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ANISOTROPIC MAGNETOHYDRODYNAMIC TURBULENCE IN A STRONG EXTERNAL MAGNETIC FIELD

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ABSTRACT

A strong external dc magnetic field introduces a basic anisotropy into incompressible magnetohydrodynamic turbulence. The modifications that this is likely to produce in the properties of the turbulence are explored for the high Reynolds number case. The conclusion is reached that the turbulent spectrum splits into two parts: an essentially two-dimensional spectrum with both the velocity field and magnetic fluctuations perpendicular to the dc magnetic field, and a generally weaker and more nearly isotropic spectrum of Alfvén waves. A minimal characterization of the spectral density tensors is given. Similarities to measurements from the Culham-Harwell Zeta pinch device and the UCLA Macrotor tokamak are remarked upon, as are certain implications for the Belcher and Davis measurements of magnetohydrodynamic turbulence in the solar wind.

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I. INTRODUCTION

We now have in hand the beginnings of a theory of high-Reynolds-number magnetohydrodynamic turbulence[1-11] that is at a level of description that is as systematic and inclusive as the corresponding theory for fluid turbulence.[12] A significant limitation of this body of theory is that it assumes a high degree of symmetry in the statistics of the turbulent fields: spatial homogeneity, rotational isotropy, and frequently, temporal stationarity and mirror (reflection) invariance. It seems clear, however, that much magnetohydrodynamic turbulence will not be so highly symmetric, both in astrophysical and laboratory situations.

The rotational isotropy assumption in particular is limiting, because many of the most interesting cases involve an externally-imposed dc magnetic field which selects a particular direction in space. While it may be reasonable to assume rotational isotropy about the direction of the mean magnetic field, it probably is not reasonable to assume it about the other two directions. Moreover, while in ordinary fluid mechanics, turbulence isotropizes itself[12] at the smaller spatial scales, it is likely that anisotropy in magnetohydrodynamics will persist over the full range of scales to which magnetohydrodynamics is applicable.

The present paper is intended to propose a qualitative picture of turbulent, homogeneous, incompressible, magnetohydrodynamic fluctuations in the presence of a strong dc magnetic field = $\underset{\sim}{B}_0$. Here, strong implies that the energy density associated with $\underset{\sim}{B}_0$ is large compared to the sum of the fluctuating

magnetic energy density and the kinetic energy density associated with the fluid motions of the magnetofluid. We will be particularly concerned with the differences between this case and the purely isotropic case in which there is no preferred direction.

Convincing data on magnetohydrodynamic turbulence are still rare. Probably the best measurements to date are those from the Culham-Harwell Zeta toroidal pinch device.[13,14,15] Strikingly similar to some of the Zeta results are some recent measurements reported for the UCLA Macrotor tokamak.[16,17] There is not very much help yet from numerical simulations: most of the published results are two-dimensional. Some recent three-dimensional isotropic results have begun to be generated by the group at the Observatoire de Nice,[3,18] but are not yet available as of the time of this writing. It appears that it is the case that existing magnetohydrodynamic turbulence computations are all spectral-method (Galerkin approximation) computations, assuming rectangular periodic boundary conditions and no externally-imposed magnetic field, and do not fully address the issues addressed here. Previous analytical calculations have addressed the subject of the anisotropies introduced by external dc magnetic fields for the case of <u>low</u> magnetic Reynolds numbers.[19,20,21]

We rely on a mixture of perturbation theory, model calculations, and physically-informed guesswork to arrive at a picture of incompressible magnetohydrodynamic turbulence, homogeneous but anisotropic due to the presence of a strong dc magnetic field $\underset{\sim}{B}_0$. The picture must be regarded as conjectural until more experiments are done. It is, however, suggestive and compatible with what is known from Zeta[13,14,15] and Macrotor.[16,17]

We may start from the observation that if the external $\underset{\sim}{B}_0$ is strong compared to the mean fluctuating field, more energy is required to bend and stretch field lines than to translate them, particularly at the larger spatial scales. This leads to the feeling, confirmed by the asymptotic analysis of Sec. II, that a strong external $\underset{\sim}{B}_0$ suppresses spatial variations of the magnetic and velocity fields along the $\underset{\sim}{B}_0$ direction. Also, the magnetic fluctuations and electric field fluctuations are primarily in a direction perpendicular to $\underset{\sim}{B}_0$. The spectrum is conjectured, in Sec. III, to consist of a highly anisotropic part of the geometry just described plus a smaller, more nearly isotropic part which can properly be described as waves. The tendency of a strong magnetic field to enforce two-dimensionality renders several recent two-dimensional calculations[4,7] and computations[5,6,9] more generally applicable than they might otherwise be. Experimental comparisons are also remarked upon, in Sec. III.

Turbulence which fulfills all four symmetries enumerated in the first paragraph requires only one scalar function to characterize it.[12] When the symmetries are relaxed, a more elaborate characterization is required. Section IV is devoted to presenting a framework in which homogeneous but anisotropic and non-mirror-symmetric turbulence may be characterized. We restrict ourselves in this section to the case in which no net electric current flows through the magnetofluid. Net currents preclude[8] spatial homogeneity and periodic boundary conditions, and are thus an additional complication in many interesting cases; we postpone some considerations associated with situations with net dc current fluxes to a later paper. Section V summarizes our conclusions.

