Supporting Research

MAXIMUM LIKELIHOOD CLUSTERING WITH DEPENDENT FEATURE TREES

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Abstract

In this report, maximum likelihood clustering for the decomposition of mixture density of the data into its normal component densities is considered. The densities are approximated with first-order dependent feature trees using criteria of mutual information and distance measures. Expressions are presented for the criteria when the densities are Gaussian. By defining different types of nodes in a general dependent feature tree, maximum likelihood equations are developed for the estimation of parameters using fixed-point iterations. The field structure of the data is also taken into account in developing maximum likelihood equations. Furthermore, experimental results from the processing of remotely sensed multispectral scanner imagery data are presented.
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DEPENDENT FEATURE TREES

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PREFACE

The techniques which are the subject of this report were developed to support the Agriculture and Resources Inventory Surveys Through Aerospace Remote Sensing program. Under Contract NAS 9-15800, Dr. C. B. Chittineni, a principal scientist for Lockheed Engineering and Management Services Company, Inc., performed this research for the Earth Resources Research Division, Space and Life Sciences Directorate, National Aeronautics and Space Administration, at the Lyndon B. Johnson Space Center.
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1. INTRODUCTION

Recently, considerable interest has been shown in developing techniques for the classification of imagery data (such as remotely sensed multispectral scanner data acquired by the Landsat series of satellites) for inventorying natural resources, monitoring crop conditions, and detecting changes in natural and manmade objects. Nonsupervised classification or clustering techniques have been found to be effective in the analysis of remotely sensed data (ref. 1). The approach of clustering for imagery data classification, in general, involves two steps: (1) partitioning the image into its inherent modes or into its homogeneous parts and (2) labeling the clusters using information from a given set of labeled patterns.

In practical applications of pattern recognition such as remote sensing, it is difficult to obtain labels for the patterns. In remote sensing imagery, an analyst-interpreter provides the labels for the picture elements (pixels) by examining imagery films and using other information (e.g., crop growth stage models and historic information). Remote sensing imagery usually has a field structure, and it is recognized that fields are easier to label than are pixels. The development of algorithms for locating fields has attracted the attention of several researchers in the recent literature (refs. 2-5).

Considerable interest has been shown in applying maximum likelihood equations for the decomposition of the mixture density of the imagery data into its normal component densities (refs. 5-9). Recently, methods have been developed (refs. 10, 11) for probabilistically labeling the modes of the data using information from a given set of labeled patterns and, also, from a given set of labeled fields.

In decomposing the mixture density of the data into its normal component densities, the parameters of the component densities and the a priori probabilities of the modes are iteratively computed using maximum likelihood equations coupled with a split and merge sequence. The updating of the parameters is usually stopped after a few iterations because of the large amount of computation.
Also, in practical problems (remote sensing imagery data of several acquisitions), a large number of parameters will be estimated. For a fixed sample size, the accuracy of estimation usually decreases (ref. 12) as the number of parameters to be estimated increases. To overcome the computational requirements and the large number of parameters to be estimated with the usual maximum likelihood clustering technique, maximum likelihood equations are obtained in this report by approximating the cluster conditional densities with first-order tree dependence (refs. 13, 14) among the features. The field structure of the data is also taken into account. Either the average mutual information between the features (ref. 13) or the probabilistic distance measures (ref. 15) can be used to construct optimal dependent feature trees for a given data type.

This paper is organized as follows. General maximum likelihood equations are presented in section 2. Section 3 concerns the problem of approximating probability density functions with dependent feature trees using the criteria of information measure and probabilistic distance measure. Expressions are derived for the criteria when the distributions of the features are Gaussian. In section 4, a general dependent feature tree and its various types of nodes are described, and expressions for the covariance between the features not connected by a single link are derived. Maximum likelihood equations for the parameters of the density functions when approximated by dependent feature trees are developed in section 5. Experimental results from the processing of remotely sensed multispectral scanner imagery data are presented in section 6. Section 7 contains the concluding remarks. Detailed derivations of maximum likelihood equations are given in appendix A. In appendix B, the field structure of the data is taken into account in developing maximum likelihood equations. An expression is derived in appendix C for the mutual information between the feature subsets when they are represented by the nodes in a dependent feature tree. Also, expressions are derived for the covariance between the feature subsets when they are connected by a path in a dependent feature tree.
2. GENERAL MAXIMUM LIKELIHOOD EQUATIONS

General maximum likelihood equations are presented in this section for the decomposition of the mixture density of the data into its component densities. It is assumed that a set $\mathcal{X} = \{X_1, \ldots, X_N\}$ of $N$ unlabeled patterns, each of dimension $n$, is given. These patterns are assumed to be drawn independently from the mixture density

$$p(X|\theta) = \sum_{j=1}^{m} p(X, \omega_j, \theta_j)p(\omega_j) \quad (2-1)$$

where $\theta$ is a fixed but unknown parameter vector, $\theta_j$ is a parameter vector for the $j$th cluster, and $m$ is the number of modes or clusters in the data. Let $P(\omega_j)$ and $p(X|\omega_j)$ be the a priori probabilities of the modes and mode conditional densities, respectively. The likelihood of the observed pattern vectors is, by definition, the joint density

$$p(\mathcal{X}|\theta) = \prod_{k=1}^{N} p(X_k|\theta) \quad (2-2)$$

Since the logarithm is a monotonic function of its argument, taking the gradient of the logarithm of equation (2-2) with respect to $\theta_j$ results in

$$\nabla_{\theta_j} \ell = \sum_{k=1}^{N} \frac{1}{p(X_k|\theta)} \nabla_{\theta_j} \left[ \sum_{j=1}^{m} p(X_k|\omega_j, \theta_j)p(\omega_j) \right] \quad (2-3)$$

where

$$\ell = \sum_{k=1}^{N} \log[p(X_k|\theta)] \quad (2-4)$$

and $\nabla_{\theta_j} \ell$ is the gradient of $\ell$ with respect to $\theta_j$. From the Bayes rule, the a posteriori probability can be written as

$$p(\omega_1|X_k, \theta) = \frac{p(X_k|\omega_1, \theta)p(\omega_1)}{p(X_k|\theta)} \quad (2-5)$$
If the elements of $\theta_i$ and $\theta_j$ are assumed to be functionally independent, using equation (2-5) in equation (2-3) yields

$$V_{\theta, \xi} = \sum_{k=1}^{N} p(\omega_i | x_k, \theta) \text{log} \left[ p(x_k | \omega_i, \theta) \right]$$

(2-6)

The following likelihood equation for the a priori probabilities can easily be obtained from equation (2-6) by introducing Lagrangian multipliers to take into account the probability constraints on $P(u)$.

$$P(\omega_i) = \frac{1}{N} \sum_{k=1}^{N} p(\omega_i | x_k, \theta)$$

(2-7)

Since $\theta_i$ is a parameter vector of the density of the $i^{th}$ cluster, equation (2-6) can be written as

$$V_{\theta, \xi} = \sum_{k=1}^{N} p(\omega_i | x_k, \theta) \text{log} \left[ p(x_k | \omega_i, \theta) \right]$$

(2-8)

From equation (2-8), general maximum likelihood equations for the parameters of the cluster conditional densities can be obtained.
3. APPROXIMATING PROBABILITY DENSITY FUNCTIONS WITH DEPENDENT FEATURE TREES

If the probability density function of the \(i\)th class is approximated by a first-order dependent feature tree, it can be written as

\[
p_i(X) = \prod_{\ell=1}^{n} p_i \left[ \frac{x_{m_{\ell}}}{{m_{j\ell}}} \right] ; \quad 0 < j(\ell) < \ell
\]  

(3-1)

where \(x_{m_{\ell}}\) is the \(m_{\ell}\)th feature of pattern vector \(X\); \((m_1, \cdots, m_n)\) is an unknown permutation of integers 1, 2, \cdots, \(n\); and \(p(x_i|x_0)\), by definition, is equal to \(p(x_i)\). Each variable in the above expansion may be conditioned upon, at most, one of the other variables. Figure 3-1 shows an example of a dependent feature tree.

![Figure 3-1](image)

Figure 3-1.- An example of a dependent feature tree.

