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TWO-DIMENSIONAL INTEGRATING MATRICES ON RECTANGULAR GRIDS

W. D. Lakin

OLD DOMINION UNIVERSITY RESEARCH FOUNDATION
Norfolk, Virginia 23508

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Langley Research Center
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## SUMMARY

Integrating matrices form the basis of an efficient numerical procedure for solving differential equations associated with rotating beam configurations. In vibration problems, by expressing the equations of motion of the beam in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the spatial dependence is removed from the governing partial differential equations and the resulting ordinary differential equations can be cast into standard eigenvalue form. Previous derivations of integrating matrices have been restricted to one spatial dimension. This report derives integrating matrices based on two-dimensional rectangular grids with arbitrary grid spacings allowed in one direction. The derivation of higher dimensional integrating matrices is the initial step in the generalization of the integrating matrix methodology to vibration and stability problems involving plates and shells.

## INTRODUCTION

The equations of motion governing the vibration and stability of rotating structures have no closed-form solutions, and approximate methods of analysis such as asymptotic techniques, finite-element methods, or direct numerical integration must be employed to effect a solution. For the special case of beamlike (i.e. one-dimensional) structures, a numerical procedure based on the use of "one-dimensional" integrating matrices (refs. 1, 2) provides a particularly efficient procedure for solution (refs. 3-5). An integrating matrix provides a means for integrating a function whose values are known on a grid composed of increments in the independent variable. Such matrices can be derived by approximating the integrand by an appropriate polynomial (refs. 2, 5). In beam vibration problems, for example, expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator and applying the boundary conditions remove dependence on the single spatial variable, and the resulting ordinary differential equations can

be put into standard eigenvalue form. Solutions can then be determined by standard methods.

The integrating matrix technique for numerically integrating a function defined by its values at a set of grid points separates data defining the function at the grid points from data defining the structure of the grid. For one-dimensional integrating matrices, i.e. matrices for integration with respect to a single space variable, data defining the grid structure uniquely determine the elements of the matrix. Thus, as long as the same grid points and same degree approximating polynomials are used, the same integrating matrix may be used to integrate any number of different functions. This property allows the integrating matrix to be used as an operator. This is in contrast to numerical integration schemes based on interpolation or divided differences where the analogous quadrature formulas involve a combination of values of the function at the grid points as well as data defining the structure of the grid.

Although the derivation and use of one-dimensional integrating matrices has been treated in the literature (refs. 1-5), the derivation of a "two-dimensional" integrating matrix for integrating a function of two variables whose values are known on a two-dimensional grid has apparently not been addressed. The purpose of this report is to derive integrating matrices based on two-dimensional rectangular grids with uniform spacing in one of the two orthogonal directions. As in the one-dimensional integrating matrix, the elements of the two-dimensional integrating matrix depend only on the data defining the grid structure and the approximating polynomials.

## PRELIMINARY CONSIDERATIONS

Derivation of a two-dimensional integrating matrix is related to numerical approximation of the multiple integral

$$A = \iint_R f(x,y)\, dx\, dy \tag{1}$$

where $R$ is the rectangular region

$$R = \left[(x,y);\ a \le x \le b,\ c \le y \le d\right]$$

and $f(x,y)$ is a continuous function on $R$. Values of $f(x,y)$ are assumed known at the $NM$ ($= N \times M$) grid points

$$G = \left\{ (x_i,\ y_j);\quad \begin{array}{l} i = 1,\ \ldots,\ N;\ j = 1,\ \ldots,\ M \\ a = x_1 < x_2 < \ldots < x_N = b \\ c = y_1 < y_2 < \ldots < y_M = d \end{array} \right\} \tag{2}$$

which lie at the intersection of lines dividing the region $R$ into $(N - 1) \times (M - 1)$ rectangles. A portion of the grid is shown in figure 1. Grid spacing may be nonuniform in either the x-direction or y-direction, but not both. In the present derivation, as shown in figure 1, nonuniformity is allowed in the x-direction.