II. THE LIMIT OF LARGE B_0

We consider an incompressible magnetofluid of uniform density. The magnetic field is $\underset{\sim}{B}_0 + \underset{\sim}{B}(\underset{\sim}{x},t)$ and the velocity field is $\underset{\sim}{v}(\underset{\sim}{x},t)$. The electric current density is $\underset{\sim}{j}(\underset{\sim}{x},t) = \nabla \times \underset{\sim}{B}(\underset{\sim}{x},t)$. Both $\underset{\sim}{v}$ and $\underset{\sim}{B}$ are solenoidal vectors. We write the dynamical equations in a common set of dimensionless variables:

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{B} \cdot \nabla \underset{\sim}{v} - \underset{\sim}{v} \cdot \nabla \underset{\sim}{B} + \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{v} \tag{1}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} = - \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} + \underset{\sim}{j} \times \underset{\sim}{B} + \underset{\sim}{j} \times \underset{\sim}{B}_0 - \nabla p_m$$

$$= - \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} + \underset{\sim}{B} \cdot \nabla \underset{\sim}{B} - \nabla p + \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{B}. \tag{2}$$

Dissipative terms involving viscosity and resistivity have been omitted for convenience, but can easily be reinserted. p_m is the mechanical part of the pressure. $p = p(\underset{\sim}{x},t)$ is the total pressure, p_m plus $(\underset{\sim}{B}_0 + \underset{\sim}{B})^2/2$. p is determined by taking the divergence of Eq. (2) and using $\nabla \cdot \underset{\sim}{v} = 0$ to get the Poisson equation

$$\nabla^2 p = \nabla \cdot [\underset{\sim}{B} \cdot \nabla \underset{\sim}{B} - \underset{\sim}{v} \cdot \nabla \underset{\sim}{v}]. \tag{3}$$

p is the solution to Eq. (3) subject to whatever boundary conditions apply. Note that for rectangular periodic geometry, p is a quadratic functional of $\underset{\sim}{B}$ and $\underset{\sim}{v}$. The ambient magnetic field is assumed to be spatially uniform, temporally constant, and in the z-direction: $\underset{\sim}{B}_0 = B_0 \hat{e}_z$. If B_0 has a spatial variation, it is assumed slow compared to the other length scales of interest.

If we assume that the mean plasma energy density, $\langle v^2 \rangle + \langle B^2 \rangle$, is small compared to $B_0^{\,2}$, we may try approximating Eqs. (1) and (2) by their linearized versions:

$$\frac{\partial \underset{\sim}{B}}{\partial t} \cong \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{v} \tag{4}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} \cong \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{B} \ . \tag{5}$$

The general solution to Eqs. (4) and (5) is a linear superposition of Alfvén waves which have both $\underset{\sim}{B}$ and $\underset{\sim}{v}$ varying as $\exp i(\underset{\sim}{k} \cdot \underset{\sim}{x} \pm \omega_{\underset{\sim}{k}} t)$, where $\omega_{\underset{\sim}{k}} = \underset{\sim}{k} \cdot \underset{\sim}{B}_0$. The equations (1) and (2) become intrinsically nonlinear, however whenever $\underset{\sim}{k}$ becomes nearly perpendicular to $\underset{\sim}{B}_0$. The neglected nonlinear terms become larger than the linear ones. As $\underset{\sim}{k}$ becomes more nearly perpendicular to $\underset{\sim}{B}_0$, the time scales associated with the evolution are no longer $\sim (kB_0)^{-1}$, but are determined by the nonlinear processes associated with the quadratic terms in Eqs. (1) and (2). Normal modes can tell us nothing about these nearly perpendicular motions, whose time scales remain finite even in the limit of infinite B_0. Note that for strictly perpendicular spatial variation, the terms involving $\underset{\sim}{B}_0$ drop out of Eqs. (1) and (2) altogether.

A somewhat more formal demonstration of the two-dimensionality of the dynamics can be given simply by assuming a well-behaved perturbation series solution to Eqs. (1) and (2) as $B_0 \to \infty$. This result is sufficiently interesting that it may perhaps be worth demonstrating in detail. We introduce a formal expansion parameter ε^{-1} into the B_0 terms of Eqs. (1) and (2) and look

at the first few orders in the assumed perturbation series $\underset{\sim}{v} = \underset{\sim}{v}^{(0)} + \varepsilon \underset{\sim}{v}^{(1)} + \varepsilon^2 \underset{\sim}{v}^{(2)} + \ldots$ and $\underset{\sim}{B} = \underset{\sim}{B}^{(0)} + \varepsilon \underset{\sim}{B}^{(1)} + \varepsilon^2 \underset{\sim}{B}^{(2)} + \ldots$. Note that $B^{(0)}$ is not the same as $\underset{\sim}{B}_0$. We demand that the time and space derivatives remain of O(1) as B_0 gets large. What is of interest is what kind of motions are enforced by the very large value of B_0. Note that what is being effected is essentially an expansion of the equations of motion in powers of $1/B_0$.

The coefficients of successively higher powers of ε are equated to zero and we require that $\underset{\sim}{v}$, $\underset{\sim}{B}$, and $\underset{\sim}{j}$ remain solenoidal at each order in the expansion. For convenience, we invoke rectangular periodic boundary conditions, so that all dynamical vector fields are representable as Fourier series.