The component of the density in the product approximation that is represented by a single link, such as the one connecting features \(x_5\) and \(x_8\) in figure 3-1, is \(p(x_8|x_5)\). The density that is approximated by the dependence tree of figure 3-1 can be written as

\[
p(X) = p(x_1)p(x_2|x_1)p(x_3|x_2)p(x_4|x_2)p(x_5|x_1)p(x_6|x_5)p(x_7|x_5)p(x_8|x_5)
\]  

(3-2)

3.1 CONSTRUCTION OF OPTIMAL DEPENDENT FEATURE TREES

This section concerns the problem of constructing dependent feature trees. The dependent feature tree, the density of which best approximates the true density, is proposed to be constructed using either the criterion of information preservation (ref. 13) or the criterion of class separability (ref. 15). An algorithm developed by Kruskal (ref. 16) provides an efficient computational procedure for constructing optimal dependent feature trees using the expressions developed in the following.
3.1.1 A CRITERION BASED ON INFORMATION MEASURE

Let \( p_{\text{it}}(X) \) be the approximate density of the \( i \)th class with the product approximation. That is,

\[
p_{\text{it}}(X) = \prod_{k=1}^{n} \left\{ p_{i} \left[ x_{m_{i}k}^{m_{j}(k)} \right] \right\}
\]  \hspace{1cm} (3-3)

Consider the following measure of closeness between the true and approximate densities (ref. 13). That is,

\[
I(p, p_{\text{t}}) = \sum_{i=1}^{C} p(\omega_{i}) \int p_{i}(X) \log \left[ \frac{p_{i}(X)}{p_{\text{it}}(X)} \right] dX
\]  \hspace{1cm} (3-4)

where \( C \) is the number of classes. From equation (3-4), it is seen that \( I(p, p_{\text{t}}) = 0 \) whenever \( p_{i}(X) \) is equal to \( p_{\text{it}}(X) \) for all \( X \) and that \( I(p, p_{\text{t}}) > 0 \) if \( p_{i}(X) \) is different from \( p_{\text{it}}(X) \) for some \( X \). To find the product approximation for the densities or the dependent feature tree that minimizes \( I(p, p_{\text{t}}) \), consider

\[
I(p, p_{\text{t}}) = - \sum_{i=1}^{C} p(\omega_{i}) \int p_{i}(X) \log[p_{\text{it}}(X)] dX
\]

\[
+ \sum_{i=1}^{C} p(\omega_{i}) \int p_{i}(X) \log[p_{i}(X)] dX
\]

\[
= - \sum_{i=1}^{n} I[x_{i}, x_{j}(\varepsilon)] + K
\]  \hspace{1cm} (3-5)

where

\[
I[x_{i}, x_{j}(\varepsilon)] = \sum_{i=1}^{C} p(\omega_{i}) I_{1}[x_{i}, x_{j}(\varepsilon)]
\]

\[
I_{1}[x_{i}, x_{j}(\varepsilon)] = \int p_{i}[x_{i}, x_{j}(\varepsilon)] \log \left( \frac{p_{i}[x_{i}, x_{j}(\varepsilon)]}{p_{i}(x_{i})p_{i}[x_{j}(\varepsilon)]} \right) dx_{i} dx_{j}(\varepsilon)
\]  \hspace{1cm} (3-6)

and

\[
K = - \sum_{i=1}^{n} I(x_{i}) + \sum_{i=1}^{C} p(\omega_{i}) \int p_{i}(X) \log[p_{i}(X)] dX
\]
The quantity $I_1[x_k,x_j(\ell)]$ is the mutual information between features $x_k$ and $x_j(\ell)$ of class 1. From equation (3-5), Kruskal's algorithm (ref. 16) can be efficiently used to construct optimal dependent feature trees.

### 3.1.2 A CRITERION BASED ON PROBABILISTIC DISTANCE MEASURES

A probability density function, like any other function, can be approximated by a number of different procedures. In the sense of preserving the separability between the classes, it is proposed that a criterion based on probabilistic distance measures such as divergence be used to construct dependent feature trees. The measure of closeness between the approximate and true densities is defined as

$$J_{12} = \int p_1(x) \log \left( \frac{p_{1t}(x)}{p_{2t}(x)} \right) dx + \int p_2(x) \log \left( \frac{p_{2t}(x)}{p_{1t}(x)} \right) dx \quad (3-7)$$

From equation (3-7), it is seen that $J_{12}$ is large whenever the ratio of $p_{1t}(x)$ to $p_{2t}(x)$ is large in the region of class 1 and the ratio of $p_{2t}(x)$ to $p_{1t}(x)$ is large in the region of class 2. By using the product approximation of equation (3-3) for the densities $p_{it}(x)$, equation (3-7) can be written as follows.

$$J_{12} = \sum_{i=1}^{n} \int p_1(x) \log \left( \frac{p_{1t}[x_m^i,x_{m_j(i)}]}{p_{2t}[x_m^i,x_{m_j(i)}]} \right) dx + \sum_{i=1}^{n} \int p_2(x) \log \left( \frac{p_{2t}[x_m^i,x_{m_j(i)}]}{p_{1t}[x_m^i,x_{m_j(i)}]} \right) dx$$

$$= \sum_{i=1}^{n} \int p_1[x_m^i,x_{m_j(i)}] \log \left( \frac{p_{1t}[x_m^i,x_{m_j(i)}]}{p_{2t}[x_m^i,x_{m_j(i)}]} \right) dx \cdot m_i \cdot dx \cdot m_j(i)$$

$$+ \sum_{i=1}^{n} \int p_2[x_m^i,x_{m_j(i)}] \log \left( \frac{p_{2t}[x_m^i,x_{m_j(i)}]}{p_{1t}[x_m^i,x_{m_j(i)}]} \right) dx \cdot m_i \cdot dx \cdot m_j(i)$$

$$= \sum_{i=1}^{n} \left\{ \Delta J_{12}[x_m^i,x_{m_j(i)}] \right\} + K \quad (3-8)$$

3-3
where
\[
\Delta J_{12}[x_{m_i},x_{m_j(i)}] = J_{12}[x_{m_i},x_{m_j(i)}] - J_{12}(x_{m_i}) - J_{12}(x_{m_j(i)})
\]  (3-9)

and
\[
K = \sum_{i=1}^{n} \int p_1(x_{m_i}) \log \left[ \frac{p_1(x_{m_i})}{p_2(x_{m_i})} \right] dx_{m_i} + \sum_{i=1}^{n} \int p_2(x_{m_i}) \log \left[ \frac{p_2(x_{m_i})}{p_1(x_{m_i})} \right] dx_{m_i}
\]  (3-10)

If more than two classes exist, the expected value of the measure of closeness defined over pairs of classes can be used to obtain optimal approximations for the densities (ref. 17). From equation (3-8), Kruskal's algorithm (ref. 16) can be efficiently used to construct optimal dependent feature trees.

3.2 EXPRESSIONS FOR THE CRITERIA WHEN THE DISTRIBUTIONS OF THE FEATURES ARE GAUSSIAN

Expressions are derived in this section for the mutual information and for \( \Delta J_{12} \) between the features, assuming that the distributions of the features are Gaussian. If \( p_\xi(x_i) \), the density of feature \( x_i \) of the \( \xi \)th class, is Gaussian, it can be written as
\[
p_\xi(x_i) = \frac{1}{\sqrt{2\pi}\sigma_\xi} \exp \left\{ -\frac{1}{2\sigma_\xi^2} [x_i - \mu_i(\xi)]^2 \right\}
\]  (3-11)
or it is denoted as \( p_\xi(x_i) \sim \mathcal{N}[\mu_i(\xi),\sigma_\xi] \). The joint and conditional densities of features \( x_i \) and \( x_j \) of the \( \xi \)th class can be written as follows.
\[
p_\xi(x_i,x_j) = \frac{1}{2\pi \sigma_i(\xi) \sigma_j(\xi) \left[1 - \rho_{ij}(\xi)^2\right]^{1/2}} \exp \left\{ -\frac{1}{2} q_\xi(x_i,x_j) \right\}
\]  (3-12)

where
\[
q_\xi(x_i,x_j) = \frac{1}{2 \left[1 - \rho_{ij}(\xi)^2\right]} \left\{ \frac{[x_i - \mu_i(\xi)]^2}{\sigma_i(\xi)} - 2\rho_{ij}(\xi) \frac{x_i - \mu_i(\xi)}{\sqrt{\sigma_i(\xi)}} \frac{x_j - \mu_j(\xi)}{\sqrt{\sigma_j(\xi)}} + \frac{[x_j - \mu_j(\xi)]^2}{\sigma_j(\xi)} \right\}
\]  (3-13)
\[ \rho_{ij}^2(\xi) = \frac{\sigma_{ij}^2(\xi)}{\sigma_i(\xi)\sigma_j(\xi)} \]  

(3-14)

\( \sigma_{ij}(\xi) \) is the covariance between features \( x_i \) and \( x_j \) of the \( \xi \)th class. From equations (3-11) and (3-12), the conditional density can be written as

\[ p_\xi(x_i | x_j) = \frac{1}{\{2\pi \sigma_i(\xi)[1 - \rho_{ij}^2(\xi)]\}^{1/2}} \exp[-q_\xi(x_i | x_j)] \]  

(3-15)

where

\[ q_\xi(x_i | x_j) = \frac{1}{2\sigma_i(\xi)[1 - \rho_{ij}^2(\xi)]} \left\{ \left[ \frac{x_i - u_i(\xi)}{\sqrt{\sigma_i(\xi)}} \right]^2 + \left[ \frac{x_j - u_j(\xi)}{\sqrt{\sigma_j(\xi)}} \right]^2 \right\} \]  

(3-16)

