Induced Drag Ideal Efficiency Factor of Arbitrary Lateral-Vertical Wing Forms

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SUMMARY

A relatively simple equation is presented for estimating the induced drag ideal efficiency factor \( e \) for arbitrary cross-sectional wing forms. This equation is based on eight basic but varied wing configurations which have exact solutions. The \( e \) function which relates the basic wings is developed statistically and is a continuous function of configuration geometry. The basic wing configurations include boxwings shaped as a rectangle, ellipse, and diamond; the V-wing; end-plate wing; 90 degree cruciform; circle dumbbell; and biplane. Example applications of the \( e \) equations are made to many wing forms such as wings with struts which form partial span rectangle, diamond, triangle, ellipse, and semi-ellipse wings; the ellipse and rectangle dumbbell wings; bow-tie, cruciform, winglet, and fan wings; and multi-wings. Derivations are presented in the appendices of exact closed form solutions found of \( e \) for the V-wing and 90 degree cruciform wing and for an asymptotic solution for multi-wings.

INTRODUCTION

With the objective of better economics through improved aerodynamic and structural shapes, many unconventional aircraft configurations are being investigated. Particularly desirable are wing forms for which both drag and structural stresses decrease. A measure of aerodynamics efficiency is the induced drag ideal efficiency factor denoted by \( e \).

Historically, exact solutions for induced drag have been found for several nonplanar wing configurations. These include the rectangle boxwing, end-plate wing, and biplane reported in reference 1; the ellipse boxwing and circle dumbbell wing from the methods of reference 2; and the diamond boxwing and V-wing in reference 3. These solutions form the basis for the present study.

The objective of this study is to develop a generalized equation for estimating the ideal \( e \) for arbitrary \( yz \)-plane wing forms. For a given wing form the ideal \( e \) is the maximum that \( e \) can become, and at ideal \( e \) all the wing form surfaces have ideal loading distributions such that the total induced drag is minimized. A generalized equation for estimating \( e \) provides the designer with
a tool for evaluating the aerodynamic efficiency for the configuration under consideration.

SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>lateral span along y-axis of wing configuration</td>
</tr>
<tr>
<td>$D_i$</td>
<td>induced drag, equation (1)</td>
</tr>
<tr>
<td>$E$</td>
<td>complete elliptic integral of the second-kind, with modulus $k$</td>
</tr>
<tr>
<td>$e$</td>
<td>induced drag ideal efficiency factor, equation (1)</td>
</tr>
<tr>
<td>$e_e$</td>
<td>$e$ for end-plate wing, equation (16)</td>
</tr>
<tr>
<td>$e_v$</td>
<td>$e$ for V-wing, equation (15)</td>
</tr>
<tr>
<td>$f$</td>
<td>function of parameter $P$, equation (20)</td>
</tr>
<tr>
<td>$G_0$</td>
<td>function of $\eta |_H$, equation (18)</td>
</tr>
<tr>
<td>$G_1$</td>
<td>function of $e_v$, $e_e$, $\eta_{af}$, equation (19)</td>
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<tr>
<td>$H$</td>
<td>dimensionless maximum height, $h_m/b$</td>
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<tr>
<td>$H_a$</td>
<td>average dimensionless height, equation (25)</td>
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<td>$h_m$</td>
<td>maximum height of wing configuration</td>
</tr>
<tr>
<td>$K$</td>
<td>complete elliptic integral of the first kind, with modulus $k$</td>
</tr>
<tr>
<td>$k$</td>
<td>modulus for $K$ and $E$, $k = \sin \alpha$ where $\alpha$ is the modular angle</td>
</tr>
<tr>
<td>$L$</td>
<td>total lift of wing system, equation (1)</td>
</tr>
<tr>
<td>$N$</td>
<td>total number of wings</td>
</tr>
<tr>
<td>$P$</td>
<td>parameter, equation (21)</td>
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<tr>
<td>$p$</td>
<td>perimeter of wing system, equation (29)</td>
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<tr>
<td>$P_e$</td>
<td>perimeter of ellipse, equation (22)</td>
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<tr>
<td>$q$</td>
<td>free stream dynamic pressure, $\frac{1}{2} \rho V^2$</td>
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The present analysis is based on a group of varied wing configurations which have exact solutions for predicting $e$. These include the wings described as rectangle boxwing, ellipse boxwing, diamond boxwing, V-wing, end-plate wing, 90 degree cruciform wing, circle dumbbell wing, and biplane. The induced drag ideal efficiency factor listed throughout the paper relates the total induced drag to the total lift of the wing system by the equation:

$$D_i = \frac{L^2}{\pi q b^2 e}$$  \hspace{1cm} (1)

where $b$ is the maximum span in the lateral plane, $q$ is the dynamic pressure, and $L$ is the sum of all the lifts, including wing, winglets, end-plates, struts, or tail. For ideal $e$, the lifts and loadings will have ideal distributions.
Rectangle and Ellipse Boxwings

From reference 1, p. 205, for the rectangle boxwing, the equations for the dimensionless maximum height for \( H \) and for \( e \) can be developed.

\[
H = \frac{h_m}{b} = \frac{E - (1 - k^2)K}{E' - k^2K'} \quad (2)
\]

\[
e = \frac{1 - k^2}{(E' - k^2K')^2} \quad (3)
\]

where \( K \) and \( E \) are complete elliptic integrals of the first and second kinds, respectively, with modulus \( k \), while \( K' \) and \( E' \) are the complementary integrals. That is, \( K' \) is \( K \) with modulus \( \sqrt{1-k^2} \), and \( E' \) is \( E \) with modulus \( \sqrt{1-k^2} \). The modulus \( k \) is related to the modular angle \( \alpha \) by \( k = \sin \alpha \). Equations (2) and (3) do not permit a direct solution of \( e \) for a specified value of \( H \). A value of \( k \) is specified and \( e \) from equation (3) is determined, then the corresponding \( H \) from equation (2) is determined using the same \( k \) value. Geometric characteristics of the rectangle include the nondimensional perimeter given by

\[
\sigma = \frac{p}{2b} = \frac{2b + 2h_m}{2b} = 1 + H
\]

(4)

The rectangle averaged cross-sectional area per wing span squared is

\[
H_a = \frac{h_m b}{b^2} = H
\]

(5)

The \( e \) for the ellipse boxwing can be determined from the work given in reference 2. The results are

\[
e = 1 + H
\]

(6)

The perimeter parameter of the ellipse is an elliptic integral of the second kind given by

\[
\sigma_e = \begin{cases} E, k = \sqrt{1 - H^2}, & H \leq 1 \\ H - E, k = \sqrt{1 - H^{-2}}, & H \geq 1 \end{cases}
\]

(7)
The ellipse averaged cross-sectional area per wing span squared is

$$H_a = \frac{\pi h_m b}{4 b^2} = \frac{\pi}{4} H$$  \hspace{1cm} (8)

Relating Rectangle and Ellipse Boxwings

Examination of equations (2) and (3) shows that at $H = 0$, $e = 1$, and at $H \to \infty$, $e \to 4H/\pi$. Then for the rectangle boxwing, $e$ can be expressed as

$$e = (1 + \frac{4}{\pi} H)(1 + f)$$  \hspace{1cm} (9)

where $f$ is a function of $H$ to be determined. Equation (9) can apply to both rectangle and ellipse boxwings by letting $H = H_a$, and $f$ be a function of perimeter difference. Then

$$e = (1 + \frac{4}{\pi} H_a)(1 + f)$$

$$f = 1.4512 P^{.841} (1 - \sqrt{P})^{1.0795} \left[1 + .585 P^3 (P - .37)^3\right]$$  \hspace{1cm} (10)

where $P = \sigma - \sigma_e$ is given in equation (7), and $\sigma$ for the rectangle is given in equation (4). The $f$ function is statistically developed from the $e$ and $H$ data of equations (2) and (3). Equation (10) applies to either the rectangle or ellipse boxwing and gives an estimate of $e$ for any shape between these two. Example values of $e$ are listed in table 1.

Diamond Boxwing

From reference 3, the added mass $M$ is given by

$$M = 2\pi \rho a$$

where (here $\gamma$ replaces $\beta$ of ref. 3)

$$\alpha = \frac{1}{\pi} (\frac{\pi}{2} - \gamma), \hspace{1cm} \gamma = \tan^{-1} H$$  \hspace{1cm} (11)
Then the dimensionless added mass is

\[ \frac{M}{\rho \pi a^2} = \frac{1 - \frac{2}{\pi} \gamma}{a^2} = \frac{\pi(1 - \frac{2\gamma}{\pi})}{\left[ \Gamma\left(\frac{1}{2} + \frac{\gamma}{\pi}\right) \Gamma(1 - \frac{\gamma}{\pi}) \right]^2} = \]

\[ \left[ \Gamma\left(\frac{1}{2} + \frac{1}{\pi} \tan^{-1} H\right) \Gamma(1 - \frac{1}{\pi} \tan^{-1} H) \right]^2 \pi(1 - \frac{2}{\pi} \tan^{-1} H) \]  

where \( a^2 \) is derived in reference 3, and \( H = h_m/b \). In reference 3, \( D_i \) and \( L \) are related by

\[ D_i = \frac{L^2}{2MV^2} = \frac{L^2}{\pi q b^2 \frac{4a^2}{b^2} \left( \frac{M}{\rho \pi a^2} \right)} \]  

where from the sketch, \( 4a^2/b^2 = 1 + H^2 \), then

\[ e = (1 + H^2) \left( \frac{M}{\rho \pi a^2} \right) = \frac{\pi(1 + H^2)(1 - \frac{2}{\pi} \tan^{-1} H)}{\left[ \Gamma\left(\frac{1}{2} + \frac{1}{\pi} \tan^{-1} H\right) \Gamma(1 - \frac{1}{\pi} \tan^{-1} H) \right]^2} \]  

where \( \Gamma \) is the gamma function. For the diamond boxwing, values of \( e \) for various values of \( H \) are presented in table 1. Tables of the gamma function are given in reference 4.

