Influence of Friction Forces on the Motion of V/TOL Aircraft During Landing Operations on Ships at Sea

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## NOMENCLATURE

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<tr>
<td>$A_a$</td>
<td>pneumatic area</td>
</tr>
<tr>
<td>$A_h$</td>
<td>hydraulic area</td>
</tr>
<tr>
<td>$A_n$</td>
<td>net orifice area</td>
</tr>
<tr>
<td>$C$</td>
<td>distance from center of gravity of ship to landing platform</td>
</tr>
<tr>
<td>$C_d$</td>
<td>coefficient of discharge</td>
</tr>
<tr>
<td>$D_{ij}$</td>
<td>direction cosines</td>
</tr>
<tr>
<td>$G_{ij}$</td>
<td>direction cosines</td>
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<tr>
<td>$d_{ij}$</td>
<td>direction cosines</td>
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<tr>
<td>$F_{wa}$</td>
<td>force acting at axle of wheel $w$</td>
</tr>
<tr>
<td>$F_g$</td>
<td>force acting at point of wheel contact</td>
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<td>$F_{Na}$</td>
<td>force acting normal to shock strut at axle of wheel $w$</td>
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<td>$F_s$</td>
<td>shock-strut force at wheel $w$</td>
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<td>$f_{wa}$</td>
<td>body axes components of axle force at wheel $w$</td>
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<tr>
<td>$f_{ig}$</td>
<td>body axes components of forces generated at point of wheel contact</td>
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<td>internal friction forces at wheel $w$</td>
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<td>gravity vector</td>
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<td>$H^w$</td>
<td>hydraulic force parameter</td>
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<td>$i,j,k$</td>
<td>triad of mutually orthogonal unit vectors in directions of platform axes</td>
</tr>
<tr>
<td>$i,j,k$</td>
<td>triad of mutually orthogonal unit vectors in directions of aircraft body axes</td>
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<tr>
<td>$I^w$</td>
<td>unit dyadic</td>
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<td>$K^w$</td>
<td>tire stiffness at wheel $w$</td>
</tr>
<tr>
<td>$l^w,m^w,n^w$</td>
<td>direction cosines of shock strut relative to aircraft axes</td>
</tr>
<tr>
<td>$M_i^w$</td>
<td>components of moment vector</td>
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</table>
\( m_w \) mass of wheel \( w \)
\( \hat{n} \) unit vector normal to plane of wheel
\( P, Q, R \) components of ship's angular velocity vector
\( p, q, r \) components of aircraft's angular velocity vector
\( \hat{p} \) unit vector in plane of wheel
\( p \) pneumatic pressure
\( p_a \) air pressure in upper chamber
\( p_h \) hydraulic pressure in lower chamber
\( r_w \) position vector of wheel \( w \) relative to origin of aircraft axes
\( R_w \) position vector of wheel \( w \) relative to ship's coordinate system
\( R_{CG} \) position vector of aircraft center of gravity relative to ship axes
\( \hat{s} \) unit vector in direction of shock strut
\( s \) shock-strut axial stroke
\( u, v, w \) components of ship's linear velocity vector
\( u, v, w \) components of aircraft linear velocity
\( \tilde{v}_w \) velocity of wheel \( w \) relative to landing platform
\( \tilde{v}_{CG} \) velocity of aircraft's center of gravity
\( X_{1} \) ship axes coordinates
\( X_{1}^p \) platform axes coordinates
\( X_{1}^w \) coordinates of wheel \( w \) relative to ship's coordinate system
\( X_{1}^{CG} \) coordinates of aircraft's center of gravity relative to ship's reference frame
\( X_{1}^0 \) initial coordinates
\( X_{A} \) coordinates of point of attachment of shock strut
\( X_{1}^w \) coordinates of wheel \( w \) relative to aircraft axes
\( X_{10}^w \) coordinates of wheel \( w \) when shock strut is in fully extended position
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\psi, \theta, \phi$</td>
<td>aircraft Euler angles</td>
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<tr>
<td>$\psi, \theta, \phi$</td>
<td>ship Euler angles</td>
</tr>
<tr>
<td>$\Theta_1$</td>
<td>Euler angles corresponding to direction cosines</td>
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<td>$\kappa^W$</td>
<td>angle in plane of landing platform between $X_1^P$ axis and velocity vector at wheel $w$</td>
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<td>$\kappa_k^W$</td>
<td>angle $\kappa$ when friction is kinetic</td>
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<td>$\kappa_s^W$</td>
<td>angle $\kappa$ when friction is static</td>
</tr>
<tr>
<td>$\tau^W$</td>
<td>angle between plane of wheel and velocity vector at wheel $w$</td>
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<td>$\mu^W$</td>
<td>coefficient of friction at wheel $w$</td>
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<td>$\delta^W$</td>
<td>tire deflection at wheel $w$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>aircraft angular velocity vector</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of hydraulic fluid</td>
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INFLUENCE OF FRICTION FORCES ON THE MOTION OF VTOL AIRCRAFT
DURING LANDING OPERATIONS ON SHIPS AT SEA