The $O(\varepsilon^{-1})$ terms simply give that

$$\underset{\sim}{j}^{(0)} \times \underset{\sim}{B}_0 = 0 \tag{6}$$

and

$$\underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{v}^{(0)} = 0 . \tag{7}$$

Equation (6) implies that the zeroth-order current must flow along $\underset{\sim}{B}_0$: $\underset{\sim}{j}^{(0)} = j_z^{(0)} \hat{e}_z$. Since $\nabla \cdot \underset{\sim}{j}^{(0)} = 0$ and

$$j_z^{(0)} = \sum_{\underset{\sim}{k}} j_z^{(0)} (\underset{\sim}{k},t) \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x}), \tag{8}$$

the non-vanishing Fourier coefficients must have $\underset{\sim}{k}$ vectors perpendicular to $\hat{e}_z$. Written in component notation,

$$\underset{\sim}{j}^{(0)} = (0, 0, j_z^{(0)} (x,y,t)), \text{ only.} \tag{9}$$

From the Fourier-transformed version of $\nabla \times \underset{\sim}{B}^{(0)} = \underset{\sim}{j}^{(0)}$ and the assumed periodic boundary conditions, we have at once that

$$\underset{\sim}{B}^{(0)} = \Sigma_k \underset{\sim}{B}^{(0)}(\underset{\sim}{k},t) \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x})$$

$$= (B_x^{(0)}(x,y,t),\ B_y^{(0)}(x,y,t),\ 0) \qquad (10)$$

only, with $\underset{\sim}{k} \cdot \underset{\sim}{B}^{(0)}(\underset{\sim}{k},t) = 0$.

Equation (7) implies that $\partial \underset{\sim}{v}^{(0)}/\partial z = 0$, or that

$$\underset{\sim}{v}^{(0)} = (v_x^{(0)}(x,y,t),\ v_y^{(0)}(x,y,t),\ v_z^{(0)}(x,y,t)). \qquad (11)$$

where

$$\frac{\partial v_x^{(0)}}{\partial x} + \frac{\partial v_y^{(0)}}{\partial y} = 0. \qquad (12)$$

The geometry implied by Eqs. (9), (10), and (11), with all variables independent of z and the variable magnetic field perpendicular to $\underset{\sim}{B}_0$, has emerged as a consequence of the assumption of a well-behaved expansion as $B_0 \to \infty$. To get the dynamical behavior of $\underset{\sim}{v}^{(0)}$ and $\underset{\sim}{B}^{(0)}$, we need to go on to the next order.

Equating the coefficients of ε^0 to each other gives

$$\frac{\partial \underset{\sim}{B}^{(0)}}{\partial t} = \underset{\sim}{B}^{(0)} \cdot \nabla \underset{\sim}{v}^{(0)} - \underset{\sim}{v}^{(0)} \cdot \nabla \underset{\sim}{B}^{(0)} + B_0 \frac{\partial \underset{\sim}{v}^{(1)}}{\partial z} \qquad (13)$$

$$\frac{\partial \underset{\sim}{v}^{(0)}}{\partial t} = -\underset{\sim}{v}^{(0)} \cdot \nabla \underset{\sim}{v}^{(0)} + \underset{\sim}{j}^{(0)} \times \underset{\sim}{B}^{(0)} + \underset{\sim}{j}^{(1)} \times \underset{\sim}{B}_0 - \nabla p^{(0)} \quad . \qquad (14)$$

The first three terms of Eq. (13) are independent of z. Integrating Eq. (13) over one periodicity length in z at fixed x, y, t, the $\underset{\sim}{v}^{(1)}$ term drops out and we are left with

$$\frac{\partial \underset{\sim}{B}^{(0)}}{\partial t} = \underset{\sim}{B}^{(0)} \cdot \nabla \underset{\sim}{v}^{(0)} - \underset{\sim}{v}^{(0)} \cdot \nabla \underset{\sim}{B}^{(0)} \quad . \tag{15}$$

The x and y components of Eq. (15) <u>are</u> the induction equation for two-dimensional magnetohydrodynamics.[4] They are equivalent to the pointwise conservation of the vector potential $\underset{\sim}{a}^{(0)} = a^{(0)}(x,y,t)\hat{e}_z$ for the two-dimensional motion of a fluid element:

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_{\perp}^{(0)} \cdot \nabla\right) \; a^{(0)}(x,y,t) = 0 \quad . \tag{16}$$

In Eq. (16) and hereafter, subscripts "$\perp$" and "$\|$" will mean components perpendicular to and parallel to $\underset{\sim}{B}_0$ respectively.

Since Eqs. (13) and (15) hold, we have $\partial \underset{\sim}{v}^{(1)}/\partial z = 0$, or $\underset{\sim}{v}^{(1)} = \underset{\sim}{v}^{(1)}(x,y,t)$ only. The z component of Eq. (15) gives $\underset{\sim}{B}^{(0)} \cdot \nabla v_z^{(0)} = 0$. There are two ways this can be achieved. Either $v_z^{(0)} = 0$, or $v_z^{(0)} = v_z^{(0)}(a^{(0)})$ only, which is a statement that $v_z^{(0)}$ is a constant along a field line of $\underset{\sim}{B}^{(0)}$.

The x and y components of Eq. (14) give

$$\frac{\partial \underset{\sim}{v}_{\perp}^{(0)}}{\partial t} = - \underset{\sim}{v}_{\perp}^{(0)} \cdot \nabla_{\perp} \underset{\sim}{v}^{(0)} + \underset{\sim}{j}^{(0)} \times \underset{\sim}{B}^{(0)} - B_0 \nabla_{\perp} B_z^{(1)} + B_0 \frac{\partial}{\partial z} \underset{\sim}{B}_{\perp}^{(1)} - \nabla_{\perp} p^{(0)} \quad . \tag{17}$$

Before manipulating Eq. (17), we can obtain one more useful piece of information on $p^{(0)}$ by returning to the z component of Eq. (14):

$$\frac{\partial v_z^{(0)}}{\partial t} + \underset{\sim}{v}^{(0)} \cdot \nabla_{\perp} v_z^{(0)} = - \frac{\partial p^{(0)}}{\partial z} \quad . \tag{18}$$

The left hand side of Eq. (18) is independent of z. Integrating it over one period in z gives

$$\frac{\partial v_z^{(0)}}{\partial t} + \underset{\sim}{v}_\perp^{(0)} \cdot \nabla_\perp v_z^{(0)} = 0, \tag{19}$$

which says that if a fluid element has a z-velocity to start with, the velocity will remain constant; no forces act in the z direction. Because of Eq. (19), Eq. (18) implies that $\partial p^{(0)}/\partial z = 0$, or that $p^{(0)} = p^{(0)}(x,y,t)$ only.

