A Theory for Predicting Boundary Impedance and Resonance Frequencies of Slotted-Wall Wind Tunnels, Including Plenum Effects

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Please make the changes listed below to p. 10.

Equation (28) should read:

\[ \beta k_1 = (2n - 1)\pi/2 \]

Equation (29) should read:

\[ \omega = (2n - 1)\pi\beta c \]

The sentence after equation (29), beginning, "This result differs . . ." should be changed to "This result is also that which is obtained from equations A17 and A27 of reference 1."

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A Theory for Predicting Boundary Impedance and Resonance Frequencies of Slotted-Wall Wind Tunnels, Including Plenum Effects

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SUMMARY

Waves may be generated in a wind-tunnel test section for various reasons. It is possible that certain frequencies of waves may be amplified because of resonance. A theory that uses acoustic impedance concepts for predicting resonance modes in a two-dimensional, slotted-wall wind tunnel with a plenum chamber is described. The equation derived is shown to be consistent with known results for limiting conditions. The computed resonance modes compare well with appropriate experimental data. When the theory is applied to perforated-wall test sections, it predicts the experimentally observed closely spaced modes that occur when the wavelength is not long compared with the plenum depth.

INTRODUCTION

Various phenomena may give rise to waves in the flow in a wind-tunnel test section. Among these phenomena are the oscillations of an airfoil, the motion of the compressor fan blades, and random fluctuations associated with flow turbulence. It is possible that certain frequencies may be amplified as a result of resonance that arises because of the geometry of the test section and the plenum and because of the flow boundary itself.

The influence of the plenum chamber on resonance phenomena in ventilated wind-tunnel test sections has been studied both theoretically and experimentally, as reported in reference 1. The experimental arrangement described in reference 1 included a small, two-dimensional test section with a box-like plenum chamber. This arrangement does not closely resemble those used in most operating wind tunnels, but the results obtained with this basic device have nevertheless provided valuable insight into the nature of wind-tunnel resonance phenomena and have provided a useful set of experimental data.

The analysis presented in reference 1 includes the derivation of equations for the resonance modes for both slotted- and perforated-wall test sections. Various combinations of steady-flow and unsteady-flow boundary conditions were used to derive the equations. The equation for the slotted-wall case yielded an inconsistent Mach number variation. Therefore it was discarded, and the perforated-wall equation was applied to the slotted-wall condition. This procedure gave much better agreement with the data than did the analysis of reference 2, which used only steady-flow boundary conditions.

The present analysis of slotted-wall test-section resonance is based on a somewhat different approach than that of reference 1. It uses the concept of the acoustic impedance of the test-section boundary and relies entirely on unsteady-flow considerations. The resulting equation for the resonance modes is characterized by a slightly different Mach number variation than that given in reference 1, and it contains the slotted-wall parameter rather than a per-
forated-wall equivalent of it. Furthermore, the consideration of the possibility of an inductive effect of the plenum air for the higher modes appears to resolve some of the previously found discrepancies with experimental data.

SYMBOLS

In all calculations, length dimensions are nondimensionalized in terms of the wind-tunnel semiheight.

A coefficient (see eq. (5))
B coefficient (see eq. (2))
b nondimensional slot spacing
c speed of sound
D nondimensional plenum-chamber depth
f cyclic frequency, kHz
\[ H = \beta^2 k_1 D \sqrt{1 - M^2 / \beta^4} \]
i \[ i = \sqrt{-1} \]
k wave number
\[ \lambda \] nondimensional slotted-wall parameter
M Mach number
n integer index
p local pressure, Pa
R reflection coefficient
r open ratio of slotted wall
s parameter, \( k_1 \cos \alpha_1 \)
t time, sec
U wave phase velocity along the boundary
V stream velocity
v velocity component in y direction
x,y coordinates in direction of stream and normal to stream, respectively
For the purpose of studying resonance phenomena, a two-dimensional model spanning a wind-tunnel test section with slotted top and bottom walls is assumed. (See fig. 1(a).) The plenum chamber, which is not shown in figure 1(a), is assumed to have solid walls parallel to the top and bottom boundaries of the test section. The oscillations of the model are forced by an external driver.