James C. Howard and David O. Chin
Ames Research Center

SUMMARY

The equations describing the friction forces generated during landing operations on ships at sea have been formulated. These forces depend on the platform reaction and the coefficient of friction. The platform reaction depends on the relative sink rate and the shock-absorbing capability of the landing gear. The coefficient of friction is assumed to vary with the surface condition of the landing platform and the angle of yaw of the aircraft relative to the landing platform. The landings contemplated are by aircraft of the VTOL type equipped with conventional oleo-pneumatic landing gears. Because aircraft of this type land with little or no forward speed, spinup forces have been neglected. Simplifications have been introduced to reduce the complexity of the mathematical description of the tire and shock-strut characteristics. It has been shown elsewhere that, for normal impact without tire bottoming, reasonable variations in the force-deflection characteristics of the tire have only a relatively small effect on the calculated behavior of the landing gear. Approximating the actual complicated force-deflection characteristics of the tire by a linear relationship appears to be adequate for practical purposes. Although the internal friction forces in the shock strut have been included in the landing gear model, experimental data suggest that these forces can often be neglected without impairing the validity of the model equations. By including only those characteristics of the tire and shock strut that contribute significantly to the generation of landing gear forces, a set of relatively simple equations is obtained. Nevertheless, the equations are considered adequate for practical purposes.

INTRODUCTION

Landing VTOL aircraft on ships at sea is complicated by the motion and surface condition of the landing platform, which moves in response to the motions of the sea, and which is exposed to a variety of atmospheric conditions. An understanding of the landing phenomena requires that a mathematical model, which includes platform reactions and friction forces, be formulated and used to simulate the landing maneuver. In assessing the platform reaction forces likely to be encountered, the actual complicated force-deflection characteristics of the tire have been approximated by a linear relationship. Moreover, in the event that tire bottoming occurs, a linear segment approximation which takes into account the increased stiffness of the tire that results from bottoming yields good results (Militsky and Cook, 1953). A limited amount of experimental data obtained during drop tests at the Langley Research
Center indicates that the behavior of the landing gear is relatively insensitive to variations in the air compression process, and that variations between isothermal and near adiabatic compression have only a secondary effect on the calculated behavior of the landing gear. Consequently, variations in the polytropic exponent have been neglected. The importance of friction forces derives from the fact that, if friction is not sufficient to prevent sliding, the aircraft may be damaged by colliding with adjacent structures. It is unlikely that VTOL aircraft landings will be attempted when the decks are awash, but during rough weather the landing platform will be exposed to rain and possibly sea spray which could reduce the coefficients of friction below safe levels. Another possibility to be considered is the likelihood that VTOL aircraft may produce a shower of spray as the jets interact with the surface of the sea during the approach. If the resulting spray were to be blown over the landing platform, the friction forces would be reduced significantly. In view of these considerations, it is important to determine the reactions and friction forces likely to be encountered during a variety of sea states and landing conditions. By including these forces in the equations of the mathematical model, the motion of the aircraft during the landing maneuver can be computed and the possibility of sliding determined. In formulating the friction force equations, bending will be neglected and it will be assumed that each landing gear has a fixed orientation relative to aircraft body axes. In many cases, the influence of elasticity may be neglected without serious error, but in some instances, particularly when the landing gear attachment points experience large displacements relative to the nodal points of the flexible system, the interaction between the deformations of the structure and the landing gear may be required to represent the system adequately. Expressions for the hydraulic, air-compression, and internal friction forces generated in the shock strut are derived in appendix B.

