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Accuracy of the Determination of Mean Anomalies and Mean Geoid Undulations from a Satellite Gravity Field Mapping Mission

by

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Abstract

Improved knowledge of the earth's gravity field can be obtained from new and improved satellite measurements such as satellite to satellite tracking and gradiometry. This improvement has been examined by estimating the accuracy of the determination of mean anomalies and mean undulations in various size blocks based on an assumed mission. In this report the accuracy is considered through a commission error due to measurement noise propagation and a truncation error due to unobservable higher degree terms in the geopotential. To do this the spectrum of the measurement has been related to the spectrum of the disturbing potential of the earth's gravity field. Equations were derived for a low-low (radial or horizontal seperation) mission and a gradiometer mission.

For a low-low mission of six month's duration, at an altitude of 160 km, with a data noise of $\pm 1\,\mu m/sec$ for a four second integration time, we would expect to determine $1^\circ \times 1^\circ$ mean anomalies to an accuracy of ± 2.3 mgals and $1^\circ \times 1^\circ$ mean geoid undulations to ± 4.3 cm. A very fast Fortran program is available to study various mission configurations and block sizes.

Foreword

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Introduction

A number of studies have recently been carried out to investigate the accuracy of the determination of the earth's gravity field from a gravity mapping satellite mission (or GRAVSAT). Such a mission has been discussed extensively for the past several years with a summary of applications and questions being found in the National Academy of Science report: Applications of a Dedicated Gravitational Satellite Mission.

The studies performed to date include the following: Douglas et al. (1980) who describe an error analysis where mean gravity anomalies are unknown to be solved for in a least squares adjustment procedure; Rapp and Hajela (1979) who describe the use of least square collocation methods to a high-low satellite-to-satellite tracking mission; Breakwell (1980a,b) who describes an optimum filter approach in the frequency domain making a reasonable flat earth approximation; Pisacane and Yionoulis (1980) who describe gravity improvement tests made with ORAN (Orbit Analysis Program) program; Lancaster et al. (1980) who describe error analysis studies for one degree mean anomalies using an adjustment procedure; and Rummel (1980) who describes the accuracies achievable from a low-low SST mission using least squares collocation methods.

The above, and other, studies have given much additional insight into what might be obtained from a gravity field mapping mission. However, the studies still have not given a wholly satisfactory picture. For example, some studies only look at SST data; some solve for anomaly accuracies only; some use limited data sets; some use flat earth approximations. We wished to develop a rapid error analysis procedure that could be used for SST missions and gradiometer missions; that could be used for mean anomaly and mean undulation accuracy estimation; and that could be used with almost continuous data. This led to the analysis and results described in this report. We do not believe this work cannot be improved upon; there will be areas where assumptions are made and missions not considered that could be (perhaps) worked with at a later time. We do feel the method presented here is reasonable and easy to use on computer implementation.

The Method

The basic method used here is the analysis of the gravity field mapping in the frequency domain with final results discussed in terms of mean gravity anomalies and mean geoid undulations on the surface of the earth. The types of measurements to be considered are velocity (or velocity differences) and vertical component gradiometer measurements.

The errors in the measurements are assumed to be uncorrelated. This type of random process can be approximated as white noise, although this is somewhat unrealistic since pure white noise is associated with infinite variance as the frequencies are affected equally throughout their infinite range. However, for the present purposes, it will be postulated that the error spectrum is flat at least over the frequencies of the measured data.

To derive the error spectrum, we suppose that the physical quantity to be measured, say F, can be expressed at the satellite altitude as a series of spherical harmonic functions. Therefore, let

$$F(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm}\, \bar{Y}_{nm}(\theta,\lambda) \tag{1}$$

where (θ,λ) are spherical coordinates and the $\bar{Y}_{nm}$ are fully normalized harmonic functions satisfying the orthogonality relationship

$$\iint_{\sigma} \bar{Y}_{nm}(\theta,\lambda)\, \bar{Y}_{kl}(\theta,\lambda)\, d\sigma = \begin{cases} 4\pi & \text{if } n=k \text{ and } m=l \\ 0 & \text{if } n\neq k \text{ or } m\neq l \end{cases} \tag{2}$$

with σ representing the unit sphere. The coefficients $\bar{f}_{nm}$ are constants given by

$$\bar{f}_{nm} = \frac{1}{4\pi} \iint_{\sigma} F(\theta,\lambda)\, \bar{Y}_{nm}(\theta,\lambda)\, d\sigma \;;\; |m| \leq n, \quad n \geq 0 \tag{3}$$

and constitute the spectrum of F. The error in the spectrum is thus

$$\epsilon(\bar{f}_{nm}) = \frac{1}{4\pi} \iint_{\sigma} \epsilon(F(\theta,\lambda))\, \bar{Y}_{nm}(\theta,\lambda)\, d\sigma \;;\; |m| \leq n, \quad n \geq 0 \tag{4}$$

There is at most a finite number of globally distributed measurements, which leads to the approximation

$$\epsilon(\bar{f}_{nm}) \approx \frac{1}{4\pi} \sum_{k} \epsilon(F_k)\, \bar{Y}_{nm}(\theta_k,\lambda_k)\, \Delta\sigma \;;\; |m| \leq n, \quad n \geq 0 \tag{5}$$

F_k is the k-th measurement in the area element $\Delta\sigma$ centered at (θ_k,λ_k). It is assumed that all measurements are made in areas (blocks) of equal size and that the blocks uniformly cover the entire sphere (requiring polar satellites). Squaring the error yields

$$\epsilon^2(\bar{f}_{nm}) = \frac{1}{16\pi^2} \sum_{k} \sum_{l} \epsilon(F_k)\, \epsilon(F_l)\, \bar{Y}_{nm}(\theta_k,\lambda_k)\, \bar{Y}_{nm}(\theta_l,\lambda_l)\, (\Delta\sigma)^2 \tag{6}$$

The variance of the error is defined as the expectation of the squared error:

$$m^2(\bar{f}_{nm}) = \frac{1}{16\pi^2} \sum_{k} \sum_{l} \mathrm{cov}(F_k, F_l)\, \bar{Y}_{nm}(\theta_k,\lambda_k)\, \bar{Y}_{nm}(\theta_l,\lambda_l)\, (\Delta\sigma)^2 \tag{7}$$

where the covariances $\mathrm{cov}(F_k, F_l)$ from the previous discussion are zero; i.e., the noise is uncorrelated:

$$\mathrm{cov}(F_k, F_l) = 0, \quad k \neq l; \quad \mathrm{cov}(F_k, F_k) = m^2(F_k) \tag{8}$$

Then

$$m^2(\overline{f}_{nm}) = \frac{1}{16\pi^2} \sum_k m^2(F_k)\, \overline{Y}_{nm}^2(\theta_k, \lambda_k)\, (\Delta\sigma)^2$$
$$= \frac{\Delta\sigma}{16\pi^2}\, m^2(F) \sum_k \overline{Y}_{nm}^2(\theta_k, \lambda_k)\, \Delta\sigma \tag{9}$$

if the variance of each measurement equals $m^2(F)$. Approximating the sum by an integral over the unit sphere and using (2), we finally arrive at the formula for the error spectrum

$$m(\overline{f}_{nm}) = \sqrt{\frac{\Delta\sigma}{4\pi}}\, m(F); \qquad |m| \le n, \;\; n \ge 0 \tag{10}$$

$\Delta\sigma$ is written in units of square radians.