From the calculus, the double integral [eq. (1)] may be written as an iterated integral

$$A = \int_c^d \left\{ \int_a^b f(x,y)\ dx \right\} dy \tag{3}$$

Hence, if

$$F(y) = \int_a^b f(x,y)\ dx \tag{4}$$

then

$$A = \int_c^d F(y)\ dy \tag{5}$$

The variable $y$ is held fixed while performing the x-integration in equation (4). Equations (4) and (5) thus effectively split the multiple integral into two single integrals of functions of one variable. This decomposition

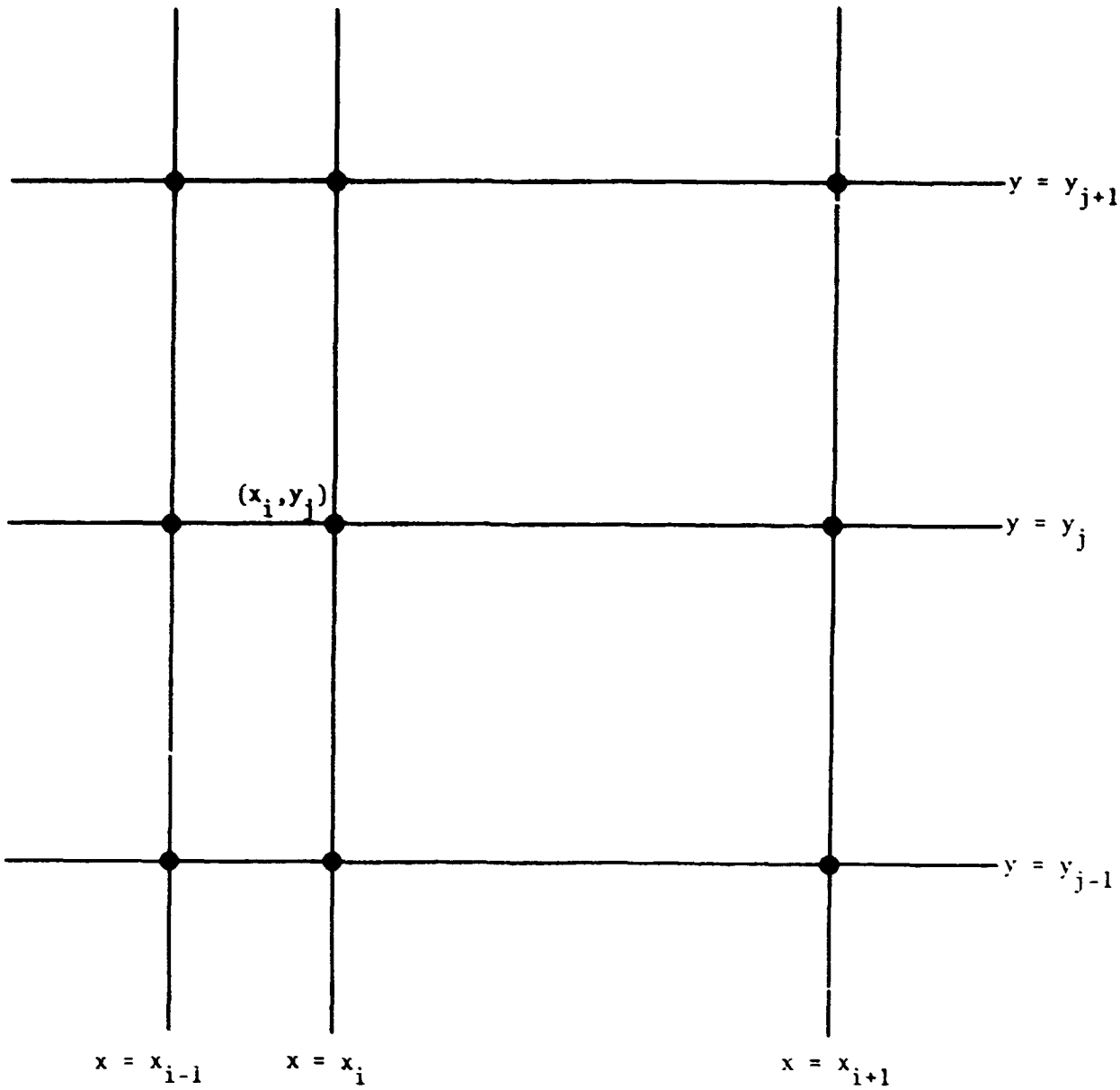


Figure 1. Portion of the grid $G$ which divides the region $R$ into $(N - 1) \times (M - 1)$ rectangles. In this sample grid, the spacing is non-uniform in the x-direction and uniform in the y-direction with $y_{j+1} = y_j + \Delta y$ where $\Delta y = (d - c)/(M - 1)$.

suggests that two one-dimensional integrating matrices based on the $x$ and $y$ grids

$$G_x: \quad a = x_1 < x_2 < \ldots < x_N = b \tag{6a}$$

$$G_y: \quad c = y_1 < y_2 < \ldots < y_M = d \tag{6b}$$

might be used to produce a two-dimensional integrating matrix based on the rectangular grid $G$. Before exploring this possibility, it is necessary to develop some notation.

