner. Through algebraic manipulations, it should be possible to further simplify the solution.

**Scheme II.** - This scheme is based on an application of the Kalman-Bucy filter. Returning to the linearized equations and denoting \( \Omega_v = (\dot{\theta}_{v1} \dot{\theta}_{v2})^T \)
Equations (30), (32) and (37) can be expressed in the form:

\[ \dot{X} = AX + B_f f_z + B_{n_1} \left( T_v + n_v \right) + B_{n_2} f_{v_z} + B_f v_a \]  

where \( X \) is the 10 x 1 state vector consisting of \( X_1 \) and \( \dot{X}_1 \), \( B_f', B_{n_1}, B_{n_2} \) are appropriately dimensioned input matrices, and \( v_a \) is the actuator noise vector. For the purpose of estimator design, \( T_v \) is assumed to be zero, and \( f_{v_z}, n_v \) and \( v_a \) are the white-noise inputs acting on the system. The Kalman-Bucy filter for this system is given by:

\[ \dot{\hat{X}} = \hat{A} \hat{X} + \hat{B}_f f_z + \hat{K} \left( g_m - \hat{C} \hat{X} \right) \]  

where \( C \) is obtained from equations (38) and (52). The actuator force vector is synthesized as

\[ f_z = G \hat{X} \]  

where \( G \) is the gain matrix obtained from equations (38), (39) and (52). The steady-state filter has constant gain matrix \( K \), and is more suitable for practical implementation than the nonsteady-state version. The filter generates estimates of \( X \), which also include estimates of \( n_v \). The sensor and actuator noise covariance matrices are needed for the design. The vehicle input torque covariance matrix \( V \) can be used as a design parameter for the Kalman-Bucy filter. Further analysis is necessary in order to evaluate the performance of this scheme using the nonlinear equations for a vehicle rotating about all three axes.
NUMERICAL RESULTS

In order to complete the analysis of RIMS, a computer program was written for its simulation using the nonlinear equations developed [eqs. (1) - (25)]. A single AMCd was used in this simulation. The rim mass was 453.6 gm (1 lb), the rim radius was 5.715 cm (2.25 in.), rim spin speed was 1,000 rpm, and the rim center was nominally located at the vehicle c.m. With zero initial conditions, a torque $T_{v_1}$ about the $X_v$ axis was used to force the vehicle for 1 sec and to obtain a rate of 0.286 rad/sec at the end of 1 sec. The system was unforced for $1 < t < 4$ sec, and a torque $-T_{v_1}$ was applied for $4 < t < 5$ sec, after which time the torque was again zero. Figures 2 to 5 show the actual and measured vehicle rate, the actual and measured vehicle attitude angle ($\Theta_{v_1}$), the rate measurement error, and the axial and radial centering errors at actuator station 1. The actuator-loop damping ratio used was 2 (at zero spin speed) in all the computer runs. In figures 2 to 5, the radial and axial actuator control-loop bandwidth $\omega_{c_1}$ used was 30 rad/sec. Figures 6 to 8 show the RIMS performance for $\omega_{c_1} = 60$ rad/sec. The measurement error as well as the actuator centering error is much smaller. The actual and measured rate for $\omega_{c_1} = 100$ rad/sec is shown in figure 9. As the actuator control loops are made tighter, the RIMS performance improves. However, since the actuator gaps (centering errors) also get smaller with higher $\omega_{c_1}$, more accurate proximity sensors are required if the implementation uses gap measurements rather than force measurements. If the rim momentum is increased, higher $\omega_{c_1}$ has to be used for getting satisfactory performance. This too results in small gaps and will need more accurate proximity sensors. Therefore, from the viewpoint of implementation, it would be better to use force measurements [in eq. (40)]. In that case it would be advisable to use higher momentum and higher $\omega_{c_1}$, since the actuator forces to be measured would be large, resulting in a more sensitive measurement scheme. It may also be advisable to use an integrator in the actuator control loop. This would result in relatively easy measurement of force (by measuring the integrator output), and also
maintain the actuator centering errors near zero. The effect of the latter would be beneficial in maintaining the actuators in the linear region. Further investigation is needed in this area in order to evaluate the dynamic response of RIMS using integrators.

In summary, the numerical results presented indicate that, with properly selected rim momentum and $\omega_{c1}$, the RIMS can provide highly accurate rate measurements.

CONCLUSIONS

Nonlinear equations of motion have been derived for the RIMS using one AMCD. The equations can be easily interfaced with a model of a carrier vehicle. The complexity of the equations can be easily increased by making additions such as the magnetic actuator models. Two methods have been presented for the measurement of vehicle angular velocities. The first method is obtained from nonlinear equations, while the second method uses linearized equations and the Kalman-Bucy filter. Results of the nonlinear simulation using the first method indicate that RIMS can offer an attractive and accurate inertial measurement system. Further analyses will be required in order to completely evaluate the second measurement method, and also to evaluate the performance of both methods in a stochastic environment.
REFERENCES


Figure 1. RIMS actuator station locations.

Figure 2. Actual and measured rate, $\omega_{cl} = 30$ rad/sec.
Figure 3. Actual and measured vehicle attitude angle, $\omega_{cl} = 30$ rad/sec.

Figure 4. Rate measurement error, $\omega_{cl} = 30$ rad/sec.
Figure 5. Axial and radial gaps at station 1, $\omega_{c1} = 30$ rad/sec.

Figure 6. Actual and measured rate, $\omega_{c1} = 60$ rad/sec.
Figure 7. Rate measurement error, $\omega_{cl} = 60$ rad/sec.

Figure 8. Axial and radial gaps at station 1, $\omega_{cl} = 60$ rad/sec.
Figure 9. Actual and measured rate, $\omega_{cl} = 100$ rad/sec.
Nonlinear equations of motion are derived for the Rim Inertial Measuring System (RIMS), which is mounted in a strapped-down configuration on a carrier vehicle. The RIMS can be used for measuring angular rates and linear accelerations and the equations derived can be easily interfaced with the dynamic model of a user-defined carrier vehicle. Two methods are presented for rate measurement, and the results of the nonlinear simulation are presented.
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