### 3.2.1 AN EXPRESSION FOR THE MUTUAL INFORMATION BETWEEN FEATURES \( x_i \) AND \( x_j \)

In this section, an expression is derived for the mutual information between Gaussian-distributed features \( x_i \) and \( x_j \) of class \( \xi \). From equations (3-11) and (3-15), the following can easily be obtained. Consider

\[ \frac{p_\xi(x_i, x_j)}{p_\xi(x_i)p_\xi(x_j)} = \frac{p_\xi(x_i | x_j)}{p_\xi(x_i)} \]

\[ = \left[ 1 - \rho_{ij}^2(\xi) \right]^{-1/2} \exp \left( \frac{\rho_{ij}(\xi)}{1 - \rho_{ij}^2(\xi)} \left[ \frac{x_i - u_i(\xi)}{\sqrt{\sigma_i(\xi)}} \right] \left[ \frac{x_j - u_j(\xi)}{\sqrt{\sigma_j(\xi)}} \right] \right) \]

\[ - \frac{\rho_{ij}^2(\xi)}{2[1 - \rho_{ij}^2(\xi)]} \left\{ \left[ \frac{x_i - u_i(\xi)}{\sqrt{\sigma_i(\xi)}} \right]^2 + \left[ \frac{x_j - u_j(\xi)}{\sqrt{\sigma_j(\xi)}} \right]^2 \right\} \]  

(3-17)

From equation (3-17), the mutual information between features \( x_i \) and \( x_j \) of class \( \xi \) can be obtained as follows.

\[ I_\xi(x_i, x_j) = \int p_\xi(x_i, x_j) \log \left[ \frac{p_\xi(x_i, x_j)}{p_\xi(x_i)p_\xi(x_j)} \right] dx_i \ dx_j \]

\[ = - \frac{1}{2} \log \left[ 1 - \rho_{ij}^2(\xi) \right] > 0 \]  

(3-18)
3.2.2 AN EXPRESSION FOR $\Delta J_{12}(x_i, x_j)$ WHEN FEATURES $x_i$ AND $x_j$ ARE NORMALLY DISTRIBUTED

From equations (3-7) and (3-9), when features $x_i$ and $x_j$ are normally distributed, an expression for $\Delta J_{12}(x_i, x_j)$ can be easily obtained as follows.

$$2\Delta J_{12}(x_i, x_j) = \left( \frac{\sigma_{x_i}(1)\sigma_{x_j}(2) - 2\sigma_{x_i}(1)\sigma_{x_j}(2) + \sigma_{x_i}(1)\sigma_{x_j}(2) - 2\sigma_{x_i}(1)\sigma_{x_j}(2) + \sigma_{x_i}(2)\sigma_{x_j}(1)}{\Delta_{ij}(2)} \right)$$

$$+ \left( \frac{\sigma_{x_i}(1)}{\Delta_{ij}(1)} + \frac{\sigma_{x_j}(2)}{\Delta_{ij}(2)} \right)[u_{x_i}(1) - u_{x_j}(2)]^2 - \left( \frac{\sigma_{x_i}(1)}{\Delta_{ij}(1)} + \frac{\sigma_{x_j}(2)}{\Delta_{ij}(2)} \right)[u_{x_i}(1) - u_{x_j}(2)]$$

$$+ \left( \frac{\sigma_{x_i}(1)}{\Delta_{ij}(1)} + \frac{\sigma_{x_j}(2)}{\Delta_{ij}(2)} \right)[u_{x_i}(1) - u_{x_j}(2)]^2 - \left[ \frac{\sigma_{x_i}(2)}{\sigma_{x_i}(1)} + \frac{\sigma_{x_j}(1)}{\sigma_{x_j}(2)} \right][u_{x_i}(1) - u_{x_j}(2)]^2$$

$$- \left( \frac{\sigma_{x_i}(2)}{\sigma_{x_i}(1)} + \frac{\sigma_{x_j}(1)}{\sigma_{x_j}(2)} \right)[u_{x_i}(1) - u_{x_j}(2)]^2$$

$$= \left( \frac{\sigma_{x_i}(1)\sigma_{x_j}(2) - 2\sigma_{x_i}(1)\sigma_{x_j}(2) + \sigma_{x_i}(1)\sigma_{x_j}(2) - 2\sigma_{x_i}(1)\sigma_{x_j}(2) + \sigma_{x_i}(2)\sigma_{x_j}(1)}{\Delta_{ij}(2)} \right)$$

where $\Delta_{ij}(1) = \sigma_{x_i}(1)\sigma_{x_j}(1) - \sigma_{x_i}(2)$

(3-19)

(3-20)
4. A GENERAL DEPENDENT FEATURE TREE

A general dependent feature tree is shown in figure 4-1.

Each node of the tree represents a feature, and the feature numbers are given in figure 4-1. In approximating the probability density functions with dependent feature trees, each feature may be conditioned upon, at most, one of the other features. Node $x_1$ is the root node of the tree. Nodes $x_2$, $x_4$, $x_5$, $x_7$, etc., are nodes on the periphery of the tree.

4.1 DIFFERENT TYPES OF NODES

For convenience in the following analysis, the nodes of the dependent feature tree are divided into the following types.

1. Type I nodes: Except for the root node, nodes on the periphery of the tree are defined as type I nodes. For example, in figure 4-1, nodes $x_2$, $x_4$, $x_5$, $x_7$, etc., are type I nodes.
2. Type II nodes: These are nodes which are one node deep from the periphery. For example, in figure 4-1, nodes $x_6$, $x_{10}$, $x_{15}$, $x_{17}$, $x_{19}$, etc., are type II nodes.

3. Type III nodes: These are nodes which are at least two nodes deep from the periphery. For example, in figure 4-1, nodes $x_3$, $x_8$, $x_9$, $x_{13}$, etc., are type III nodes.

4. Type IV nodes: The root node of the tree is defined as a type IV node. The types of root nodes are divided into type IVa and type IVb. Examples of the types of root nodes are described in the following.

a. Type IVa node: The type IVa node is the root node of a tree with a single link. As an example, node $x_1$ of figure 4-2 is a type IVa node.

![Figure 4-2.- Illustration of a type IVa node.](image)

b. Type IVb node: The type IVb node is the root node of a tree with two or more links. As an example, node $x_1$ of figure 4-3 is a type IVb node.

![Figure 4-3.- Illustration of a type IVb node.](image)

It is noted that the type IVb node is different from the type IVa node in that more than one node links directly with the root node of the tree.
4.2 AN EXPRESSION FOR THE COVARIANCE BETWEEN THE FEATURES CONNECTED BY A PATH IN A DEPENDENT FEATURE TREE

An expression for the covariance between the features when a path connects their representative nodes in a dependent feature tree is developed in this section. For example, features $x_{11}$ and $x_{16}$ are connected through features $x_{10}$, $x_9$, $x_8$, $x_{13}$, and $x_{15}$ in the dependent feature tree of figure 4-1. For the following analysis, consider the dependent feature tree shown in figure 4-4.

![Example Dependent Feature Tree](image)

Figure 4-4.- An example dependent feature tree.

The probability density represented by the dependent feature tree of figure 4-4 can be written as follows.

$$p(X) = p(x_1)p(x_2|x_1)p(x_3|x_2)p(x_4|x_2)p(x_5|x_2)p(x_6|x_5)p(x_7|x_4)$$  \hspace{1cm} (4-1)

In the following, an expression for the covariance between features $x_6$ and $x_7$ of figure 4-4 is derived.

$$E[(x_6 - u_6)(x_7 - u_7)] = \int (x_6 - u_6)(x_7 - u_7)p(X) dX$$

$$= \int (x_6 - u_6)p(x_2)p(x_5|x_2)p(x_6|x_5)dx_2 \cdot dx_5 \cdot dx_6$$

$$\cdot \int p(x_4|x_2)dx_4 \int (x_7 - u_7)p(x_7|x_4)dx_7$$  \hspace{1cm} (4-2)

From equation (3-15), the following equations are obtained.

$$\int (x_7 - u_7)p(x_7|x_4)dx_7 = \rho_{47}\sqrt{\frac{\sigma_7}{\sigma_4}}(x_4 - u_4)$$  \hspace{1cm} (4-3)

$$\int (x_4 - u_4)p(x_4|x_2)dx_4 = \rho_{42}\sqrt{\frac{\sigma_4}{\sigma_2}}(x_2 - u_2)$$  \hspace{1cm} (4-4)

4-3
Using equations (4-3) and (4-4) in equation (4-2) yields

\[
E[(x_6 - u_6)(x_7 - u_7)] = \rho_{74} \sqrt{\frac{\sigma_7}{\sigma_4}} \rho_{42} \sqrt{\frac{\sigma_4}{\sigma_2}} \int (x_2 - u_2)p(x_2)dx_2 \\
\int p(x_5|x_2)dx_5 \int (x_6 - u_6)p(x_6|x_5)dx_6
\]  

(4-5)

Similar to equations (4-3) and (4-4), which were developed from equation (3-15), the following are obtained.

\[
\int (x_6 - u_6)p(x_6|x_5)dx_6 = \rho_{65} \sqrt{\frac{\sigma_6}{\sigma_5}} (x_5 - u_5)
\]

\[
\int (x_5 - u_5)p(x_5|x_2)dx_5 = \rho_{52} \sqrt{\frac{\sigma_5}{\sigma_2}} (x_2 - u_2)
\]

\[
\int (x_2 - u_2)(x_2 - u_2)p(x_2)dx_2 = \sigma_2
\]

(4-6)

From equations (4-5) and (4-6), the covariance between features \(x_6\) and \(x_7\) can be obtained as follows.

\[
E[(x_6 - u_6)(x_7 - u_7)] = \frac{\sigma_{74}}{\sigma_4} \cdot \frac{\sigma_{42}}{\sigma_2} \cdot \frac{\sigma_{65}}{\sigma_5} \cdot \sigma_{52}
\]  

(4-7)

For a general case, the following theorems can easily be established.

