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THE STRUCTURE OF CORRELATION TENSORS IN HOMOGENEOUS ANISOTROPIC TURBULENCE

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ABSTRACT

The study of turbulence with spatially homogeneous but anisotropic statistical properties has applications in space physics and laboratory plasma physics. The first step in the systematic study of such fluctuations is the elucidation of the kinematic properties of the relevant statistical objects, which are the correlation tensors. We review the theory of isotropic tensors, developed by Robertson, Chandrasekhar and others, and extend it to cover the general case of turbulence with a pseudo-vector preferred direction, without assuming mirror-reflection invariance. Attention is focused on two point correlation functions and it is shown that the form of the decomposition into proper and pseudo-tensor contributions is restricted by the homogeneity requirement. It is also shown that the vector and pseudovector preferred direction cases yield different results. We present an explicit form of the two point correlation tensor which is appropriate for analyzing interplanetary magnetic fluctuations. A procedure for determining the magnetic helicity from experimental data is presented.

I. INTRODUCTION

The essential ingredients of a statistical treatment of homogeneous turbulence are correlation functions of the components of the relevant fields, measured at various temporal and spatial separations[1,2]. The structure of correlation matrices is the subject of the theory of isotropic tensors, developed by Robertson[3], Von Karman and Howarth[4] and Batchelor[5]. The input to this theory is the set of symmetries possessed by the system. The most common assumption is that of complete isotropy, in which the turbulent fields are taken to be invariant under rigid rotations and coordinate inversion. In recent years interest has developed in systems for which neither of these assumptions are necessarily good ones. Closer correspondence of theory to the physical situation may be achieved by imposing less restrictive assumptions on the statistical framework. Particularly in the theory of plasma behavior, it is of interest to study turbulence which is nearly homogeneous and invariant under rotations about a single axis. Turbulent plasmas often persist in the presence of mean magnetic and fluid velocity fields. To the extent that such a plasma can be described in terms of magnetohydrodynamics, the mean velocity may be removed by a Galilean transformation, but the mean magnetic field and its associated preferred direction cannot be so removed[6]. Plasmas with anisotropic statistical properties are found, for example, in interplanetary space and in certain laboratory devices. Experiments have shown that interplanetary magnetic fluctuations exhibit a strongly anisotropic energy spectrum which is nearly axisymmetric about the local mean field direction[7,8]. Many of the interesting properties and outstanding problems of the interplanetary magnetic field have been reviewed by Barnes[9].

The evolution of a plasma in a reversed field pinch confinement device (such as the Los Alamos ZT-40 or the Culham Zeta) is characterized by a strong mean toroidal magnetic field. Magnetic fluctuations in Zeta have been observed to preferentially lie in the plane perpendicular to the mean field[10,11]. Evidently the analysis applied to these anisotropic magnetic fluctuations has not made use of the most general form of the correlation tensors, which will be presented in this paper.

The assumption of coordinate inversion or mirror-reflection invariance of the correlations in a turbulent plasma rules out the possibility of non-zero values of several interesting pseudo-scalar mean values. The magnetic helicity density, an important quantity in the Taylor relaxation theory[12], which has also been conjectured to participate in dual-cascade[13] and selective decay[14] processes, is such a pseudo-scaler. The mean value of the inner product of electric current density and magnetic field is a pseudo-scalar which is related to the time derivative of the magnetic helicity. Large values of the pseudo-scalar cross helicity[15] are to be expected when field fluctuations are largely "Alfvenic" such as in the solar wind.

The correlations most easily accessible experimentally are the two point correlations $\langle B_i(\underline{x})B_j(\underline{y})\rangle$ where $\underline{B}$ is the field of interest and $\underline{x}$ and $\underline{y}$ are the measurement positions at a single time. The brackets denote ensemble averaging, which is taken as equivalent to time averaging[1]. Often $\underline{B}$ has a stationary non zero mean value which induces a preferred direction on the system. The mean value $\langle\underline{B}\rangle$ is usually not a statistical quantity, and it is convenient to deal directly with the fluctuating part of $\underline{B}$[16]. A somewhat surprising result is that the structure of the two-point correlations depends on whether $\langle\underline{B}\rangle$ is a proper or pseudo-vector.

The theory of correlation tensors for fields with a preferred proper vector direction has been developed in detail by Chandrasekhar, with the assumption of inversion invariance[17]. The simplest case of isotropic correlations without inversion invariance was presented by Betchov[18]. Tensor representations for magnetohydrodynamic (MHD) correlations in the isotropic case were developed by Chandrasekhar[19] and further discussed by Frisch et al.[13] Recently Montgomery and Turner have investigated the structure of magnetic correlations in terms of their Fourier transforms[20].

The principal goals of this paper are to present the complete structure of two point magnetic correlations for axisymmetric MHD, and to discuss how some physically interesting quantities may be extracted from this model. We begin in section II by presenting some general developments which are applicable to two point correlations of solenoidal vector fields in situations in which no symmetry except homogeneity is assumed. In particular it is shown that for the usual sorts of correlation tensors a simple theorem can be derived which immediately restricts the form of the tensors to about half of the general set described by Batchelor[21]. In section III the theory of isotropic tensors is reviewed. In section IV the form of two point correlations with a single pseudo-vector preferred direction is presented. In section V the form of the correlation tensor is given for the proper vector preferred direction case. Section VI briefly introduces the Fourier space version of the above results. Particular attention is addressed to comparison of the present results with those of Montgomery and Turner. We emphasize discussion of the pseudo-tensor correlation and associated measures of helicity which play an important role in the MHD description of a turbulent plasma. The results are summarized in section VII. The Appendix outlines a procedure for calculating the magnetic helicity in an axisymmetric plasma from experimental data.