We define the maximum resolution of the data by the frequency at which the signal to noise ratio is 1:1. Hence, the spectrum of the signal must be known. Consider first the case of gradiometer measurements of the vertical gradient of the gravity disturbance, $(\partial^2 T/\partial r^2)$, where T is the disturbing potential and r is the geocentric radius ($r = R + h$, h is the altitude of the satellite). In spherical harmonics,

$$T(\theta, \lambda, r) = R\gamma \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \overline{A}_{nm}\, \overline{Y}_{nm}(\theta, \lambda) \tag{11}$$

R is the mean radius of the earth, $\gamma = (kM/R^2)$ is an average value of gravity, kM is the product of the gravitational constant and the earth's mass, and the $\overline{A}_{nm}$ are harmonic coefficients representing the spectrum of $T/R\gamma$ on the sphere of radius R. The zero- and first-degree harmonics are zero under the assumptions of identical centers, masses, and potentials of the level ellipsoid and geoid. Applying the radial derivatives, we find

$$\frac{\partial^2 T(\theta, \lambda, r)}{\partial r^2} = \frac{\gamma}{R} \sum_{n=2}^{\infty} (n+1)(n+2) \left(\frac{R}{r}\right)^{n+3} \sum_{m=-n}^{n} \overline{A}_{nm}\, \overline{Y}_{nm}(\theta, \lambda) \tag{12}$$

Averaging the square of the n-th degree component of this signal over the sphere yields the degree variance (taking note of equation (2)):

$$\sigma_n^2 \left(\frac{\partial^2 T}{\partial r^2}\right) = \frac{\gamma^2}{R^2} (n+1)^2 (n+2)^2 \left(\frac{R^2}{r^2}\right)^{n+3} \sum_{m=-n}^{n} \overline{A}_{nm}^2 \quad ; \quad n \ge 2 \tag{13}$$

The sum of squared coefficients $\overline{A}_{nm}$ is modeled by Tscherning and Rapp (1974) as:

$$\left.\begin{aligned} \sum_{m=-n}^{n} \overline{A}_{nm}^2 = \frac{c_n}{\gamma^2 (n-1)^2} \; ; \quad c_n &= \frac{425.28\,(n-1)}{(n-2)(n+24)} (.999617)^{n+2}\ \mathrm{mgal}^2, \\ & \qquad n > 2 \\ c_2 &= 7.5\ \mathrm{mgal}^2 \end{aligned}\right\} \tag{14}$$

The c_n are gravity anomaly degree variances. The degree and order variance of the gradient is then approximated by

$$\sigma_{nm}^2 \left(\frac{\partial^2 T}{\partial r^2}\right) = \frac{1}{2n+1} \sigma_n^2 \left(\frac{\partial^2 T}{\partial r^2}\right) \; ; \; |m| \leq n, \quad n \geq 2 \tag{15}$$

The spectrum of the signal is consequently modeled as

$$\sigma_{nm} \left(\frac{\partial^2 T}{\partial r^2}\right) = \frac{1}{\sqrt{2n+1}} \frac{(n+1)(n+2)}{R} \left(\frac{R}{r}\right)^{n+3} \frac{\sqrt{c_n}}{n-1} \; ; \; |m| \leq n, \quad n \geq 2 \tag{16}$$

The degree n_{max} for which the equality of $\sigma_{nm}(\partial^2 T/\partial r^2)$ and $m((\partial^2 T/\partial r^2)_{nm})$, given by (10), is attained then indicates the highest resolution in the data.

The spectrum of the velocity can be determined by utilizing the law of conservation of energy:

$$\frac{1}{2} \hat{v}^2 - V = E \tag{17}$$

where V is the total gravitational potential (the minus sign is a matter of convention), $\hat{v}$ is the velocity, and E is the total (constant) energy. Decomposing V and $\hat{v}$ into a systematic (reference) component and a signal part, we have

$$\frac{1}{2} (v_m + v)^2 - U - T = E$$

$$\frac{1}{2} (v_m^2 + 2 v_m v + v^2) - U - T = E \tag{18}$$

where the reference velocity is defined by

$$\frac{1}{2} v_m^2 - U = E \tag{19}$$

U is, for example, the potential of a reference ellipsoid which at the satellite's altitude we approximate by the potential of a homogeneous sphere of mass M. Then from celestial mechanics we may show that

$$v_m = \sqrt{\frac{kM}{r}} = R\sqrt{\frac{\gamma}{r}} \tag{20}$$

Neglecting the second-order term, equation (18) becomes

$$v = \frac{1}{v_m} T \tag{21}$$

The spectrum of the velocity signal, v, is then simply

$$\sigma_{nm}(v) = \frac{1}{v_m} \sigma_{nm}(T) \; ; \; |m| \leq n, \quad n \geq 2 \tag{22}$$

where from (11) and (14)

$$\sigma_{nm}(T) = \frac{1}{\sqrt{2n+1}} R \frac{\sqrt{c_n}}{(n-1)} \left(\frac{R}{r}\right)^{n+1} \; ; \; |m| \leq n, \quad n \geq 2 \tag{23}$$

We note that the corresponding degree variances are

$$\sigma_n^2(v) = \frac{1}{v_m^2}\sigma_n^2(T) \; ; \; n > 2 \tag{24}$$

$$\sigma_n^2(T) = \frac{R^2 c_n}{(n-1)^2}\left(\frac{R^2}{r^2}\right)^{n+1} \; ; \; n \geq 2 \tag{25}$$

When comparing the velocity spectrum to the noise spectrum $m(v_{nm})$, equation (10), to determine the resolution of the data, it is assumed that the total velocity signal vector is measured. In practical situations, this ideal case is usually not realized, as for instance with satellite-to-satellite tracking in the high-low mode, only the component of the velocity signal vector along the line connecting the two satellites is detected. For a more thorough discussion of velocity measurements from satellite-to-satellite tracking, see e.g. Rummel (1978) or Hajela (1974).