## Notation for Data and Integrals

To describe data defining $f(x,y)$ at the grid points, let

$$f_{ij} = f(x_i, y_j) \quad i = 1, \ldots, N; \; j = 1, \ldots, M \tag{7}$$

Integrals with respect to $x$ from $a = x_1$ to $x_k$ for fixed $y$ will now be denoted by

$$F_k(y) = \int_{x_1}^{x_k} f(x,y)\, dx \qquad k = 1, \ldots, N \tag{8}$$

These integrals play the role of $F(y)$ in equation (4). Similarly, the role of $A$ in equation (5) will be played by NM integrals of the form

$$A_{jk} = \int_{y_1}^{y_j} F_k(y)\, dy \qquad j = 1, \ldots, M; \; k = 1, \ldots, N \tag{9}$$

where $y_1 = c$. Combining equations (8) and (9) gives

$$A_{jk} = \int_{y_1}^{y_j} \int_{x_1}^{x_k} f(x,y)\, dx\, dy \tag{10}$$

The one-dimensional N by N integrating matrix based on the $x$ grid $G_x$ will be denoted by $[I]$, while the one-dimensional M by M integrating matrix based on the $y$ grid $G_y$ will be denoted by $[J]$.

To approximate integrals of $f(x,y)$ with respect to $x$ for fixed $y$, the N by N matrix $[I]$ premultiplies the N by 1 column vector $\{f_\ell\}$ which, for the fixed $y$ value $y = y_\ell$ say, contains data defining $f(x, y_\ell)$ on $G_x$. The data required by $[I]$ is shown in figure 2. In particular, if the column vector $\{f_\ell\}$ is defined by

$$\{f_\ell\} = (f_{1\ell}\ f_{2\ell}\ \ldots\ f_{N\ell})^T \tag{11}$$

and

$$\{F_\ell\} = \left((0\ F_2(y_\ell)\ F_3(y_\ell)\ \ldots\ F_N(y_\ell)\right)^T \tag{12}$$

then

$$[I]\ \{f_\ell\} = \{F_\ell\} \tag{13}$$

To approximate integrals of $F_k(y)$ with respect to $y$ for fixed $k$, the M by M matrix $[J]$ premultiplies the M by 1 column vector $\{H_k\}$ which contains data defining $F_k(y)$ on the grid $G_y$. The data required by $[J]$ is shown in figure 3. For fixed $k = 1, \ldots, N$, defining the column vector $\{H_k\}$ by

$$\{H_k\} = \left(F_k(y_1)\ F_k(y_2)\ \ldots\ F_k(y_M)\right)^T \tag{14}$$

gives

$$[J]\ \{H_k\} = (0\ A_{2k}\ A_{3k}\ \ldots\ A_{Mk})^T \tag{15}$$

A comparison of figures 2 and 3, or equivalently comparing equation (14) with the right-hand side of equation (13), shows that the output from approximating an $x$ integration is not, by itself, suitable input for approximating a $y$ integration. To obtain the data required to form any of the vectors $\{H_k\}$, $x$ integrations must be performed at all M of the $y$ levels, and the resulting values must be rearranged from