II. PROPERTIES OF THE CORRELATION MATRIX

The basic quantity under consideration is the matrix of correlations between two spatially separated components of a solenoidal vector field $\underline{B}(\underline{x}, t)$. We consider only the homogeneous case, where the statistical properties are invariant under bulk translation of the measurement apparatus. In that situation the correlation matrix, $\underline{\underline{R}}$ depends only on the vector separation of the two measurement points. We define,

$$R_{ij}(\underline{r}) = \langle B_i(\underline{x})\, B_j(\underline{x}+\underline{r})\rangle \tag{1}$$

where the time dependence has been suppressed and R_{ij} is explicitly independent of $\underline{x}$. Homogeneity implies the additional property:

$$R_{ij}(\underline{r}) = R_{ji}(-\underline{r}) \tag{2}$$

which follows by letting $\underline{x} \rightarrow \underline{x}-\underline{r}$ in (1). The solenoidal nature of $\underline{B}$ requires that

$$\frac{\partial}{\partial r_j} R_{ij}(\underline{r}) = \frac{\partial}{\partial r_i} R_{ij}(\underline{r}) = 0. \tag{3}$$

Here, and subsequently, repeated tensor indices imply summation over all coordinate directions, unless specified otherwise.

A useful property of the matrix elements is

$$|R_{ij}(\underline{r})|^2 \le R_{ii}(0)\, R_{jj}(0) \text{ (no sum implied)} \tag{4}$$

which follows from a Schwarz inequality, treating the ij-th element of $\underline{\underline{R}}$ as the inner product of $B_i(\underline{x})$ and $B_j(\underline{x}+\underline{r})$. The above properties are discussed in detail in reference 1.

In the most general case the matrix $\underline{\underline{R}}$ must transform under coordinate rotations in a way which reflects its tensor character[3,4,5,6]. $\underline{\underline{R}}$ is basically a time averaged dyadic product of two vectors (or perhaps pseudo-vectors), so contributions to $\underline{\underline{R}}$ must transform as proper on pseudo-tensors. The homogeneity property allows certain conclusions to be drawn concerning the decomposition of $R_{ij}(\underline{r})$. Coordinate inversion will be designated by $\mathcal{J}$ and taken in the usual sense to be that improper rotation for which

$$\mathcal{J}\, \underline{r} = -\, \underline{r} \tag{5}$$

where $\underline{r}$ is the position vector. Under inversion coordinate basis vectors are mapped into their opposites and a right-handed system becomes left-handed. A proper tensor T of rank n transforms as

$$\mathcal{J}\, T = (-1)^n\, T$$

whereas an nth rank pseudo-tensor P satisfies

$$\mathcal{J}\, P = (-1)^{n+1}\, P.$$

Now the correlation matrix can always be decomposed into symmetric and anti-symmetric parts:

$$R_{ij}(\underline{r}) = R^S_{ij}(\underline{r}) + R^A_{ij}(\underline{r})$$

were $R^S_{ij} = R^S_{ji}$ and $R^A_{ij} = - R^A_{ji}$ =. Letting $\mathcal{J}$ operate on $R^S_{ij}(\underline{r})$ and using the homogeneity property (2) yields

$$\mathcal{J}\, R^S_{ij}(\underline{r}) = R^S_{ij}(-\underline{r}) \text{ (by definition of inversion)}$$

$$= R^S_{ji}(+\underline{r}) \text{ (by homogeneity)}$$

$$= R^S_{ij}(+\underline{r}) \text{ (since } R^S \text{ is symmetric),}$$

showing that R^S_{ij} is a symmetric proper tensor form. Similarly $R^A_{ij}(\underline{r}) = -R^A_{ij}(\underline{r})$ and R^A_{ij} is an antisymmetric pseudo-tensor. The argument is invertible. A proper tensor part of the correlation can be shown to be symmetric and a pseudo-tensor part shown to be antisymmetric. Thus, any homogeneous correlation matrix $\underline{\underline{R}}$ of the form (1) consists of the sum of a symmetric proper tensor and an antisymmetric pseudo-tensor. This result, which will be subsequently referred to as theorem A, follows for either proper or pseudo-vector field $\underline{B}$.

In addition to excluding symmetric pseudo-tensors and antisymmetric proper tensors from consideration as possible components of $\underline{\underline{R}}$, the above result suggests several corollaries. Let $R_{ij}(\underline{r}) = T_{ij}(\underline{r}) + P_{ij}(\underline{r})$ where $T_{ij} = T_{ji}$, $P_{ij} = -P_{ji}$, $\mathcal{J}\underline{\underline{T}} = +\underline{\underline{T}}$ and $\mathcal{J}\underline{\underline{P}} = -\underline{\underline{P}}$. Several useful properties are easy to show:

a) The proper tensor part of $\underline{\underline{R}}$ is an even function of $\underline{r}$, i.e.,

$$T_{ij}(\underline{r}) = T_{ij}(-\underline{r})$$

while the pseudo-tensor is odd,

$$P_{ij}(\underline{r}) = -P_{ij}(-\underline{r}).$$

If one writes down a power series for $R_{ij}(\underline{r})$, supposed to be valid for some region near $\underline{r} = 0$, the above property shows that the even order terms are part of T_{ij} and the odd order terms part of P_{ij}. If the symmetries of the system allow dependence on $|\underline{r}| \equiv r$, then analyticity requires that such terms only appear in even powers. If dependence is allowed on a projection of $\underline{r}$ on a preferred direction $\underline{\lambda}$, then this property determines the parity of the power series dependence on $\underline{r} \cdot \underline{\lambda}$.