Although the waves emitted by a two-dimensional model tend ultimately to become cylindrical waves propagating outward at all angles from the model, only the part of the wave surface that propagates in certain critical directions contributes to the resonance condition. This effect is illustrated in figure 1(b) (which is similar to fig. 3(a) of ref. 1). Resonance occurs when a segment of the wave front returns, in phase, to the driving source after...
reflection from the boundary. This condition is obtained only when the upstream component of the sound velocity exactly equals the stream velocity, so that the ray path is always perpendicular to the stream direction. (See fig. 1(b).) That is, if \( \alpha \) is the angle of propagation relative to this perpendicular direction, then \( \sin \alpha = M \).

In order to determine the combinations of frequency and stream Mach number for which resonance occurs, the change in phase of the wave (at the wall) must be calculated. This phase change is determined by the acoustic impedance of the boundary.

**Calculation of Boundary Impedance**

For the purpose of studying the wall impedance, a plane wave incident at angle \( \alpha \) on a wall situated at \( y = 0 \) will be considered. The velocity potential for such a wave (see ref. 3) can be written as

\[
\phi_i = \exp \{ik_1[y \cos \alpha_1 + (x - Vt) \sin \alpha_1 - ct]\} \tag{1}
\]

The wave reflected from the boundary has, in general, a different amplitude:

\[
\phi_r = (1 + B) \exp \{ik_1[-y \cos \alpha_1 + (x - Vt) \sin \alpha_1 - ct]\} \tag{2}
\]

When a wave propagates through a slotted wall in the absence of a flow, the wall has the effect of a uniform acoustic inductance on the wave as it appears some distance from the wall, provided that the slot spacing is small compared with the wavelength (ref. 4). On the other hand, when a wave is incident on an interface of relative motion in the absence of a wall, then, depending on the magnitude of the relative velocity and the angle of incidence, it may be totally reflected or it may be partly reflected and partly transmitted with refraction. For the latter situation, the refraction angle \( \alpha_2 \) and the wave number \( k_2 \) in the outside stationary air are determined by the conditions of compatibility of the waves at the boundary (see ref. 3):

\[
c \csc \alpha_1 + V = c \csc \alpha_2 = U \tag{3}
\]

and

\[
k_1 \sin \alpha_1 = k_2 \sin \alpha_2 = k_b \tag{4}
\]

where \( k_b \) and \( U \) are the wave number and the phase velocity of the boundary displacement.
When both a slotted wall and a flow interface exist coincidentally, a wave is propagated through this boundary when the conditions represented by equations (3) and (4) can be satisfied. At some distance from the wall, the wave will appear to have been transmitted through a homogeneous boundary, with a definite phase velocity $U$ along the boundary. The wave will also be subject to a definite phase change consistent with the known acoustic inductance effect of the slotted wall (refs. 4 and 5). The velocity potential of the transmitted wave will therefore be

$$\phi_t = A \exp[i k_2(y \cos \alpha_2 + x \sin \alpha_2 - ct)]$$

If the region outside the flow boundary is not infinite in extent but is a chamber with a solid wall at a distance $D$ beyond the ventilated boundary, then, in order to satisfy the boundary condition at the solid wall, the solution for the waves in the chamber must contain a term representing the wave reflected from the solid boundary in addition to that for the transmitted wave:

$$\phi_c = A \exp[i k_2(y \cos \alpha_2 + x \sin \alpha_2 - ct)] + A \exp[i k_2(-(y - 2D) \cos \alpha_2 + x \sin \alpha_2 - ct)]$$

At the boundary $y = 0$, the acceleration of the air through the wall is proportional to the pressure difference across the wall:

$$p_i + p_r - p_c = \lambda \frac{\partial}{\partial y}(p_i + p_r + p_c)$$

where the slotted-wall parameter $\lambda$ is

$$\lambda = \left(\frac{b}{\pi}\right) \ln \csc \left(\frac{\pi b}{2}\right)$$

The condition that the slopes of the streamlines be the same on both sides of the boundary yields the equation

$$\frac{1}{U - v} \frac{\partial (\phi_i + \phi_r)}{\partial y} = \frac{1}{U} \frac{\partial \phi_c}{\partial y}$$

or, with equation (3)
\[
\sin \alpha_1 \frac{\partial(\phi_i + \phi_r)}{\partial y} = \sin \alpha_2 \frac{\partial \phi_c}{\partial y}
\]  