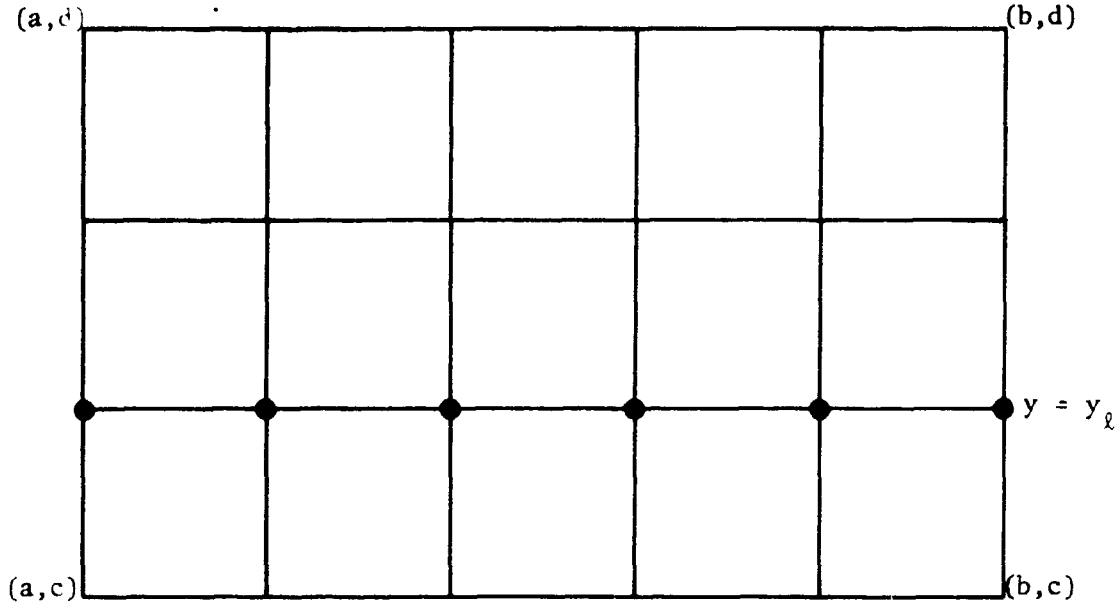


Figure 2. A grid $G$ with $N = 6$ and $M = 4$. Grid points marked with ● denote points at which data is required for computation of the x-integrals $F_k(y_\ell)$ $(k = 1, \ldots, 6)$ using the integrating matrix $[I]$ based on the grid $G_x$.

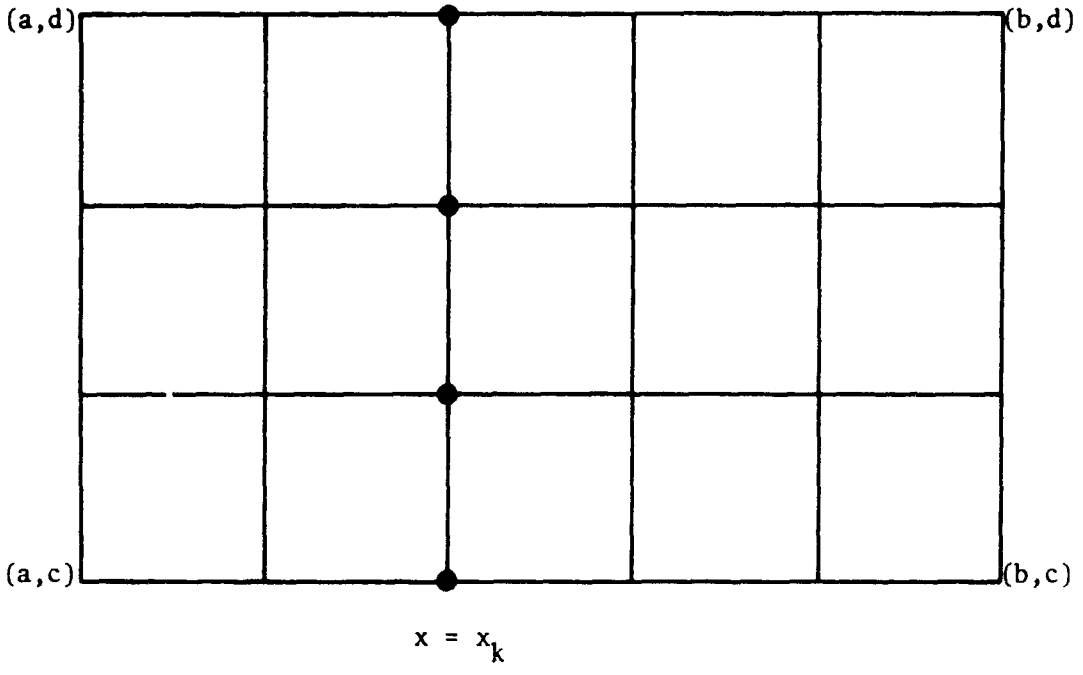


Figure 3. A grid G with N = 6 and M = 4. Grid points marked with ● denote points at which data is required for integrating the function $F_k(y)$ with respect to y using the integrating matrix [J] based on the grid $G_y$.