Now assume that two satellites are being tracked and the observations are the velocity differences between the satellites; for example, $\Delta v_H = v_P - v_Q$, where P, Q are the locations of the satellites travelling at the same altitude and on the same orbit in the low-low mode. However, we are now faced with a problem because, while the velocity v_P, or v_Q, under the approximations leading to equation (21) can be expanded in a spherical harmonic series, the difference Δv_H is a function depending also on a direction (azimuth of P with respect to Q) and on the central angle between P and Q and thus does not admit to an expansion of the form of equation (1). If we fix a direction and a separation, then it is possible to find, at least in theory, the spectrum of Δv_H (equ. (3)) despite its singularities at the poles (Hobson, 1965, p. 344). However, this spectrum is not merely proportional to the spectrum of the disturbing potential. A simpler and not necessarily less accurate course will be followed here. Consider the root mean square (RMS) velocity difference over all possible directions

$$\Delta\bar{v}_H = \left[\frac{1}{2\pi}\int_0^{2\pi} (v_P - v_Q)^2 \, d\alpha\right]^{\frac{1}{2}} \tag{26}$$

where α is the azimuth. This function is continuous over the sphere and for a fixed separation between P and Q, $\Delta\bar{v}_H$ can be expanded in a series of spherical harmonic functions. To obtain the power spectrum of $\Delta\bar{v}_H$, we form the square of its RMS value over the sphere:

$$\begin{aligned}\sigma^2(\Delta\bar{v}_H) &= \frac{1}{4\pi}\iint_\sigma \Delta\bar{v}_H^2 \, d\sigma \\ &= \frac{1}{8\pi^2}\int_\sigma\int_0^{2\pi} (v_P^2 + v_Q^2 - 2v_P v_Q) \, d\alpha \, d\sigma \\ &= 2\sigma^2(v) - 2\,\mathrm{cov}(v_P, v_Q)\end{aligned} \tag{27}$$

The last equation follows from the definition of the (physical) covariance function (Moritz, 1972, p .83) which, when the signal v is isotropic and homogeneous on the sphere, may be expanded as

$$\mathrm{cov}(v_P , v_Q) = \sum_{n=2}^{\infty} \sigma_n^2(v) P_n(\cos\psi_{PQ}) \tag{28}$$

Substituting (24) into (28), equation (27) becomes (noting that $\sigma^2(v) \equiv \mathrm{cov}(v_P , v_P)$)

$$\sigma^2(\Delta\overline{v}_H) = \sum_{n=2}^{\infty} \frac{2}{v_n^2}(1 - P_n(\cos\psi_{PQ}))\, \sigma_n^2(T) \tag{29}$$

This is the sum of the degree variances of $\Delta\overline{v}_H$, i.e. its power spectrum components. The spectrum of $\Delta\overline{v}_H$ may be estimated from

$$\sigma_{nm}(\overline{\Delta v}_H) = \frac{1}{v_n} \sqrt{\frac{2(1 - P_n(\cos\psi_{PQ}))}{2n+1}}\, \sigma_n(T)$$

$$= \sqrt{\frac{r}{\gamma}}\, \frac{1}{n-1} \sqrt{\frac{2(1 - P_n(\cos\psi_{PQ}))c_n}{2n+1}} \left(\frac{R}{r}\right)^{n+1}, \quad |m| \le n, \quad n \ge 2 \tag{30}$$

where (20) and (25) have been used. We then assume that the spectrum of the horizontal velocity difference in any one direction does not substantially differ from that of its RMS value over all directions; i.e., $\sigma_{nm}(\Delta v_H) \approx \sigma_{nm}(\overline{\Delta v}_H)$.

The error of the velocity difference will be considered, not as the difference of two errors, but as a directionally independent quantity to be represented by the uncertainty of a single measurement, since the velocity difference in satellite-to-satellite tracking is derived from a single Doppler measurement. Thus equation (10) is applicable if we assume that this uncertainty is a good approximation of the standard error of $\overline{\Delta v}_H$, that is $m(\overline{\Delta v}_H)$.

None of the theoretical problems of the horizontal velocity difference occurs for the radial velocity difference; that is, when P and Q lie on the same radius. For the case of two orbiting satellites, such a radial velocity difference signal is a fictional quantity, however, since no two satellies at different altitudes could maintain an essentially constant separation. Nevertheless, the analysis of the radial velocity difference may provide some confirmation of that of the RMS horizontal difference if the signal is fairly homogeneous locally in space.

The physical variance of the vertical velocity difference is

$$\sigma^2(\Delta v_r) = \frac{1}{4\pi}\iint_\sigma (\Delta v_r)^2\, d\sigma$$

$$= 2\sigma^2(v) - 2\,\mathrm{cov}(v_P , v_Q) \tag{31}$$

by following the derivation of equation (27). The covariance between the disturbing potentials at two different altitudes is

$$\operatorname{cov}(T_P, T_Q) = \sum_{n=2}^{\infty} \left(\frac{R^2}{r_P\, r_Q}\right)^{n+1} \sigma_n^2(T_R)\, P_n(\cos\psi_{PQ}) \tag{32}$$

where $\sigma_n^2(T_R)$ is the degree variance of T on the sphere of radius R (taking $r = R$ in equation (25)). Noting that at the altitude of point P, the degree variance is $\sigma_n^2(T) = \left(\frac{R^2}{r_P^2}\right)^{n+1} \sigma_n^2(T_R)$, and using equation (24), we find

$$\operatorname{cov}(v_P, v_Q) = \sum_{n=2}^{\infty} \left(\frac{r_P}{r_Q}\right)^{n+1} \sigma_n^2(v_P)\, P_n(\cos\psi_{PQ}), \tag{33}$$

Now $\psi_{PQ} = 0$, since r_P and r_Q are colinear in the case of radial velocity difference measurements. The subsequent substitution of (33) and (24), (25) into (31) then yields the degree variance of the radial velocity difference:

$$\sigma_n^2(\Delta v_r) = \frac{1}{v_m^2} R^2 \frac{c_n}{(n-1)^2}\left[\left(\frac{R^2}{r_P^2}\right)^{n+1} + \left(\frac{R^2}{r_Q^2}\right)^{n+1}\right] - 2\frac{1}{v_m^2} R^2 \frac{c_n}{(n-1)^2}\left(\frac{R^2}{r_P\, r_Q}\right)^{n+1}$$

$$= \frac{1}{v_m^2} R^2 \frac{c_n}{(n-1)^2}\left[1 - \left(\frac{r_P}{r_Q}\right)^{n+1}\right]^2 \left(\frac{R^2}{r_P^2}\right)^{n+1} \tag{34}$$

where $r_P < r_Q$. The spectrum of the signal may be estimated from

$$\sigma_{nm}(\Delta v_r) = \frac{1}{\sqrt{2n+1}}\, \sigma_n(\Delta v_r) \tag{35}$$

Since the radial velocity difference, being independent of direction, is readily expanded in an harmonic series, equation (10) provides the corresponding error spectrum.

The Quantities to Be Estimated

The maximum resolution of the data, indicated by a signal-to-noise ratio of 1:1, is assumed to be also the maximum resolution of the subsequent estimated mean gravity anomaly or mean geoid undulation. Thus, the recovered mean anomaly or undulation contains no gravimetric information beyond the degree n_{max} that corresponds to the maximum resolution. The error propagated into these estimates, due to the measurement noise, (i.e. the commission error) can be determined on the basis of the error spectrum model (equation (10)). If we desire to estimate mean quantities, where "mean" denotes a simple average over a block (or spherical cap), then a further error, the truncation error, is introduced, since such a mean quantity by its definition comprises the entire spectrum, although the averaging process dampens the magnitudes of the high frequency components. If, on the other hand, we desire to estimate a "filtered" quantity which by our definition contains no gravimetric information above degree n_{max}, then obviously (by definition) no truncation error enters. The former case is adopted here.