(9b)

Define \( \phi_o \) as follows:

\[
\phi_o = \exp[ik_1(x - Ut) \sin \alpha] \sin \alpha
\]

Then, at the wall \( y = 0 \),

\[
p_i = -\rho \frac{\partial \phi}{\partial t} |_{y,x-Vt} = ik_1 \rho c \phi_o
\]

(11a)

where the subscripted vertical bar indicates that the quantities \( y \) and \( x - Vt \) are held constant in the differentiation. Similarly,

\[
p_r = (1 + B)ik_1 \rho c \phi_o
\]

(11b)

and

\[
p_c = Ai k_2 \rho c [\exp(2Dk_2 \cos \alpha_2) + 1] \phi_o
\]

(11c)

Differentiating these equations yields the following expressions:

\[
\frac{\partial p_i}{\partial y} = -k_1^2 \rho c \cos \alpha \phi_o
\]

(12a)

\[
\frac{\partial p_r}{\partial y} = (1 + B)k_1^2 \rho c \cos \alpha \phi_o
\]

(12b)

and

\[
\frac{\partial p_c}{\partial y} = -\rho A k_2^2 \cos \alpha [1 - \exp(2Dk_2 \cos \alpha)] \phi_o
\]

(12c)
Substituting equations (11) and (12) into equation (7) yields

\[
\rho c k_1 (2 + B) \phi_o - \rho c k_2 [1 + \exp(2Dik_2 \cos \alpha_2)]A\phi_o
\]
\[
= \rho c k_1^2 \cos \alpha_1 B \phi_o - \rho c k_2^2 \cos \alpha_2 [1 - \exp(2Dik_2 \cos \alpha_2)]A\phi_o
\]

(13)

Computing the derivatives in equation (9b) from equations (1), (2), and (6) yields

\[-ik_1 \sin \alpha_1 \cos \alpha_1 \phi_o = ik_2 \sin \alpha_2 \cos \alpha_2 [1 - \exp(2Dik_2 \cos \alpha_2)]A\phi_o
\]

(14)

The coefficient \(A\) can be eliminated between equations (13) and (14) to give

\[
k_1 (2 + B) + ik_1 \frac{\sin 2\alpha_1}{\sin 2\alpha_2} \frac{1 + \exp(2Dik_2 \cos \alpha_2)}{1 - \exp(2Dik_2 \cos \alpha_2)} B
\]
\[
= lk_2 \cos \alpha_1 B + lk_1 k_2 \frac{\sin \alpha_1 \cos \alpha_1}{\sin \alpha_2} B
\]

(15)

Equation (15) determines the impedance \(z\), which is given by

\[
\frac{z}{ipc} = \frac{1 + R}{1 - R}
\]

(16)

where the reflection coefficient \(R\) is \(1 + B\). Thus

\[
\frac{z}{ipc} = \frac{2 + B}{-B} = \frac{\sin 2\alpha_1}{\sin 2\alpha_2} \cot (Dk_2 \cos \alpha_2)
\]
\[
- \ell \left( k_1 \cos \alpha_1 + k_2 \cos \alpha_2 \frac{\sin \alpha_1}{\sin \alpha_2} \right)
\]

(17a)

In terms of \(k_1\), \(z\) can be written

\[
\frac{z}{ipc} = \frac{\sin 2\alpha_1}{\sin 2\alpha_2} \cot (Dk_1 \cot \alpha_2 \sin \alpha_1) - \ell \left[ k_1 \cos \alpha_1 + k_1 \cos \alpha_1 \left( \frac{\sin \alpha_1}{\sin \alpha_2} \right)^2 \right]
\]

(17b)
Equation (17a) (or, similarly, eq. (17b)) is the formula for the boundary impedance. For various limiting conditions, equation (17a) reduces to known results. For $\alpha \to 0$ and $M \to 0$, it reduces to the results given in reference 5. If the outside plenum wall is removed, the exponential term in equation (11c), which represents the wave reflected from the plenum wall, is absent. Consequently, the cotangent in equations (17a) or (17b) is replaced by $i$, yielding the result given in equation (25) of reference 6 for the impedance of a slotted wall at a flow interface.