**Theorem 1:** Suppose the features \(x_1\) and \(x_{r+s}\) in a dependent feature tree are connected by a path as shown in figure 4-5. Then, the covariance between features \(x_1\) and \(x_{r+s}\) is given by equation (4-8).

\[\text{Figure 4-5. - A path between features } x_1 \text{ and } x_{r+s} \text{ in a dependent feature tree.}\]
\[
E[(x_1 - u_1)(x_{r+s} - u_{r+s})] = \frac{\sigma_{12}}{\sigma_2} \cdot \frac{\sigma_{23}}{\sigma_3} \cdot \frac{\sigma_{r-1,r}}{\sigma_r} \cdot \frac{\sigma_{r+s,r+s-1}}{\sigma_{r+s-1}} \cdot \frac{\sigma_{r+s-1,r+s-2}}{\sigma_{r+s-2}} \cdot \frac{\sigma_{r+2,r+1}}{\sigma_{r+1}} \cdot \sigma_{r+1,r}
\] (4-8)

**Theorem 2:** Suppose the features \(x_1\) and \(x_r\) in a dependent feature tree are connected by a path as shown in figure 4-6. Then, the covariance between features \(x_1\) and \(x_r\) is given by equation (4-9).

\[
E[(x_1 - u_1)(x_r - u_r)] = \frac{\sigma_{12}}{\sigma_2} \cdot \frac{\sigma_{23}}{\sigma_3} \cdot \frac{\sigma_{34}}{\sigma_4} \cdot \frac{\sigma_{r-2,r-1}}{\sigma_{r-1}} \cdot \sigma_{r-1,r}
\] (4-9)

Figure 4-6.- A path between features \(x_1\) and \(x_r\) in a dependent feature tree.
In this section, maximum likelihood equations are developed for estimating the parameters of the cluster conditional densities when approximated by the first-order dependent feature trees. In practice, such as in the classification of remotely sensed multispectral scanner imagery data, considerable interest has been shown in applying maximum likelihood clustering for the decomposition of the mixture density of the data into its normal component densities. The mixture density $p(X)$ can be written as

$$p(X) = \sum_{i=1}^{m} P(\omega_i)p(X|\omega_i) \tag{5-1}$$

where $m$ is the number of clusters, and $P(\omega_i)$ and $p(X|\omega_i)$ are the a priori probabilities of the modes and mode conditional densities, respectively. If the cluster conditional densities are Gaussian [i.e., $p(X|\omega_i) \sim N(\mu_i, \Sigma_i)$], by using a given set of $N$ independent observations from the mixture density, from equation (2-6), the maximum likelihood equations for the estimates of the parameters of the mixture density can easily be shown to be the following (ref. 6).

$$P(\omega_i) = \frac{1}{N} \sum_{k=1}^{N} p(\omega_i | X_k) \quad \left\{ \begin{array}{l}
U_i = \frac{\sum_{k=1}^{N} X_k p(\omega_i | X_k)}{\sum_{k=1}^{N} p(\omega_i | X_k)} \\
\Sigma_i = \frac{\sum_{k=1}^{N} (X_k - U_i)(X_k - U_i)^T p(\omega_i | X_k)}{\sum_{k=1}^{N} p(\omega_i | X_k)}
\end{array} \right. \tag{5-2}$$

In maximum likelihood clustering, equation (5-2) is used for updating the parameters of the densities, and this computation is coupled with a split and merge sequence. The updating is usually stopped after a few iterations because of the large amount of computation in clustering data such as imagery data.
For practical problems, the number of parameters to be estimated is large. Using equation (5-2), the number of parameters to be estimated for each mode is $\frac{n(n + 3)}{2}$, where $n$ is the dimensionality of the patterns. It is known that, with a fixed sample size, the accuracy of estimation usually decreases as the number of parameters to be estimated increases (ref. 12).

In this paper, the cluster conditional densities are approximated with first-order dependent feature trees to reduce the number of parameters to be estimated. In the product approximation for the densities discussed in the previous sections, it is noted that each feature is conditioned upon, at most, one of the other features. The number of parameters to be estimated for each mode is obtained as follows: the means $n$, the variances $n$, and the covariances $(n - 1)$, or a total of $(3n - 1)$, where $n$ is the dimensionality of the patterns. When the product approximation is used for the probability densities, with an increase in the dimensionality of the patterns, the reduction in the number of parameters for each mode is as shown in figure 5-1.

![Figure 5-1: Reduction in the number of parameters with the dimensionality.](image)

In the following, maximum likelihood equations are developed for estimating the parameters of the cluster conditional densities when approximated with first-order dependent feature trees. It is assumed that the structure of the dependent feature tree is determined using the techniques discussed in section 3. The different types of nodes described in section 4 are considered separately.
5.1 MAXIMUM LIKELIHOOD EQUATIONS FOR THE A PRIORI PROBABILITIES, MEANS, AND VARIANCES

In this section, maximum likelihood equations similar to equation (5-2) are obtained for the a priori probabilities of the clusters and for the means and variances of features in each cluster when the cluster conditional densities are approximated with dependent feature trees. The different types of nodes discussed in section 4 are treated separately. It is assumed that a set \( \mathcal{X} = \{X_1, \ldots, X_N\} \) of \( N \) unlabeled patterns, each of dimension \( n \), drawn independently from the mixture density \( p(X) \) is given. When the cluster conditional densities are approximated by first-order dependent feature trees, the density of the \( i \)th cluster can be written as

\[
p(X|\omega_i) = \prod_{j=1}^{n} \left\{ p_i[x_j|x_j(\xi)] \right\}
\]  

The maximum likelihood equations for the a priori probabilities of the clusters can easily be shown to be the following.

\[
P(\omega_i) = \frac{1}{N} \sum_{k=1}^{N} p(\omega_i|X_k)
\]  

If \( \theta_i \) is a parameter of the \( i \)th cluster, using equation (5-3) in equation (2-8) results in

\[
\frac{\partial \xi}{\partial \theta_i} = \sum_{k=1}^{N} p(\omega_i|X_k, \theta) \left[ \sum_{j=1}^{n} \frac{\partial \theta_i}{\partial \theta_j} \left( \log p_i[x_j|x_j(\xi)] \right) \right]
\]  

In the following, it is assumed that the distributions of pattern features in each cluster are Gaussian. That is,

\[
p_i(x_\xi) \sim N[u_i(\xi), \sigma_i(\xi)]
\]  

5.1.1 MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF TYPE I NODES

Consider a link in a dependent feature tree containing a type I node, as shown in figure 5-2.
Since each feature is conditioned upon, at most, one of the other features, equation (5-5) becomes

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta) \left( \frac{\partial}{\partial \phi_1} \left[ \log[p_1(x_2, x_3) \mid x_k] \right] \right)$$

for $\phi_1 = u_3(i)$

and $\phi_1 = \sigma_3(i)$ (5-7)

When $\phi_1 = u_3(i)$, from equation (5-7), the following is obtained.

$$u_3(i) = \frac{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta) \left( x_3^{[k]} - \frac{\sigma_23(i)}{\sigma_2(i)} \left[ x_2^{[k]} - u_2(i) \right] \right)}{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta)}$$

(5-8)

In equation (5-7), letting $\phi_1 = \sigma_3(i)$ and $\phi_1 = \sigma_{23}(i)$ and eliminating $\left[ x_3^{[k]} - u_3(i) \right]^2$ from the resulting equations yields, after simplification, an expression for the covariance between type I and type II nodes. That is,

$$\sigma_{23}(i) = \sigma_2(i) \cdot \frac{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta) \left[ x_2^{[k]} - u_2(i) \right] \left[ x_3^{[k]} - u_3(i) \right]}{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta) \left[ x_2^{[k]} - u_2(i) \right]^2}$$

(5-9)

Letting $\phi_1 = \sigma_3(i)$ in equation (5-7) and using equation (5-9) yields the following.

$$\sigma_3(i) = \frac{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta) \left( \left[ x_3^{[k]} - u_3(i) \right]^2 + \sigma_{23}(i) \left( \frac{\sigma_{23}(i)}{\sigma_2(i)} \left[ x_2^{[k]} - u_2(i) \right] \right)^2 \right)}{\sum_{k=1}^{N} p(x_k \mid x_{k+}, \theta)}$$

(5-10)
5.1.2 MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF TYPE II NODES

A typical type II node, as defined in section 4, in a general dependent feature tree is shown in figure 5-3.