With the spherical approximation to the fundamental equation of geodesy (Heiskanen and Moritz, 1967, pp. 86-87)

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r} T \tag{36}$$

the gravity anomaly, Δg, can be expressed in terms of spherical harmonics using (11) as follows:

$$\Delta g(\theta,\lambda,r) = \gamma \sum_{n=2}^{\infty} (n-1) \left(\frac{R}{r}\right)^{n+2} \sum_{m=-n}^{n} \bar{A}_{nm} \bar{Y}_{nm}(\theta,\lambda) \tag{37}$$

Also, from Bruns' formula, $N = T/\gamma$, the geoid undulation becomes (setting $r = R$ in (11)):

$$N(\theta,\lambda) = R \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \bar{A}_{nm} \bar{Y}_{nm}(\theta,\lambda) \tag{38}$$

We define the mean gravity anomaly as the simple average of Δg over a block on the sphere (which corresponds to current practice) and approximate this by a simple average over a spherical cap having the same area as the block:

$$\overline{\Delta g}(\theta,\lambda,R) = \frac{1}{2\pi(1-\cos\psi_0)} \iint_{\sigma_c} \Delta g \, d\sigma \tag{39}$$

where ψ_0 is the radius (spherical distance) of the cap σ_c and $2\pi(1-\cos\psi_0)$ is its area. Substituting (37) (with $r = R$) into (39) yields the corresponding spherical harmonic expansion (see Meissl, 1971)

$$\overline{\Delta g}(\theta,\lambda) = \gamma \sum_{n=2}^{\infty} (n-1)\, \beta_n \sum_{m=-n}^{n} \bar{A}_{nm} \bar{Y}_{nm}(\theta,\lambda) \tag{40}$$

Similarly, the mean geoid undulation is expanded as

$$\bar{N}(\theta,\lambda) = R \sum_{n=2}^{\infty} \beta_n \sum_{m=-n}^{n} \bar{A}_{nm} \bar{Y}_{nm}(\theta,\lambda) \tag{41}$$

The coefficients β_n effect a smoothing of the spectrum and are given recursively (Sjöberg, 1980) by

$$\left.\begin{aligned} \beta_n &= \frac{2n-1}{n+1} \cos\psi_0 \; \beta_{n-1} - \frac{n-2}{n+1} \beta_{n-2} \\ \beta_0 &= 1, \quad \beta_1 = (1+\cos\psi_0)/2 \end{aligned}\right\} \tag{42}$$

From (12), the harmonic coefficient of degree n and order m of the vertical gradient of gravity at the satellite's altitude is given by

$$\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm} = \frac{\gamma}{R}(n+1)(n+2)\left(\frac{R}{r}\right)^{n+3}\bar{A}_{nm}\ ;\quad |m| \le n,\quad n \ge 2 \tag{43}$$

The corresponding coefficient of the mean gravity anomaly, from (40), is

$$(\overline{\Delta g})_{nm} = \gamma\,\beta_n(n-1)\,\bar{A}_{nm}\ ; \tag{44}$$

and for the mean geoid undulation it is (see (41))

$$(\bar{N})_{nm} = R\,\beta_n\,\bar{A}_{nm}\ ; \tag{45}$$

Substituting (43) into (44) and (45), we obtain

$$(\overline{\Delta g})_{nm} = \frac{R\,\beta_n\,(n-1)}{(n+1)\,(n+2)}\left(\frac{r}{R}\right)^{n+3}\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm}\ ; \tag{46}$$

$$(\bar{N})_{nm} = \frac{R^2\,\beta_n}{\gamma\,(n+1)\,(n+2)}\left(\frac{r}{R}\right)^{n+3}\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm}\ ; \tag{47}$$

The truncated mean anomaly and mean undulation are then

$$\overline{\Delta g}' = R\sum_{n=2}^{n_{max}}\sum_{m=-n}^{n}\frac{\beta_n\,(n-1)}{(n+1)\,(n+2)}\left(\frac{r}{R}\right)^{n+3}\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm}\bar{Y}_{nm} \tag{48}$$

$$\bar{N}' = \frac{R^2}{\gamma}\sum_{n=2}^{n_{max}}\sum_{m=-n}^{n}\frac{\beta_n}{(n+1)\,(n+2)}\left(\frac{r}{R}\right)^{n+3}\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm}\bar{Y}_{nm} \tag{49}$$

Propagating the error in $\left(\frac{\partial^2 T}{\partial r^2}\right)_{nm}$, given by (10), into these equations and averaging over the sphere (using (2)) yields the total commission error:

$$m_c(\overline{\Delta g}) = R\sqrt{\frac{\Delta\sigma}{4\pi}}\ m\left(\frac{\partial^2 T}{\partial r^2}\right)\left[\sum_{n=2}^{n_{max}}\frac{\beta_n^2\,(n-1)^2\,(2n+1)}{(n+1)^2\,(n+2)^2}\left(\frac{r^2}{R^2}\right)^{n+3}\right]^{\frac{1}{2}} \tag{50}$$

$$m_c(\bar{N}) = \frac{R^2}{\gamma}\sqrt{\frac{\Delta\sigma}{4\pi}}\ m\left(\frac{\partial^2 T}{\partial r^2}\right)\left[\sum_{n=2}^{n_{max}}\frac{\beta_n^2\,(2n+1)}{(n+1)^2\,(n+2)^2}\left(\frac{r^2}{R^2}\right)^{n+3}\right]^{\frac{1}{2}} \tag{51}$$

Similarly, from (21) and (11), we find the (n, m) - harmonic coefficient of the velocity:

$$(v)_{nm} = \frac{1}{v_m}\cdot R\gamma\left(\frac{R}{r}\right)^{n+1}\bar{A}_{nm}\ ;\quad |m| \le n,\quad n \ge 2 \tag{52}$$

which when substituted into (44) and (45) gives

$$(\overline{\Delta g})_{nm} = v_m \frac{\beta_n (n-1)}{R}\left(\frac{r}{R}\right)^{n+1} (v)_{nm} ; \tag{53}$$

$$(\overline{N})_{nm} = \frac{v_m \beta_n}{\gamma}\left(\frac{r}{R}\right)^{n+1} (v)_{nm} ; \tag{54}$$

The corresponding commission errors, using (10) and (20), are

$$m_c(\overline{\Delta g}) = \sqrt{\frac{\gamma}{r}}\sqrt{\frac{\Delta\sigma}{4\pi}}\, m(v)\left[\sum_{n=2}^{n_{max}} \beta_n^2 (n-1)^2 (2n+1)\left(\frac{r^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \tag{55}$$

$$m_c(\overline{N}) = \frac{R}{\sqrt{\gamma r}}\sqrt{\frac{\Delta\sigma}{4\pi}}\, m(v)\left[\sum_{n=2}^{n_{max}} \beta_n^2 (2n+1)\left(\frac{r^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \tag{56}$$