### Table 1. - Ideal \( e \) Values for Rectangle, Ellipse, and Diamond Boxwings

<table>
<thead>
<tr>
<th>( H )</th>
<th>rectangle eq. (10)</th>
<th>ellipse eq. (6)</th>
<th>diamond eq. (14)</th>
<th>( H )</th>
<th>rectangle eq. (10)</th>
<th>ellipse eq. (6)</th>
<th>diamond eq. (14)</th>
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<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>.8</td>
<td>2.47997</td>
<td>1.8</td>
<td>1.29518</td>
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<tr>
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<td>1.15178</td>
<td>1.05</td>
<td>1.01270</td>
<td>1.0</td>
<td>2.78642</td>
<td>2.0</td>
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<td>1.26814</td>
<td>1.10</td>
<td>1.02622</td>
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<td>1.49715</td>
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<td>1.37327</td>
<td>1.15</td>
<td>1.04056</td>
<td>1.4</td>
<td>3.38041</td>
<td>2.4</td>
<td>1.60562</td>
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<td>.20</td>
<td>1.47189</td>
<td>1.20</td>
<td>1.05573</td>
<td>1.6</td>
<td>3.67066</td>
<td>2.6</td>
<td>1.71751</td>
</tr>
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<td>.25</td>
<td>1.56610</td>
<td>1.25</td>
<td>1.07171</td>
<td>1.8</td>
<td>3.96086</td>
<td>2.8</td>
<td>1.83206</td>
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<td>1.65709</td>
<td>1.3</td>
<td>1.08850</td>
<td>2.0</td>
<td>4.24189</td>
<td>3.0</td>
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</tr>
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<td>.4</td>
<td>1.83204</td>
<td>1.4</td>
<td>1.12442</td>
<td>3.0</td>
<td>5.63505</td>
<td>4.0</td>
<td>2.55051</td>
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<td>.5</td>
<td>2.00026</td>
<td>1.5</td>
<td>1.16330</td>
<td>4.0</td>
<td>6.99909</td>
<td>5.0</td>
<td>3.16840</td>
</tr>
<tr>
<td>.6</td>
<td>2.16362</td>
<td>1.6</td>
<td>1.20490</td>
<td>( \infty )</td>
<td>( \rightarrow 4H/\pi )</td>
<td>( \rightarrow H )</td>
<td>( \rightarrow 2H/\pi )</td>
</tr>
</tbody>
</table>
V-Wing

An exact closed function solution was found for the \( e \) of a V-wing. The derivation is given in appendix A. From equation (A15)

\[
e_v = \frac{1}{2} \cos^{-\gamma} \left( \frac{1 - \frac{2\gamma}{\pi}}{1 + \frac{2\gamma}{\pi}} \right) \tan^{-1} \frac{2\gamma}{\pi}, \quad \gamma = \tan^{-1} \frac{2H}{b/2}
\]

(15)

where \( \gamma \) is the dihedral angle in radians. Values of \( e \) for various \( H \) values are listed in table 2.

End-Plate Wing

An accurate but simplified \( e \) function for end plate wings is developed in appendix B. From equation (B14)

\[
e_e = 1 + \left[ \frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + .732H - .0612H(\frac{b/2}{b/2})^2} \right] H
\]

(16)

Values of \( e \) for the end-plate wing computed from equation (16) are presented in table 2.

90° Cruciform Wing and Circle Dumbbell Wing

The 90° cruciform wing has \( H = 1 \), and as shown in appendix C, \( e = 2 \) for the unbanked wing. Also shown in appendix C is the variation of \( e \) with bank angle, and that when induced drag is taken as relative to wing length (and not lateral span) the \( e \) is unity at all bank angles.

The \( e \) for the circle dumbbell wing can be developed from induced drag methods presented in reference 2.
The objective is to develop an \( e \) function, with continuity, which becomes the proper \( e \) function of each of the basic configurations when that configuration geometry is inserted. Because there are many basic configurations and the \( e \) function is continuous, the application to general configurations should be within an acceptable order of accuracy. The general \( e \) function is statistically developed so that each basic configuration is taken into account.

\[
e = \left\{ 1 + \frac{4}{\pi} H_a \left[ 1 - \left( 1 - \frac{16}{\pi^2} \eta_a \frac{H_a}{H} \right) \left( 1 - \eta_a \frac{H}{H_a} \right) G_0 \right] \right\} (1 + f) + \\
\left[ 1 - \left( \frac{H_a}{\eta_a H} \right)^{1.7} \right] G_1
\]

\[
G_0 = \frac{(1 - 0.19 \eta_{10} H)(1 + 0.4826 \eta_{10}^2 H^2)}{(1 + 1.81 \eta_{10} H)(1 + 0.255 \eta_{10}^{5/2} H^{5/2})}
\]

\[
G_1 = \frac{(6 - 4\sqrt{2})(e_e - 1)(e_v - 1) n_a^2 f}{\left[ \sqrt{e_e - 1} - \sqrt{e_v - 1} - (\sqrt{e_e - 1} - 2\sqrt{e_v - 1}) \sqrt{n_a f} \right]^2}
\]

\[
n_a f = \frac{1}{e_e - 1} \quad \eta_{af} = \left( \frac{1}{2} \right)^{1/2} \quad e_v - 1
\]

where \( e_v \) and \( e_e \) are given in equations (15) and (16) and in table 2.

\[
f = 1.4512 \frac{P}{|P|} \cdot 0.841 (1 - |P|^{1/2})^{1.0795} \left[ 1 + 0.585 |P|^{3/2} (|P| - 0.37)^3 \right]
\]

\[
P = \left[ \frac{\eta_{10} H_a}{1 + 3(1 - \eta_{10} \frac{H_a}{H}) (1 - 2 \eta_{10} \frac{H_a}{H})} \right]^{3/2} (\sigma - \sigma_e)
\]
\[ \sigma_e = \text{perimeter of full or of partial span symmetric ellipse boxwing} \]
\[ \frac{\sigma_e}{\eta_1} = \begin{cases} \sqrt{E}, & k = \sqrt{1 - \left(\frac{H}{\eta_1}\right)^2}, \frac{H}{\eta_1} \leq 1 \\ \frac{HE}{\eta_1}, & k = \sqrt{1 - \left(\frac{H}{\eta_1}\right)^2}, \frac{H}{\eta_1} \geq 1 \end{cases} \] (22)

Also functionally

\[ \sigma_e \left(\frac{H}{\eta_1}\right) = \frac{H}{\eta_1} \sigma_e \left(\frac{\eta_1}{H}\right) \]

Numerical values of \( \sigma_e/\eta_1 \) are presented in table 3.

An approximation for \( \sigma_e \) is

\[ \sigma_e = \eta_0 + (\eta_1 - \eta_0) \begin{cases} E, & k = \sqrt{1 - \left(\frac{H}{\eta_1 - \eta_0}\right)^2} \\ \frac{HE}{\eta_1 - \eta_0}, & k = \sqrt{1 - \left(\frac{H}{\eta_1 - \eta_0}\right)^2} \end{cases} \] (23)

For a semi-ellipse in a \( \eta_1-\eta_0 \) interval on the wing semi-span

\[ \sigma_e = \eta_0 + (\eta_1 - \eta_0) \begin{cases} E, & k = \sqrt{1 - \left(\frac{H}{\eta_1 - \eta_0}\right)^2} \\ \frac{HE}{\eta_1 - \eta_0}, & k = \sqrt{1 - \left(\frac{H}{\eta_1 - \eta_0}\right)^2} \end{cases} \] (24)

\[ H_a = \frac{\text{cross-section area of boxwing}}{b^2} \] (25)

\[ H = \frac{h_m}{b}, \] where \( h_m \) is the maximum height of wing configuration. (26)
\[ n_{10} = n_1 - n_0, \text{ where } n_1 \text{ and } n_0 \text{ are the spanwise stations of the outboard and inboard sides of the boxwing at the wing.} \] (27)

\[ n_a = \int_0^1 \eta d\left(\frac{h}{h_m}\right), \text{ averaged } \eta \text{ of outboard face of wing.} \] (28)

\[ \sigma = \frac{p}{2b_0}, \text{ where } p \text{ is perimeter of full or of partial span boxwing in which } h \text{ is constant or decreases with } \eta. \] (29)

For bow-tie and dumbbell type wing configurations, the effective perimeter is determined from the shape with the maximum dumbbell height translated one-half dumbbell width towards the inboard side but limited to the inboard side of the dumbbell which is \( n_0 \). Some examples are

\[ n_{af} = n_a, \text{ for full or for partial span boxwings in which } h \text{ is constant or decreases with } \eta. \] (30)

\[ n_{af} = \frac{1 + 0.05 \frac{H_2}{H_1} \eta_{me}^2}{1 + 0.05 \eta_{me}^2} n_a, \text{ for bow-tie, dumbbell type, and vertical fin wing configurations,} \] (31)

where \( \eta_{me} \) is the translated shape needed to determine \( p \) and \( \sigma \) for this type of wing. The value of \( \sigma_e \) of the dumbbell wing with symmetrical ellipses is given in equation (24). This \( \sigma_e \) and the dumbbell type wing value of \( \sigma \) are used in equation (21) for evaluating \( P \).
For fin wings $S_1 = S_2$. For a vertical fin wing, $\eta_{\text{me}} = \eta_0 = \eta_1 = \eta_2$.