KINEMATICS AND DYNAMICS OF AIRCRAFT-SHIP INTERACTIONS

Forces and Moments

Friction forces and platform reactions—Relative to the ship's coordinate system \((X_1, X_2, X_3)\), which originates at the center of gravity, the landing platform lies in the plane \((X_3 = C)\) = 0, where \(C\) is negative constant. The \(X_1X_3\) plane of the ship's coordinate system corresponds to the plane to symmetry of the vessel and the \(X_2\) axis is normal to it. A platform coordinate system \((X'_1, X'_2, X'_3)\) having axes parallel to the axes of the ship's coordinate system is used to specify vector components relative to the platform (see fig. 1).

Subsequent to the instant at which wheel \(w\) makes contact with the landing platform, the aircraft is subjected to a reaction force \(F'_3\) which is normal to the landing platform and a friction force which is coplanar with the platform. Relative to platform axes, the friction force has components \(F'_1\) and \(F'_2\). Hence, subsequent to wheel contact, the force vector \(F'_w\) generated by wheel \(w\) has components

\[
F'_w = F'_1 + F'_2 + F'_3
\]  

(1)
where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are a triad of mutually orthogonal unit vectors in the directions of the platform axes and the superscript $w$ denotes the wheel being considered.

Therefore, if the dynamic response of the aircraft subsequent to the wheel contact is being computed, the wheel reactions and friction forces of equation (1) must be added to the inertia, thrust, aerodynamic, gravitational, and shock-strut forces to complete the mathematical description of the force system.

The corresponding wheel forces relative to aircraft body axes are $f_1^w$ where

$$f_1^w = d_{_i^{_j^{^j}}} f^w_1 , \quad i,j = 1,2,3 \quad (2)$$

and $d_{_i^{_j^{^j}}}$ are direction cosines (appendix A). The force generated by each wheel is obtained by assigning the appropriate wheel number.

In this and subsequent equations, the summation convention is assumed; that is, if in any term an index occurs twice, the term is to be summed with respect to that index for all admissible values of the index.

**Moments produced by friction forces and platform reactions** - The $i$th moment component produced by the platform reaction and the friction force at wheel $w$ is the vector cross-product of the position vector of wheel $w$ and the friction force vector:

$$M_i^w = (x_i^w f^w - x_i^w f^w) \quad (3)$$

where $x_i^w$ are the aircraft body axes positional components of wheel $w$. It should be noted that $i$, $j$, and $k$ must be in cyclic order in equation (3).
From equation (2),

\[ f^W_i = d_{kn} F^W_n; \quad f^W_j = d_{jn} F^W_n \]

Substitution of these values in equation (3) yields the following moment equation:

\[ M^W_1 = (x^W_j d_{kn} - x^W_k d_{jn}) F^W_n \]

(4)

Therefore, the individual components are

\[ M^W_1 = (x^W_2 d_{in} - x^W_3 d_{2n}) F^W_n \]
\[ M^W_2 = (x^W_3 d_{1n} - x^W_1 d_{3n}) F^W_n \]
\[ M^W_3 = (x^W_1 d_{2n} - x^W_2 d_{1n}) F^W_n \]

When these equations are summed on \( n \), the moment components assume the form

\[ M^W_1 = [x^W_2 (d_{31} F^W_1 + d_{32} F^W_2 + d_{33} F^W_3) - x^W_3 (d_{21} F^W_1 + d_{22} F^W_2 + d_{23} F^W_3)] \]

(5)

\[ M^W_2 = [x^W_3 (d_{11} F^W_1 + d_{12} F^W_2 + d_{13} F^W_3) - x^W_1 (d_{31} F^W_1 + d_{32} F^W_2 + d_{33} F^W_3)] \]