We now use the fact that all the terms in Eq. (17) except those involving $\underset{\sim}{B}^{(1)}$ have been shown to be z-independent. Applying $\partial/\partial z$ to Eq. (17) gives

$$-\nabla_\perp \frac{\partial B_z^{(1)}}{\partial z} + \frac{\partial^2 \underset{\sim}{B}_\perp^{(1)}}{\partial z^2} = 0. \tag{20}$$

Since $\partial B_z^{(1)}/\partial z + \nabla_\perp \cdot \underset{\sim}{B}_\perp^{(1)} = 0$, Eq. (20) implies

$$\nabla_\perp \nabla_\perp \cdot \underset{\sim}{B}_\perp^{(1)} + \frac{\partial^2 \underset{\sim}{B}_\perp^{(1)}}{\partial z^2} = 0. \tag{21}$$

Periodic boundary conditions are then invoked again to assert that $\underset{\sim}{B}^{(1)}$ is a super-position of linearly independent plane wave contributions of the form $\underset{\sim}{B}_{\underset{\sim}{k}} \exp(i\, \underset{\sim}{k} \cdot \underset{\sim}{x})$ with $\underset{\sim}{k} \cdot \underset{\sim}{B}_{\underset{\sim}{k}} = 0$. Eq. (21) implies that

$$\underset{\sim}{k}_\perp \underset{\sim}{k}_\perp \cdot \underset{\sim}{B}_{\underset{\sim}{k}\perp}^{(1)} + k_z^2 \underset{\sim}{B}_{\underset{\sim}{k}\perp}^{(1)} = 0. \tag{22}$$

From Eq. (22) we can see that $\underset{\sim}{k}_\perp$ and $\underset{\sim}{B}_{\underset{\sim}{k}\perp}^{(1)}$ are parallel or else both terms in Eq. (22) are zero. We must examine these possibilities separately. If both terms and $k_z = 0$, then $\underset{\sim}{B}_\perp^{(1)} = 0$. If the individual terms of Eq. (22) are not zero, then

$$(k_\perp^2 + k_z^2)\, \underset{\sim}{k}_\perp \cdot \underset{\sim}{B}^{(1)}_{\underset{\sim}{k}_\perp} = 0 \tag{23}$$

which, since $k^2 \neq 0$, implies $\underset{\sim}{k}_\perp \cdot \underset{\sim}{B}^{(1)}_{\underset{\sim}{k}_\perp} = 0$, contrary to hypothesis. Therefore both terms in Eq. (22) vanish separately. For the first term to vanish, either $\underset{\sim}{k}_\perp = 0$ or $\underset{\sim}{k}_\perp \cdot \underset{\sim}{B}^{(1)}_{\underset{\sim}{k}_\perp} = 0$. If $\underset{\sim}{k} = 0$, $\underset{\sim}{B}^{(1)}_{\underset{\sim}{k}_\perp} = 0$ and $\underset{\sim}{B}^{(1)}$ is purely in the z-direction and is a function only of z. This cannot be ($\underset{\sim}{B}^{(1)}$ is solenoidal), so $\underset{\sim}{k}_\perp \cdot \underset{\sim}{B}^{(1)}_{\underset{\sim}{k}_\perp} = 0$ is the only possibility left. This implies $k_z = 0$, so that $\underset{\sim}{B}^{(1)}_\perp$ can only be a function of x,y, and t. Returning to Eq. (17), the only terms which survive the above conclusions are

$$\frac{\partial \underset{\sim}{v}_\perp^{(0)}}{\partial t} = -\underset{\sim}{v}_\perp^{(0)} \cdot \nabla_\perp \underset{\sim}{v}_\perp^{(0)} + \underset{\sim}{j}^{(0)} \times \underset{\sim}{B}^{(0)} - \nabla_\perp \tilde{p}^{(0)} \tag{24}$$

where $\tilde{p}^{(0)} \equiv p^{(0)} - B_0 B_z^{(1)}$ is still formally a pressure. It is not necessary to know $B_z^{(1)}$ to determine $\tilde{p}^{(0)} = \tilde{p}^{(0)}(x,y,t)$. The Poisson equation for $\tilde{p}^{(0)}$ which results from taking the divergence of Eq. (24) and using $\nabla_\perp \cdot \underset{\sim}{v}_\perp^{(0)} = 0$ determines $\tilde{p}^{(0)}$ in the usual way.

Equation (24) and the transverse components of Eq. (15),

$$\frac{\partial \underset{\sim}{B}_\perp^{(0)}}{\partial t} = \underset{\sim}{B}_\perp^{(0)} \cdot \nabla_\perp \underset{\sim}{v}_\perp^{(0)} - \underset{\sim}{v}_\perp^{(0)} \cdot \nabla_\perp \underset{\sim}{B}_\perp^{(0)} , \tag{25}$$

are identical to the two-dimensional magnetohydrodynamic system whose turbulence properties have been investigated in considerable detail by Fyfe et al.,[4,5,6] by Orszag and Tang,[9] and by Pouquet,[7] from different viewpoints, for the case $v_z^{(0)} = 0$.

The procedure can be iterated to higher order in ε, but the amusing collapse of the geometry to two dimensional magnetohydrodynamics does not change.