For fin wings

\[
\eta_{af} = \left\{ \begin{array}{l}
1 + \frac{1}{2} (n_{a2} - n_{01})^2 \sqrt{H_2^2 + (n_{12} - n_{01})^2} \left( 1 - \frac{\sin \gamma_2}{\sin \gamma_1} \right)
\end{array} \right.
\]

\[
= \left[ 0.05 + \frac{(n_{11} - n_{01})|n_{11} - n_{01}|^{1/2}}{2.1527} \right] + \frac{1.77 + 0.9H}{1 + 0.67H} (n_{13} - n_{01})
\]

\[
= \left\{ \begin{array}{l}
\frac{n_{a1}}{1 + 0.05 n_{01}^2} + \frac{0.77 + 0.9H}{1 + 0.67H}
n_{a,2n-1}
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\frac{H_{2n-1} + H_{2n}}{1.15 N H_1} (n_{a,2n} - n_{01}) \sqrt{H_2^2 + (n_{1,2n} - n_{0,n-1})^2} \left( 1 - \frac{\sin \gamma_{2n}}{\sin \gamma_{2n-1}} \right)
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
0.05 + \frac{(n_{1,2n-1} - n_{0,n-1})^{3/2}}{2.1527}
\end{array} \right\} n_{a,2n-1}
\]

(32)

where the variables are defined through the following fin wing semispan sketch:
### TABLE 2. - IDEAL $e$ OF V-WING AND END-PLATE WING

<table>
<thead>
<tr>
<th>$H$</th>
<th>$V$-wing eq. (15)</th>
<th>$e_\nu$</th>
<th>$H$</th>
<th>$V$-wing eq. (15)</th>
<th>$e_\nu$</th>
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### TABLE 3. - PERIMETER PARAMETER OF THE ELLIPSE BOXWING

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<thead>
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<th>$H_{\nu}$</th>
<th>$H_{\nu}$</th>
<th>$H_{\nu}$</th>
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<td>1.117941</td>
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<td>.94</td>
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<tr>
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<td>.39</td>
<td>1.144932</td>
<td>.59</td>
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<td>.79</td>
<td>1.410704</td>
<td>.99</td>
<td>1.562952</td>
</tr>
</tbody>
</table>
APPLICATION TO PARTIAL SPAN BOXWINGS

When a strut is designed so that it contributes to the lift of the vehicle and since the strut is attached to the wing, the configuration is the same as that of the boxwing, in particularly, partial span boxwings. One objective of this section is to demonstrate usage of the generalized equation (17) for estimating e for this type of wing.

For the rectangle boxwing, \( H_a = H, \eta_0 = 0, \eta_1 = \eta_{10} = \eta_0 = 1, \sigma = 1 + H \), and equations (17) and (21) reduce to

\[
e = (1 + \frac{4}{\pi} H)(1 + f)
\]

\[
P = \sigma - \sigma_e
\]

Equation (33) is the same as equation (10) which is a solution for the rectangle boxwing.

For the ellipse boxwing, \( H_a = \pi H/4, \eta_a = \pi H/4, \sigma = \sigma_e \), then equation (17) becomes \( e = 1 + H \), which is equal to equation (6).

For the diamond boxwing, \( \eta_a = 1/2, H_a = H/2, \eta_0 = 0, \eta_1 = 1, \eta_{10} = 1, \) and \( \sigma = \sqrt{1 + H^2} \), then equations (17) with (18) and (21) reduce to

\[
e = \left[1 + \frac{2}{\pi} \mathbb{I} (1 - 0.44604 \frac{1}{1 + 1.81H} - 0.19H + 0.4826H^2 \frac{1}{1 + 0.255H^{5/2}}) \right](1 + f)
\]

\[
P = 0.35355 (\sqrt{1 + H^2} - \sigma_e)
\]

where with the P of equation (35), f is determined from equation (20). These f values are negative since P is negative. Values of e computed from equation (34) compare within one-tenth of one percent accuracy with those of the diamond boxwing listed in table 1.
Reflexed Ellipse Boxwing

For this wing, \( \eta_a = \eta_{af} = 1 - \pi/4 \),
\( H_a/H = 1 - \pi/4 \), \( \eta_{10} = 1 \), \( \sigma = \sigma_e \), then
\( P = 0 \), \( f = 0 \), and equation (17) becomes

\[
e = 1 + \left( \frac{4}{\pi} - 1 \right) H \left[ 1 - \left( 5 - \frac{\pi}{2} - \frac{8}{\pi} \right) G_0 \right]
\]

\[
e = 1 + 0.27324 H (1 - 0.881725 G_0)
\]

where \( G_0 \) is given in equation (18) with \( \eta_{10} = 1 \). Some typical values of \( e \) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1</td>
<td>1.00721</td>
<td>1.02008</td>
<td>1.07519</td>
<td>1.19110</td>
<td>1.46881</td>
</tr>
</tbody>
</table>

Partial Span Ellipse Boxwing

For this wing, \( \eta_a = \eta_{af} = (\pi/4) \eta_1 \), \( H_a/H = (\pi/4) \eta_1 \),
\( \eta_{10} = \eta_1 \), \( \eta_1 \) is a specified value, \( 0 \leq \eta_1 \leq 1 \),
\( \sigma = \sigma_e \), then \( P = 0 \), \( f = 0 \), and equation (17) becomes

\[
e = 1 + \eta_1 H \left[ 1 - (1 - \eta_1^2)(1 - \frac{\pi^2}{16} \eta_1^2) G_0 \right]
\]

When \( H = \eta_1 \), the central wing is circular. For \( \eta_1 = 0.5 \), the equation and some values of \( e \) are

\[
e = 1 + 0.5 H (1 - 0.63434 G_0)
\]

where \( G_0 \) is given in equation (18) with \( \eta_{10} = 0.5 \).

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1</td>
<td>1.02116</td>
<td>1.04710</td>
<td>1.14372</td>
<td>1.33842</td>
<td>1.78399</td>
</tr>
</tbody>
</table>
Partial Span Semi-Ellipse Boxwing

For this wing, \( \eta_a = \eta_{af} = (\pi/4) \eta_1 \), \( H_a/H = (\pi/4) \eta_1 \), \( \eta_{10} = \eta_1 \), then equations (17) and (21) become

\[
e = \left\{ 1 + \eta_1 H \left[ 1 - (1 - \eta_1^2)(1 - \frac{\eta_1^2}{16} G_0) \right] \right\} (1 + f) \tag{38}
\]

\[
P = \left[ -\frac{\pi \eta_1^2}{4} \right]^{3/2} \left[ 1 + 3(1 - \frac{\pi \eta_1^2}{4}) (1 - \frac{\pi \eta_1^2}{2})^2 \right]^{3/2} (\sigma - \sigma_e) \tag{39}
\]

The semi-ellipse perimeter parameter is

\[
\sigma = \frac{\eta_1}{2} \left\{ 1 + \begin{cases} E, & k = \sqrt{1 - \left( \frac{2H}{\eta_1} \right)^2}, H \leq \frac{\eta_1}{2} \\ \frac{2H}{\eta_1} E, & k = \sqrt{1 - \left( \frac{2H}{\eta_1} \right)^2}, H \geq \frac{\eta_1}{2} \end{cases} \right\} \tag{40}
\]

and \( \sigma_e \) is given in equation (24). With \( \eta_1 = 0.5 \), equations (38) through (40) reduce to

\[
e = \left[ 1 + \frac{H}{2} (1 - 0.63434 G_0) \right] (1 + f) \tag{41}
\]

\[
P = 0.033507 (\sigma - \sigma_e) \tag{41}
\]

\[
\sigma = \frac{1}{4} + \frac{1}{4} \begin{cases} E, & k = \sqrt{1 - (4H)^2}, H \leq \frac{1}{4} \\ 4HE, & k = \sqrt{1 - (4H)^2}, H \geq \frac{1}{4} \end{cases} \tag{41}
\]
Using equation (41) with equations (18) and (22), for \( \eta_1 = 0.5 \), some typical values for \( e \) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e  )</td>
<td>1</td>
<td>1.02324</td>
<td>1.05142</td>
<td>1.15340</td>
<td>1.35489</td>
<td>1.81216</td>
</tr>
</tbody>
</table>

For the full-span semi-ellipse boxwing, \( \eta_1 = 1 \), then equations (38) through (40) reduce to

\[
\begin{align*}
\sigma &= \frac{1}{2} + \frac{1}{2} \left[ E, \quad k = \sqrt{1 - (2H)^2}, H < \frac{1}{2} \right] \\
&\quad \left[ 2HE, \quad k = \sqrt{1 - (2H)^2}, H > \frac{1}{2} \right] \\
\end{align*}
\]

Using equation (42), for \( \eta_1 = 1 \), some typical values of \( e \) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e  )</td>
<td>1</td>
<td>1.11669</td>
<td>1.23959</td>
<td>1.61191</td>
<td>2.22938</td>
<td>3.45462</td>
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</tbody>
</table>

Partial Span Rectangle Boxwing

For this wing, \( \eta_a = \eta_{af} = \eta_1 \), \( H_a/H = \eta_1 \), \( \sigma = \eta_1 + H \), \( \eta_{10} = \eta_1 \), then equations (17) and (21) become

\[
\begin{align*}
e &= \left[ 1 + \frac{4\eta_1}{\pi} \left[ 1 - \left( 1 - \frac{16}{\pi^2} \eta_1^2 \right)^2 \right] \right] (1 + f) \\
\sigma &= \eta_1 + H - \sigma_e \\
\end{align*}
\]

\[
\begin{align*}
P &= \frac{\eta_1^2}{1 + 3(1 - \eta_1^2)(1 - 2\eta_1^2)^2} (\eta_1 + H - \sigma_e) \\
\end{align*}
\]
With $n_f = 0.5$, equations (43) and (44) reduce to

$$e = \left[ 1 + \frac{2H}{\pi} (1 - .44604 G_0) \right] (1 + f)$$

$$P = .064 (H + .5 - \sigma_e)$$

where $\sigma_e$ is determined from equation (22) with $n_f = 0.5$. For $n_f = 0.5$, some typical values of $e$ are

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
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<td>$e$</td>
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<td>1.10433</td>
<td>1.26529</td>
<td>1.55646</td>
<td>2.18486</td>
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</table>

Partial Span Diamond Boxwing

For this wing, $n_{af} = (1/2) n_f$, $H_a/H = (1/2) n_1$, $\sigma = \sqrt{n_1^2 + H^2}$, $n_{10} = n_1$, then equations (17) and (21) become

$$e = \left\{ 1 + \frac{2n_1 H}{\pi} \left[ 1 - \left( 1 - \frac{4}{\pi^2} n_1^2 \right) \left( 1 - \frac{1}{4} n_1^2 G_0 \right) \right] \right\} (1 + f)$$

$$P = \left[ \frac{\frac{1}{2} \frac{2}{\pi} n_1^2}{1 + 3(1 - \frac{1}{2} n_1^2)(1 - n_1^2) \sigma_e} \right]^{3/2}$$