b) The solenoidal property of $\underline{\underline{R}}$ (equation (3)) implies that $\underline{\underline{P}}$ and $\underline{\underline{T}}$ are separately solenoidal, i.e.,

$$\frac{\partial}{\partial r_j} P_{ij}(\underline{r}) = 0$$

$$\frac{\partial}{\partial r_i} P_{ij}(\underline{r}) = 0$$

$$\frac{\partial}{\partial r_i} T_{ij}(\underline{r}) = 0$$

$$\frac{\partial}{\partial r_i} T_{ij}(\underline{r}) = 0$$

In the application of the theory of isotropic tensors this property serves to reduce the number of independent scalar functions which multiply tensor forms.

c) The diagonal elements of $\underline{\underline{R}}$ are purely proper tensor and all pseudo-tensor contributions vanish as $\underline{r} \to 0$. Thus,

$$R_{ii}(\underline{r}) = T_{ii}(\underline{r}) \quad \text{(no sum)},$$

$$R_{ij}(0) = T_{ij}(0)$$

and, from the Schwarz inequality we have

$$T_{ii}(0) \geq T_{ii}(\underline{r}) \text{ (no sum).}$$

d) The Schwarz inequality, (4), takes on the slightly more restrictive form:

$$T_{ii}(0)\, T_{jj}(0) \geq (T_{ij}(\underline{r}))^2 + (P_{ij}(\underline{r}))^2 + 2|P_{ij}(\underline{r})\, T_{ij}(\underline{r})|$$

where the bars denote absolute value and no sum is implied. In cases where there is a large 'helicity' this form may become relevant.

These properties as well as theorem A are useful formulae for constructing minimal tensor representations for homogeneous systems with arbitrary symmetries. It should be noted that in related circumstances similar results can be shown. For example. The correlation of $\underline{B}$ with $\underline{J} = \nabla\times\underline{B}$ can be treated by considering the tensors

$$\langle \underline{B}(\underline{x})\underline{J}(\underline{x}+\underline{r}) \pm \underline{J}(\underline{x})\underline{B}(\underline{x}+\underline{r})\rangle .$$

In each case there is a homogeneity property which along with the transformation properties of $\underline{B}$ and $\underline{J}$, implies a theorem which eliminates certain tensor forms from admissibility.

III. THE THEORY OF ISOTROPIC TENSORS

Here we briefly review the theory of the construction of correlation tensors, due to Robertson and others[3,4,5,17,19]. We focus on two point correlation tensors, but the generalization to arbitrary rank is straightforward.

Consider a correlation $\underline{\underline{R}}$ which is assumed to possess an invariance property with respect to an orthogonal symmetry transformation which has a (matrix) representation O^+ (Hermitian adjoint of O) and maps a vector $\underline{r}$ into $\underline{r}'$ so that $\underline{r}' = O^+\underline{r}$. We define a scalar F as,

$$F = F(\underline{a},\underline{b},\underline{r}) = \underline{a}\cdot\underline{\underline{R}}(\underline{r})\cdot\underline{b}$$

and require that

$$F(\underline{a}',\underline{b}',\underline{r}') = F(\underline{a},\underline{b},\underline{r}) \tag{6}$$

where the primes denote action of the transformation O^+ and $\underline{a}$ and $\underline{b}$ are arbitrary vectors. Equation (6) is just the requirement that the correlation measured between the $\underline{a}$ and $\underline{b}$ direction with separation $\underline{r}$ be invariant when $\underline{a}$, $\underline{b}$ and $\underline{r}$ are all rigidly transformed under O^+. A simple manipulation shows that the requirement on the structure of $\underline{\underline{R}}(\underline{r})$ is that it satisy the matrix equation

$$\underline{\underline{R}}(\underline{r}) = O\underline{\underline{R}}(O^+\underline{r})O^+ \tag{7}$$

Robertson[3] has shown that this requirement is satisfied if $\underline{\underline{R}}$ consists of a sum of all possible dyadic products of the vectors fundamental to the problem and the fundamental invariants δ_{ij} and ϵ_{ijk}, each multiplied by a scalar function of the invariant scalars allowed by O^+. Thus in the case of one preferred direction ($\underline{\lambda}$, say) O^+ is an arbitrary rotation about the $\underline{\lambda}$ direction and the scalar functions may depend on $\underline{r}\cdot\underline{r}$ and $\underline{r}\cdot\underline{\lambda}$.

The results of the previous section further restrict the candidates. Each tensor form (dyadic products and combinations including δ_{ij} and ϵ_{ijk}) may be symmetrized and antisymmetrized. If $\underline{\lambda}$ is a proper vector there are no intrinsically pseudo-scalar quantities in the problem, each scalar function is a proper scalar, and the transformation properties of each form depends only on the "part with indices".

If the preferred direction is a pseudo-vector $\underline{B}$, such as a mean magnetic field, then $z \equiv \underline{r}\cdot\underline{B}$ is pseudo-scalar and there is the possibility of forming a proper symmetric tensor by taking the product of a symmetric pseudo-tensor form with a pseudo-scalar function of z. There is no guarantee that such correlat ons exist in physical situations, but there is nothing in the mathematics to rule them out.

IV. TWO POINT CORRELATION WITH A MEAN MAGNETIC FIELD

Here we consider the correlations of the fluctuating part of a pseudo vector field $\underline{B}^* = \langle\underline{B}^*\rangle + \underline{B}(\underline{x},t)$. We assume that the only preferred direction is

$$\hat{b} = \frac{\langle\underline{B}^*\rangle}{|\langle\underline{B}^*\rangle|}$$

which is a pseudo-vector. The fluctuating part has zero mean so $\langle\underline{B}\rangle = 0$. This situation is probably the one most relevant to interplanetary magnetic fluctuations viewed in a frame moving with the plasma. The correlation of interest is $R_{ij}(\underline{r}) = \langle B_i(\underline{x})\, B_j(\underline{x}+\underline{r})\rangle$ which as above, is taken to be independent of $\underline{x}$. This correlation tensor contains fluctuations of the same field as the one responsible for producing the preferred direction, but the results obtained hold for correlations with a pseudo-vector axisymmetry direction of any origin.