(6)

\[ M^W_3 = [x^W_1 (d_{21} F^W_1 + d_{22} F^W_2 + d_{23} F^W_3) - x^W_2 (d_{11} F^W_1 + d_{12} F^W_2 + d_{13} F^W_3)] \]

(7)

Orientation of Friction Force Vector

Relative velocities- To determine the orientation of the friction force vector in the plane of the landing platform, it is necessary to know the relative velocity of each landing wheel at the instant of touchdown. The friction force at each wheel will be in a direction opposite to the direction of the aircraft velocity relative to the landing platform at the point of wheel contact (Ijff, 1972). The point at which wheel \( w \) makes contact with the landing platform has a position vector \( \vec{R}^W \) relative to the origin of the ship's coordinate system. The angle in the plane of the landing platform between the direction of the \( X^F_p \) axis and the velocity vector at wheel \( w \) is denoted by \( \kappa^W \) (see fig. 2). When the magnitude of the wheel velocity relative to the landing platform is known, the angle \( \kappa^W \) is obtained as follows. The velocity of wheel \( w \) relative to the landing platform is \( \vec{V}^W \), where

\[ \vec{V}^W = V^W_{1X} + V^W_{2Y} + V^W_{3Z} \]
Figure 2. - Systems of platform reference axes.

This velocity is obtained by subtracting the velocity of the point of wheel contact on the platform from the wheel's velocity relative to the platform:

$$\vec{V'} = D(\vec{V}_{cg} + \vec{\omega}_x\vec{R'}) - (\vec{V}_{ship} + \vec{\omega}_{ship}\times\vec{R'})$$

In matrix notation these equations assume the form

$$
\begin{pmatrix}
V_1' \\
V_2' \\
V_3'
\end{pmatrix} =
\begin{pmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{pmatrix}
\begin{pmatrix}
V_1' \\
V_2' \\
V_3'
\end{pmatrix} -
\begin{pmatrix}
U + QX_1' - RX_2' \\
V + RX_1' - PX_2' \\
W + PX_2' - QX_1'
\end{pmatrix}
$$

where $U, V, W$ and $P, Q, R$ are the components of the ship's linear and angular velocity vectors, respectively.

For experiments of this type, the most important components of ship motion are heaving, pitching, and rolling. Hence, by assuming that

$$U = V = R = 0$$

the amount of computation is reduced, and equation (9) assumes the simpler form

$$
\begin{pmatrix}
V_1' \\
V_2' \\
V_3'
\end{pmatrix} =
\begin{pmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{pmatrix}
\begin{pmatrix}
V_1' \\
V_2' \\
V_3'
\end{pmatrix} -
\begin{pmatrix}
QX_1' \\
-PX_2' \\
W + PX_2' - QX_1'
\end{pmatrix}
$$

(10)
Moreover,

\[
\begin{bmatrix}
\nu_1^W \\
\nu_2^W \\
\nu_3^W
\end{bmatrix} = (\nu_{cg}^W + \omega \times v^W) = 
\begin{bmatrix}
u + rx_1^W - px_3^W \\
v + rx_2^W - px_4^W \\
w + px_2^W - qx_1^W
\end{bmatrix}
\]

(11)

where \(\nu_{cg}^W\) is the velocity of the aircraft's center of gravity and \(F^W\) is the position vector of wheel \(w\) relative to the origin of the aircraft body axes system (Etkin, 1972). Hence, in terms of the velocity components \(V_1^W\) and \(V_2^W\), the angle \(\kappa^W\) is (see fig. 2)

\[
\kappa^W = \tan^{-1}\left(\frac{V_2^W}{V_1^W}\right)
\]

(12)

Equation (10) can be solved for the velocity components \(V_1^W\) when the \(X_1^W\) coordinates of the point of wheel contact on the platform are known. To determine these coordinates, it is first necessary to compute the position vector \(\mathbf{R}^{CS}\)

\[
\mathbf{R}^{CS} = \mathbf{x}_1^{CS} + \mathbf{x}_2^{CS} + \mathbf{x}_3^{CS}
\]