Derivation of Equation for Resonant Modes

Equation (17a) (or (17b)) for the boundary impedance can now be used to determine the resonant frequencies. In the wind-tunnel test section both boundaries must be considered. Therefore the $x$-axis is now taken on the test-section centerline, with the boundaries at $y = \pm 1$. If attention is again restricted to waves inclined at the angle $\alpha$ for which resonance occurs, the velocity potential for this symmetric set of waves is

$$\phi_T = \sin (k_1 y \cos \alpha_1) \exp[k_1((x - Vt) \sin \alpha_1 - ct)]$$  \hspace{1cm} (18)

Then,

$$p_T = -\rho \frac{\partial \phi_T}{\partial t} = k_1 \rho c \sin (k_1 \cos \alpha_1) \exp[k_1((x - Vt) \sin \alpha_1 - ct)]$$  \hspace{1cm} (19)

and

$$v = \frac{\partial \phi_T}{\partial y} = k_1 \cos \alpha_1 \cos (k_1 y \cos \alpha_1) \exp[k_1((x - Vt) \sin \alpha_1 - ct)]$$  \hspace{1cm} (20)

By definition, at the boundary,

$$z = \frac{p_T}{v}$$  \hspace{1cm} (21)

or, setting $y = 1$ and using equation (17b),
\[
\frac{\tan (k_1 \cos \alpha_1)}{\cos \alpha_1} = \frac{\sin 2\alpha}{\cos \alpha_2} \cot (k_1 \cos \alpha_1 \tan \alpha_1 \cot \alpha_2) \\
- \ell k_1 \cos \alpha_1 \left[ 1 + \left( \frac{\sin \alpha_1}{\sin \alpha_2} \right)^2 \right] \quad (22)
\]

The angular frequency is given by

\[
\omega = k_1 (c + V \sin \alpha_1)
\]

(See ref. 3, eq. (2.4).) But since the waves considered are at the resonant angle \( \alpha_1 \) for which \( \sin \alpha_1 = -M \), then

\[
\omega = k_1 c (1 - M^2) = k_1 c \cos^2 \alpha_1
\]

Therefore, if equation (22) is solved for the parameter \( s = k_1 \cos \alpha_1 \), the resonant frequency is \( sc \cos \alpha_1 \).

For the resonant condition,

\[
\begin{align*}
\cos \alpha_1 &= \beta \\
\sin \alpha_2 &= \frac{-M}{\beta^2}
\end{align*}
\]

and

\[
\cos \alpha_2 = \sqrt{1 - (M^2/\beta^4)}
\]

Making these substitutions in equation (22) yields

\[
\tan (\beta k_1) = \frac{\beta^4}{\sqrt{1 - (M^2/\beta^4)}} \cot \left[ \beta^2 k_1 \sqrt{1 - (M^2/\beta^4)} - \beta^2 k_1 \ell (1 + \beta^4) \right] \quad (26)
\]

Equation (26), when solved for \( s = \beta k_1 \), gives the resonant frequencies from equation (24):
Although equation (26) is somewhat different from the corresponding equation derived in reference 1 for the perforated-wall tunnel, it reduces to the same equation for $M = 0$, provided that the wall parameter $\ell$ is taken to be one-half the empirical parameter used in reference 1 (since the factor $(1 + \beta^4)^{-2}$ approaches zero as $M \to 0$). The term containing $\ell$ represents acoustically the inductive effect of the ventilated wall.

The other term on the right is generally considered to represent the capacitative effect of the air in the plenum. (See ref. 5.) For small values of the argument $H = \beta^2 k_D \sqrt{1 - M^2/\beta^4}$, which gives $\cot H = 1/H$ so that, in this case, the term corresponds to a spring effect. However, as $H$ increases, the approximation becomes invalid and eventually negative values of the cotangent have to be considered. In the latter case this term adds to the inductive effect. This change in sign of the cotangent term at multiples of $\pi/2$ for the argument sometimes leads to closely spaced resonant modes.