![Figure 5-3. A typical type II node in a general dependent feature tree.](image)

In figure 5-3, node $x_s$ is a type II node, and nodes $x_1, x_2, \ldots, x_r$ are type I nodes. The terms in the product approximation of the probability density function of cluster $i$ containing feature $x_s$ are

$$p(x|\omega_i) = \cdots p(x_s|x_t)p(x_1|x_s) \cdots p(x_r|x_s) \cdots \tag{5-11}$$

If $\theta_i$ is a parameter of the $i^{th}$ cluster, using equation (5-11) in equation (2-8) yields

$$\frac{\partial \ell}{\partial \theta_i} = \sum_{k=1}^{N} p(x_k|x_s, \theta) \left( \sum_{s=1}^{r} p_i(x_s|x_s) + p_i(x_s|x_t) \right) = 0 \tag{5-12}$$

From equation (5-12), letting $\theta_i = u_s(i)$ results in the following maximum likelihood equation for the mean of feature $x_s$ of cluster $i$.

$$u_s(i) = \frac{\sum_{k=1}^{N} p(x_k|x_s, \theta) x_k + \sum_{k=1}^{r} \frac{1}{\sigma_{s(t)}^2} \sum_{s=1}^{r} \sigma_{s(t)}^2 \left[ \sigma_{s(s)}(i) \left( x_k - u_{s(t)}(i) \right) + \sum_{s=1}^{r} \sigma_{s(s)}(i) \left[ x_k - u_{s(t)}(i) \right] \right] x_s }{\sum_{k=1}^{N} p(x_k|x_s, \theta)} \tag{5-13}$$

where

$$\Delta_{s(t)}(i) = \sigma_s(i)\sigma_s(i) - \sigma_{s(t)}(i) \tag{5-14}$$
In equation (5-12), letting \( \theta_i = \sigma_{s(i)}, \sigma_{s1(i)}, \ldots, \sigma_{sr(i)} \), and \( \sigma_{st(i)} \) yields the following after simplification.

\[
\sigma_{s(i)} = \frac{\sum_{k=1}^{N} p(\omega_k|X_k, \theta) \left( r \left[ x_s^k - u_s(1) \right]^2 + \sigma_{s1(i)}^2 \left( \sum_{t=1}^{r} \frac{[x_t^k - u_t(1)]^2}{\sigma_{st(t)}^2} \right) \right)}{\sum_{k=1}^{N} p(\omega_k|X_k, \theta) \left( r + \left[ x_s^k - u_s(1) \right]^2 \left( \sum_{t=1}^{r} \frac{\sigma_{s1(t)}^2}{\sigma_{st(t)}^2} \right) \right)}
\]

(5-15)

### 5.1.3 Maximum Likelihood Equations for the Parameters of Type III Nodes

A typical type III node in a general dependent feature tree is shown in figure 5-4.

![Diagram of a typical type III node in a general dependent feature tree.](5-6)

In figure 5-4, \( x_s \) is a type III node; and nodes \( x_1, x_2, \ldots, x_r \) and \( x_t \) may be type III nodes or other types of nodes. Proceeding as in section 5.1.2, the maximum likelihood equations for the variance and mean of feature \( x_s \) of cluster \( i \) can be shown to be the following (see appendix A).

\[
\sigma_{s(i)}^2 = \frac{\sum_{k=1}^{N} p(\omega_k|X_k, \theta) \left( \left[ x_s^k - u_s(1) \right]^2 + \frac{\sigma_{s1(i)}^2}{\sigma_{st(i)}^2} \left( \sum_{t=1}^{r} \frac{[x_t^k - u_t(1)]^2}{\sigma_{st(t)}^2} \right) \right)}{\sum_{k=1}^{N} p(\omega_k|X_k, \theta) \left( r + \left[ x_s^k - u_s(1) \right]^2 \left( \sum_{t=1}^{r} \frac{\sigma_{s1(t)}^2}{\sigma_{st(t)}^2} \right) \right)}
\]

(5-16)
and

\[
\sum_{k=1}^{N} \frac{p(u_i|X_k, \theta)}{p(u_i|X_k, \theta)} x_k^k + \frac{1}{\sum_{k=1}^{N} \frac{p(u_i|X_k, \theta)}{p(u_i|X_k, \theta)} \sum_{k=1}^{N} \frac{p(u_i|X_k, \theta)}{p(u_i|X_k, \theta)} \sum_{k=1}^{N} \frac{p(u_i|X_k, \theta)}{p(u_i|X_k, \theta)} x_k^k}
\]

\[
\sum_{k=1}^{N} \frac{p(u_i|X_k, \theta)}{p(u_i|X_k, \theta)}
\]

(5-17)

5.1.4 MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF TYPE IVA NODES

A typical type IVa node in a general dependent feature tree is shown in figure 5-5.

![Figure 5-5](image)

**Figure 5-5.** A typical type IVa node in a general dependent feature tree.

In figure 5-5, \(x_1\) is a root node of type IVa, and node \(x_2\) may be of type I, type II, or type III. The maximum likelihood equations for the variance and mean of feature \(x_1\) of cluster \(i\) are given in the following.

\[
\sigma_{21}(1) = \frac{\sum_{k=1}^{N} p(u_i|X_k, \theta) \left( \frac{x_k^k - u_1(1)}{\sum_{k=1}^{N} p(u_i|X_k, \theta)} \right)^2 + \frac{\sigma_{21}(1)}{\sigma_{22}(1) \left( \sigma_{21}(1) - \left[ \frac{x_k^k - u_1(1)}{\sum_{k=1}^{N} p(u_i|X_k, \theta)} \right]^2 \right)}{\sigma_{22}(1)}}{\sum_{k=1}^{N} p(u_i|X_k, \theta)}
\]

(5-18)

and

\[
\sigma_{1}(1) = \frac{\sum_{k=1}^{N} p(u_i|X_k, \theta) \left( \frac{x_k^k - u_1(1)}{\sum_{k=1}^{N} p(u_i|X_k, \theta)} \right)^2 + \frac{\sigma_{21}(1)}{\sigma_{22}(1) \left( \sigma_{21}(1) - \left[ \frac{x_k^k - u_1(1)}{\sum_{k=1}^{N} p(u_i|X_k, \theta)} \right]^2 \right)}{\sigma_{22}(1)}}{\sum_{k=1}^{N} p(u_i|X_k, \theta)}
\]

(5-19)

5.1.5 MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF TYPE IVB NODES

A typical type IVb node in a general dependent feature tree is shown in figure 5-6.
Figure 5-6.- A typical type IVb node in a general dependent feature tree.

In figure 5-6, \( x_s \) is a root node of type IVb, and nodes \( x_1, x_2, \ldots, x_r \) are of type I, type II, or type III. The maximum likelihood equations for the variance and mean of feature \( x_s \) of cluster i can be shown to be the following.

\[
\sigma_s(i) = \frac{\sum_{k=1}^{N} p(\omega_i | x_k, \theta) \left( (r - 1) \left[ \frac{x^k_s - u_s(i)}{\sigma_s(i)} \right]^2 + \left\{ \sum_{k=1}^{r} \frac{\sigma_s(i) \sigma_{S}(1)}{\sigma_S(i) \Delta_{S}(1)} \left[ \frac{x^k_s - u_s(i) \Delta_{S}(1)}{\sigma_S(i)} \right]^2 \right\} \right)}{\sum_{k=1}^{N} p(\omega_i | x_k, \theta) (r - 1) + \left\{ \sum_{k=1}^{r} \frac{\sigma_s(i) \sigma_{S}(1)}{\sigma_S(i) \Delta_{S}(1)} \left[ \frac{x^k_s - u_s(i) \Delta_{S}(1)}{\sigma_S(i)} \right]^2 \right\}}
\]

and

\[
u_s(i) = \frac{\sum_{k=1}^{N} p(\omega_i | x_k, \theta) x^k_s + \frac{1}{\sigma_s(i)} \sum_{k=1}^{r} \frac{\sigma_s(i) \sigma_{S}(1)}{\sigma_S(i) \Delta_{S}(1)} \sum_{k=1}^{N} p(\omega_i | x_k, \theta) \left[ \frac{x^k_s - u_s(i) \Delta_{S}(1)}{\sigma_S(i)} \right]}{\sum_{k=1}^{N} p(\omega_i | x_k, \theta)}
\]

5.2 MAXIMUM LIKELIHOOD EQUATIONS FOR THE COVARIANCES BETWEEN FEATURES

In this section, maximum likelihood equations are developed for the covariances between the features when the probability density functions of the clusters are approximated by first-order dependent feature trees.

5.2.1 MAXIMUM LIKELIHOOD EQUATIONS FOR THE COVARIANCE BETWEEN TYPE I AND TYPE II NODES

In this section, maximum likelihood equations for the covariance between type I and type II features are derived. A typical link connecting type I and type II nodes is shown in figure 5-7.
In figure 5-7, node $x_s$ is of type I, and node $x_r$ is of type II. The maximum likelihood equation for the covariance between features $x_r$ and $x_s$ of cluster $i$ is given in the following.

$$
\sigma_{rs}(i) = \frac{\sum_{k=1}^{N} p(\omega_1|X_k, \theta) [x^k_r - u_r(i)][x^k_s - u_s(i)]}{\sum_{k=1}^{N} p(\omega_1|X_k, \theta) [x^k_r - u_r(i)]^2}
$$

(5-22)

5.2.2 MAXIMUM LIKELIHOOD EQUATIONS FOR THE COVARIANCE BETWEEN TYPE IVA AND TYPE II OR TYPE III NODES

A typical link connecting type IVA and type II or type III nodes in a general dependent feature tree is shown in figure 5-8.