(13)

and \(X_1^{CS}\) constitutes the locus of points on the trajectory of the aircraft center of gravity projected onto the plane \((X_1 - C) = 0\), which is the plane of the landing platform. Hence the distance of any point on the projected trajectory from the origin of the ship's coordinate system is

\[
\mathbf{R}^{CS} = \mathbf{x}_1^{CS} + \mathbf{x}_2^{CS} + \mathbf{C}
\]

(14)

where \(C\) is a negative constant. By substituting \(X_1^W = X_1^{CS}\) and \(X_1 = 0\) in equations (10) and (11), respectively, the velocity components of the aircraft's center of gravity relative to the landing platform are obtained:

\[
\begin{bmatrix}
V_1^{CS} \\
V_2^{CS} \\
V_3^{CS}
\end{bmatrix} = \begin{bmatrix}
d_{11} & d_{21} & d_{31} \\
d_{12} & d_{22} & d_{32} \\
d_{13} & d_{23} & d_{33}
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w
\end{bmatrix} - \begin{bmatrix}
QC \\
-PC \\
W + P\mathbf{x}_2^{CS} - Q\mathbf{x}_1^{CS}
\end{bmatrix}
\]

(15)

Therefore, the velocity components \(V_1^{CS}\) and \(V_2^{CS}\) are

\[
V_1^{CS} = (d_{11}u + d_{21}v + d_{31}w - QC)
\]

(16)

\[
V_2^{CS} = (d_{12}u + d_{22}v + d_{32}w + PC)
\]

(17)
The velocities $V_{1G}$ and $V_{2G}$ can be integrated to yield the coordinates $X_{1G}$ and $X_{2G}$. Therefore, the required coordinates are

$$X_{1G} = X_{10} + \int V_{1G} \, dt$$  \hspace{0.5cm} (18)
$$X_{2G} = X_{20} + \int V_{2G} \, dt$$  \hspace{0.5cm} (19)

where $X_{1G}$ and $X_{2G}$ are initial coordinates.

The velocity of the aircraft center of gravity normal to the landing platform is given by

$$V_{3G} = \left[ (d_{11}u + d_{22}v + d_{13}w - (W + Px_{1G} - Qx_{1G}) \right]$$  \hspace{0.5cm} (20)

Since $X_{1G}$ and $X_{2G}$ are known from equations (18) and (19), respectively, equation (20) can be integrated to yield

$$X_{3G} = X_{30} + \int V_{3G} \, dt$$  \hspace{0.5cm} (21)

where $X_{3G}$ is the initial distance from the $X_1X_2$ plane of the ship's coordinate system. The initial values $X_{10}$, $X_{20}$, and $X_{30}$ are ship axes components.

When initial conditions are given in Earth-fixed axes, a transformation from these axes to ship axes is required. Given that $X_{10}$ are initial values in Earth-fixed axes, the corresponding ship axes components are (from appendix A)

$$\begin{pmatrix} x_{1G} \\ x_{2G} \\ x_{3G} \end{pmatrix} = [T]_{ES} \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix}$$  \hspace{0.5cm} (22)

Where equations (18) through (21) are modified in accordance with this transformation, the coordinates of the aircraft trajectory assume the form:

$$\begin{pmatrix} x_{1G} \\ x_{2G} \\ x_{3G} \end{pmatrix} = [T]_{ES} \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} + \int \begin{pmatrix} V_{1G} \\ V_{2G} \\ V_{3G} \end{pmatrix} \, dt$$  \hspace{0.5cm} (23)
Let the origin location of the landing platform with respect to the ship's coordinate system be $X^P = (X_{10}^P, X_{20}^P, C)$; then the height of the aircraft center of gravity above the landing platform is $X_3^P$, where

$$X_3^P = (X_{3}^CG - C)$$

Moreover,

$$X_1^P = (X_{1}^CG - X_{10}^P)$$

$$X_2^P = (X_{2}^CG - X_{20}^P)$$

Wheel $w$ will make contact with the landing platform when (Howard, 1977)

$$X_3^P = -(d_{13}x_1^w + d_{23}x_2^w + d_{33}x_3^w)$$

(see fig. 3).