Recalling that $\sigma_{nm}(\overline{\Delta v_H})$, equation (30), is an estimate of the (n, m)-harmonic coefficient of $\overline{\Delta v_H}$ and that from (14), an estimate of $\overline{A}_{nm}$ is

$$A_{nm} \approx \frac{1}{\gamma(n-1)} \sqrt{\frac{c_n}{2n+1}} \quad , \quad |m| \le n, \quad n \ge 2 \tag{57}$$

we obtain the approximation

$$(\Delta v_H)_{nm} = \sqrt{\gamma r}\ \sqrt{2(1 - P_n(\cos\psi_{PQ}))}\ \left(\frac{R}{r}\right)^{n+1} \overline{A}_{nm} \ , \ |m| \le n, \ n \ge 2 \tag{58}$$

Substituting (58) into (44) and (45), it is not difficult to derive the commission errors as before:

$$m_c(\overline{\Delta g}) = \sqrt{\frac{\gamma}{r}}\ \sqrt{\frac{\Delta\sigma}{4\pi}}\ \ m(\Delta v_H)\left[\sum_{n=2}^{n_{max}} \frac{\beta_n^2 (n-1)^2 (2n+1)}{2(1 - P_n(\cos\psi_{PQ}))}\left(\frac{r^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \tag{59}$$

$$m_c(\overline{N}) = \frac{R}{\sqrt{\gamma r}}\ \sqrt{\frac{\Delta\sigma}{4\pi}}\ \ m(\Delta v_H)\left[\sum_{n=2}^{n_{max}} \frac{\beta_n^2 (2n+1)}{2(1 - P_n(\cos\psi_{PQ}))}\left(\frac{r^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \tag{60}$$

It is assumed in these equations that $m(\Delta v_H) \approx m(\overline{\Delta v_H})$. By not expanding $\overline{\Delta v_H}$ explicitly as a series of spherical harmonic functions, we were led to conjecture (equation (58)) that the spectral components of $\overline{\Delta v_H}$ are directly proportional to those of the disturbing potential. This unsubstantiated detail is not essential for the derivation of the commission errors, since we again average over the orders m. We require only that the corresponding degree variances are proportional to each other, this being the case as seen from equation (29).

Finally, it is evident from (11) and (21) that the harmonic coefficient of the radial velocity difference is given by

$$(\Delta v_r)_{nm} = \frac{R\gamma}{v_m}\left[\left(\frac{R}{r_P}\right)^{n+1} - \left(\frac{R}{r_Q}\right)^{n+1}\right]\bar{A}_{nm}$$

$$= \frac{R\gamma}{v_m}\left[1 - \left(\frac{r_P}{r_Q}\right)^{n+1}\right]\left(\frac{R}{r_P}\right)^{n+1}\bar{A}_{nm}\ ;\quad |m| \le n,\quad n \ge 2 \qquad (61)$$

where $r_Q > r_P$. Upon substituting (61) into (44) and (45), the total commission errors are readily found to be

$$m_C(\overline{\Delta g}) = \sqrt{\frac{\gamma}{r_P}}\sqrt{\frac{\Delta\sigma}{4\pi}}\ m(\Delta v_r)\left[\sum_{n=2}^{n_{max}} \frac{\beta_n^2 (n-1)^2 (2n+1)}{(1-(r_P/r_Q)^{n+1})^2}\left(\frac{r_P^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \qquad (62)$$

$$m_C(\bar{N}) = \frac{R}{\sqrt{\gamma r_P}}\sqrt{\frac{\Delta\sigma}{4\pi}}\ m(\Delta v_r)\left[\sum_{n=2}^{n_{max}} \frac{\beta_n^2 (2n+1)}{(1-(r_P/r_Q)^{n+1})^2}\left(\frac{r_P^2}{R^2}\right)^{n+1}\right]^{\frac{1}{2}} \qquad (63)$$

The truncation errors, in each case, are simply (see equations (44) and (45)):

$$\epsilon_T(\overline{\Delta g}) = \gamma \sum_{n=n_{max}+1}^{\infty} \beta_n (n-1) \sum_{m=-n}^{n} \bar{A}_{nm}\bar{Y}_{nm} \qquad (64)$$

$$\epsilon_T(\bar{N}) = R \sum_{n=n_{max}+1}^{\infty} \beta_n \sum_{m=-n}^{n} \bar{A}_{nm}\bar{Y}_{nm} \qquad (65)$$

which when squared and averaged over the sphere become the RMS truncation errors:

$$m_T(\overline{\Delta g}) = \left[\sum_{n=n_{max}+1}^{\infty} \beta_n^2 c_n\right]^{\frac{1}{2}} \qquad (66)$$

$$m_T(\bar{N}) = \frac{R}{\gamma}\left[\sum_{n=n_{max}+1}^{\infty} \frac{\beta_n^2 c_n}{(n-1)^2}\right]^{\frac{1}{2}} \qquad (67)$$

where (2) and (14) have been utilized.

A summary of the myriad of equations derived in this section is in order. The purpose was to obtain expressions for the errors in mean gravity anomalies and mean geoid undulations, the mean being defined as a simple average over a surface block. There are two types of errors: the commission error due to the uncertainties (noise) of the data, and the truncation error due to the finite resolution of the data (the mean by our definition contains all frequencies, though the higher ones are subdued). Equations (50) and (51) give the commission error if the data consist of gradiometer measurements; equations (55) and (56) correspond to the case of velocity measurements; if horizontal velocity differences constitute the data, then equations (59) and (60) describe the commission errors; while for the radial velocity difference, equations (62) and (63) are to be used. The truncation errors for each type of data are given by expressions (66) and (67). The total RMS errors are simply

$$m_{TOT}(\overline{\Delta g}) = \left[m_c^2(\overline{\Delta g}) + m_T^2(\overline{\Delta g})\right]^{\frac{1}{2}} \qquad (68)$$

$$m_{TOT}(\bar{N}) = \left[m_c^2(\bar{N}) + m_T^2(\bar{N})\right]^{\frac{1}{2}} \qquad (69)$$

The Data Density

The parameter $\Delta\sigma$ which enters the commission error is the least area element containing a single measurement. Considered as a block on the unit sphere, its dimensions are determined by the time interval between measurements along a track and by the (presumably uniform) track spacing on the equator after the duration of the mission. In the derivation of equation (10) it was assumed that the blocks $\Delta\sigma$ are all equal in size and uniformly cover the entire sphere. Obviously, this assumption deviates from reality since the measurements increase in density with increasing latitude (for polar satellites). If h is the altitude of the satellite, then its period is approximately (Kepler's third law)

$$P = \frac{2\pi (R+h)^{3/2}}{\sqrt{kM}} \tag{70}$$

Given the duration of the mission, D, the number of orbits is

$$q = \frac{D}{P} \tag{71}$$

Assuming uniformly spaced orbital tracks after the duration of the mission, the across-track interval at the equator is

$$d_1 = \frac{\pi}{q} \quad \text{(radians)} \tag{72}$$

noting that each orbit crosses the equator twice. The along-track interval is simply

$$d_2 = \frac{2\pi P}{s} \quad \text{(radians)} \tag{73}$$

where s is the sampling interval measured in units of time. The area element becomes (on the unit sphere)

$$\Delta\sigma = d_1 \cdot d_2 \quad \text{(radians)}^2 \tag{74}$$

To obtain the intervals d_1 and d_2 in units of length, one simply scales them to refer to a particular sphere, e.g. on the earth, the intervals are Rd_1 and Rd_2.