With $n_1 = 0.5$

$$e = \left[ 1 + \frac{H}{\pi} (1 - .84251 G_0) \right] (1 + f)$$

$$P = .011339 (\sqrt{.25 + H^2} - \sigma_e)$$

For $n_1 = 0.5$, some typical values of $e$ are

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>1</td>
<td>1.00645</td>
<td>1.01685</td>
<td>1.06521</td>
<td>1.17650</td>
<td>1.44816</td>
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</table>
Partial Span Right Triangle Boxwing

For this wing, \( \eta_a = \eta_{af} = (1/2) \eta_1 \),

\[ \frac{H_a}{H} = \left( \frac{1}{2} \right) \eta_1 \],

\( \sigma = (1/4) \eta_1 + (1/4) \eta_2^2 + H^2 \),

\( \eta_{10} = \eta_1 \), then equations (17) and (21)

become

\[
e = \left\{ 1 + \frac{2 \eta_H}{\pi} \left[ 1 - \left(1 - \frac{4}{\pi^2} \eta_1^2 \right) \left(1 - \frac{1}{4} \eta_1^2 \right)^2 \right] \right\} (1 + f) \tag{50}
\]

\[
P = \left[ \frac{\frac{1}{2} \eta_1^2}{1 + 3 \left(1 - \frac{1}{2} \eta_1^2 \right) \left(1 - \frac{1}{2} \eta_1^2 \right)} \right]^{3/2} \cdot \left( \frac{1}{2} \eta_1 + \sqrt{\frac{1}{4} \eta_1^2 + H^2 - \sigma_e} \right) \tag{51}
\]

For \( \eta_1 = 0.5 \), typical values of \( e \) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.01853</td>
<td>1.07080</td>
<td>1.18579</td>
<td>1.46190</td>
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</tbody>
</table>

For \( \eta_1 = 1 \), typical values of \( e \) are

<table>
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<tr>
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<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1</td>
<td>1.03178</td>
<td>1.07181</td>
<td>1.23909</td>
<td>1.60153</td>
<td>2.37390</td>
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</tbody>
</table>

Ellipse Dumbbell Boxwing

In this wing

\[
\eta_a = \eta_{af} = \frac{1}{2} \left( \eta_1 + \eta_0 \right) + \frac{\pi}{8} \left( \eta_1 - \eta_0 \right)
\]

\[
\frac{H_a}{H} = \frac{\pi}{4} \left( \eta_1 - \eta_0 \right), \ \eta_{10} = \eta_1 - \eta_0 \tag{52}
\]

and \( \sigma = \sigma_e \), so \( P = f - 0 \)

where \( \sigma_e \) is from equation (24).
With these values and using equations (17) through (19), $e$ can be determined for arbitrary ellipse and position in the wing semispan. For the special case of circular boxwings with $\eta_1 = 1$, the wing appears as shown below.

For the circle dumbbell boxwing with $\eta_1 = 1$, the height parameter is $H = (1 - \eta_0)/2$, and the wing terms simplify to

$$
\begin{align*}
\eta_a &= \frac{1}{2} (1 + \eta_0) + \frac{\pi}{8} (1 - \eta_0) \\
\frac{H_a}{H} &= \frac{\pi}{4} (1 - \eta_0) \\
\eta_{10} &= 1 - \eta_0 \\
\eta_{10} H &= \frac{1}{2} (1 - \eta_0)^2
\end{align*}
$$

(53)

With the values of equation (53) and equations (17) through (19) typical values of $e$ are determined at $\eta_0 = .8, .7, .5, and 0$, which correspond to $H$ values of $.1, .15, .25, and .50$, respectively.

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>.10</th>
<th>.15</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>1</td>
<td>1.17494</td>
<td>1.24393</td>
<td>1.39304</td>
<td>1.64077</td>
</tr>
<tr>
<td>ref. 2, $e$</td>
<td>1</td>
<td>1.175</td>
<td>1.254</td>
<td>1.393</td>
<td>1.630</td>
</tr>
</tbody>
</table>

These circular dumbbell boxwing values compare well with the theory results from the application of the methods of reference 2.
In this wing,

\[ \eta_a = \eta_{af} = \eta_1 \]
\[ \eta_{10} = \eta_1 - \eta_0 = \frac{H_a}{H} \]
\[ \sigma = \eta_1 + H \]

With the terms of equation (54), \( e \) is determined by using equations (17) through (21) and (24). For the special case of a square dumbbell boxwing with \( \eta_1 = 1 \), then \( H = (1 - \eta_0)/2 \) and the equations simplify to

\[ e = \left\{ 1 + \frac{8}{\pi} H^2 \left[ 1 - (1 - \frac{32}{\pi^2} H)(1 - 2H)G_0 \right] \right\} (1 + f) + \]
\[ \left[ 1 - (2H)^{1.7} \right] (e_e - 1) \]

\[ G_0 = \frac{(1 - .38 H^2)(1 + 1.9304 H^4)}{(1 + 3.62 H^2)(1 + 1.4425 H^5)} \]
\[ \sigma = 1 + H, \sigma_e = 1 + .422112H \]
\[ p = \frac{8(0.577888)H^4}{\left[ 1 + 3(1 - 4 H^2)(1 - 8 H^2)^2 \right]^{3/2}} \]

where \( H = (1 - \eta_0)/2 \). With equation (55), typical values of \( e \) of the square dumbbell boxwing are determined at \( \eta_0 = .8, .7, .5, \) and 0, which correspond to \( H \) values of \( .1, .15, .25, \) and \( .50 \), respectively.

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>.10</th>
<th>.15</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1</td>
<td>1.20451</td>
<td>1.29238</td>
<td>1.50320</td>
<td>2.00026</td>
</tr>
</tbody>
</table>
For this wing,
\[ e = (1 + \frac{\pi}{4} H)(1 + f) + 0.560149 \left( e_e - 1 \right) \]

With the terms of equation (56), \( e \) can be determined by using equations (17) through (21) and (24). For the special case of \( e_0 = 0 \), and \( H_0/H = (\pi^2 - 8)/8 = 0.2337 \), the equations for determining \( e \) simplify to

\[ e = \frac{H_0}{H} + \frac{1}{2} \left[ 1 + e_0 + H + \frac{H_0}{H} \right] \]

Typical values of \( e \) for this bow-tie boxwing computed from equation (57) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1</td>
<td>1.24446</td>
<td>1.45077</td>
<td>2.03665</td>
<td>2.94646</td>
<td>4.69481</td>
</tr>
</tbody>
</table>

**APPLICATION TO FIN-WINGS**

Fin-wing configurations include end-plate wings, winglets, V-wings, cross-wings, and fan wings. Since in this class of wings \( H_a = 0 \), equation (17) simplifies to

\[ e = 1 + G \]

where $G_1$ as a function of $\eta_{af}$ is given in equation (19), and $\eta_{af}$, in equation (32). The prediction of $e$ for fin-wings can be demonstrated by application of the method for estimation of $\eta_{af}$.

**End-Plate Wings and V-Wings**

For the end-plate wing

\[ n_{a1} = n_{a2} = n_{11} = n_{01} = n_{13} = 1, \]
\[ \gamma_2 = -\gamma_1, \]

then equation (32) reduces to

\[ \eta_{af} = \frac{1 + 0.05 \frac{H_2}{H_1}}{1.05}, \text{ at } H_2 = H_1, \eta_{af} = 1 \]  

(59)

At $\eta_{af} = 1$, $G_1 = e_e - 1$, so by equation (58), $e = e_e$, (eq. 16).

For the V-wing

\[ n_{a1} = n_{a2} = 0.5, n_{11} = 1, n_{01} = n_{13} = 0, \gamma_2 = \gamma_1, \]

then equation (32) reduces to $\eta_{af} = 1/2$. At this value of $\eta_{af}$, $G_1 = e_v - 1$, so by equation (58), $e = e_v$, (eq. 15).

**Cruciform or Cross-Wing**

For the cross-wing

\[ n_{a1} = n_{a2} = 0.5, n_{11} = n_{12} = 1, \]
\[ n_{01} = n_{13} = 0, \]

then equation (32) reduces to
When \( \gamma_2 = \gamma_1 \), then \( \eta_{af} = \frac{1}{2} \), the configuration becomes the V-wing, and as before, \( e = e_v \). When \( \gamma_2 = -\gamma_1 \) and \( H_2 = H_1 \), the configuration becomes a symmetrical cross-wing, and equation (60) leads to \( \eta_{af} = 0.757267 \). With this value of \( \eta_{af} \), and equation (19), equation (58) gives \( e \) for the symmetrical cross-wing as

\[
e = 1 + \frac{1.84922 (e_v - 1)}{(.397872 + \sqrt{\frac{e_v - 1}{e_e - 1}})^2}
\]  

(61)

where \( e_v \) and \( e_e \) are given in equations (15) and (16), and in table 2. Some typical values of \( e \) are

<table>
<thead>
<tr>
<th>( H )</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e  )</td>
<td>1.0000</td>
<td>1.03997</td>
<td>1.11847</td>
<td>1.43415</td>
<td>2.00000</td>
<td>3.09219</td>
</tr>
</tbody>
</table>

where the \( H = 1 \) configuration is a 90 degree symmetrical cross-wing which as shown in appendix D has a value of \( e = 2 \).