Figure 3.- Aircraft attitude relative to landing platform.
When this condition is satisfied, the computed values of $X_{1}^{cG}$ and $X_{2}^{cG}$ may be used to determine the point of contact of wheel $w$ from

\[ X_{1}^{w} = X_{1}^{cG} + d_{11}x_{1}^{w} + d_{21}x_{2}^{w} + d_{31}x_{3}^{w} \]  

(25)

\[ X_{2}^{w} = X_{2}^{cG} + d_{12}x_{1}^{w} + d_{22}x_{2}^{w} + d_{32}x_{3}^{w} \]  

(26)

The relative velocity components can now be evaluated by substituting from equations (25) and (26) in equation (10), and the angle $\kappa^{w}$ determined from equation (12).

**Kinetic friction force** - The friction force vector $F_{w}$ at wheel $w$ is opposite the resultant velocity vector and is equal to the product of the coefficient of friction and the platform reaction. It is assumed that the coefficient of friction is a function of the angle $\tau^{w}$, which is the angle between the resultant velocity vector at wheel $w$ and the line of intersection of the rolling plane of the wheel with the landing platform. Denoting the direction of this line by the unit vector $\hat{p}$ and the perpendicular direction by the unit vector $\hat{n}$, the velocity components relative to these directions are (see sketch (a))

\[ v_{p}^{w} = v_{R}^{w} \cos(\kappa^{w} - \theta^{w}) = v_{R}^{w} \cos \tau^{w} \]  

(27)

\[ v_{n}^{w} = v_{R}^{w} \sin(\kappa^{w} - \theta^{w}) = v_{R}^{w} \sin \tau^{w} \]  

(28)
where $k^w$ is defined by equation (12) and $\theta^w_3$ is the aircraft angle of yaw with respect to platform axes. The velocity components at wheel $w$ relative to platform axes are $V^w_1$ and $V^w_2$ and the resultant velocity in the plane of the platform is $V^w_R$.

The equation for angle $\tau^w$ may be written in the following alternative forms (see eq. (A19)):

$$\begin{align*}
\tau^w &= \left[ \tan^{-1} \frac{V^w_2}{V^w_1} - \theta^w_3 \right] \\
\tau^w &= \tan^{-1} \left( \frac{V^w_2 - V^w_1 \tan \theta^w_3}{V^w_1 + V^w_2 \tan \theta^w_3} \right) = (k^w - \theta^w_3)
\end{align*}$$

Relative to these directions, the kinetics friction force vector $\vec{F}^w_f$ assumes the form

$$\vec{F}^w_f = \mu^w (\cos \tau^w \hat{p} + \sin \tau^w \hat{u}) F^w_3$$

(29)

Relative to aircraft body axes, the kinetic friction force and the platform reaction have components $f^w_{ig}$, where

$$\begin{pmatrix} f^w_{1g} \\ f^w_{2g} \\ f^w_{3g} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} F^w_3 \cos \tau^w \\ F^w_3 \sin \tau^w \\ F^w_3 \end{pmatrix}$$

(30)

and

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ \sin \theta_1 \sin \theta_2 & \cos \theta_1 \sin \theta_2 & \cos \theta_1 \cos \theta_2 \\ \cos \theta_1 \sin \theta_2 & -\sin \theta_1 \cos \theta_2 \\ \cos \theta_1 \sin \theta_2 & -\sin \theta_1 \cos \theta_2 \end{pmatrix}$$

**Kinetic friction moments** - When the components of the force vector from equation (30) are substituted in equations (5) through (7), and the coefficients $D_{ij}$ substituted for $d_{ij}$, the moment components are obtained as functions of the platform reactions. These are
\[ M_1^w = [x_2^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w) \\
- x_3^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w)]F_3^w \] (31)

\[ M_2^w = [x_3^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w) \\
- x_1^w(D_{31}^w \cos \tau^w + D_{32}^w \sin \tau^w + D_{33}^w)]F_3^w \] (32)

\[ M_3^w = [x_1^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w) \\
- x_2^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w)]F_3^w \] (33)