In figure 5-8, node $x_1$ is of type IVA, and node $x_2$ may be of type II or type III. The maximum likelihood equation for the covariance between features $x_1$ and $x_2$ of cluster $i$ is as follows.

$$
\sigma_{12}(i) = \sigma_{2}(i) \cdot \frac{\sum_{k=1}^{N} p(\omega_1|X_k, \theta) [x^k_1 - u_1(i)][x^k_2 - u_2(i)]}{\sum_{k=1}^{N} p(\omega_1|X_k, \theta) [x^k_2 - u_2(i)]^2}
$$

(5-23)
5.2.3 MAXIMUM LIKELIHOOD EQUATIONS FOR THE COVARIANCE BETWEEN TYPES OF NODES OTHER THAN THOSE CONSIDERED IN SECTIONS 5.2.1 AND 5.2.2

Let there be a link between nodes $x_2$ and $x_3$ in a general dependent feature tree whose types are other than those considered in sections 5.2.1 and 5.2.2. The maximum likelihood equation for the covariance between features $x_2$ and $x_3$ of cluster $i$ is given in the following.

\[ \sigma_{23}(i) = \frac{\sum_{k=1}^{N} p(\omega_{1}|x_{k},0) \left\{ \sigma_{2}(1)\sigma_{3}(1) + \sigma_{23}(1)\sigma_{23}(1) + \sigma_{23}(1)\left[x_k - u_2(1)\right]\left[x_k - u_3(1)\right] \right\}}{\sum_{k=1}^{N} p(\omega_{1}|x_{k},0) \left\{ \sigma_{3}(1)\left[x_k - u_2(1)\right]^2 + \sigma_{2}(1)\left[x_k - u_3(1)\right]^2 \right\}} \]  

(5-24)
6. EXPERIMENTAL RESULTS

In this section, some results from processing remotely sensed Landsat multispectral scanner imagery data are presented. The images are of a 5- by 6-nautical-mile area called a segment. The image is divided into a rectangular array of pixels, 117 rows by 196 columns. The image is overlaid with a rectangular grid of 209 grid intersections. Two classes are considered: class 1 is wheat, and class 2 is "other." The true (ground truth) labels for the pixels at the grid intersections are acquired. The locations of the segments and the individual acquisitions used for each of the segments are listed in table 6-1. The a priori probabilities of the classes are estimated as sample estimates. Equations (3-6) and (3-18) are used to compute the weighted mutual information between the features, assuming in each class that the features are Gaussian distributed. Kruskal's algorithm (ref. 16) is used to construct optimal dependent feature trees by minimizing $I(p,p_t)$ of equation (3-5). The optimal dependent feature trees of segments 1648 and 1739 are shown in figures 6-1 and 6-2, respectively.

![Figure 6-1.- Optimal dependent feature tree of segment 1648.](image)

Generally, it is known that, for each acquisition, a strong dependency exists between channels 1 and 2 and between channels 3 and 4. From figures 6-1 and 6-2, it is seen that these dependencies appear in the optimal dependent feature trees.
TABLE 6-1.- CONFOUND MATRICES AND CLASSIFICATION ACCURACIES OF BAYES CLASSIFIER

<table>
<thead>
<tr>
<th>Segment</th>
<th>Location (county, state)</th>
<th>Acquisition dates</th>
<th>Full covariance matrix</th>
<th>Independent features</th>
<th>Optimal dependent feature tree</th>
<th>Arbitrary dependent feature tree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Training set</td>
<td>Test set</td>
<td>Training set</td>
<td>Test set</td>
</tr>
<tr>
<td>1520</td>
<td>Big Stone, Minnesota</td>
<td>77174</td>
<td>0.9423</td>
<td>0.7809</td>
<td>0.8269</td>
<td>0.8476</td>
</tr>
<tr>
<td></td>
<td></td>
<td>77156</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>77120</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1604</td>
<td>Renville, North Dakota</td>
<td>77143</td>
<td>0.75</td>
<td>0.5809</td>
<td>0.6827</td>
<td>0.6000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>77125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1648</td>
<td>Bowman, North Dakota</td>
<td>77179</td>
<td>0.7211</td>
<td>0.6571</td>
<td>0.6923</td>
<td>0.6095</td>
</tr>
<tr>
<td></td>
<td></td>
<td>77125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1739</td>
<td>Teton, Montana</td>
<td>77222</td>
<td>0.9327</td>
<td>0.6571</td>
<td>0.7212</td>
<td>0.6762</td>
</tr>
<tr>
<td></td>
<td></td>
<td>77168</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td>77132</td>
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<tr>
<td></td>
<td></td>
<td>76263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1853</td>
<td>Ness, Kansas</td>
<td>77193</td>
<td>0.9231</td>
<td>0.7905</td>
<td>0.8365</td>
<td>0.7905</td>
</tr>
<tr>
<td></td>
<td></td>
<td>77067</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>76253</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean classification accuracy

<table>
<thead>
<tr>
<th>Segment</th>
<th>Location (county, state)</th>
<th>Acquisition dates</th>
<th>Full covariance matrix</th>
<th>Independent features</th>
<th>Optimal dependent feature tree</th>
<th>Arbitrary dependent feature tree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Training set</td>
<td>Test set</td>
<td>Training set</td>
<td>Test set</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.8554</td>
<td>0.6933</td>
<td>0.7519</td>
<td>0.7047</td>
</tr>
</tbody>
</table>

\*Confusion matrix.
\*Probability of correct classification.
To investigate the effectiveness of the optimal dependent feature trees in classification, an experiment is performed to compare the classification accuracies of the Bayes classifier (1) when the densities are approximated with optimal dependent feature trees, (2) when no approximation is used for the densities (full covariance matrix), (3) assuming the features are independent, and (4) when the densities are approximated with arbitrary dependent feature trees. Spectral vectors of 104 labeled pixels are used as the training pattern set, and the spectral vectors of the remaining 105 labeled pixels are used as the test pattern set. The structure of the arbitrary dependent feature tree used in this experiment is shown in figure 6-3.

The computed confusion matrices and the classification accuracies on the training and test sets for each of the segments processed are listed in table 6-1. From table 6-1, it is seen that, in general, better classification accuracies...
are obtained on the training set when the full covariance matrix is used without approximating the densities. Improved classification accuracies are obtained on the test set when the densities are approximated with optimal dependent feature trees. This might be due to the fact that a large number of parameters are estimated when the full covariance matrices are used.

One of the important objectives in the classification of remotely sensed agricultural imagery data is to estimate the proportion of the class of interest in the image. The ratio of the variance of the estimated proportion using machine classification to the variance of the estimated proportion using simple random sampling is called variance reduction factor $R$ (ref. 1). The quantity $R$ can be viewed as an indication of how much the machine classification improves the proportion estimation. The computed variance reduction factors for each of the segments processed are listed in table 6-2. From table 6-2, it is seen that the variance reduction factor consistently improves when the densities are approximated with dependent feature trees, compared to the other cases.
### TABLE 6-2.- COMPARISON OF VARIANCE REDUCTION FACTORS

<table>
<thead>
<tr>
<th>Segment</th>
<th>Location (county, state)</th>
<th>Acquisition dates</th>
<th>Full covariance matrix</th>
<th>Independent features</th>
<th>Optimal dependent feature tree</th>
<th>Arbitrary dependent feature tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1520</td>
<td>Big Stone, Minnesota</td>
<td>77174, 77156, 77120</td>
<td>0.8194</td>
<td>0.6405</td>
<td>0.5532</td>
<td>0.7540</td>
</tr>
<tr>
<td>1604</td>
<td>Renville, North Dakota</td>
<td>77143, 77125</td>
<td>0.9786</td>
<td>0.9637</td>
<td>0.8475</td>
<td>0.9889</td>
</tr>
<tr>
<td>1648</td>
<td>Bowman, North Dakota</td>
<td>77179, 77125</td>
<td>0.9325</td>
<td>0.9509</td>
<td>0.8896</td>
<td>0.9679</td>
</tr>
<tr>
<td>1739</td>
<td>Teton, Montana</td>
<td>77222, 77168, 77132, 76263</td>
<td>0.9432</td>
<td>0.9242</td>
<td>0.8724</td>
<td>0.9450</td>
</tr>
<tr>
<td>1853</td>
<td>Ness, Kansas</td>
<td>77193, 77067, 76253</td>
<td>0.8034</td>
<td>0.7701</td>
<td>0.6576</td>
<td>0.7701</td>
</tr>
</tbody>
</table>
7. CONCLUDING REMARKS

In the classification of imagery data, such as in the machine processing of remotely sensed multispectral scanner data, unsupervised classification techniques have been found to be effective. The application of clustering techniques for the analysis of imagery data essentially involves two steps: (1) clustering the data or partitioning the image into its inherent modes and (2) giving the probabilistic class labels to the resulting clusters. In practice, it is observed that fields are relatively easy to label when compared to pixels.