It should be noted that, though p is an O(1) quantity, the mechanical pressure $p_m = p - \frac{(\underline{B}_0+\underline{B})^2}{2}$ contains formally both $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ terms. The first of these makes no contribution to ∇p_m in Eq. (2), but the second must be checked in detail to see that there is no $O(\varepsilon^{-1})$ contribution, for consistency's sake. This is not difficult to do.

III. QUALITATIVE PICTURE OF THE SPECTRUM

For turbulence which obeys spatial homogeneity (i.e., the statistics of the fluctuating field are translation-invariant), the most basic variables are the covariance tensors of the fields and their Fourier transforms, the spectral densities. For example, for the fluctuating part of the magnetic field, $\underset{\sim}{B}$, the covariance

$$\langle \underset{\sim}{B}(\underset{\sim}{x},t)\underset{\sim}{B}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau)\rangle = \int d\underset{\sim}{k}\ \underset{\approx}{S}(\underset{\sim}{k},\tau)\ \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r}), \tag{26}$$

where the spectral density tensor $\underset{\approx}{S}(\underset{\sim}{k},\tau)$ may also depend upon the time t in non-steady situations, is the most physically revealing quantity to consider. The trace of $\underset{\approx}{S}(\underset{\sim}{k},0)$ measures the amount of magnetic energy per unit wavenumber space, and various moments of it give such covariances as $\langle \underset{\sim}{j}\cdot\underset{\sim}{B}\rangle$ and $\langle \underset{\sim}{A}\cdot\underset{\sim}{B}\rangle$ where $\underset{\sim}{A}$ is the vector potential for which $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$. Comparable spectral density tensors exist for the correlation matrices $\langle \underset{\sim}{v}\underset{\sim}{v}\rangle$ and $\langle \underset{\sim}{v}\underset{\sim}{B}\rangle$.

The physics of turbulence is most satisfactorily discussed in terms of the spectral densities, such as $\underset{\approx}{S}(\underset{\sim}{k},0)$. These are always the result of a balance among three competitive processes: (1) injection of excitations due to whatever is driving the turbulence; (2) dissipation due to viscosity, resistivity, or other decay processes at high wavenumbers; and (3) modal transfer, due to the nonlinear interactions, between one spatial mode and another. The range of injection or driving mechanisms is even greater for magnetohydrodynamic turbulence than for Navier-Stokes fluids, and the various possibilities

for modal transfer are also greater; so, depending upon the situation, more than one qualitative character for $\underset{\approx}{S}$ may be expected. Nevertheless, the results of Section II suggest a crude qualitative picture of the spectrum of the fluctuations of homogeneous magnetohydrodynamic turbulence with an anisotropy introduced by a large value of B_0. In Section IV we give a minimal characterization of this field without, however, being able to provide theoretically-derived expressions for its elements.

We suggest that a typical turbulent $\underset{\sim}{B}$ spectrum will consist of two parts. First, the greater part of the energy in $\underset{\sim}{k}$ space will reside near the plane $\underset{\sim}{k}\cdot\underset{\sim}{B}_0 = 0$, will involve polarizations such that $\underset{\sim}{B}(\underset{\sim}{k})\cdot\underset{\sim}{B}_0 \cong 0$, and will approximate the conditions of two-dimensional[4,5,6,7,9] magnetohydrodynamic turbulence. It will be non-oscillatory, with time scales which are determined by the degree of nonlinearity with which the fields are excited. A second part of the spectrum will be more nearly isotropic and can be properly called Alfvén waves: their time scales will be, predominantly, their length scales divided by the Alfvén speed constructed from B_0. As B_0 gets large, they will be separated in frequency from the quasi-two-dimensional part of the spectrum and will have a slower transfer rate, determined by the amplitude of the spectrum itself. In the quasi-two-dimensional part of the spectrum, an Alfvén-wave-like motion can exist, in which the small-scale fluctuations run along the <u>local</u> mean field lines provided by the larger spatial scale components.[22] The dynamical role of these pseudo-Alfvén waves remains uncertain, but they seem to be effective at enforcing equipartition[4,5,6] between magnetic and kinetic energies at the small spatial scales. In any case, their frequency

scales, in the present situation, do not vary proportionately to B_0, and they are to be sharply distinguished from the true Alfvén waves being alluded to here, which are basically the three-dimensional solutions of Eqs. (4) and (5). For what we are calling Alfvén waves, these two linear equations are a good approximation; for what we are calling the two-dimensional part of the spectrum, the linear terms in Eqs. (4) and (5) are identically zero, and Eqs. (24) and (25) are required. Real life is also likely to involve a transition region, with $\underset{\sim}{k} \cdot \underset{\sim}{B}_0$ small but non-zero, where linear and non-linear effects will be of comparable magnitude. The direction of flow of excitations in $\underset{\sim}{k}$ space across this transition region is one of the major unanswered questions remaining.

The partition of the excitations between the two types of turbulence, two-dimensional and Alfvén wave, is bound to be situation dependent and will depend upon the excitation mechanism for the turbulence. In laboratory experiments on confined plasmas, the candidates for excitation mechanisms are very numerous. Literally hundreds of plasma instabilities (growing linear perturbations about quiescent laminar states) have been catalogued; at a more elementary level, the large radial gradients that are maintained in such fundamental parameters as temperature and pressure, (and frequently, magnetic field, density, and fluid velocity) loom immediately, to anyone familiar with fluid turbulence, as potential drivers for turbulent motions. Because of the relatively rapid variation of the mean properties with the transverse coordinates, compared to typically slower axial variations, one may well imagine a selective excitation of the two dimensional part of the spectrum. For many turbulence-producing agents (impressed changes in the boundary conditions or electrical circuitry

supporting the system, for example) the time scales will be finite and will not speed up as the external magnetic field strength is made larger. One may reasonably expect some matching between the time scales of the excited turbulence and the time scales of the processes which drive it. Likewise, it is reasonable that the more rapidly varying components of the turbulence will usually arise parasitically, as a result of nonlinear transfer from modes involved in the low-frequency (i.e. << the reciprocal of an Alfvén transit time) part of the spectrum. In this case, the Alfvén wave component will be regarded as derived from the two-dimensional magnetohydrodynamic component.