grouping by grid row to grouping by grid column, that is, data must be rearranged from variable x-fixed y to fixed x-variable y. This places a restriction on the format for presenting data defining $f(x,y)$ at the grid points. In particular, to derive a two-dimensional integrating matrix on the rectangular grid G, the data for $f(x,y)$ cannot be given as an M by N matrix. This result was unexpected as the most "natural" way of representing data on an M by N rectangular grid is an M by N matrix. While sequential application of [I] and [J] to an M by N matrix of data does produce an M by N matrix [A] whose elements approximate $A_{jk}$ in equation (10), the resulting matrix [A] is not an integrating matrix. As this deviation of [A] provides motivation for choosing the proper format of data defining $f(x,y)$ at the grid points, a brief account of its deviation will be given below.

## A Natural Derivation Which Does Not Yield an Integrating Matrix

Let $[D]^T$ be the N by M matrix of data defining $f(x,y)$ on G, i.e. the ijth entry of $[D]^T$ is $f_{ij}$. The $\ell$th column in the matrix $[D]^T$ is thus the column vector $\{f_\ell\}$ ($\ell = 1, \ldots, M$) in equation (11), i.e. values of $f(x,y)$ on $G_x$ for fixed $y = y_\ell$. Hence, by equation (13), the $\ell$th column in the N by M matrix [E] defined by

$$[E] = [I]\ [D]^T \tag{16}$$

is the column vector $\{F_\ell\}$ in equation (12).

The matrix [E] contains data for the functions $F_k(y)$, $k = 1, \ldots, N$ (i.e. x-integrals) on the grid $G_y$. As previously discussed, this data must be rearranged into a form appropriate for y-integration using the integrating matrix [J]. In particular, the kth column of the M by N matrix $[E]^T$ is the column vector $\{H_k\}$ in equation (14) ($k = 1, \ldots, N$). Consequently, if [A] is as previously defined, then

$$[A] = [J]\ [E]^T. \tag{17}$$

Combining equation (17) with equation (16) now gives

$$[A] = [J]\ [D]\ [I]^T \tag{18}$$

While equation (18) uses the two one-dimensional integrating matrices $[I]$ and $[J]$ to obtain $[A]$, it is clear that the right-hand side of equation (18) cannot be recast to give the result desired, that is,

$$[A] = [P]\ [D] \tag{19}$$

which would allow $[P]$ to be interpreted as a two-dimensional integrating matrix.

The above argument shows that data defining $f(x,y)$ on $G$ cannot be expressed as an N by M matrix $[D]^T$ if the desired end product of the procedure approximating the multiple integral [eq. (1)] is to be a two-dimensional integrating matrix. In the next section, it will be shown that a column vector format for the data defining $f(x,y)$ on $G$ will lead to a two-dimensional integrating matrix.

## THEORY FOR TWO-DIMENSIONAL INTEGRATING MATRICES

Results from previous sections show that representing data defining $f(x,y)$ on $G$ as an N by M matrix does not lead to an approximation to the double integral [eq. (1)] which can be interpreted as an integrating matrix times the matrix of function data. In this section an alternate approach is considered where data defining $f(x,y)$ on $G$ is represented in column vector form.

Let $\{f\}$ be the NM by 1 column vector of data obtained by "stacking" the M, N by 1 column vectors $\{f_\ell\}$, which by equation (11) give data for fixed values of $y = y_\ell$ ($\ell = 1, \ldots, M$), that is,

$$\begin{aligned}\{f\} &= (\{f_1\}, \{f_2\}, \ldots, \{f_M\})^T \\ &= (f_{11}, f_{21}, \ldots, f_{N1}, f_{12}, \ldots, f_{N2}, \\ &\quad \ldots, f_{1M}, \ldots, f_{NM})^T\end{aligned} \tag{20}$$

Similarly, let $\{H\}$ be the NM by 1 column vector obtained by "stacking" the N, M by 1 column vectors $\{H_m\}$ in equation (14), that is,

$$\{H\} = (\{H_1\}, \{H_2\}, \ldots, \{H_N\})^T$$

$$= \big(F_1(y_1), \ldots, F_1(y_M), F_2(y_1), \ldots, F_2(y_M), \ldots, F_N(y_1), \ldots, F_N(y_M)\big)^T \tag{21}$$