Several researchers have investigated methods for locating fields in the imagery data. Recently, considerable interest has been shown in developing techniques for probabilistically labeling the clusters using information from a given set of labeled patterns and, also, from a given set of labeled fields.

In decomposing the mixture density of the data into its normal component densities, the parameters of the component densities and the a priori probabilities of the modes are iteratively computed using maximum likelihood equations coupled with a split and merge sequence. The updating of the parameters is usually stopped after a few iterations; and for practical data, a large number of parameters must be estimated. For a fixed sample size, the accuracy of estimation usually decreases as the number of parameters to be estimated increases.

To overcome the above shortcomings, it is proposed in this paper that the densities be approximated with first-order dependent feature trees. The dependent feature trees can be constructed using criteria based on information measure and, also, based on class separability measure. Expressions are derived for the criteria when the distributions of the features are Gaussian. Expressions also are derived for the covariances between features not connected by a single link in the dependent feature tree.

Different types of nodes are defined in a general dependent feature tree. Maximum likelihood equations are derived for the parameters of the mixture.
density of the data by approximating the cluster conditional densities with first-order dependent feature trees. The field structure of the data is also taken into account in the decomposition of the mixture density of the data into its normal component densities. Furthermore, experimental results from the processing of remotely sensed multispectral scanner imagery data are presented.
8. REFERENCES


APPENDIX A

DERIVATIONS OF MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF A TYPE III FEATURE
APPENDIX A

DERIVATIONS OF MAXIMUM LIKELIHOOD EQUATIONS FOR THE PARAMETERS OF A TYPE III FEATURE

In this appendix, maximum likelihood equations for the parameters of a typical type III feature are derived. A typical type III node in a general dependent feature tree is illustrated in figure A-1.

Figure A-1.- Illustration of a typical type III node in a general dependent feature tree.

In figure A-1, $x_2$ is a type III feature. The following is obtained from equation (5-5) by keeping only the terms that involve feature $x_2$ in the product approximation of the density of the $i^{th}$ cluster.

\[
\frac{\partial \log L}{\partial \theta_i} = \sum_{k=1}^{N} p(\omega_k | x_k, \theta) \frac{\partial}{\partial \theta_i} \left[ \log[p_i(x_4 | x_2)] + \log[p_i(x_3 | x_2)] \right] + \log[p_1(x_2 | x_1)]
\]

\[
= \sum_{k=1}^{N} p(\omega_k | x_k, \theta) \frac{\partial}{\partial \theta_i} \left[ \log[p_i(x_2, x_4)] + \log[p_i(x_2, x_3)] \right] + \log[p_1(x_1, x_2)] - 2 \log[p_i(x_2)] - \log[p_1(x_1)]
\]  

(A-1)
It is assumed that the features of the $i^{th}$ cluster are Gaussian distributed. That is,

$$\log[p_i(x_r, x_s)] = -\log(2\pi) - \frac{1}{2} \log[\Delta_{rs}(i)] - \frac{1}{2\Delta_{rs}(i)} \{\sigma_s(i)[x_r - u_r(i)]^2$$

$$- 2\sigma_{rs}(i)[x_r - u_r(i)][x_s - u_s(i)] + \sigma_r(i)[x_s - u_s(i)]^2\}$$

(A-2)

where

$$\Delta_{rs}(i) = \sigma_r(i)\sigma_s(i) - \sigma_{rs}(i)^2$$

(A-3)

Using equation (A-2) in equation (A-1) yields

$$\frac{\partial \log[p_i]}{\partial \theta_i} = \sum_{k=1}^{N} p(\omega_k | x_k, \theta) \frac{\partial C_k}{\partial \theta_i}$$

(A-4)

where

$$C_k = \log[\Delta_{24}(i)] + \frac{1}{\Delta_{24}(i)} \{\sigma_4(i)[x_r^k - u_r(1)]^2 - 2\sigma_{24}(1)[x_r^k - u_r(1)][x_s^k - u_4(1)] + \sigma_4(i)[x_s^k - u_4(1)]^2\}$$

$$+ \log[\Delta_{23}(i)] + \frac{1}{\Delta_{23}(i)} \{\sigma_3(i)[x_r^k - u_2(1)]^2 - 2\sigma_{23}(1)[x_r^k - u_2(1)][x_s^k - u_3(1)] + \sigma_3(i)[x_s^k - u_3(1)]^2\}$$

$$+ \log[\Delta_{21}(i)] + \frac{1}{\Delta_{21}(i)} \{\sigma_1(i)[x_r^k - u_2(1)]^2 - 2\sigma_{21}(1)[x_r^k - u_2(1)][x_s^k - u_1(1)] + \sigma_1(i)[x_s^k - u_1(1)]^2\}$$

$$- 2 \log[\sigma_2(i)] - \frac{2}{\sigma_2(i)} [x_r^k - u_2(1)]^2$$

(A-5)

Letting $\theta_i = u_2(1)$, from equation (A-5), the following is obtained.

$$\frac{\partial C_k}{\partial u_2(1)} = \left\{ -\frac{2\sigma_4(i)}{\Delta_{24}(i)} [x_r^k - u_2(1)] + \frac{2\sigma_{24}(1)}{\Delta_{24}(i)} [x_s^k - u_4(1)] \right\}$$

$$+ \left\{ -\frac{2\sigma_3(i)}{\Delta_{23}(i)} [x_r^k - u_2(1)] + \frac{2\sigma_{23}(1)}{\Delta_{23}(i)} [x_s^k - u_3(1)] \right\}$$

$$+ \left\{ -\frac{2\sigma_1(i)}{\Delta_{21}(i)} [x_r^k - u_2(1)] + \frac{2\sigma_{21}(1)}{\Delta_{21}(i)} [x_s^k - u_1(1)] \right\}$$

$$+ \left\{ \frac{4}{\sigma_2(i)} [x_r^k - u_2(1)] \right\}$$

(A-6)
Substituting equation (A-6) in equation (A-4) and equating the results to zero yields the following maximum likelihood equation for the mean of feature $x_2$ of cluster $i$.

$$\sum_{k=1}^{N} p(x_i | k, \theta) \left( \sum_{i=1}^{N} \frac{1}{\sigma_2(i)} \frac{\sigma_2(i)}{2} \left[ x_i^2 - u_2(i) \right]^2 \cdot \left[ \frac{\sigma_2(i)}{\sigma_2(i)^2} \left[ x_i^2 - u_2(i) \right]^2 + \frac{\sigma_2(i) \sigma_2(i)}{\sigma_2(i)^2} \left[ x_i^2 - u_2(i) \right] \right] \right)$$

$$= 0$$

\[u_2(i) = \frac{\sum_{k=1}^{N} p(x_i | k, \theta) \left[ x_i^2 - u_2(i) \right]}{\sum_{k=1}^{N} p(x_i | k, \theta)}\]  

(A-7)

Letting $\theta_1 = \sigma_2(1)$ in equation (A-4) and equating the result to zero yields the following after simplification.

$$\sum_{k=1}^{N} p(x_i | k, \theta) \left( \frac{\sigma_2(1)}{\sigma_2(1)^2} \left[ x_i^2 - u_2(i) \right]^2 - \frac{\sigma_2(1)}{\sigma_2(1)^2} \left[ x_i^2 - u_2(i) \right]^2 \cdot \frac{2\sigma_2(1)\sigma_2(1)}{\sigma_2(1)} \left[ x_i^2 - u_2(i) \right] \right) = 0$$

(A-8)

Similarly, differentiating $\ell$ with respect to $\sigma_2(j)(i)$ for $j = 1,3,4$ and equating the resulting expression to zero yields

$$\sum_{k=1}^{N} p(x_i | k, \theta) \left( \frac{2\sigma_2(1)}{\sigma_2(1)^2} \left[ x_i^2 - u_2(i) \right] \left[ x_i^2 - u_3(i) \right] + \frac{2\sigma_2(1)\sigma_2(1)}{\sigma_2(1)} \left[ x_i^2 - u_2(i) \right] \left[ x_i^2 - u_3(i) \right] \right) = 0$$

(A-9)

Using equation (A-9) for $j = 1,3,4$ in equation (A-8) yields, after simplification, the following maximum likelihood equation for $\sigma_2(i)$.

$$\sum_{k=1}^{N} p(x_i | k, \theta) \left( \frac{\sigma_2(i)}{\sigma_2(i)^2} \left[ x_i^2 - u_2(i) \right]^2 \cdot \frac{2\sigma_2(i)\sigma_2(i)}{\sigma_2(i)} \left[ x_i^2 - u_2(i) \right] \right) = 0$$

(A-10)
Proceeding, similar to equations (A-9) and (A-10), it can easily be shown that
the maximum likelihood equations for the mean $u_s(i)$ and variance $\sigma_s(i)$ of
feature $x_s$ of cluster $i$ of figure 5-4 are of equations (5-16) and (5-15).
In practical applications of pattern recognition, such as in the classification of remotely sensed agricultural imagery data, one of the difficult problems is to obtain labels for the training patterns. The labels for the training patterns are usually provided by an analyst-interpreter by examining imagery films and using some other information such as historic information and crop calendar models. Agricultural imagery data usually have a field-like structure, and it is observed that fields are relatively easy to label when compared to pixels. Recently, considerable interest has been shown in developing techniques for locating fields in the imagery data (ref. 2-4) and in developing methods for the probabilistic labeling (refs. 10, 11) of cluster distributions using information from a given set of labeled fields. Once the fields are located by a field-finding algorithm, the problem of fitting a mixture of Gaussian density functions to the data by taking into account the field structure of the data is considered in this appendix.