Keeping in mind the results of sections II and III we may catalogue the tensor forms allowed in the construction of $\underline{\underline{R}}$. First we consider proper tensors. There are four independent symmetric proper tensors which may appear multiplied by scalar functions. They are

$$\delta_{ij},\ r_i r_j,\ b_i b_j \tag{8}$$

and

$$r_i \epsilon_{jk\ell}\, b_k r_\ell + r_j \epsilon_{ik\ell}\, b_k r_\ell.$$

Note that the last of these is a proper tensor since $\hat{b} \times \underline{r}$ is a proper vector. There are two independent symmetric pseudo-tensor forms which may be taken to be

$$b_i r_j + r_i b_j \tag{9}$$

and

$$b_i\epsilon_{jkl}b_kr_l+b_j\epsilon_{ikl}b_kr_l.$$

These two forms are admissible provided they are each multiplied by pseudo-scalar function, which is possible since z is a pseudo-scalar.

Turning to the pseudo-tensor contributions, we see that there are a total of three independent possibilities; two antisymmetric pseudo-tensors and one antisymmetric tensor which must be multiplied by pseudo-scalar functions. The antisymmetric pseudo-tensors are the forms

$$\epsilon_{ijk}r_k \text{ and } b_ir_j-b_jr_i. \tag{10}$$

The only independent antisymmetric proper tensor form is

$$\epsilon_{ijk}b_k. \tag{11}$$

Other forms such as $b_i\epsilon_{jkl}b_kr_l-b_j\epsilon_{ikl}b_kr_l$ and $r_i\epsilon_{jkl}b_kr_l-r_j\epsilon_{ikl}b_kr_l$ are not independent due to certain identities[22]. Each of these contributes to the pseudo-tensor part of $\underline{R}$ when multiplied by a function which is odd in z.

Assembling the terms from (8)-(11) and applying the solenoidal condition, we arrive at the representation for $R_{ij}(\underline{r})$:

$$R_{ij}(\underline{r}) = T_{ij}(\underline{r})+P_{ij}(\underline{r}) \tag{12a}$$

$$\begin{aligned} T_{ij}(\underline{r}) = {} & A\delta_{ij}+Br_ir_j+C(b_ir_j+b_jr_i) \\ & + Db_ib_j \\ & + E(r_i\epsilon_{jkl}b_kr_l+r_j\epsilon_{ikl}b_kr_l) \\ & + F(b_i\epsilon_{jkl}b_kr_l+b_j\epsilon_{ikl}b_kr_l) \end{aligned} \tag{12b}$$

$$P_{ij}(\underline{r}) = G\epsilon_{ijk}r_k+K\epsilon_{ijk}b_k \tag{12c}$$

Here A, B, D, E and G are functions of even powers of r and z. C, F and K are odd functions of z and contain even powers of r. Note that the form $b_i r_j - b_j r_i$ has disappeared entirely due to the solenoidal property. Clearly not all of the above functions are independent. The solenoidal condition gives one constraint between G and K, one between E and F and two constraints among A, B, C and D. These equations yield sets of partial differential equations relating the scalar functions to each other. Equivalently these constraints may be solved in principle by relating them to four "basic" functions. The problem of presenting the solution of these equations in a form which is both compact and experimentally meaningful will be discussed in a later section. For now, we note that the equation connecting the scalars in the pseudo-tensor does take a simple form. Using a subscript to denote explicit differentiation, so that

$$\frac{\partial}{\partial r_j} G\ (r,z) = G_r \frac{r_j}{r} + G_z b_j$$

We find that the solenoidal property implies

$$G_z = \frac{K_r}{r}$$

and thus

$$G = \phi_r(r,z)/r$$

$$K = \phi_z(r,z)$$

where ϕ is a function which contains even powers of r and z in its Taylor series.

Several important physical quantities can be extracted from equation 12, 12a and 12b. In the appropriate units, the energy density of the field $\underline{B}$ is just

$$\langle B^2(\underline{x})\rangle = \text{Trace } \underline{\underline{R}}(0).$$

$$= 3\ A(0,0)+D(0,0)\quad .$$

Note that throughout this paper regularity of the scalars at $\underline{r}\to 0$ is assumed. Evidently A(0,0) is the isotropic component of the energy while B (0,0) represents the influence of the preferred direction $\hat{\delta}$ on the distribution of energy in the fluctuations. In the case where almost all of the energy is in fluctuations perpendicular to $\hat{\delta}$, $D(0,0)\to -A(0,0)$ achieves the necessary projection, while in the isotropic limit $D(0,0)\to 0$. Other aspects of these limiting cases will be discussed in later sections. The "helicity" H_J of the field $\underline{B}$ is the pseudo-scalar correlation of, say, the magnetic field with the electric current $\underline{J} = \nabla\times\underline{B}$. A simple calculation shows

$$H_J = \langle \underline{B}\cdot\underline{\nabla}\times\underline{B}\rangle$$

$$= -6\ G(0,0) - 2\ K_z(0,0)$$

Thus, in the case of magnetohydrodynamic configurations which are nearly "force free" (i.e., $\underline{J}\times\underline{B} \simeq 0$ and $\underline{J} \simeq \lambda\ \underline{B}$) one would expect G(0,0) to take on large values subject to the limitation

$$H_J \lesssim \langle B^2\rangle\ \langle J^2\rangle$$

or, $-6\ G(0,0) - 2\ K_z(0,0) \leq (3\ A(0,0) + D(0,0))\ \langle J^2\rangle$.

We shall return to discussion of the pseudo-vector preferred direction case in section VI. In the following we briefly discuss the "mean flow" case, wherein the preferred direction transforms under inversion as a proper vector.