Experimental evidence indicates that, in the case of normal impact without tire bottoming, the assumption of a linear force-deflection relationship for the tire is adequate for practical purposes (Milwitzky and Cook, 1953). Hence, by assuming that each tire is a linear spring with an effective stiffness \( K_e \), the force resulting from a deflection \( \delta^w \) can be expressed in the simple form

\[ F_3^w = -K_e^w \delta^w \] (34)

Moreover, in the event that tire bottoming occurs, a linear segment approximation which takes into account the increased stiffness of the tire that results from bottoming yields good results.

Subsequent to tire and shock-strut deflection, the moment arms become functions of the tire and shock-strut deflections; that is

\[ x_1^w = (x_{10}^w - l^w s^w) + (l^w R^w - D_{13}\delta^w) \]
\[ x_2^w = (x_{20}^w - m^w s^w) + (m^w R^w - D_{23}\delta^w) \]
\[ x_3^w = (x_{30}^w - n^w s^w) + (n^w R^w - D_{33}\delta^w) \] (35)

where \( x_{10}^w \) are the coordinates of the axle at wheel \( w \) when the shock strut is in the fully extended position; \( l^w, m^w, \) and \( n^w \) are direction cosines; \( s^w \) is the shock-strut deflection, and \( R^w \) is the radius of wheel \( w \) (see fig. 4). Hence, in terms of these moment arms and tire forces, the moment equations are

\[ M_1^w = [x_2^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w) \\
- x_3^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w)]K_e^w \delta^w \] (36)
\[
\begin{align*}
\mathbf{x}_w^{10} &= \text{COORDINATES OF AXLE AT WHEEL } w \\
&\quad \text{WHEN SHOCK STRUT IS IN FULLY EXTENDED POSITION} \\
\mathbf{R}_w &= \text{RADIUS OF WHEEL } w
\end{align*}
\]

Figure 4.- Orientation of landing gear relative to aircraft body axes.

\[
\begin{align*}
M_2^w &= \left[ x_1^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w) \\
&\quad - x_3^w(D_{11}^w \cos \tau^w + D_{12}^w \sin \tau^w + D_{13}^w) \right] \mathbf{K}_w^w \delta^w \\
M_3^w &= \left[ x_2^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w) \\
&\quad - x_1^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w) \right] \mathbf{K}_w^w \delta^w \\
M_3^w &= \left[ x_2^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w) \\
&\quad - x_1^w(D_{21}^w \cos \tau^w + D_{22}^w \sin \tau^w + D_{23}^w) \right] \mathbf{K}_w^w \delta^w
\end{align*}
\] (37)

The forces relative to aircraft body axes are

\[
\begin{bmatrix}
\mathbf{f}_{1g}^w \\
\mathbf{f}_{2g}^w \\
\mathbf{f}_{3g}^w
\end{bmatrix} = -K_w^w \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{bmatrix} \begin{bmatrix}
\mu^w \cos \tau^w \\
\mu^w \sin \tau^w \\
1
\end{bmatrix}
\] (39)
Static friction forces—Subsequent to wheel contact, wheel \( w \) is exposed to a static friction force. If this force is not sufficient to prevent sliding, a kinetic friction force will be generated. The kinetic friction force will persist until the velocity of wheel \( w \) in the plane of the landing platform is dissipated, at which point a static friction force is again encountered. During static friction conditions, the velocity of the point of wheel contact in the plane of the landing platform is zero. In this case, the wheel axle will move relative to the point of wheel contact by stretching the tire spring and the aircraft will oscillate under the restoring influence of the tire springs.

If torsional stiffness is neglected, each tire may be treated as a three-spring system: a lateral spring that acts to restore deformations of the tire normal to the plane of the wheel, a spring that resists displacements in the direction of the strut axis, and a spring normal to these two directions. That is, if \( \hat{n} \) is a unit vector normal to the plane of the wheel and \( \hat{s} \) a unit vector coaxial with the shock strut, then the tire resists motions in the directions of \( \hat{n}, \hat{s}, \) and \( \hat{n} \times \hat{s} \).