It is assumed that there are $f$-fields in the data. Let the $j$th field be denoted by $F_j$; let it contain $N_j$ pixels; and let $X_{jk}$, $k = 1, 2, \ldots, N_j$, be their spectral vectors. Let $m$ be the number of clusters in the data. Let $P(\omega_i)$ and $p(X|\omega_i)$ be the a priori probability that a field belongs to cluster $\omega_i$ and cluster conditional densities, respectively. Let $\widetilde{X}_j$ be the concatenated vector of spectral vectors of the pixels in the $j$th field. That is,

$$\widetilde{X}_j = \begin{bmatrix} X_{j1} \\ X_{j2} \\ \vdots \\ X_{jN_j} \end{bmatrix}$$  \hspace{1cm} (B-1)

It is assumed that the fields are independent. Then, the joint density of $f$-fields is given by

$$p(\widetilde{X}_1, \ldots, \widetilde{X}_f) = \prod_{j=1}^{f} p(\widetilde{X}_j)$$  \hspace{1cm} (B-2)
The mixture density \( p(\tilde{x}_j) \) can be written as

\[
p(\tilde{x}_j) = \sum_{\omega} p(\omega) p(\tilde{x}_j | \omega) \quad (B-3)
\]

If it is assumed that the spectral vectors of the pixels in each field are cluster conditionally independent, then

\[
p(\tilde{x}_j | \omega) = \prod_{k=1}^{N} p(x_{jk} | \omega) \quad (B-4)
\]

Using equations (B-3) and (B-4), the joint density of equation (B-2) can be written as follows.

\[
p(\tilde{x}_1, \ldots, \tilde{x}_f) = \prod_{j=1}^{f} \left\{ \sum_{\omega} \left[ \prod_{k=1}^{N} p(x_{jk} | \omega) \right] \right\} \quad (B-5)
\]

Since the logarithm is a monotonic function of its argument, taking the log of both sides of equation (B-5) and denoting it by \( \ell \) results in

\[
\ell = \sum_{j=1}^{f} \log \left\{ \sum_{\omega} \left[ \prod_{k=1}^{N} p(x_{jk} | \omega) \right] \right\} \quad (B-6)
\]

From equation (B-6), which is similar to equation (2-7), the maximum likelihood equation for the probability that a field belongs to a cluster can easily be obtained as the following.

\[
P(\omega_r) = \frac{1}{f} \sum_{j=1}^{f} \frac{p(\omega_r)p(\tilde{x}_j | \omega_r)}{p(\tilde{x}_j)} \quad (B-7)
\]

If \( \theta_i \) is a parameter of the density function of the \( i \)th cluster, differentiating \( \ell \) of equation (B-6) with respect to \( \theta_i \) yields the following.

\[
\frac{\partial \ell}{\partial \theta_i} = \sum_{j=1}^{f} \frac{P(\omega_r)p(\tilde{x}_j | \omega_r)}{p(\tilde{x}_j)} \sum_{k=1}^{N} \frac{\partial}{\partial \theta_i} \left\{ \log[p(x_{jk} | \omega_i)] \right\} \quad (B-8)
\]
If the probability density functions of the clusters are Gaussian [i.e., \( p(X|\omega_i) \sim N(U_i, \Sigma_i) \)], from equation (B-8), the maximum likelihood equations for the mean and covariance matrix of the densities of the clusters can be shown to be the following.

\[
U_i = \frac{\sum_{j=1}^{f} N_j p(\omega_i | \bar{X}_j) \bar{X}_j}{\sum_{j=1}^{f} N_j p(\omega_i | \bar{X}_j)} \quad (B-9)
\]

and

\[
\Sigma_i = \frac{\sum_{j=1}^{f} p(\omega_i | \bar{X}_j) \left( \sum_{k=1}^{N_i} (X_{jk} - U_i)(X_{jk} - U_i)^T \right)}{\sum_{j=1}^{f} N_j p(\omega_i | \bar{X}_j)} \quad (B-10)
\]

where

\[
\bar{X}_{j\cdot} = \frac{1}{N_j} \sum_{k=1}^{N_j} X_{jk} \quad (B-11)
\]

If the probability density functions of the clusters are approximated with first-order feature dependence trees, maximum likelihood equations for the parameters of the densities (similar to those developed in section 5) can easily be obtained from equations (B-8) and (3-1) by taking into account the field structure of the data.
APPENDIX C

DEPENDENT FEATURE TREES WITH THE NODES REPRESENTING FEATURE SUBSETS
APPENDIX C

DEPENDENT FEATURE TREES WITH THE NODES REPRESENTING FEATURE SUBSETS

Very often it is necessary to have each node of a dependent feature tree represent a set of features instead of one feature. For example, in remote sensing, the satellite makes multiple passes over a given area and, at each acquisition, gathers several channels of data. In some instances, it is desirable to have each node of a dependent feature tree represent a set of features (e.g., the set of features corresponding to an acquisition). In this appendix, expressions are developed for the mutual information between the feature subsets and for the covariance between the feature subsets when a path connects them in a dependent feature tree. It is assumed that the features are Gaussian distributed.

Let the components of feature vectors \( X_i \) and \( X_j \) be the sets of features represented by nodes \( i \) and \( j \), respectively. Let \( n_i \) and \( n_j \) be the dimensionality of vectors \( X_i \) and \( X_j \), respectively. If the feature vector \( X_i \) is normally distributed, its probability density \( p(X_i) \) can be represented as

\[
p(X_i) \sim N(\mu_i, \Sigma_i)
\]  

(C-1)

where \( \mu_i \) is the mean vector and \( \Sigma_i \) is the covariance matrix.

Let \( Z = (x_i^T, x_j^T)^T \). Then,

\[
p(Z) \sim N(\mu_z, \Sigma_z)
\]  

(C-2)

where

\[
\Sigma_z = \begin{bmatrix}
\Sigma_i & \Sigma_{ij} \\
\Sigma_{ji} & \Sigma_j
\end{bmatrix}
\]  

(C-3)
The mutual information between feature vectors $X_i$ and $X_j$ can be written as

$$I(X_i, X_j) = \int p(X_i, X_j) \log \left( \frac{p(X_i, X_j)}{p(X_i)p(X_j)} \right) \, dX_i \, dX_j$$

$$= -\frac{1}{2} \log \left( \frac{|\Sigma_i|}{|\Sigma_j|} \right)$$

(C-4)

$$I(X_i, X_j) = -\frac{1}{2} \log \left( \frac{|\Sigma_i - \Sigma_{ij} \Sigma_{ij}^{-1} \Sigma_{ij}|}{|\Sigma_j|} \right)$$

(C-5)

where $|\Sigma_z|$ is the determinant of the matrix $\Sigma_z$. Let $y$ and $v$ be the zero mean normal random vectors. That is, $p(y) \sim N(0, \Sigma_y)$ and $p(v) \sim N(0, \Sigma_v)$. Let $Z = (y^T, v^T)$ and $p(Z) \sim N(0, \Sigma_Z)$. Let

$$C_Z = \begin{bmatrix}
C_y & C_{vy} \\
C_{vy}^T & C_v
\end{bmatrix}$$

(C-6)

and

$$C^{-1}_Z = \begin{bmatrix}
Q_y & Q_{vy} \\
Q_{vy}^T & Q_v
\end{bmatrix}$$

Consider

$$p(y|v) = \frac{p(Z)}{p(v)}$$

$$= \text{constant} \cdot \exp \left( -\frac{1}{2} A \right)$$

(C-7)

where

$$A = \left( y + Q_{y}^{-1} Q_{vy} \right)^T Q_y \left( y + Q_{y}^{-1} Q_{vy} v \right)$$

(C-8)
Thus, the density \( p(y|v) \) is Gaussian with the mean \(-Q^{-1}_yQ_{yv}v\) and the covariance matrix \( Q^{-1}_y \). Following a similar argument, it can easily be shown that, if \( X_i \) is normally distributed, \( p(X_i|X_j) \) is normally distributed with the mean
\[
\left[ U_i - Q^{-1}_{i}O_{ij}(X_j - U_j) \right]
\] and the covariance matrix \( Q_{ii}^{-1} \). Now expressions for the covariance between the feature subsets, when a path connects their representative nodes in a dependent feature tree, can be derived as in section 4.2. For example, if \( X_4 \) and \( X_7 \) are Gaussian random vectors, similar to equation (4-3), the following can easily be obtained.

\[
\int (X_7 - U_7)p(X_7|X_4)dX_7 = -Q_{74}^{-1}Q_{4}(X_4 - U_4)
\]

Thus, expressions similar to equations (4-8) and (4-9) can easily be obtained.
"Maximum Likelihood Clustering with Dependent Feature Trees"

SUPPORTING RESEARCH
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