Results

A great number of tests can be conducted with the procedures outlined in the previous section. For initial considerations we examine the signal spectrum of the velocity difference and the vertical gravity gradient. The spectrum (signal / coefficient) is shown in Figure 1 (velocity) and Figure 2 (gradient) for two different heights. In Figure 1 the horizontal and vertical velocity spectrum is shown for the case of the two satellites separated by 300 km. Note that these two spectra differ by a factor of about 1.4.

Next we consider some specific GRAVSAT type missions. First consider a six month SST mission with a data spacing of 8 seconds and a data noise of ± 1 um/sec. For a radial separation of the two satellites of 300 km the anomaly and undulation accuracies to be expected are shown in Table 1 for two different altitudes. In this table accuracy estimates from both a horizontal and a radial separation mission are given. It is clear that there is not a significant difference in the results.

Similar results are seen from a gradiometer mission as shown in Table 2.

The results presented in both Table 1 and 2 are based on the Tscherning/Rapp anomaly degree variance model. Computation made with the more recent two component model of Rapp (1979) yield values about 20% larger than what is given in the tables.

Table 1
Accuracy of Mean Anomaly and Mean
Geoid Undulations as Determined from
SST Tracking (Noise = $\pm 1 \mu$m/sec, $\left\{\begin{matrix}\text{Radial Mission} \\ \text{Horizontal Mission}\end{matrix}\right\}$

Separation = 300 km); T/R, 208 Degree Variance Model

	H = 150 km		H = 180 km	
Block Size	Anomaly	Undulation	Anomaly	Undulation
2°	0.9 mgal 0.8	1.6 cm 1.4	1.2 mgal 1.1	2.6 cm 2.4
1°	2.2 mgal 2.0	4.1 cm 3.6	3.3 mgal 3.0	7.3 cm 6.3
30'	9.1 mgal 8.7	16.2 cm 15.2	10.6 mgal 10.3	21.1 cm 19.8

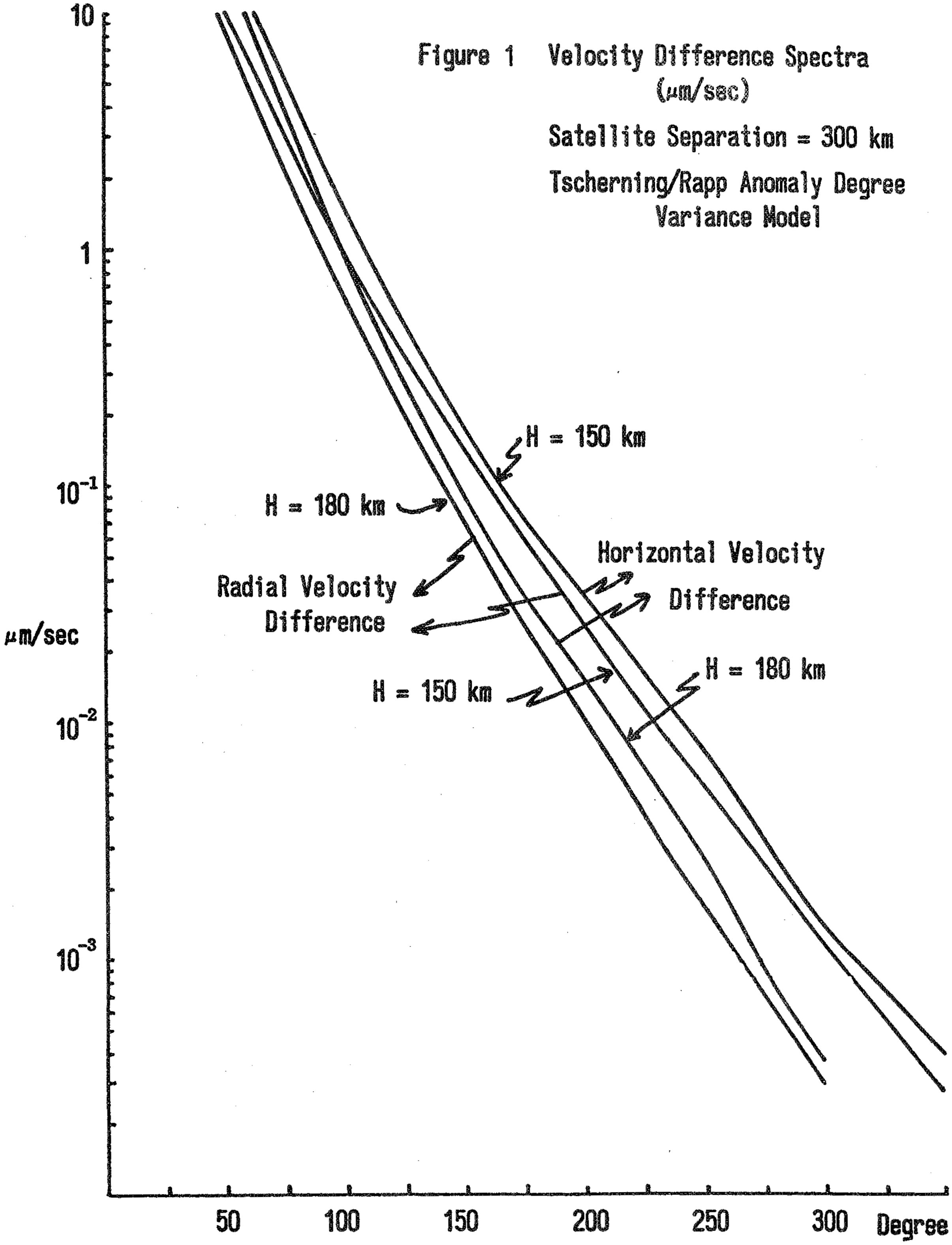
Figure 1 Velocity Difference Spectra
(μm/sec)
Satellite Separation = 300 km
Tscherning/Rapp Anomaly Degree
Variance Model
10
1
10^{-1}
10^{-2}
10^{-3}
μm/sec
H = 150 km
H = 180 km
Radial Velocity
Difference
Horizontal Velocity
Difference
H = 150 km
H = 180 km
50
100
150
200
250
300
Degree