One other reason for imagining the Alfvén wave component to be weaker or less energetic than the two-dimensional magnetohydrodynamic component is that resonance conditions mitigate against a rapid or effective transfer to the Alfvén waves and among the Alfvén waves.[22] Speaking loosely for a moment, two modes with wave numbers $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, and frequencies ω_1, ω_2 transfer most effectively to a third mode with wavenumber $\underset{\sim}{k}_3$ and frequency ω_3 when modal matching conditions are met:

$$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 \tag{27a}$$

$$\omega_1 + \omega_2 = \omega_3 \ . \tag{27b}$$

For Navier-Stokes turbulence and two-dimensional magnetohydrodynamic turbulence, all three of ω_1, ω_2, and ω_3 are always zero, so every triad satisfying (27a) is always "resonant". On the other hand, in three dimensions, some of the modes involved may be Alfvén waves with $\omega = \pm \underset{\sim}{k} \cdot \underset{\sim}{B}_0$, and (27b) then considerably restricts the possibilities for transfer. In particular two two-dimensional magnetohydrodynamic Fourier modes cannot combine resonantly

to feed an Alfvén wave with $\omega_3 = \underset{\sim}{k}\cdot\underset{\sim}{B}_0 \neq 0$. Two Alfvén waves with $\omega_2 = \omega_3$ and $\underset{\sim}{k}_2\cdot\underset{\sim}{B}_0 = \underset{\sim}{k}_3\cdot\underset{\sim}{B}_0$ can resonantly drain a two-dimensional magnetohydrodynamic mode with $\omega_1 = 0$, but this is a higher-order process, and therefore a slower one, than transfer among three two-dimensional magnetohydrodynamic modes. Of course these arguments from the "weak turbulence" perspective are less than rigorous, and it is likely that higher-order processes will drain two-dimensional magnetohydrodynamic turbulence into the Alfvén wave part of the spectrum. Nevertheless, there seems to be qualitative reason to regard the Alfvén wave coupling to be relatively weak.

To summarize the above picture, we conjecture several features of magnetohydrodynamic turbulent fields existing in a strong dc magnetic field $\underset{\sim}{B}_0$: (1) velocity-field and magnetic-field fluctuations are perpendicular to $\underset{\sim}{B}_0$, or nearly so; (2) the correlation lengths along $\underset{\sim}{B}_0$ are much longer than those transverse to $\underset{\sim}{B}_0$, since the fluctuating components have little variation along $\underset{\sim}{B}_0$; (3) since the electric field $\underset{\sim}{E} \cong -\underset{\sim}{v} \times \underset{\sim}{B}_0$, the electric field fluctuations are also largely perpendicular to $\underset{\sim}{B}_0$; and (4) on top of the essentially two-dimensional magnetohydrodynamic spectrum is superposed a weaker three-dimensional Alfvén wave spectrum with frequency scales which scale as B_0.

One may inquire into the extent to which the above predictions are borne out by existing measurements. As far as laboratory measurements go, the answer appears to be, rather well. Although the measurements performed were in neither case exactly what a theorist would have wished for, two rather different sets of measurements on the Zeta toroidal Z pinch[13,14,15] and the UCLA Macrotor Tokamak[16,17] substantiate the above picture in several respects. First, the correlation lengths in the direction of the mean field (toroidal)

direction were measured to at least an order of magnitude greater than the correlation lengths in the two transverse (radial and poloidal) directions, for the magnetic field fluctuations, the electric field fluctuations ($\underset{\sim}{E} \cong -\underset{\sim}{v} \times \underset{\sim}{B}_0$, so for the transverse components, these are essentially velocity-field measurements), and the electrostatic potential fluctuations. Thus, a high degree of two-dimensionality was indicated in all cases. Second, the rms transverse magnetic fluctuations were always larger than the toroidal fluctuations: by more than an order of magnitude in the Macrotor measurements, and by an uncertain factor in the Zeta experiments, partially because apparently no distinction was made between parallel fluctuations and toroidal fluctuations (the poloidal mean field was large enough in Zeta that it makes a difference). Finally, the single-point frequency measurements for both the magnetic and electric field fluctuations showed frequency spectra which were rather featureless, were well fit by power laws, and fell off so steeply as to be essentially zero below either the ion gyrofrequency or the reciprocal of the correlation length divided by the toroidal Alfvén speed. This last fact indicates the surprising result that not only is the higher-frequency Alfvén wave part of the spectrum "weak", as per our conjecture, but that it is in effect nearly absent in these two situations. We have no satisfactory explanation for this absence. In both sets of measurements, the received fluctuating signals were filtered of their first several kilohertz before any statistical processing was carried out, making an assessment of the absolute levels of the rms fluctuations impossible, since the frequency spectra rise steeply towards zero frequency.

As far as space physics measurements are concerned, only some of the above conjectures are testable, with data currently available. The best measurements available are for solar wind magnetometer data; these are well exemplified by the work of Belcher and Davis[23] and are summarized in the review of Barnes[24]. Some ninety percent of the fluctuating magnetic energy is associated with fluctuations perpendicular to the mean field,[23] but the one measurement of parallel versus perpendicular correlation lengths[25] to date indicates that the perpendicular correlation length is about a factor of two <u>greater</u> than the parallel one. (The ensemble chosen was, admittedly, a rather specialized one.) The solar wind situation is not as well fit by the above analysis as the laboratory one, however, since the mean field is comparable to the fluctuation level.