Winglet Wings

For this wing,

\[
\eta_{a1} = \frac{1}{2} (1 + \eta_{01}), \quad \eta_{a2} = \frac{1}{2} (\eta_{12} + \eta_{01}),
\]

\( \eta_{11} = \frac{1}{2}, \quad \eta_{13} = \eta_{01} \)
then with equation (32)
\[
\eta_{af} = \left\{ 1 + \frac{1}{2} \eta_{12}^2 \sqrt{\frac{H_2^2 + (\eta_{12} - \eta_0)^2}{H_1^2 + (1 - \eta_0)^2}} (1 - \frac{\sin \gamma_2}{\sin \gamma_1}) \right\}
\]
\[\times \left[ 0.05 + \frac{(1 - \eta_0)(1 - \eta_0)^{1/2}}{2.1527} \right] \frac{1 + \eta_0}{2(1 + 0.05 \eta_0)} \]  
(62)

For a single winglet, \(\gamma_2 = \gamma_1\), then
\[
\eta_{af} = \frac{1 + \eta_0}{2(1 + 0.05 \eta_0)} \]  
(63)

At \(H = 1\), equation (58), with (19) and the \(e_e\) and \(e_v\) values at \(H = 1\), results in
\[
e = 1 + \frac{2.14961 \eta_{af}^2}{(1 + 0.1267 \sqrt{\eta_{af}})^2} \]  
(64)

Typical values of \(e\) for various locations of a single winglet for \(H = 1\), are

<table>
<thead>
<tr>
<th>(\eta_{01})</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1.00</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta_{af})</td>
<td>.50000</td>
<td>.62305</td>
<td>.74074</td>
<td>.85106</td>
<td>.95238</td>
<td>.97145</td>
</tr>
<tr>
<td>(e)</td>
<td>1.45265</td>
<td>1.68961</td>
<td>1.95892</td>
<td>2.24811</td>
<td>2.54421</td>
<td>2.60315</td>
</tr>
</tbody>
</table>

where the \(\eta_{01} = 0\) value is the same as for the V-wing.

For vertically symmetric winglets,
\(\gamma_2 = -\gamma_1\), \(\eta_{12} = 1\), \(H_2 = H_1\), then

equation (62) becomes
\[
\eta_{af} = \left\{ 1.05 + \frac{(1 - \eta_0)(1 - \eta_0)^{1/2}}{2.1527} \right\} \frac{1 + \eta_0}{2(1 + 0.05 \eta_0)} \]  
(65)
Then using equation (64), typical values of \( e \) for various \( \eta_{01} \) values of a wing with vertically symmetric winglets, for \( H = 1 \), are

<table>
<thead>
<tr>
<th>( \eta_{01} )</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1.00</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_{af} )</td>
<td>.757267</td>
<td>.84219</td>
<td>.89943</td>
<td>.94303</td>
<td>1.00</td>
<td>1.01498</td>
</tr>
<tr>
<td>( e )</td>
<td>2.00000</td>
<td>2.22356</td>
<td>2.38588</td>
<td>2.51568</td>
<td>2.69328</td>
<td>2.74146</td>
</tr>
</tbody>
</table>

where the \( \eta_{01} = 0 \) value is the same as for the symmetric cross-wing of \( H = 1 \), and the \( \eta_{01} = 1 \) value is the same as the end-plate wing of \( H = 1 \).

For the case of \( \gamma_2 = 0, \eta_{12} = 1, \) then \( H_2 = 0 \), and equation (62) becomes

\[
\eta_{af} = \left\{ \frac{1}{2} \left[ 1 + \frac{1}{2} \frac{\sqrt{(1 - \eta_{01})^2}}{H_1^2 + (1 - \eta_{01})^2} \right] \left[ .05 + \frac{(1 - \eta_{01})(1 - \eta_{01})^{1/2}}{2.1527} \right] \right\}^{1/3}
\]

\[
\frac{1}{2} \frac{\eta_{01}}{1 + .05 \eta_{01}^2}
\]

(66)

Then using equation (64), typical values of \( e \) for various \( \eta_{01} \) values of a wing with upper surface winglets, for \( H = H_1 = 1 \), are

<table>
<thead>
<tr>
<th>( \eta_{01} )</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_{af} )</td>
<td>.59096</td>
<td>.68880</td>
<td>.77623</td>
<td>.86222</td>
<td>.95238</td>
</tr>
<tr>
<td>( e )</td>
<td>1.62335</td>
<td>1.83499</td>
<td>2.04810</td>
<td>2.27930</td>
<td>2.54421</td>
</tr>
</tbody>
</table>
To analyze a \(2N - 1\) fan wing where

\[
\begin{align*}
\eta_{1,2n-1} &= \eta_{1,2n} = 1, \\
\gamma_{2n} &= -\gamma_{2n-1}, \\
H_{2n} &= H_{2n-1}, \\
\eta_{a,2n} &= \frac{1}{2}(1 + \eta_{01})
\end{align*}
\]

then from equation (32)

\[
\eta_{af} = \left[ 1.05 + \frac{(1 - \eta_{01})^{3/2}}{2.1527} + \frac{2}{9} \left( \frac{.77 + .9H}{1 + .67H} \right) (1 - \eta_{01})^2 \right] \left( 1 + \eta_{01} \right) + \frac{.77 + .9H}{1 + .67H} \left( \frac{1 - \eta_{01}^2}{1.15} \right) \sum_{n=3}^{N} \frac{H_{2n-1}}{H_1} \]

(67)

For equally spaced fins along \(H\)

\[
\begin{align*}
\frac{H_{2n-1}}{H_1} &= \frac{n - 2}{N - 1} \\
\sum_{n=3}^{N} \frac{H_{2n-1}}{H_1} &= \frac{1}{2} (N - 2)
\end{align*}
\]

(68)

For a three-fin fan wing, \(N = 2\), then from the first part of equation (67), the \(\eta_{af}\) values for various \(\eta_{01}\) span stations and typical \(e\) values at \(H = 1\) from the equation (64) are

<table>
<thead>
<tr>
<th>(\eta_{01})</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta_{af})</td>
<td>.86838</td>
<td>.92008</td>
<td>.94059</td>
<td>.95486</td>
<td>1</td>
</tr>
<tr>
<td>(e)</td>
<td>2.29667</td>
<td>2.44668</td>
<td>2.50828</td>
<td>2.55182</td>
<td>2.69328</td>
</tr>
</tbody>
</table>
For other values of $H$, $e$ can be determined from equation (58) with $G_1$ given in equation (19), where $\eta_{af}$ are those listed above.

For a five-fin fan wing, $N = 3$, and with equations (67) and (68), the $\eta_{af}$ values and typical $e$ values at $H = 1$ are

<table>
<thead>
<tr>
<th>$n_01$</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_{af}$</td>
<td>.94295</td>
<td>.98166</td>
<td>.98352</td>
<td>.97640</td>
<td>1</td>
</tr>
<tr>
<td>$e$</td>
<td>2.51544</td>
<td>2.63513</td>
<td>2.64098</td>
<td>2.61862</td>
<td>2.69328</td>
</tr>
</tbody>
</table>

For an infinite-fin fan wing, $N = \infty$, then with $H = 1$

<table>
<thead>
<tr>
<th>$n_01$</th>
<th>0</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_{af}$</td>
<td>1.0921</td>
<td>1.1048</td>
<td>1.0694</td>
<td>1.0195</td>
<td>1</td>
</tr>
<tr>
<td>$e$</td>
<td>2.9992</td>
<td>3.0433</td>
<td>2.9216</td>
<td>2.7561</td>
<td>2.6933</td>
</tr>
</tbody>
</table>

The $n_{01} = 0$ value of $e$ is the same as the bow-tie wing with $n_0 = H_0 = 0$, and $H = 1$. All the above $n_{01} = 1$ values of $e$ correlate with the $e$ of the symmetric end-plate wing with $H = 1$.

For the infinite-fin wing with $n_{01} = 0$, equation (67) simplifies to

$$\eta_{af} = 0.75727 + 0.3348 \frac{.77 + .9H}{1 + .67H}$$

Then with various $H$ values, typical values of $e$ from equations (58) and (19), are

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_{af}$</td>
<td>1.01507</td>
<td>1.02712</td>
<td>1.03775</td>
<td>1.06324</td>
<td>1.09207</td>
<td>1.12498</td>
</tr>
<tr>
<td>$e$</td>
<td>1</td>
<td>1.2525</td>
<td>1.4575</td>
<td>2.0677</td>
<td>2.9992</td>
<td>4.7419</td>
</tr>
</tbody>
</table>
These values of \( e \) for the infinite-fin wing with \( n_{01} = 0 \), are close to those listed for the bow-tie boxwing where \( H_0/H = .2337, n_0 = 0 \). It is apparent the \( H_0 = n_0 = 0 \) bow-tie and infinite-fin cross-wing have the same \( e \).

An example usage of the fin wing equations for other shapes is the case where

\[
\begin{align*}
n_{11} &= n_{12} = 1, \
\gamma_2 &= -\gamma_1, \
H_2 &= H_1, \\
n_{01} &= n_{13} = 0, \
\gamma_6 &= -\gamma_5 = -\pi/2, \
H_5 &= H_6 = H_1, \\
\eta_{a5} &= \eta_{a6} = 1
\end{align*}
\]

then equation (32) reduces to

\[
\eta_{af} = .75727 + .3344 \frac{.77 + .9H}{1 + .67H}
\]  \( (70) \)

Equation (70) is the same as equation (69), hence \( e \) is equal to that listed under equation (69).