V. TWO POINT CORRELATION WITH A MEAN FLOW DIRECTION

When a proper vector $\underline{\lambda}$ ($\mathcal{J}\ \underline{\lambda} = -\underline{\lambda}$) induces a preferred direction on a turbulent system, the general form of the correlation $\underline{\underline{R}}$ is simplified. Chandrasekhar has given an elegant presentation of the inversion symmetric part of $\underline{\underline{R}}$ for this case.[17] The notable difference between the result of the last section and the present case is the absence of the tensor forms

$r_i\epsilon_{jk\ell}\lambda_k r_\ell + r_j\epsilon_{ik\ell}\lambda_k r_\ell$ and $\lambda_i\epsilon_{jk\ell}b_k r_\ell + \lambda_j\epsilon_{ik\ell}b_k r_\ell$. These forms may not appear because $\underline{r}\times\underline{\lambda}$ is now a pseudo-vector and $\underline{r}\cdot\underline{\lambda}$ is a proper scalar. The tensor R_{ij} now has the form

$$R_{ij}(\underline{r}) = T_{ij}(\underline{r}) + P_{ij}(\underline{r}) \tag{13}$$

$$T_{ij}(r,\ z) = A\delta_{ij} + Br_i r_j + C(\lambda_i r_j + r_i\lambda_i) + D\lambda_i\lambda_j$$

$$P_{ij}(r,z) = G\epsilon_{ijk}r_k + H\epsilon_{ijk}\lambda_k$$

where $z = \underline{r}\cdot\underline{\lambda}$.

The dependence of the scalar functions on z must again satisfy the properties discussed in section II. A summary of these properties and those of section IV may be found in Table I.

VI. FOURIER TRANSFORM OF THE CORRELATION TENSOR AND FORMAL SIMPLICATION OF $\underline{\underline{R}}$

The Fourier transform of the two point correlation[1], also known as the energy spectrum tensor is defined as

$$S_{ij}(\underline{k}) = \frac{1}{(2\pi)^3}\int d\underline{r}\ R_{ij}(\underline{r})\ e^{-i\underline{k}\cdot\underline{r}} \tag{14}$$

For our purposes the important properties of $\underline{\underline{S}}$ are reality ($S_{ij}(\underline{r}) = S_{ij}{}^*(-\underline{k})$) homogeneity ($S_{ij}(\underline{k}) = S_{ji}(-\underline{k})$) and hermiticity ($S_{ij}(\underline{k}) = S_{ji}{}^*(\underline{k})$). Since theorem A applies equally well in $\underline{k}$ space, it follows that the real part of $\underline{S}$, which is a symmetric matrix and even under $\underline{k}\leftrightarrow-\underline{k}$ can contribute only to the proper tensor part of $\underline{\underline{S}}$. Im$\underline{\underline{S}}$ is antisymmetric, odd under $\underline{k}\leftrightarrow-\underline{k}$ and is a pseudo-tensor. With these properties in mind, the application of the same arguments leading to equations (12) and (13) yields the general form of $\underline{\underline{S}}$ (for the case of a preferred pseudo-vector direction $\underline{\beta}$)

$$S_{ij}(\underline{k}) = \mathcal{A}\delta_{ij} + \mathcal{B}k_i k_j + \mathcal{C}(b_i k_j + b_j k_i)$$
$$+ \mathcal{D}b_i b_j + \mathcal{E}(k_i \epsilon_{jk\ell} b_k k_\ell + k_j \epsilon_{ik\ell} b_k k_\ell) \tag{15}$$
$$+ \mathcal{F}(b_i \epsilon_{jk\ell} b_k k_\ell + b_j \epsilon_{ik\ell} b_k k_\ell)$$
$$+ \mathcal{G}\epsilon_{ij\ell} k_\ell$$

The solenoidal condition $k_i S_{ij}(\underline{k}) = k_j S_{ij}(\underline{k}) = 0$ has been implicity used in this equation, yielding algebraic constraints among the scalars which appear. These constraints have eliminated terms proportional to $\epsilon_{ijk} b_k$ and $r_i b_j - r_j b_i$. The functions $\mathcal{A}$ through $\mathcal{G}$ depend on $|\underline{k}| = k$ and $k_z = \underline{k} \cdot \hat{b}$. Only $\mathcal{C}$ and $\mathcal{F}$ are odd under $k_z \to -k_z$. Again, if the preferred direction were a proper vector we have $\mathcal{E} = \mathcal{F} = 0$.

In order to solve the constraints implicit in equation (14), we make use of the form for $S_{ij}(\underline{k})$ introduced by Montgomery and Turner, which includes the solenoidal property at the onset[20]. Defining $\hat{e}_3 = \hat{k}$, $\hat{e}_2 = \underline{k} \times \hat{b}/|\underline{k} \times \hat{b}|$ and $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3$, the most general form of $\underline{\underline{S}}(\underline{k})$ is

$$\underline{\underline{S}}(\underline{k}) = \sum_{\alpha,\beta = 1}^{2} A_{\alpha\beta}\, \hat{e}_\alpha\, \hat{e}_\beta \tag{16}$$

which automatically satisfies the property $k_i S_{ij} = k_j S_{ij} = 0$[23]. The four independent scalar functions are A_{11}, A_{22}, A^s and A^a where $A_{12} = A^s + i A^a$; all are functions of k and $\underline{k} \cdot \hat{b} \equiv k_z$. The scalar functions in (14) may be evaluated in terms of the $A_{\alpha\beta}$ by use of various vector identities. The results are:

$$\mathcal{A} = A_{22}$$

$$\mathcal{B} = \frac{-1}{k^2}\left(\frac{k^2 A_{22} - k_z^2 A_{11}}{k^2 - k_z^2}\right)$$