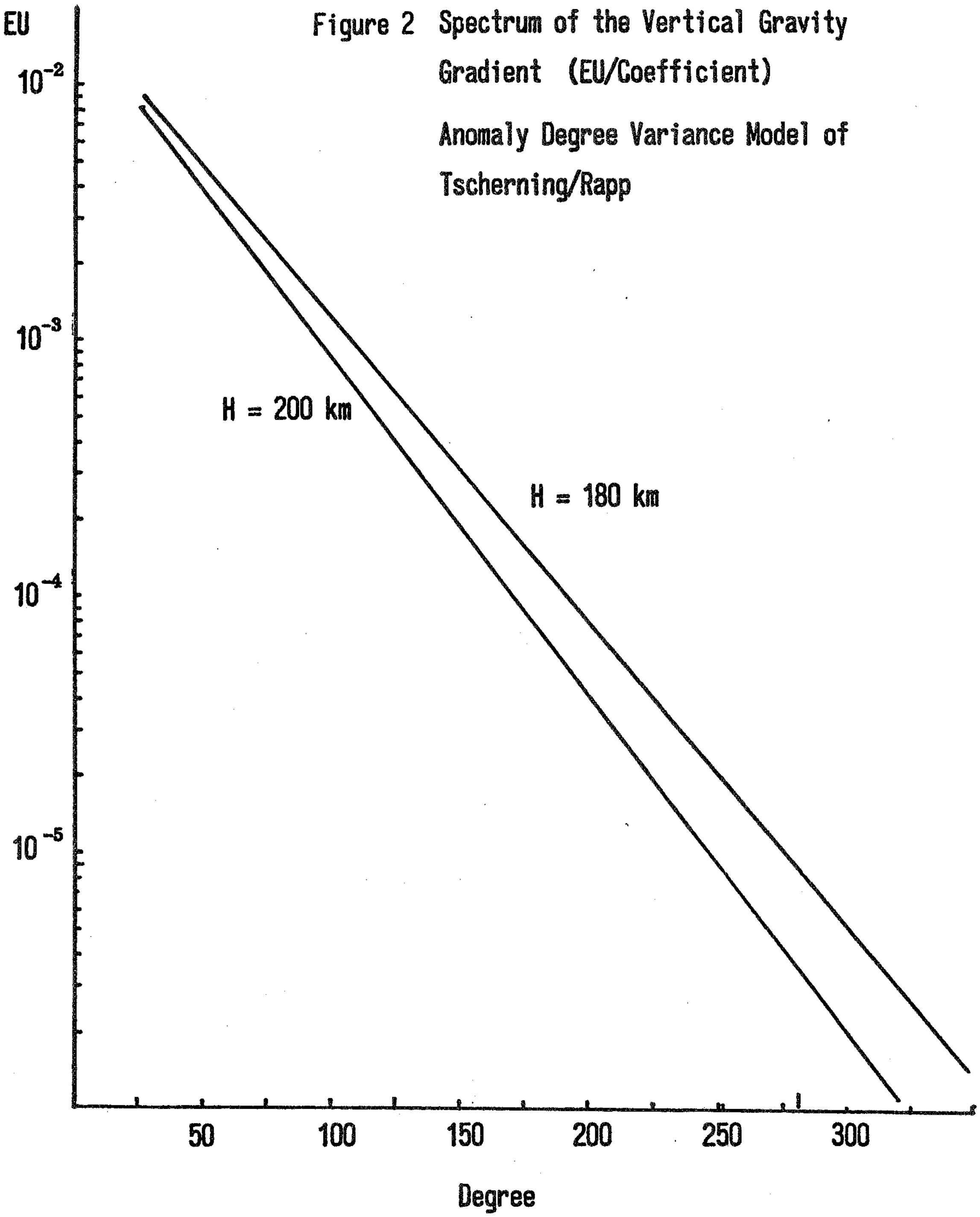

Figure 2 Spectrum of the Vertical Gravity Gradient (EU/Coefficient)

Anomaly Degree Variance Model of Tscherning/Rapp

Table 2
Accuracy of Mean Anomaly and Mean Geoid Undulations as Determined from Radial Component Gradiometry

(Noise = ±0.01 EU.)

Block Size	H = 150 km		H = 180 km	
	Anomaly	Undulation	Anomaly	Undulation
2°	0.8 mgal	1.9 cm	1.2 mgal	2.9 cm
1°	1.9 mgal	3.7 cm	3.1 mgal	7.2 cm
30'	8.5 mgal	14.5 cm	10.4 mgal	20.6 cm

Table 3
Truncation and Commission Error for H = 180 km and 1° Blocks

Type of Measurement	N_{max}	Commission Error		Truncation Error		Total Error	
		$\overline{\Delta g}$	$\overline{N}$	$\overline{\Delta g}$	$\overline{N}$	$\overline{\Delta g}$	$\overline{N}$
Gradiometer	275	1.9 mgal	5.4 cm	2.5 mgal	4.7 cm	3.1 mgal	7.2 cm
Radial Velocity Diff.	266	1.8 mgal	4.8 cm	2.7 mgal	5.5 cm	3.3 mgal	7.3 cm
Horizontal Velocity Diff.	273	1.3 mgal	3.9 cm	2.5 mgal	4.9 cm	3.0 mgal	6.3 cm

Additional tests have also been made with a mission having the satellites at an altitude of 160 km with various range rate accuracies specified over a 4 second integration time. Some representative numerical results are given in Table 4 for the accuracy of the $1° \times 1°$ anomaly and undulation recovery.

Table 4
$1^\circ \times 1^\circ$ Anomaly and Undulation Accuracies
for a Low-Low 6 Month Mission
(Horizontal Separation = 300 km), Height = 160 km

Noise	m(Anomaly)	m(Undulation)
1 μm/sec	$\pm$ 2.1 mgal	$\pm$ 3.7 cm
2	2.6	5.2
4	3.2	7.0
5	3.4	7.7
6	3.6	8.4
8	3.9	9.5
10	4.2	10.4
50	6.9	21.1

Comparing Table 1 and Table 2 we see very similar results. We also give in Table 3 a comparison of the commission error and the truncation error for the radial and horizontal measurement and the radial gradiometer measurement for H = 180 km.

From the table we see that the estimated accuracies are not linearly dependent on the data noise. This was also seen in the results of Rapp and Hajela (1979) using the method of least squares collocation for anomaly and undulation accuracy estimates.

Computations were also made to obtain results to compare to that given for a high/low case in Table 3 of Rapp and Hajela (1979). The accuracies obtained from the method described in this paper were approximately 20 % smaller than those found through the least squares collocation techniques used by Rapp and Hajela (1979). This is not unreasonable as the collocation results used less data than implied by the techniques of this paper.

In these results we have concentrated on looking at accuracy estimates in blocks. It is also possible to look at the accuracy estimates by degree n. To do this we can look at the percentage error up to the cut off degree. This is done by dividing the accuracy at a given degree by the expected signal. This relative accuracy estimate will hold for any quantity of interest (potential coefficients, anomalies, undulations etc.).