APPLICATION TO MULTI-WINGS

Multi-Wings with Equal Spans

A vertical stack of wings with equal spans is pictured as
then \( H_a = H, n_a = 1 \), and equations (17) and (21) become

\[
e = (1 + \frac{4}{n} H)(1 + f)
\] (71)

\[
P = \sigma_M - \sigma_e
\] (72)

Comparing equations (71) and (10) shows, for small \( H \) so that \( P \) and \( f \) are small, that the \( e \) for multi-wings of equal spans approaches that of the rectangle boxwing. This is in accord with multi-plane thoery. The perimeter \( p \) for multi-wings requires a redefinition here since there are no vertical panels. In effect the \( h \) distances in \( p \) are subdued so that for equal span wings

\[
\sigma_M = 1 + \frac{H}{1 + 3.06 \frac{N-1}{N} H^{1/6} \sum_{n=2}^{N} \frac{H_n^2}{H}}
\] (73)

where \( \sigma_M \) denotes \( \sigma \) of a multi-wing aircraft and \( N \) is the number of wings and \( N - 1 \) is the number of spaces between wings. For equally vertically spaced wings, \( h_2 = h_3 = \ldots h_N \), then

\[
\sum_{n=2}^{N} \left( \frac{H_n}{H} \right)^2 = (N-1) \left( \frac{H_2}{H} \right)^2 = \frac{N-1}{(N-1)^2} \left[ \frac{H_2}{H/(N-1)} \right]^2 = \frac{1}{N-1}
\]

thus for wings of equal spans which are vertically equally spaced, the effective perimeter parameter is

\[
\sigma_M = 1 + \frac{H}{1 + 3.06 \frac{N}{N} H^{1/6}}
\] (74)

The \( e \) is determined by inserting \( \sigma_M \) of equation (73) or (74) into equation (72) and using equations (20) and (71). Equation (71) is accurate in the range \( 0 < H < 1 \), but somewhat higher as \( N \) becomes large. As mathematically proven in appendix D, in the limit for \( H \) very large, the \( e \) of the multi-wing and the lifts of wings of equal spans are simply
\[ e = N, \text{ and } \frac{L_n}{L_1} = 1 \]  \hspace{1cm} (75)

which is independent of \( H_n \) distribution. As the number \( N \) of wings becomes large, then as shown in equation (74), \( \sigma = 1 + H \), which is also the value for the rectangle boxwing. Thus the \( e \) of the multi-wing with an infinite number of equal span wings is identical to the rectangle boxwing of the same \( H \).

Multi-Wings with Arbitrary Spans

A vertical stack of wings with various spans is pictured as

The effective integrated value for \( \eta_a \) is the sum of the rectangle areas formed by each of the multiwings, then

\[ \eta_a = \frac{H_a}{H} = \sum_{n=2}^{N} \frac{H_n}{H}, \text{ where } H = \sum_{n=2}^{N} H_n \]  \hspace{1cm} (76)

where \( \frac{H_n}{H} = \left( \frac{h_n/b}{h_m/b} \right) \), and \( b \) is the span of the widest wing. The effective perimeter of the multiwing is approximated by

\[ \sigma = \frac{\eta_2}{2} + H_2 + \sum_{n=2}^{N} \sqrt{\left( \frac{\eta_n - \eta_{n+1}}{2} \right)^2 + H_{n+1}^2} \]  \hspace{1cm} (77)
The multiwing perimeter parameter is

\[ \sigma_M = \eta_2 + \frac{\sqrt{((\sigma - \eta_2)^2 + (1 - \eta_2)^2B^2 - (1 - \eta_2)B}}{1 + B} \]  

(78)

where

\[ B = 3.06 \frac{N - 1}{N} H^{1/6} \sum_{n=2}^{N} (2 - \eta_n)^2 \left( \frac{H}{H_n} \right)^2 \]  

(79)

As N approaches infinity the multi-wings outline a boxwing and equations (76) and (77) become the values of the boxwings. Then since at N = \( \infty \), B = 0, the prediction of \( e \) for the multi-wing of infinite wings is identical to that of the boxwing formed by the outline of the infinite wings.

The \( \eta_{10} \) parameter is \( \eta_2 \) which represents the span extent of the wing with second largest span, then

\[ \eta_{10} = \eta_2 \]  

(80)

The elliptic \( \varphi_e \) parameter is for a partial span to the \( \eta_2 \) or \( \eta_{10} \) span station. The values for \( \varphi_e \) are determined from using \( \sigma_e/\eta_{10} \) instead of \( \sigma_e/\eta_1 \) in equation (22) or in table 2. Thus with \( \eta_a \) and \( H_a/H \) from equation (76), and \( \sigma_M \) from equation (78), then P is evaluated from equation (21) which becomes

\[ P = \left[ \frac{\eta_{10}\eta_a}{1 + 3(1 - \eta_{10}\eta_a)(1 - 2\eta_{10}\eta_a)} \right]^{3/2} \left( \sigma_M - \eta_{10} \frac{\sigma_e}{\eta_{10}} \right) \]  

(81)

With P, f is found from equation (20), then e computed from equations (17) and (18).

Accuracy is ensured to H values such that \( e/(1 + f) \) is the same as the \( H = \infty \) value of e. The \( H = \infty \) value is derived in appendix D where from equation (D9), for maximum e at \( H = \infty \)

\[ e = 1 + \sum_{n=2}^{N} n^2, \quad \frac{L_n}{L_1} = n^2 \]  

(82)
For finite height parameter $H$, $e$ and $L_n$ are less than these values.

The biplane solution has $N = 2$, then

$$H_2 = H, \eta_a = H_a/H = \eta_{10} = \eta_N = \eta_2$$

and equations (77), (78), and (79) become

$$\sigma = \eta_2 + H, B = 1.53(2 - \eta_2)^2 H^{1/6}$$

$$\sigma^2 = \eta_2 + \sqrt{H^2 + (1 - \eta_2)^2 B^2} - (1 - \eta_2)B$$

With these values $P$ is determined from equation (81) and $e$ from equations (17), (18), and (20).

**Diamond Shaped Multi-Wings**

A diamond shaped multi-wing can be represented by letting wing gaps and wing spans be

$$H_n = \frac{H}{N - 1}$$

$$\eta_n = \frac{(1/2(N + 1)) + 1 - n}{(1/2)(N + 1)}$$

where $N$ is an odd integer. Then from equation (76)

$$\eta_a = \frac{H_a}{H} = 2 \sum_{n=2}^{N} \frac{1}{n^2} \frac{H_n}{H} = \frac{4}{N^2 - 1} \sum_{n=2}^{N} \frac{(1/2)(N + 1)}{n^2} \left(\frac{N + 3}{2} - n\right) = \frac{1}{2} \left[\frac{N + 1}{2} - 1\right] \frac{N + 1}{2} \left(\frac{N + 1}{2} \left(\frac{N + 1}{2} + 1\right) + 1\right) = \frac{1}{2}$$
or for all odd integer \( N \), \( \eta_a = H_a/H = 1/2 \). The \( \eta_{10} \) value is

\[
\eta_{10} = \frac{N - 1}{N + 1} \quad (86)
\]

The perimeter is

\[
\sigma = \frac{n_{N+1}}{2} + 2H_2 + \sqrt{(\eta_2 - \frac{N + 1}{2})^2 + (H - 2H_2)^2}
\]

\[
= \frac{2}{N + 1} + \frac{2H}{N - 1} + \sqrt{\left(\frac{N - 3}{N + 1}\right)^2 + \left(\frac{N - 3}{N - 1}\right)^2 H^2} \quad (87)
\]

The multi-wing perimeter parameter \( \sigma_M \) is determined from equation (78) in which \( \eta_2 \) is given by equation (86). \( B \) from equation (79) summed for this diamond shape is

\[
B = \frac{6.12}{N} \frac{H^{1/6}}{1 + 7/\sqrt{N}} \quad (88)
\]

With \( \sigma_M \) evaluated, \( P \) is computed from equation (81) in which \( \eta_a = 1/2 \), \( \eta_{10} \) is from equation (86), and \( \sigma_e \) is with the ellipse extending to \( \eta_{10} \).

When \( N \) becomes infinite, these parameters become \( \eta_a = H_a/H = 1/2 \) (for any \( N \)), \( \eta_{10} = \eta_2 = 1 \), \( B = 0 \), \( \sigma_M = \sigma = \sqrt{1 + H^2} \). These values are the same as those listed in equations (34) and (35) for the diamond boxwing, hence \( e \) is the same as those listed in table 1 for the diamond.

**Numerical Values of \( e \) for Multi-Wings**

For equal span wings and vertically equally spaced wings, for the biplane \( N = 2 \), and for the triplane \( N = 3 \), equation (74) becomes

\[
\sigma_M = 1 + \frac{H}{1 + 1.53 H^{1/6}} \quad (\text{biplane})
\]

\[
= 1 + \frac{H}{1 + 1.02 H^{1/6}} \quad (\text{triplane}) \quad (89)
\]
With equation (89), P is evaluated from equation (72) with \( \sigma_e \) of the ellipse extending to \( \eta_1 = 1 \). With P, f is evaluated from equation (20), then e from equation (71). These e's are listed in table 4 for various H values with the \( H = \infty \) values from equation (75).

For a partial span wing on the biplane, \( N = 2, H_2 = H, \eta_a = H_a/H = \eta_{10} = \eta_2 \), and \( \sigma_M \) is given in equation (83), where \( \eta_2 \) denotes the spanwise extent of the lower wing. Equation (81) becomes

\[
P = \left[ \frac{\eta_2^2}{1 + 3(1 - \eta_2^2)(1 - 2\eta_2^2)^2} \right]^{3/2} (\sigma_M - \sigma_e)
\]

(90)

where \( \sigma_e \) is with the ellipse extending to the \( \eta_{10} = \eta_2 \) span station. With P, f is determined from equation (20). Equation (17) becomes

\[
e = \left\{ 1 + \frac{4}{\pi} \eta_2 H \left[ 1 - \left( 1 - \frac{16}{\pi^2} \eta_2^2 \right)^2 \right](1 - \eta_2^2)G_0 \right\}(1 + f)
\]

(91)

where \( G_0 \) is given in equation (18) in which \( \eta_{10} = \eta_2 \). The e for the biplane with \( \eta_2 = 0.6 \) computed from equation (91) is listed in table 4 for various H values. The \( H = \infty \) value of e is from equation (82).