$$\mathcal{C} = \frac{k_z}{k^2 - k_z^2}(A_{22} - A_{11})$$

$$\hat{D} = \frac{k^2}{k^2 - k_z^2} (A_{11} - A_{22})$$

$$\hat{E} = - \frac{k_z}{k\,(k^2 - k_z^2)} A^s \qquad (17)$$

$$\hat{F} = \frac{k}{k^2 - k_z^2} A^s$$

$$\hat{G} = \frac{i}{k} A^a$$

Consistent with the discussion of Montgomery and Turner, we note that when $A_{22} \to A_{11}$ and $A^s = \mathrm{Re}\, A_{12} \to 0$, equation (14) becomes

$$S_{ij}(\underline{k}) = (\delta_{ij} - \frac{k_i k_j}{k^2}) A_{22}(\underline{k}) + i \frac{A^a(\underline{k})}{k} \epsilon_{ijk} k_\ell$$

which is the familiar isotropic result if A_{22} and A^a depend only on $|\underline{k}|$[13].

From the above discussion one can easily deduce that A_{11}, A_{22} and Im A_{12} are even in k_z, while ReA_{12} is odd. Furthermore since $\hat{e}_1\, \hat{e}_2$ is a pseudo-tensor dyadic but contributes symmetrically through Re A_{12}, we must have Re $A_{12} = 0$ for the case where $\hat{b} \to \hat{z}$ (a proper vector direction). As expected this would insure $\hat{E} = \hat{F} = 0$ in (14).

The decomposition of $\underline{\underline{S}}$ due to Montgomery and Turner suggests several ways of expressing $\underline{\underline{R}}$ in explicit terms of four functions. Equations (15) and (17) may be used in the inverse of (14) to show that

$$R_{ij}(\underline{r}) = \delta_{ij}\, \nabla^2 Q^{(1)}$$

$$- \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} (Q^{(1)} - \frac{\partial^2}{\partial z^2} Q^{(2)})$$

$$- \left(b_i \frac{\partial}{\partial r_j} + b_j \frac{\partial}{\partial r_i}\right) \frac{\partial}{\partial z} \nabla^2 Q^{(2)}$$

$$+ b_i b_j \nabla^2 \nabla^2 Q^{(2)} \tag{18}$$

$$+ \left(b_m \epsilon_{j\ell m} \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_\ell} + b_m \epsilon_{i\ell m} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_\ell}\right) \frac{\partial}{\partial z} Q^{(3)}$$

$$- \left(b_i \epsilon_{j\ell m} b_m \frac{\partial}{\partial r_\ell} + b_j \epsilon_{i\ell m} b_m \frac{\partial}{\partial r_\ell}\right) \nabla^2 Q^{(3)}$$

$$+ \epsilon_{ijk} \frac{\partial}{\partial r_k} \Phi$$

Equation (18) may be further reduced by expanding the derivatives, but the result is unwieldy. The four independent functions $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$ and Φ are functions of r and z, $Q^{(3)}$ being the only one which is odd in z. Isotropy is achieved when $Q^{(2)} \to 0$ and $Q^{(3)} \to 0$ and Φ and $Q^{(1)}$ depend only on r. Mirror reflection invariance is recovered when $\Phi \to 0$. The fluctuations concentrate in the plane perpendicular to $\hat{B}$ when $\nabla^2 Q^{(2)} = -Q^{(1)}$ and $Q_z^{(2)} = 0$.

An alternative formulation of the real space correlation $\underline{R}$ may be written in terms of four scalar functions $W^{(1)}$, $W^{(2)}$, $W^{(3)}$ and Φ:

$$R_{ij}(\underline{r}) = \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)_i \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)_j W^{(1)}(r,z)$$

$$+ \left(\frac{\partial}{\partial \underline{r}} \times \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)\right)_i \left(\frac{\partial}{\partial \underline{r}} \times \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)\right)_j W^{(2)}(r,z) \tag{19}$$

$$+ \left[\left(\frac{\partial}{\partial \underline{r}} \times \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)\right)_i \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)_j + \left(\frac{\partial}{\partial \underline{r}} \times \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)\right)_j \left(\frac{\partial}{\partial \underline{r}} \times \hat{B}\right)_i\right] W^{(3)}(r,z)$$

$$+ \epsilon_{ijk} \frac{\partial}{\partial r_k} \Phi(r, z).$$

This is the most compact form we have found for expressing the structure of $\underline{\underline{R}}$ in terms of four functions. The solution of the constraints implicit in equations (12a) and (12b) may be found by expanding the derivatives in equation (19) but again the form is cumbersome. Various formulae connecting the W functions with the Q's in (18) and the Fourier transforms of the $A_{\alpha\beta}$ may be easily derived. In the isotropic limit $W^{(1)} \to$

$\nabla^2 W^{(2)}$, $\phi_z \to 0$, $W^{(3)} \to 0$. Mirror reflection symmetry implies $\phi = 0$. The two dimensional limit (fluctuations $\perp \hat{B}$) is approached when only $W^{(1)}$ is non-zero and $W_z^{(1)} = 0$.