Such a computation has been made for a low-low 6 month radial mission with H = 160km, and a 4 sec integration time. Results are presented in Figure 5 for two noise estimates 1 μm/sec and 10 μm/sec. These percentages can also be converted to undulation and anomaly errors. Using the Tscherning/Rapp anomaly degree variance model, and the 1 μm/sec noise level of the mission in the above paragraph, the undulation error at degree 300 was 2.1 cm and the corresponding anomaly error was 1.0 mgal.

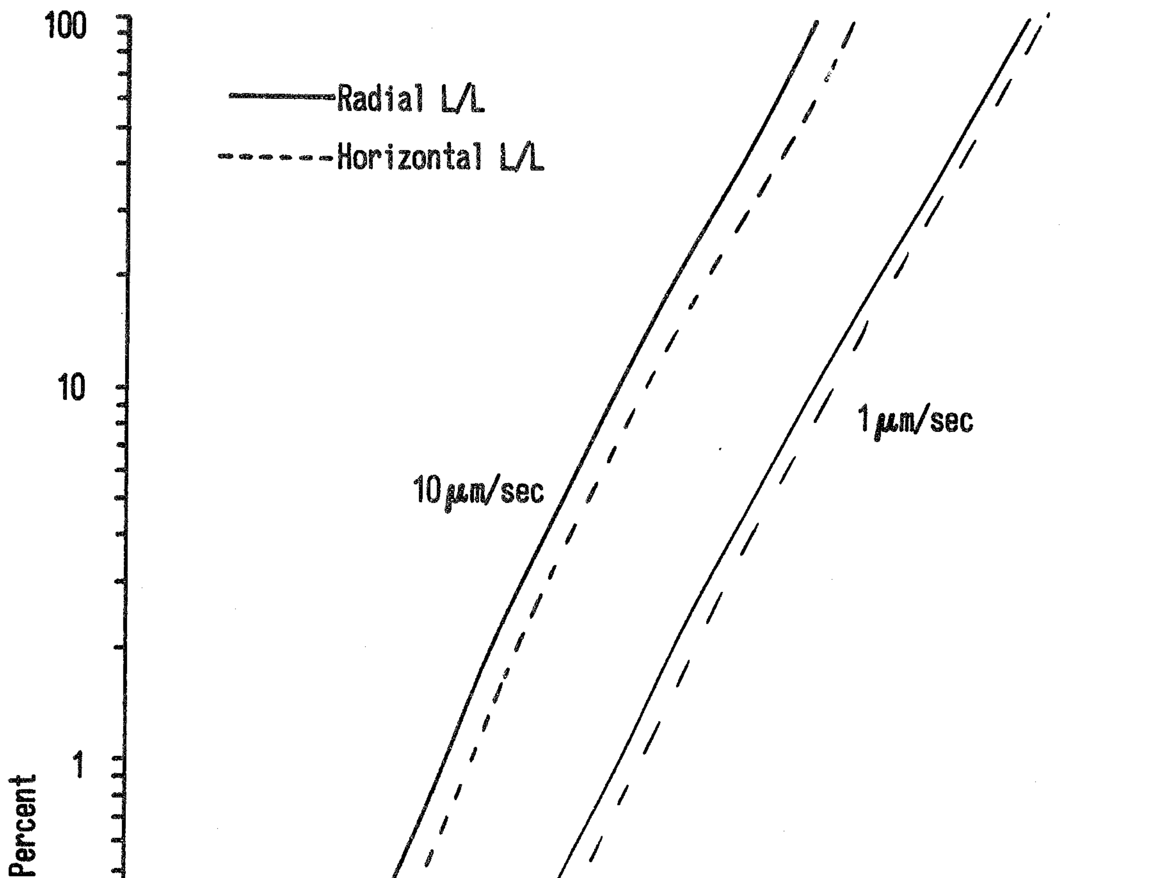

Figure 3

Percentage Error By Degree in Gravity Field Determination Based on a Low-Low 6 Month Mission at H = 160km with a 4 Second Integration Time

The percentage errors by degree were also computed for the low-low horizontal mission. These results are also plotted in Figure 3 where we see only a small shift to the higher spherical harmonic degree.

One additional way to look at the results is to consider the cumulative commission error out to the maximum degree that can be determined. This procedure would correspond to the accuracy of the computation of a point quantity with a truncated series. Some results for the 160 km, 6 month low-low horizontal mission are shown in Table 5 using the Tscherning/Rapp anomaly degree variance model and a noise of $1 \mu m/sec$.

Table 5
RMS Error of "Point" Computation
Assuming No Truncation Error

N_{max}	m(Anomaly)	m(Undulation)
50	$\pm$.000mgals	$\pm$.007cm
100	.003	.022
150	.026	.123
200	.128	.460
250	.584	1.636
300	3.195	7.327

Summary and Conclusions

This report has examined the accuracy in the determination of the earth's gravity field, to be expected from low-low range rate or a gradiometer satellite mission. The theory used is one of error propagation and truncation effect. We started with the type of measurement and computed the noise effect of that measurement on the spectrum of the earth's gravity field. The maximum degree that could be determined was set when the propagated noise equaled the estimated signal based on some model of the spectrum. We then examined the computation of mean anomalies and mean geoid undulations by considering two error components. The first is the propagated commission error due to the noise in the measurements and the second is the effect of the neglected higher degree terms on the mean anomaly and mean undulation. In doing this we used the Pellinen smoothing operator to relate a spherical harmonic expansion to a mean value. Such a procedure closely approximates that actual procedure used to determine a mean quantity in a block.

It should be noted that the procedure developed in this report is not an optimum estimation procedure such as is used by Breakwell. However if our results are compared with those obtained by Breakwell using the same filter or weighting parameter, they are within 15% showing that our procedure and the optimum estimation procedure do not yield significantly different results.

Our procedure basically ignores earth rotation and simply assumes that data will be obtained on a global basis on some interval dependent on the lifetime of the mission. In addition we have assumed no orbit error is present in the data.

In carrying out these developments we assume that the spherical harmonic expansions are valid everywhere on and above the geoid. In fact this is not the case because of the topography and so some additional attention needs to be placed on the downward continuation problem. However we have not discussed in this report a data reduction procedure but an error determination procedure. When the actual data reduction is carried out the downward continuation question must be considered if results are to be given on the earth's surface or the geoid.

The equations derived were applied to several test missions. We found that a low-low mission with the two satellites having a horizontal separation of 300 km at a height of 150 km and a data noise of ± 1 um/sec gave essentially the same results as a radial component gradiometer (± 0.01E) at the same altitude.

Tests were also done with a radial low-low 6 month mission, H = 160 km, with various data noise. For $1 \mu m$/sec noise (4 second integration time), the $1^\circ \times 1^\circ$ anomaly would be determined to ± 2.3 mgal while the corresponding undulation would be ± 4.3 cm. If the noise was raised to 6 μm/sec the corresponding values would be ± 4.1 mgals and 10.0 cm.

The results presented in this report clearly show that significant improvement in the determination of the earth's gravity field can be expected if a gravity mapping satellite mission could be flown. The equations developed here have been programmed in a very fast error analysis study program that can be used for many other test cases. We now need to develop a suitable data reduction procedure.

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