IV. MINIMAL CHARACTERIZATION OF THE SPECTRA

If the turbulence were homogeneous, isotropic, etc., the spectral density tensor in Eq. (26) would reduce at $\tau = 0$ to

$$\underset{\approx}{S}(\underset{\sim}{k}) \equiv \underset{\approx}{S}(\underset{\sim}{k},0) = (\underset{\approx}{1} - \hat{k}\hat{k})\, E_B(k)/k^2 \tag{28}$$

and would be characterized by the single scalar function $E_B(k)$, the magnetic energy spectrum. It is to the mechanical analogue[12] of this spectral function that Kolmogoroff similarity arguments are usually applied, leading to the well-known $k^{-5/3}$ behavior. In case the symmetries are suspended, a more elaborate set of dependences upon $\underset{\sim}{k}$ is necessary.

For $S_{ij}(\underset{\sim}{k})$, the ijth element of $\underset{\approx}{S}$, homogeneity and reality alone give the conditions $S_{ij}(\underset{\sim}{k}) = S^*_{ji}(\underset{\sim}{k}) = S_{ji}(-\underset{\sim}{k})$. From this it follows that the real parts of the $S_{ij}(\underset{\sim}{k})$ are even under $\underset{\sim}{k} \to -\underset{\sim}{k}$, the imaginary parts of the $S_{ij}(\underset{\sim}{k})$ are odd under $\underset{\sim}{k} \to -\underset{\sim}{k}$, and the diagonal elements are real.

This restricts the number of independent functions involved in $\underset{\approx}{S}(\underset{\sim}{k})$ considerably, but there are still several. It is clear that the most economical choices are desirable in order to represent $\underset{\approx}{S}(\underset{\sim}{k})$, since any future theory, such as a generalization of the Kolmogoroff similarity-variable arguments, will probably be done on the elements of $\underset{\approx}{S}$.

After considerable trial and error, we have determined the most economical representation of $\underset{\approx}{S}(\underset{\sim}{k})$ to be possible in a $\underset{\sim}{k}$-dependent set of basis vectors. A set of <u>configuration-space</u> elements of the tensor have been given by Matthaeus and Smith,[26] who elaborate several points beyond those considered here. In particular, if we choose a set of basis vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, where

$$\hat{e}_3 \equiv \underset{\sim}{k}/k$$

$$\hat{e}_2 \equiv \underset{\sim}{k} \times \underset{\sim}{B}_0/|\underset{\sim}{k} \times \underset{\sim}{B}_0|.$$

$$\hat{e}_\perp \equiv \hat{e}_2 \times \hat{e}_3, \tag{29}$$

then the minimal representation of $\underset{\approx}{S}(\underset{\sim}{k})$ is

$$\underset{\approx}{S}(\underset{\sim}{k}) = \sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}\hat{e}_\alpha\hat{e}_\beta \quad . \tag{30}$$

The absence of $\hat{e}_3$ contributions to the dyadics in Eq. (30) is due to the condition that the fields be solenoidal:

$$\underset{\sim}{k}\cdot\underset{\approx}{S}(\underset{\sim}{k}) = \underset{\approx}{S}(\underset{\sim}{k})\cdot\underset{\sim}{k} = 0 \ . \tag{31}$$

Even in this $\underset{\sim}{k}$-dependent coordinate system, it can easily be shown that A_{11} and A_{22} are real, while $A_{12} = A_{21}^*$. Thus there are four independent real scalar functions involved in this representation of $\underset{\approx}{S}(\underset{\sim}{k})$ and there are no additional constraints provided by the requirement that the fields be solenoidal. These functions are functions of the scalars k, cos θ (where θ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{B}_0$), and ϕ (where ϕ is the azimuthal angle between the projection of $\underset{\sim}{k}$ onto a plane perpendicular to $\underset{\sim}{B}_0$ and some fixed direction in that plane). For turbulence which is isotropic with respect to rotations about $\underset{\sim}{B}_0$, there is no ϕ- dependence, and the $A_{\alpha\beta}$ are functions only of k and cos θ. The limiting two-dimensional spectrum described in the previous two sections is represented by a dominant A_{22} which is sharply peaked near $\theta = \pi/2$. Isotropic turbulence with no helicity is represented by $A_{12} = A_{21} = 0$ and with $A_{11} = A_{22}$ being independent of cos θ and ϕ. The presence of magnetic helicity $\langle\underset{\sim}{A}\cdot\underset{\sim}{B}\rangle$ where $\underset{\sim}{A}$ is

the vector potential fluctuation for which $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$, is signalled by non-zero imaginary parts of A_{12}. In fact, the magnetic helicity per unit wave number space is

$$H_M(\underset{\sim}{k}) = (i/k^2)\ \varepsilon_{\alpha\beta\gamma} k_\beta S_{\alpha\gamma}$$

$$= (i/k)\ (A_{21} - A_{12})$$

$$= 2\mathrm{Im}\ A_{12}/k \quad . \tag{32}$$

Applying $\int d\underset{\sim}{k}$ to the expression (32) gives the magnetic helicity $\langle \underset{\sim}{A} \cdot \underset{\sim}{B} \rangle$. Non-helical but anisotropic turbulence is represented by real but unequal values of A_{11} and A_{22}, with $A_{12} = 0$. The magnetic modal energy spectrum can be written as $A_{11} + A_{22}$.