Of particular interest in equation (19) is the function ϕ which generates all pseudo-tensor correlations. Comparison with the discussion following equation 12b shows that the pseudo-tensors involving ϕ in (18) and (19) are equivalent to $(r_k \epsilon_{ijk} G + b_k \epsilon_{ijk} K)$ which is guaranteed to be solenoidal when $G = \phi_r/r$ and $K = \phi_z$. It can be shown that

$$\phi(r,z) = \operatorname{Im} \int d\underline{k} \; \frac{A_{12}(\underline{k})}{k} \, e^{i\underline{k}\cdot\underline{r}}$$

and from equation (32) of Montgomery and Turner it follows that

$$\begin{aligned} 2\phi(0,0) &= \text{"Magnetic Helicity Density"} = H_M \\ &= \langle \underline{A}\cdot\underline{B} \rangle \end{aligned} \tag{20}$$

where $\underline{B} = \nabla \times \underline{A}$. Also, recalling the discussion of section IV we have

$$\begin{aligned} H_J &= \text{"helicity density"} = \langle \underline{B}\cdot\nabla\times\underline{B} \rangle \\ &= \lim_{\underline{r}\to 0} \left(-2\phi_{rr} - 4\phi_r/r - 2\phi_{zz}\right) \\ &= -6G(0,0) - 2\,K_z(0,0) \end{aligned} \tag{21}$$

Equation (20) is perhaps the most significant result of this paper and can be derived by the following procedure: $\underline{\underline{R}}$ may be expressed as the inverse transform of $\underline{\underline{S}}$ and may be uncurled in Fourier space, yielding the symmetric part of the correlation $\langle \underline{B}\,\underline{A} \rangle$. Here $\underline{A}$ is the magnetic vector potential satisfying
$\nabla\cdot\underline{A} = 0$. The Fourier transform of $\langle \underline{B}\,\underline{A} \rangle$ is defined to be $H_{ij}(\underline{k}) = ik^{-2} \epsilon_{jrs} k_r S_{is}$ which satisfies $\langle \underline{A}\cdot\underline{B} \rangle = \int d\underline{k}\, H(\underline{k})$ where $H(\underline{k}) \equiv H_{ii}(\underline{k})$. Then it follows that the k-space pseudo-tensor (equation 15) may be written

$$\epsilon_{ij\ell} k_\ell \hat{G} = \frac{i}{2} \epsilon_{ij\ell} H(\underline{k}) k_\ell$$

It is easy to verify that Φ, the generator of the real space pseudo-tensor satisfies

$$\Phi(r,z) = -i \int d\underline{k}\, e^{i\underline{k}\cdot\underline{r}} \hat{G}(\underline{k})$$
$$= \frac{1}{2} \int d\underline{k}\, e^{i\underline{k}\cdot\underline{r}} H(\underline{k})$$

which yields equation (20) when evaluated at $\underline{r} = 0$.

The conclusion, then is that knowledge of the real space pseudo-tensor correlation near the origin can be used to extract the helicity of the field $\underline{B}$, while knowledge of Φ at the origin is necessary to find H_M. In this formulation, there is no decomposition of magnetic helicity into isotropic and anisotropic contributions, but the helicity density H_J has an isotropic contribution G(0,0) and an anisotropic contribution $K_z(0,0)$.

VII. SUMMARY AND DISCUSSION

In this paper we have examined the structure of homogeneous correlation tensors with an emphasis on the conclusions which may be drawn concerning axisymmetric magnetohydrodynamic fluctuations in the presence of a mean magnetic field. The formal results of section II are helpful in the construction of the appropriate tensors with any assumed symmetry. In section III we reviewed the technique for constructing "isotropic tensors" and presented the axisymmetric results in sections IV and V. When the fluctuations have a pseudo-vector axisymmetry direction (mean magnetic field), four scalar functions are required to determine $\underline{\underline{R}}$, but when the axisymmetry direction is a "proper" vector (mean flow direction) only three such functions exist. In each case the pseudo-tensor part of the correlation is antisymmetric in its spatial indices, odd under $\underline{r} \to -\underline{r}$ and is completely determined by one scalar function. The relatively simple structure of the pseudo-tensor correlation is used in Appendix A to give a prescription for measurement of the magnetic helicity. It is important to note that we have calculated the magnetic helicity of the fluctuations. If inhomogeneities were present, there would be an additional contribution which may be thought of as the magnetic helicity of the mean field. In section VI we considered $\underline{\underline{S}}$, the Fourier transform of $\underline{\underline{R}}$ and showed its equivalence to the form deduced by Montgomery and Turner. The Fourier representation was used to derive explicitly solenoidal versions of $\underline{\underline{R}}$ containing derivatives up to fourth order. Chandrasekhar[17] has solved the solenoidal constraints for the proper tensor part of the correlation (for the mean flow direction case) in terms of derivatives up to second order on two functions. Complications arise when applying his technique to the more general case treated here, but we believe that it may be possible to develop a representation, similar to (18), and (19) containing only second derivatives of four scalar functions.

The tensor structures presented here have obvious application in the further development of the kinematic theory of axisymmetric turbulence. Each of the four functions discussed in sections IV and VI implicitly contain microscales and correlation lengths which characterize certain moments of the energy spectrum tensor. Another application lies in the calculation of transport coefficients. For example, use of the form of $\underline{\underline{R}}$ we have presented may impact calculations of the scattering of cosmic rays induced by interplanetary magnetic fluctuations[24]. Hasselman[25] has studied this problem using axisymmetric correlations which are not as general as those presented here.

At this point in time any claim of the existence of a universal equilibrium range for magnetohydrodynamic turbulence must be viewed as conjectural[26]. However, one may envision the intriguing possibility that the four scalar functions characterizing axisymmetric MHD may possess universal limiting behavior at large magnetic Reynolds number.

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23. See equations 29 and 30 of reference 20. We differ slightly from that presentation by letting $A_{\alpha\beta}$ depend on $\underline{k}\cdot\hat{B}$ rather than $\cos\theta = \hat{k}\cdot\hat{B}$. The subscripts representing the elements of $A_{\alpha\beta}$ should not be confused with the explicit differentation introduced earlier.