The quantity $\cos \alpha_2 = \sqrt{1 - (M^2/\beta^4)}$ vanishes at $M = 0.618$. Consequently, near this Mach number the cotangent term tends to dominate the impedance. For $M > 0.618$, the reflection from the boundary is complete, since in this case $\cos \alpha_2$ becomes imaginary. When this occurs, no energy is transmitted into the plenum, according to the plane-wave analysis; however, the boundary does not behave like a solid wall, since in general the reflection coefficient is not 1.0 (ref. 3).

Equation (26) differs somewhat in its Mach number dependence from the corresponding equation derived in reference 1 for the perforated-wall tunnel. Equation (26) differs more markedly from the relation for the slotted-wall tunnel derived in reference 1. That relation led to an inconsistent Mach number variation and was not used in the calculations of reference 1.

If the slotted wall becomes fully closed, $\ell \to \infty$. In this case, the solutions of equation (26) are

$$\beta k_1 = (2n - 1)\pi/2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Thus, the solutions for the resonant frequencies are

$$\omega = (2n - 1)\pi \beta c \quad \text{for} \quad n = 1, 2, 3, \ldots$$

which is the result given in reference 7. This result is also that which is obtained from equations A17 and A27 of reference 1. Similar results are obtained if the plenum-chamber depth approaches zero.

At the beginning of the "Analysis" section, some justification was given for limiting the analysis to plane waves inclined at the resonance angle $\alpha$.
The weaknesses incurred by applying this restricted theory to waves excited by a source at the origin have not yet been discussed. If one considers a cylindrical wave front to be the envelope of a family of plane waves propagating outward, it would appear to be valid in an analysis of resonance phenomena to ignore the waves propagating in directions other than the resonance direction, except at one point in the analysis. In the derivation of equation (6) the wave reflected from the solid back wall of the plenum was assumed to be inclined at angle $\alpha_2$. Such would be the case if the wave pattern in the test section consisted only of waves inclined at angle $\alpha_1$ throughout the test section, as was assumed in the analysis of reference 1. However, if the emitted waves are cylindrical, segments of the wave front are incident on the boundary at various angles, each associated with a different refraction angle $\alpha_2$ and, consequently, a different angle of reflection from the plenum wall. The wave pattern in the plenum is therefore rather complex. Furthermore, some form of wave pattern will exist in the plenum when, for $M > 0.618$, the plane-wave analysis predicts the absence of plenum waves. Also, for the data available, problems associated with the experimental arrangement arise when the plane-wave results are compared with the data for $M > 0.618$, as discussed in the next section.

For $M < 0.618$, the plane-wave analysis appears to give a reasonable approximation in most cases.

RESULTS AND DISCUSSION

As already mentioned, equation (26) (for the slotted-wall-tunnel resonance frequencies) resembles somewhat the equation derived in reference 1 for perforated-wall tunnels. However, several differences should be noted. First, there is an additional factor in the expression for the wall impedance. Second, in the present analysis, the solution of equation (26) for $B_{kl}$ is multiplied by $\Delta c$ to obtain the resonant frequency, in accordance with equation (24), whereas in reference 1 the corresponding solution is multiplied only by $c$. Finally, the inductive term in equation (26) contains the actual slotted-wall parameter $\kappa$ rather than the perforated-wall equivalent of it.

Thus, in the calculation of reference 1 for slotted walls at $M = 0$, the resonant frequencies are plotted against a semiempirical parameter which is a perforated-wall counterpart of the slotted-wall parameter $2\kappa$. In figure 2, these results are compared with the corresponding results from equation (26) using the empirical values for $2\kappa$ taken from reference 1. This comparison indicates that equation (26) with the slotted-wall parameter is appropriate for predicting the slotted-wall configuration resonant modes at $M = 0$.