**TABLE 4. - IDEAL e OF VARIOUS MULTI-WINGS**

<table>
<thead>
<tr>
<th>H</th>
<th>equal span biplane eq. (89)</th>
<th>equal span triplane eq. (89)</th>
<th>( \eta_2 = 0.6 ) span biplane eq. (91)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>.1</td>
<td>1.2021</td>
<td>1.2176</td>
<td>1.0456</td>
</tr>
<tr>
<td>.2</td>
<td>1.3529</td>
<td>1.3834</td>
<td>1.0876</td>
</tr>
<tr>
<td>.3</td>
<td>1.4810</td>
<td>1.5315</td>
<td>1.1301</td>
</tr>
<tr>
<td>.4</td>
<td>1.5842</td>
<td>1.6654</td>
<td>1.1769</td>
</tr>
<tr>
<td>.5</td>
<td>1.6407</td>
<td>1.7839</td>
<td>1.2154</td>
</tr>
<tr>
<td>( \infty )</td>
<td>2</td>
<td>3</td>
<td>1.36</td>
</tr>
</tbody>
</table>

34
CONCLUSIONS

It has been demonstrated that the application of the generalized equations (17) through (32) and equations (76) through (82) provides a direct method for estimates of ideal $e$ for arbitrary lateral-vertical wing forms. Example predictions of $e$ are made for a wide range of wing configurations, from multi-wings to fan and dumbbell wings. Several check points were verified with other methods including the cruciform at $H = 1$ (eq. 61), the circle dumbbell (eq. 53), and biplane and triplane (table 4 with ref. 1). Also, multi-wing solutions converge to those boxwings as the number of wings increase, and fan wing solutions converge to those of bow-tie wings as the number of fins increase. For the V-wing, an exact closed function was developed. An asymptotic unique maximization solution for multi-wings was found which resulted in exact expressions for ideal $e$ and ideal lift of each wing.

Hampton Technical Center, Kentron International, Inc.
Hampton, Virginia 23666
June 20, 1980
From reference 3, the added mass $M$ is given by

$$M = \pi \rho \frac{1 - \alpha}{\alpha}$$

where (here $\gamma$ notation replaces $\beta$ of ref. 3)

$$\alpha = \frac{1}{2} - \frac{\gamma}{\pi} \tag{A1}$$

The dimensionless added mass is given by

$$\frac{M}{\rho \pi \frac{a}{2}^2} = \frac{1 - \alpha}{\alpha \frac{a}{2}^2} \tag{A2}$$

where $a$ is determined from the definite integral

$$a = \int_0^1 \frac{\xi d\xi}{(1 - \xi)^{1-\alpha}(c + \xi)^\alpha} \tag{A3}$$

where

$$c = \frac{1 - \alpha}{\alpha}, \quad 0 \leq \alpha \leq \frac{1}{2} \tag{A4}$$

This integral was evaluated numerically in reference 3 for various values of $\gamma$, then dimensionless added mass was computed and tabulated. However, as will be shown here, the integration in equation (A3) can be obtained by expanding the second parenthesis term into a binomial series. Thus

$$\frac{\xi}{(c + \xi)^\alpha} = \frac{1}{\alpha c} \left[ \xi - \frac{\alpha}{c} \xi^2 + \frac{\alpha}{2!c^2} (\alpha + 1) \xi^2 + \ldots + \right.$$

$$\left. \frac{(-1)^n\alpha}{n!c^n} (\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1) \xi^{n+1} + \ldots \right] \tag{A5}$$
The definite integrals of the following type can be evaluated:

\[
\int_0^1 \frac{\xi^{n+1}}{(1 - \xi)^{1-\alpha}} d\xi = \frac{1}{\alpha(\alpha + 1)} - \frac{2}{\alpha(\alpha + 1)(\alpha + 2)} + \frac{3}{\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)} - \ldots + \left\{ \begin{array}{ll}
\frac{n+1}{(\alpha + n)(\alpha + n + 1)c^n} & \text{for } \alpha > -1, n = 0, 1, 2, \ldots
\end{array} \right. \]

Inserting equation (A5) into equation (A3) and applying equation (A6) leads to

\[
a = \frac{1}{c^\alpha} \left[ \frac{1}{\alpha(\alpha + 1)} - \frac{2}{(\alpha + 1)(\alpha + 2)c} + \frac{3}{(\alpha + 2)(\alpha + 3)c^2} - \ldots + \frac{(-1)^n(n + 1)}{(\alpha + n)(\alpha + n + 1)c^n} \right] + \ldots \tag{A7}
\]

With \( c \) given in equation (A4), equation (A7) can be written as

\[
a = \left( \frac{1 - \frac{\alpha}{\alpha}}{\alpha} \right)^{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n(n + 1)}{(n + \alpha)(n + 1 + \alpha)} \left( \frac{\alpha}{1 - \alpha} \right)^{n+1} \tag{A8}
\]

For the range of \( 0 \leq \alpha \leq 1/2 \), this summation is identically unity. Thus the exact solution for \( a \) is simply

\[
a = \left( \frac{1 - \alpha}{\alpha} \right)^{1-\alpha} \tag{A9}
\]

With equation (A2) the dimensionless added mass is

\[
M = \frac{2}{\rho \pi a} \left( \frac{\alpha}{1 - \frac{\alpha}{\alpha}} \right)^{1-2\alpha} \tag{A10}
\]
From reference 3, the induced drag is given by

\[ D_i = \frac{L^2}{2MV^2} \]  

(A11)

which equated to equation (1) results in

\[ e = \frac{4a^2}{b^2} \left( \frac{M}{\rho a^2} \right) \]  

(A12)

From the V-wing sketch the geometrical a and \( \gamma \) are

\[ \frac{2a}{b} = \frac{1}{\cos \gamma} = (1 + 4H^2)^{1/2}, \gamma = \tan^{-1} 2H \]  

(A13)

With equation (A1), the dimensionless added mass and factor \( e \), in terms of the \( \gamma \) angle in radians, in closed functions, are

\[ \frac{M}{\rho a^2} = \left( \frac{1 - \frac{2 \gamma}{\pi}}{1 + \frac{2 \gamma}{\pi}} \right)^{\frac{2 \gamma}{\pi}} \]  

(A14)

\[ e = \frac{1}{\cos^2 \gamma} \left( \frac{1 - \frac{2 \gamma}{\pi}}{1 + \frac{2 \gamma}{\pi}} \right)^{\frac{2 \gamma}{\pi}} \]  

(A15)

In terms of the height parameter, \( H = \frac{h_m}{b} \)

\[ \frac{M}{\rho a^2} = \left( \frac{1 - \frac{2}{\pi} \tan^{-1} \frac{2H}{\pi}}{1 + \frac{2}{\pi} \tan^{-1} \frac{2H}{\pi}} \right)^{\frac{2}{\pi} \tan^{-1} 2H} \]  

(A16)

\[ e = (1 + 4H^2) \left( \frac{1 - \frac{2}{\pi} \tan^{-1} \frac{2H}{\pi}}{1 + \frac{2}{\pi} \tan^{-1} \frac{2H}{\pi}} \right)^{\frac{2}{\pi} \tan^{-1} 2H} \]  

(A17)

Equations (A15) and (A17) give \( e \) in terms of elementary functions of either the dihedral angle or the height parameters of the V-wing.
APPENDIX B

END-PLATE WING e FACTOR

From page 211 of reference 1,

\[ \gamma = \beta k \text{ sn} n, \quad \frac{\gamma}{\beta} = \left(\frac{K - E}{K}\right)^{1/2} \quad (B1) \]

then

\[ \text{sn} n = \frac{1}{k} \left(\frac{K - E}{K}\right)^{1/2} = \sin \phi \quad (B2) \]

From the definition of snn related to \( \sin \phi \), \( n \) is

\[ n = F(\phi \alpha) = \int_{0}^{\phi} \frac{d\phi}{(1 - \sin^{2} \alpha \sin^{2} \phi)^{1/2}} \quad (B3) \]

that is, the incomplete elliptic integral of the first kind. In addition, from reference 1,

\[ h_m = 2 \beta \left[ E(n) - (1 - \frac{\gamma^2}{\beta^2}) n \right], \quad b = \frac{\beta \pi}{k}, \]

\[ e = \frac{4 \beta^2}{b^2} \left(\frac{2E}{K} - k' \right) \quad (B4) \]

where \( k = \sin \alpha \), \( k' = (1 - k^2)^{1/2} = \cos \alpha \). With equations (B1), (B2), (B3), the \( h_m \) of equation (B4) becomes

\[ H = \frac{2K}{\pi} \left[ E(\phi \alpha) - \frac{E}{K} F(\phi \alpha) \right] \quad (B5) \]

\[ \phi = \sin^{-1} \left[ \frac{1}{\sin \alpha} \left(\frac{K - E}{K}\right)^{1/2} \right] \quad (B6) \]

where \( F(\phi \alpha) \) and \( E(\phi \alpha) \) are incomplete elliptic integrals of the first and second kinds with amplitude \( \phi \) and modular angle \( \alpha \), and \( K \) and \( E \) are complete elliptic integrals of first and second kind with modular angle \( \alpha \).
within the brackets of equation (B5) is the Jacobian Zeta Function. Then

\[ H = \frac{2}{\pi} KZ(\phi \alpha) \]  

(B7)

where the function \( KZ(\phi \alpha) \) is tabulated in reference 4 for various values of \( \phi \) and \( \alpha \), and has a maximum value at the \( \phi \) given in equation (B6). From equation (B4)

\[ e = \frac{4}{\pi^2} k^2 \left( \frac{2E}{K} - \cos^2 \alpha \right) \]  

(B8)

As can be seen from equations (B5) through (B8), the factor \( e \) is not a direct function of \( H \) but is related to \( H \) only through the parameter \( \alpha \). That is, a value for \( \alpha \) is specified, then \( \phi \) is determined from equation (B6), then \( H \) from either equation (B5) or (B7), and \( e \) from equation (B8). Some typical results for a range of \( \alpha \) values are shown as follows:

<table>
<thead>
<tr>
<th>( \alpha ) deg</th>
<th>( H )</th>
<th>( e )</th>
<th>( r )</th>
<th>( \alpha ) deg</th>
<th>( H )</th>
<th>( e )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>.07194</td>
<td>1.14138</td>
<td>.6996</td>
<td>80</td>
<td>.89782</td>
<td>2.53705</td>
<td>.7312</td>
</tr>
<tr>
<td>45</td>
<td>.17351</td>
<td>1.33322</td>
<td>.7082</td>
<td>82</td>
<td>1.0165</td>
<td>2.71842</td>
<td>.7296</td>
</tr>
<tr>
<td>60</td>
<td>.34824</td>
<td>1.64573</td>
<td>.7200</td>
<td>84</td>
<td>1.1718</td>
<td>2.95205</td>
<td>.7263</td>
</tr>
<tr>
<td>68</td>
<td>.49550</td>
<td>1.89566</td>
<td>.7268</td>
<td>85</td>
<td>1.2713</td>
<td>3.10002</td>
<td>.7233</td>
</tr>
<tr>
<td>75</td>
<td>.68710</td>
<td>2.20712</td>
<td>.7318</td>
<td>88</td>
<td>1.7821</td>
<td>3.84313</td>
<td>.7048</td>
</tr>
</tbody>
</table>