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APPENDIX A: Evaluation of the Magnetic Helicity

Here we develop a simple procedure for calculating the magnetic helicity, $\langle \underline{A} \cdot \underline{B} \rangle$ from experimental knowledge of the two-point magnetic correlation $R_{ij}(\underline{r})$. We assume that to good approximation the statistics are homogeneous, axisymmetric about $\hat{b}$ and time-stationary. The two-point correlation is given for values of $\underline{r} = \hat{V}S$ where $\hat{V}$ is a time independent unit vector and S is a parameter. In section VI we found that $\langle \underline{A} \cdot \underline{B} \rangle = 2\phi(\underline{r} = 0)$. If ϕ is single valued and sufficiently well behaved, it may be evaluated at $\underline{r} = 0$ by

$$\phi(0) = \int_R^0 \nabla\phi \cdot \hat{V} dS + \phi(\hat{V}R) \tag{22}$$

If $\phi(\hat{V}R) \to 0$ rapidly as R becomes large, the second term in (22) may be dropped and the lower limit replaced by ∞. This will introduce little error if the relevant correlation lengths are small compared with both the range of available measurements and the lengths characterizing departure from the symmetries we have assumed.

Consider a unit vector $\hat{c}$ in the direction of $\hat{b} \times \underline{r} = s\, \hat{b} \times \hat{V}$, and an angle ψ which satisfies $\cos \psi = \hat{V} \cdot \hat{b}$ and $\hat{b} \times \underline{r} = r \sin\psi\, \hat{c}$. From equation (12c) and the discussion following equation (19) we may deduce that

$$\phi_r \sin \psi = \hat{c} \cdot \underline{\underline{P}} \cdot \hat{b}$$

and

$$\phi_z \sin \psi = -\hat{c} \cdot \underline{\underline{P}} \cdot \hat{V}$$

where $\underline{\underline{P}}$ is the pseudo-tensor part of $\underline{\underline{R}}$. Using $\nabla\phi \cdot \hat{V} = \phi_r + \cos \psi \phi_z$ the magnetic helicity may be evaluated as

$$\langle \underline{A} \cdot \underline{B} \rangle = \frac{1}{\sin\psi} \int_0^\infty [\hat{c} \cdot \underline{\underline{P}} \cdot \hat{v} \cos \psi - \hat{c} \cdot \underline{\underline{P}} \cdot \hat{b}]\, dS \tag{23}$$

This form may be useful for calculating $\langle \underline{A} \cdot \underline{B} \rangle$ in laboratory plasmas from a series of single time two-point measurements, making use of $P_{ij} = 1/2\ (R_{ij} - R_{ji})$.

A slightly different approach is appropriate to extract the magnetic helicity from interplanetary magnetic data which, at this time are limited to time series of single point field measurements. First we extend our definition of $\underline{\underline{R}}$ to include two-time measurements, so

$$R_{ij}(\underline{r},\tau) = \langle B_i(\underline{x},t)\ B_j(\underline{x}+\underline{r},\ t+\tau)\rangle$$

where the new argument represents time. Then, we adopt the "frozen flux" approximation,

$$R_{ij}(0,\tau) = R_{ij}(-\ u\hat{v}\tau, 0)$$

where $u\hat{v}$ is the solar wind streaming velocity. This approximation is expected to be valid when the streaming speed u is much greater than the local Alfven speed, the magnetic Reynolds number is very large and the correlation lengths are small compared with the scales of spatial variation of uv and b. Since the pseudo-tensor P_{ij} is equivalent to $1/2\ (R_{ij}(\underline{r},t) - R_{ij}\ (-\underline{r},t))$, we may rearrange (23) using the frozen flux property resulting in

$$\langle \underline{A} \cdot \underline{B} \rangle = \frac{1}{2\sin\psi} \int_0^{\infty} - \int_0^{-\infty} (\hat{c} \cdot \underline{\underline{R}} \cdot \hat{b} - \cos\psi\ \hat{c} \cdot \underline{\underline{R}}\hat{v})\ d\tau \qquad (24)$$

where $\underline{\underline{R}}$ is evaluated at $\underline{r} = 0$ but with temporal argument τ.

Many data sets may disallow use of the above prescriptions (23) and (24) by virtue of strong inhomogeneities. The question of whether useful numbers can be obtained from other candidates remains to be decided by a posteriori consistency tests. Only simple modifications of (23) and (24) are necessary to accommodate data with spatial separations parallel to the mean field.

TABLE I

Summary of Properties of the Scalars in the Tensor

$$
\begin{aligned}
R_{ij} = {} & A\,\delta_{ij} + B\,r_i r_j + C(b_i r_j + b_j r_i) \\
& + D\,b_i b_j + E(r_i \epsilon_{jk\ell} b_k r_\ell + r_j \epsilon_{ik\ell} b_k r_\ell) \\
& + F(b_i \epsilon_{jk\ell} b_k r_\ell + b_j \epsilon_{ik\ell} b_k r_\ell) \\
& + G\,\epsilon_{ijk} r_k + H\,\epsilon_{ijk} b_k
\end{aligned}
$$

Function	Pseudo-vector $\hat{b}$ (pseudo-scalar $\hat{b}\cdot\underline{r}$)	Proper Vector ($\hat{b} \rightarrow \hat{\lambda}$)	Dependence on Basic Functions
A	Even in z	Even	Q_1, Q_2
B	Even	Even	Q_1, Q_2
C*	Odd in z	Odd	Q_1, Q_2
D	Even	Even	Q_1, Q_2
E	Even	Does not appear	Q_3
F	Odd	Does not appear	Q_3
G	Even	Even	Φ
H*	Odd	Odd	Φ

Notes to Table I

The asterisk (*) signifies that the tensor character of the form multiplying the function changes when $\hat{b} \rightarrow \hat{\lambda}$.

Even and odd refer to the powers of z in the series representing the function.

The dependence on basic functions column lists symbolically which of the underlying functions determines each form. Only forms depending on the same basic function are 'mixed' in the Fourier transformed representation (see section VI).