Figure 3 shows the theoretical and experimental resonant frequencies for a slotted-wall parameter of 0.55. The presence of some resonant modes that decrease in the middle Mach number range is a result of the change in sign of the cotangent term in the wall impedance in the higher modes. These decreasing modes are indicated in the data but were not predicted by the theory of reference 1.
The contribution of the plenum space to the acoustic impedance of the test-section boundary is determined by the phase shift of the wave reflected from the solid plenum wall. This phase shift is $2k_1D\sqrt{\beta^2 - (M^2/\beta^2)}$. When this quantity is near $2\pi$, the wave approaching the slotted boundary in the plenum is nearly in phase with the wave leaving it. In this case, closely spaced resonant modes can occur. When the phase shift is small, the plenum space acts as a pure capacitor and the resonant modes vary gradually with Mach number. When $k_1D$ is large, small changes in Mach number cause rapid variation in the phase of the plenum waves, and consequently in the resonant modes. Thus, for the higher modes, the resonant frequencies are not smoothly distributed.

No attempt was made to correlate the theoretical values with experimental data for $M > 0.618$. It would be interesting to determine if the plane-wave analysis provided a reasonable approximation for the higher Mach numbers. However, this theory for plane waves in the test-section flow predicts no transmitted wave energy in the plenum. The data of reference 1 for $M > 0.618$, however, was obtained with waves generated by speakers located in the plenum chamber. The two conditions are therefore inconsistent and any correlation, or lack of correlation, would appear to be fortuitous.

Figure 4 shows the comparison of theoretical and experimental frequencies for a slotted wall with deep slots (i.e., a thick wall). The value of the wall parameter used for the calculation is obtained by adding the nondimensional wall thickness (0.31 in this case) to the value of $l$ obtained from equation (8). (See eq. (18) of ref. 5.) This procedure yields a value of the parameter of 0.95, which is so different from the value of 2.04 given in reference 1 that, in this case, the theoretical value was used for the calculation.

The question arises as to the possibility of applying equation (26) to the calculation of resonant modes for perforated-wall tunnels. The theory should be applicable if the slotted-wall parameter is replaced by the empirically determined perforated-wall parameter and if the structure of the perforated wall does not destroy the character of the transmitted wave. That is, the effect of the boundary must at some distance appear as if it were disturbed in accordance with the boundary conditions (eqs. (7) and (9b)). Under the assumption that such is the case, calculations were performed for one value of the perforated-wall parameter (0.11). Comparison of the calculations with data from reference 1 is shown in figure 5. It is seen that, at least for this small value of the wall parameter, the values agree to the same order as that obtained for the slotted-wall termed calculations. The presence of multiple modes for larger values of $k_1D$ is apparent both in the data and the theoretical values.

CONCLUDING REMARKS

A theory for predicting resonant modes in a two-dimensional, slotted-wall wind tunnel with a plenum chamber has been presented. The analysis used the concept of the acoustic impedance of the boundaries and unsteady-flow relations throughout the development. The equation derived was shown to be consistent
with known results for limiting conditions. The computed resonant modes compared well with appropriate experimental data.

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REFERENCES


(a) Test section with top and bottom slotted walls and oscillating airfoil as acoustic source.

Figure 1.- Basic geometry.
Figure 1:- Concluded.
Figure 2.- Comparison of present-theory and theory of reference 1 with experimental data for slotted walls at $M = 0$. 
Figure 3.—Theoretical and experimental resonant frequencies for thin slotted walls. \( 2\ell = 0.55 \); Tunnel height = 10.2 cm.
Figure 4. - Theoretical and experimental resonant frequencies for deeply slotted walls. $2\ell = 0.95$; Tunnel height = 10.2 cm.
Figure 5.- Theoretical and experimental resonant frequencies for 26-percent open perforated walls. Wall parameter = 0.11; Tunnel height = 10.2 cm.
Wave-induced resonance associated with the geometry of wind-tunnel test sections can occur. A theory that uses acoustic impedance concepts to predict resonance modes in a two-dimensional, slotted-wall wind tunnel with a plenum chamber is described. The equation derived is consistent with known results for limiting conditions. The computed resonance modes compare well with appropriate experimental data. When the theory is applied to perforated-wall test sections, it predicts the experimentally observed closely spaced modes that occur when the wavelength is not long compared with the plenum depth.