This list includes the parameter \( r \) which is evaluated from

\[ r = \frac{1 + 2H - e}{(e - 1 - \frac{4}{\pi} H)H} \]  

(B9)

Equations (B5) through (B8) are quite complicated functions and do not permit \( e \) predictions for a specified value of \( H \). It has been found that a relatively simple function exists between \( e \) and \( H \). In the limit as \( \alpha \to 0 \) equations (B7) and (B8) become

\[ H \to \frac{1}{4} k^2, \quad e \to 1 + \frac{1}{2} k^2 \]
where \( k = \sin \alpha \rightarrow 0 \). Eliminating \( k^2 \) results in

\[
H \rightarrow 0 \\
e = 1 + 2H
\]

For \( \alpha \rightarrow \pi/2 \), equations (B7) and (B8) become

\[
H \rightarrow \frac{2}{\pi} \ln \frac{4}{k'} \quad , \quad e \rightarrow \frac{8}{\pi^2} \ln \frac{4}{k'}
\]

where \( k' = (1 - k^2)^{1/2} = \cos \alpha \). Eliminating the log term, \( e \) becomes

\[
H \rightarrow \infty \\
e = \frac{4}{\pi} H
\]

A combination of equations (B10) and (B11) indicates a function of the following type

\[
e = 1 + \left( \frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + \frac{rH}{H}} \right) H
\]

where \( r \) is a quasi-constant. The solution of equation (B12) for \( r \) is equation (B9). As can be seen from the listed \( H \) and \( r \) values, \( r \) is nearly constant with \( H \), but remains a weak function of \( H \). An approximation for \( r \) which duplicates those in the list is given by

\[
r = .732 - \frac{.0612 (.8 - H)^2}{1 + .3688 H^2}
\]

Then equation (B12) gives \( e \) with a four to five digit accuracy for the wing with end plates as

\[
e = 1 + \left[ \frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + .732 H - \frac{.0612 H(.8 - H)^2}{1 + .3688 H^2}} \right] H
\]

\[
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APPENDIX C

90° CRUCIFORM WING

As discussed in reference 5, page 39, in a 90° cruciform wing there is no induced velocity at one vorticity sheet due to the other. In this case each wing acts independently of the other, thus for minimum induced drag the spanwise loading distribution is elliptical along \( b_w \). With \( L_{n1} \) and \( L_{n2} \) representing the lift normal to wings 1 and 2, with equation (1)

\[
D_i = \frac{L_{n1}^2}{\pi q b_w^2} + \frac{L_{n2}^2}{\pi q b_w^2} = \frac{L^2}{\pi q b_w^2}
\]

since \( L_{n1}^2 + L_{n2}^2 = L^2 \). Thus for this case \( e = 1 \). In reference 5 when the \( C_L \), \( C_{D_i} \), and \( A \) are reduced to dimensional quantities with \( b_w \), the solution is also \( e' = 1 \).

For a 90° cruciform wing banked at angle \( \phi \), the loading is elliptical along wings 1 and 2, and, similarly to equation (C1), the induced drag is

\[
D_i = \frac{L_{n1}^2}{\pi q b_w^2} + \frac{L_{n2}^2}{\pi q b_w^2} = \frac{L^2}{\pi q b_w^2}
\]

since again \( L_{n1}^2 + L_{n2}^2 = L^2 \), and hence \( e = 1 \). Thus when the induced drag of a 90° cruciform is in terms of the wing width, \( b_w \), the efficiency factor is unity at any bank angle.

Generally the induced drag of a lifting configuration is in terms of the lateral span of the configuration, that is, \( b \). Then from equation (1), the
induced drag of the 90° cruciform wing is

\[ D_i = \frac{L^2}{\pi q b^2 e} \quad (C3) \]

Equating equations (C3) and (C2) results in

\[ e = \left( \frac{b_{W}}{b} \right)^2 = \frac{1}{\cos^2\left(\frac{\pi}{4} - |\phi|\right)} \quad (C4) \]

where \( \phi \) is the angle of bank of the 90° cruciform wing. When \( \phi = 0, \pm \pi/8, \pm \pi/4 \), then \( e = 2, 1.176, \) and 1, respectively. In summary for the 90° cruciform wing

\[ e = 1, \text{ for all bank angles when } D_i \text{ is defined by equation (C1)}, \]

\[ e = \text{equation (C4), for all bank angles when } D_i \text{ is defined by equation (C3)}. \]
APPENDIX D

ASYMPTOTIC SOLUTION FOR OPTIMUM e AND LIFTS FOR MULTI-WINGS WITH LARGE GAPS

If the gap height $h_n$ of a multi-wing is large enough such that the influence of one wing on another is negligible, then the ideal loading for each wing will be elliptic and induced drag is given by $D_{ln} = \frac{L_n^2}{\pi q b_n^2}$. The sum of these induced drags equals that of the multi-wing having the sum of all the lifts. Thus for a multi-wing with $N$ planar wings

$$D_i = \frac{L_i^2}{\pi q b_1 e} = \sum_{n=1}^{N} \frac{L_n^2}{\pi q b_n^2}, \quad L = \sum_{n=1}^{N} L_n$$  \hspace{1cm} (D1)

where $b_1$ is the wing with maximum span. In dimensionless coordinates,

$$\varphi_n = \frac{L_n}{L_1}, \quad \eta_n = \frac{b_n}{b_1}$$  \hspace{1cm} (D2)

then equation (D1) becomes

$$\left(1 + \sum_{n=2}^{N} \varphi_n\right)^2 e = 1 + \sum_{n=2}^{N} \frac{\varphi_n^2}{\eta_n^2}$$

solving for $e$

$$e = \frac{(1 + \sum_{n=2}^{N} \varphi_n)^2}{1 + \sum_{n=2}^{N} \frac{\varphi_n^2}{\eta_n^2}}$$  \hspace{1cm} (D3)

Equation (D3) is valid for any lift variation among the wings, however, there is some lift variation for which $e$ is maximum. For equal loading $e$ can be less than unity when $1 + \eta_2^{-2} + \eta_3^{-2} + \ldots + \eta_N^{-2} > N^2$. The maximization technique of reference 5 can be applied to equation (D3). A partial derivative
of $e$ with respect to each of the loading ratios gives

$$\frac{\partial e}{\partial \eta_m} = \frac{2(1 + \sum_{n=2}^{N} \frac{\eta_n^2}{n^2})}{(1 + \sum_{n=2}^{N} \frac{\eta_n^2}{n^2})^2} \left[ 1 + \sum_{n=2}^{N} \frac{\eta_n^2}{n^2} - (1 + \sum_{n=2}^{N} \eta_n) \frac{\eta_m}{n_m} \right]$$

The conditions for maximum $e$ are that these derivatives be zero for each of the loading ratios. Then

$$1 + \sum_{n=2}^{N} \frac{\eta_n^2}{n^2} - (1 + \sum_{n=2}^{N} \eta_n) \frac{\eta_m}{n_m} = 0$$

These are non-linear simultaneous equations in $\eta_n$ with $n = 2, 3, \ldots N$ unknowns, and $m = 2, 3, \ldots N$ equations. For the biplane, $N = 2$, then equation (D4) with (D3) leads to

$$1 + \frac{\eta_2^2}{n_2^2} - (1 + \eta_2) \frac{\eta_2}{n_2} = 0, \text{ or } \eta_2 = \frac{n_2}{e} = 1 + \eta_2$$

$$e = 1 + \eta_2$$

For the triplane, $N = 3$, then the solution of two equations of (D4) leads to

$$1 + \frac{\eta_3^2}{n_3^2} - (1 + \eta_3) \frac{\eta_3}{n_3} = 0$$

which involves the solution of a cubic equation. As can be seen, the solution of equation (D4) becomes complicated when $N \geq 3$. What is needed is to have $\eta_n$ as some function of $\eta_n$ so that equation (D4) is satisfied. Such a function has been discovered by the hint given in equation (D5). Let
\[ \lambda_n = \eta_n^2 \]  

then equation (D4) becomes

\[ 1 + \sum_{n=2}^{N} \eta_n^2 - \left(1 + \sum_{n=2}^{N} \eta_n^2 \right) \frac{\eta_m^2}{\eta_m^2} = 0 \]  

which completely satisfies the maximum conditions, and hence equation (D7) gives the individual wing lifts for maximum \( e \). With equation (D7) inserted into equation (D3) the maximum or ideal \( e \) is

\[ \lambda_n = \eta_n^2, \ e = e_{\text{max}} = 1 + \sum_{n=2}^{N} \eta_n^2 \]  

For equal span wings, \( \eta_n = 1 \), so

\[ \lambda_n = 1, \ e = N \]  

For biplanes

\[ \lambda_2 = \eta_2^2, \ e = 1 + \eta_2^2 \]  

For triplanes

\[ \lambda_2 = \eta_2^2, \lambda_3 = \eta_3^2, \ e = 1 + \eta_2^2 + \eta_3^2 \]  

and so forth.
REFERENCES


A relatively simple equation is presented for estimating the induced drag ideal efficiency factor $e$ for arbitrary cross-sectional wing forms. This equation is based on eight basic but varied wing configurations which have exact solutions. The $e$ function which relates the basic wings is developed statistically and is a continuous function of configuration geometry. The basic wing configurations include boxwings shaped as a rectangle, ellipse, and diamond; the V-wing; end-plate wing; 90 degree cruciform; circle dumbbell; and biplane. Example applications of the $e$ equations are made to many wing forms such as wings with struts which form partial span rectangle dumbbell wings; bowtie, cruciform, winglet, and fan wings; and multi-wings. Derivations are presented in the appendices of exact closed form solutions found of $e$ for the V-wing and 90 degree cruciform wing and for an asymptotic solution for multi-wings.