Note that each vector $\{f_\ell\}$ in the stacked vector $\{f\}$ gives values of $f(x,y)$ at the fixed values $y = y_\ell$ and, hence, is exactly the input necessary to approximate the N x-integrals $F_k(y_\ell)$ $(k = 1, \ldots, N)$. Similarly, each stacked vector $\{H_m\}$ in $\{H\}$ for fixed m contains data appropriate for approximating integrals of $F_m(y)$ with respect to y. As seen previously, going from $\{f\}$ to $\{H\}$ requires not only integrations with respect to x for each fixed y level, but also rearrangement of the resulting data $F_k(y_\ell)$ from a row format to a column format. In the present devivation, the integration with respect to x for fixed y and the data rearrangement will be accomplished simultaneously by multiplying $\{f\}$ by an appropriate NM by NM matrix $[T]$ so that

$$\{H\} = [T]\,\{f\} \tag{22}$$

A key observation here is that $F_k(y_\ell)$ may be obtained from the N by N integrating matrix $[I]$ based on the grid $G_x$ and the column vector $\{f_\ell\}$ by taking the usual dot product of the kth row vector $\lfloor I_k \rfloor$ of $[I]$ with $\{f_\ell\}$: that is,

$$F_k(y_\ell) = \int_{x_1}^{x_k} f(x, y_\ell)\, dx = \lfloor I_k \rfloor \cdot \{f_\ell\} \tag{23}$$

Hence, each vector $\{H_m\}$ in $\{H\}$ has the form

$$\{H_m\} = \big(\lfloor I_m \rfloor \cdot \{f_1\}, \lfloor I_m \rfloor \cdot \{f_2\}, \ldots, \lfloor I_m \rfloor \cdot \{f_M\}\big)^T \tag{24}$$

(m = 1 ,..., N), i.e., only the mth row of [I] enters here. Equation (24) and the stacked nature of the vectors {f} and {H} now show that the required NM by NM matrix [T] in equation (22) may itself be written as a "stack" of N, M by NM matrices $[T_s]$ (s = 1 ,..., N) where the nonzero portion of each row of $[T_s]$ is simply the sth row vector $\lfloor I_s \rfloor$ of the N by N matrix [I]. Symbolically, we have that, if $\lfloor 0 \rfloor$ denotes the 1 by N zero-row vector, then

$$[T_s] = \begin{bmatrix} \lfloor I_s \rfloor & \lfloor 0 \rfloor & & & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor I_s \rfloor & \lfloor 0 \rfloor & \cdots & \lfloor 0 \rfloor \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \cdots & \cdots & \lfloor 0 \rfloor & \lfloor I_s \rfloor \end{bmatrix} \tag{25}$$

and

$$[T] = \begin{bmatrix} [T_1] \\ [T_2] \\ \vdots \\ [T_N] \end{bmatrix} \tag{26}$$

It is worth noting at this stage that as $\lfloor I_1 \rfloor = \lfloor 0 \rfloor$, $[T_1]$ is simply the M by NM zero matrix. Also, in the stacked vector {H}, the entries

$$F_1(y_\ell) = \int_{x_1}^{x_1} f(x, y_\ell)\, dx \equiv 0 \tag{27}$$

for all $\ell$ = 1 ,..., M.

Having derived the matrix [T] which transforms the column vector {f} of data into the column vector {H}, it is now a relatively easy

matter to derive an NM by NM matrix $[S]$ which gives a column vector of double integrals, that is,

$$\{A\} = [S]\ \{H\} \tag{28}$$

where the NM by 1 column vector $\{A\}$ is a stack of N, M by 1 column vectors $\{A_m\}$ $(m = 1, \ldots, N)$ and

$$\{A_m\} = \left(0, \int_{y_1}^{y_2}\int_{x_1}^{x_m} f(x,y)\ dx\ dy, \ldots, \int_{y_1}^{y_M}\int_{x_1}^{x_m} f(x,y)\ dx\ dy\right)^T = (0, A_{2m}, A_{3m}, \ldots, A_{Mm})^T \tag{29}$$