The corresponding representation of the scalars in $\underset{\sim}{x}$ space is much more complicated, because the differential equations which result from the condition that the fields be solenoidal have no obvious solutions. Connections exist between the various configuration-space-dependent functions which are highly implicit[27,28] (however, cf. Ref. 26).

It must be regarded as an open question as to whether there exists any universal anisotropic spectrum comparable to the Kolmogoroff spectrum[12] for the isotropic case. This is as true for anisotropic Navier-Stokes turbulence as it is for magnetohydrodynamics, and it may be a long time before the question is answered definitely. It seems likely that if any such universality exists, it will be phrased in terms of statements about A_{11}, A_{22}, and $\mathrm{Im}A_{12}$.

A quantity directly expressible in terms of the $A_{\alpha\beta}$ which has been measured in the solar wind is the covariance matrix $\langle \underset{\sim}{B}(\underset{\sim}{x})\underset{\sim}{B}(\underset{\sim}{x}) \rangle$. In terms of the $A_{\alpha\beta}$,

$$\langle B_i(\underset{\sim}{x})B_j(\underset{\sim}{x})\rangle = \hat{a}_i \cdot \int d\underset{\sim}{k} \sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}\hat{e}_\alpha\hat{e}_\beta \cdot \hat{a}_j \ , \tag{33}$$

where $\hat{a}_i$, $\hat{a}_j$ are any two unit basis vectors in a set of $\underset{\sim}{k}$-independent coordinates. If we assume that the turbulence is rotationally symmetric about the direction of $\underset{\sim}{B}_0$, so that the $A_{\alpha\beta}$ are functions only of k and cos θ, symmetries lead to several cancellations of the integrals in Eq. (33). The end result is an expression for the covariance matrix

$$\langle \underset{\sim}{B}(\underset{\sim}{x})\underset{\sim}{B}(\underset{\sim}{x})\rangle = A\,\underset{\approx}{1} + B\,\hat{b}\hat{b}$$

$$= \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A+B \end{pmatrix} \ , \tag{34}$$

where $\underset{\approx}{1}$ is the unit dyadic and $\hat{b} = \underset{\sim}{B}_0/B_0$ is a unit vector along the dc magnetic field. A and B are numbers, integrals of the $A_{\alpha\beta}$. Thus it follows that for turbulence that is isotropic with respect to rotations about a fixed direction, $\underset{\sim}{B}_0$, the principal axis transformation can be used to diagonalize the covariance matrix with two principal moments equal and one principal axis along $\underset{\sim}{B}_0$.

A matrix of the form (34) was <u>not</u> found by Belcher and Davis[23,24] for the covariance matrix $\langle \underset{\sim}{B}\underset{\sim}{B}\rangle$ in the solar wind. Rather, the three principal moments typically stood in the ratios 5:4:1, indicating perhaps a two-dimensionality to a first approximation, but also a non-zero departure from isotropy with respect to rotations about <u>any</u> direction. The direct measurement of such tensors as $\langle \underset{\sim}{B}(\underset{\sim}{x})\underset{\sim}{B}(\underset{\sim}{x})\rangle$ is probably the first single-point measurement that is reasonable to attempt for a magnetohydrodynamic turbulent field. Ion

counters which permit measurements of the velocity field $\underset{\sim}{v}(\underset{\sim}{x})$ permit measurement of the covariances matrices $\langle \underset{\sim}{v}(\underset{\sim}{x})\underset{\sim}{B}(\underset{\sim}{x})\rangle$ and $\langle \underset{\sim}{v}(\underset{\sim}{x})\underset{\sim}{v}(\underset{\sim}{x})\rangle$. The two-point matrices $\langle \underset{\sim}{B}(\underset{\sim}{x})\underset{\sim}{B}(\underset{\sim}{x}+\underset{\sim}{r})\rangle$ are in general necessary to establish the spectral densities such as $\underset{\approx}{S}(\underset{\sim}{k})$, unless some version of Taylor's "frozen flow" hypothesis[12] is applicable.

V. SUMMARY

Externally imposed dc magnetic fields $\underset{\sim}{B}_0$ introduce a basic anisotropy into homogeneous, incompressible magnetohydrodynamic turbulence. In the limit of large B_0, the fluctuation spectrum splits into an essentially two dimensional part with magnetic and velocity fluctuations nearly normal to $\underset{\sim}{B}_0$ and nearly independent of the coordinate along $\underset{\sim}{B}_0$, plus a more nearly isotropic, weaker, higher-frequency Alfvén-wave part. For many imaginable turbulence-driving mechanisms, the quasi-two dimensional part may be expected to dominate. Such a spectrum seems to have characterized the Zeta[13,14,15] pinch device, well in advance of any satisfactory theory, as well as the UCLA Macrotor tokamak.[16,17]

The spectral density tensors of the covariance matrices can be characterized by at most four independent scalar functions which, roughly speaking, characterize the turbulence by saying how energetic it is, how anisotropic, and how helical. It is unknown as to what extent limiting forms exist at high Reynolds numbers for these $A_{\alpha\beta}$ functions, as they are believed to exist for isotropic Navier-Stokes turbulence.

The most useful future direction for the subject to take might well be more thorough-going measurements of the kind that were carried out in Zeta. Intuitive pictures, involving perhaps no more than dimensional considerations, need to be developed for relating measured properties of the fluctuations to the $A_{\alpha\beta}$ functions.[26] Finally, the qualitative effects of the different kinds of spectral shapes upon such properties as transport need to be assessed. It

may be that many of the observed particle confinement properties of toroidal devices can be satisfactorily explained in terms of simple random-walk models for the magnetic field lines, using measured fluctuation levels and correlation lengths, and without reference to the underlying dynamics.

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