In particular, if $[J]$ is the M by M one-dimensional integrating matrix based on the grid $G_y$, then

$$\{A_m\} = [J]\ \{H_m\} \tag{30}$$

Equations (28) and (30) now show that $[S]$ will be a banded matrix with N integrating matrices $[J]$ along its diagonal, i.e., if $[0]$ is an M by M zero matrix, then

$$[S] = \begin{bmatrix} [J] & [0] & \cdots & [0] \\ [0] & [J] & \ddots & \vdots \\ \vdots & \ddots & \ddots & [0] \\ [0] & \cdots & [0] & [J] \end{bmatrix} \tag{31}$$

Combining equations (28) and (22) now shows that the required two-dimensional interating matrix $[K]$, such that

$$\{A\} = [K]\ \{f\} \tag{32}$$

is the NM by NM matrix

$$[K] = [S]\ [T] \tag{33}$$

As $[S]$ and $[T]$ involve only $[J]$ and the row vectors of $[I]$, it is clear that $[K]$ will be independent of data defining the function $f(x,y)$ at the grid points.

Sample matrices for $[S]$, $[T]$, and $[K]$ associated with a 4 by 3 rectangular grid are given in appendix A.

## CONCLUDING REMARKS

This report documents the derivation of two-dimensional integrating matrices for approximating integrals of a function of two variables on a rectangular region. The analysis shows that data defining the function at points on a rectangular grid which corresponds to the region must be used in a "stacked" column vector format. An appropriate two-dimensional integrating matrix is then obtained as the product of two auxiliary matrices which are formed from the one-dimensional integrating matrices corresponding to the two orthogonal directions on the grid.

## APPENDIX A

This appendix contains sample matrices [I], [J], [S], [T], and [K] based on a 2-dimensional, rectangular grid G with 12 points (NM = 12, N = 4, M = 3). The x-grid $G_x$ and y-grid $G_y$ which define the grid G are

$$G_x = \{0., 12., 24., 36.\}$$

and

$$G_y = \{0., 2., 4.\}$$

In these examples, the four by four integrating matrix [I] on the one-dimensional grid $G_x$ is based on quadratic interpolating polynomials, while the three by three integrating matrix [J] on the one-dimensional grid $G_y$ is based on linear interpolating polynomials.

Table A1. The four by four integrating matrix [I] on the one-dimensional grid $G_x$ based on quadratic interpolating polynomials.

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 5.0 & 8.0 & -1.0 & 0.0 \\ 10.0 & 16.0 & -2.0 & 0.0 \\ 10.0 & 15.0 & 6.0 & 5.0 \end{bmatrix}$$

Table A2. The 12 by 12 matrix [T] formed from the row vectors of the four by four matrix [I]. Row vectors of [I] are enclosed in boxes.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 5.0 | 0.0 | -1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 5.0 | 0.0 | -1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 5.0 | 0.0 | -1.0 | 0.0 |
| 10.0 | 16.0 | -2.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 10.0 | 16.0 | -2.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 10.0 | 16.0 | -2.0 | 0.0 |
| 10.0 | 15.0 | 6.0 | 5.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 10.0 | 15.0 | 6.0 | 5.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 10.0 | 15.0 | 6.0 | 5.0 |

Table A3. The three by three integrating matrix [J] on the one-dimensional grid $G_y$ based on linear interpolating polynomials.

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

Table A4. The 12 by 12 matrix [S] formed from the 3 by 3 integrating matrix [J] based on the grid $G_y$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1.0 | 2.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 1.0 | 2.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 2.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 2.0 | 1.0 |

Table A5. The 12 by 12 integrating matrix [K] based on the two-dimensional rectangular grid G.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 5.0 | 8.0 | -1.0 | 0.0 | 5.0 | 8.0 | -1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 5.0 | 8.0 | -1.0 | 0.0 | 10.0 | 16.0 | -2.0 | 0.0 | 5.0 | 8.0 | -1.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 10.0 | 16.0 | -2.0 | 0.0 | 10.0 | 16.0 | -2.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 10.0 | 16.0 | -2.0 | 0.0 | 20.0 | 32.0 | -4.0 | 0.0 | 10.0 | 16.0 | -2.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 10.0 | 15.0 | 6.0 | 5.0 | 10.0 | 15.0 | 6.0 | 5.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 10.0 | 15.0 | 6.0 | 5.0 | 20.0 | 30.0 | 12.0 | 10.0 | 10.0 | 15.0 | 6.0 | 5.0 |

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