In order to determine the spring forces, it is necessary to know the velocity components of the wheel axle in the directions of \( \hat{n}, \hat{s}, \) and \( \hat{n} \times \hat{s} \),

\[
\hat{n} = L\hat{i} + M\hat{j} + NK
\]
\[
\hat{s} = L\hat{i} + M\hat{j} + NK
\]
\[
\hat{n} \times \hat{s} = (nM - mN)\hat{i} + (mN - nL)\hat{j} + (mL - 2M)\hat{k}
\]  

and the direction cosines \( l^W, m^W, n^W, L^W, N^W, N^W \) must be ascertained from the geometry of a particular aircraft. The tire displacements are \( \delta^W \) normal to the plane of the wheel, \( \delta^V \) in the direction of the strut axis, and \( \delta^S \) normal to these two directions. Before touch-down, the displacement of the wheel axle relative to the point of wheel contact is zero. After touch-down, the velocity components of the wheel axle are

\[
\begin{pmatrix}
d^W_{10} \\
d^W_{11}
\end{pmatrix} =
\begin{pmatrix}
L^W - n^W N^W & L^W - n^W N^W \\
L^W - n^W N^W & L^W - n^W N^W
\end{pmatrix} =
\begin{pmatrix}
v^W_a + qx^W_{3a} - rx^W_{2a} \\
v^W_a + qx^W_{3a} - px^W_{2a}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathbf{D}
\end{pmatrix} =
\begin{pmatrix}
d_{11} & d_{12} & d_{13} \\
d_{12} & d_{22} & d_{23} \\
d_{13} & d_{23} & d_{33}
\end{pmatrix}
\]

\[
\begin{pmatrix}
QX^W_1 \\
-FX^W_1
\end{pmatrix}
\]
and

\[ u_a^w = (u - l^w s^w) ; \quad v_a^w = (v - m^w s^w) ; \quad w_a^w = (w - n^w s^w) \]
\[ x_{1a}^w = (x_{10}^w - l^w s^w) ; \quad x_{2a}^w = (x_{20}^w - m^w s^w) ; \quad x_{3a}^w = (x_{30}^w - n^w s^w) \]

The \( x_{10}^w \) are the coordinates of the axle when the shock strut is in the fully extended position and \( s^w \) is the shock-strut deflection at wheel \( w \).

The forces generated by the tire displacements are

\[
\begin{pmatrix}
  f_p^w \\
  f_n^w \\
  f_v^w
\end{pmatrix} =
\begin{pmatrix}
  k_p^w & 0 & 0 \\
  0 & k_n^w & 0 \\
  0 & 0 & k_v^w
\end{pmatrix}
\begin{pmatrix}
  \delta_p^w \\
  \delta_n^w \\
  \delta_v^w
\end{pmatrix}
\]

where

\[ k_n^w = k_n^w(p^w, \delta^w) ; \quad k_v^w = k_v^w(p^w, \delta^w) ; \quad k_p^w = k_p^w(p^w, \delta^w) \]

and \( p^w \) and \( \delta^w \) are the pressure and tire deflection, respectively, at wheel \( w \).

If after touch-down wheel \( w \) has a velocity component \( V_p^w \) in the direction of unit vector \( \hat{p} \), then a kinetic friction force will be generated. The magnitude of the force will be

\[ F_p^w = \mu F_p^w \cos \tau_w ; \quad \mu = \mu(\tau_w) \]

Likewise, a velocity component \( V_n^w \) perpendicular to \( V_p^w \) will be accompanied by a kinetic friction force \( F_n^w \), where

\[ F_n^w = \mu F_p^w \sin \tau_w ; \quad \mu = \mu(\tau_w) \]

Generally, these two components will exist simultaneously and give rise to the resultant kinetic friction force:

\[ F_f^w = \mu F_p^w \]

The tire force components \( (f_{iT}^w) \) relative to the landing platform are obtained by transforming the tire forces from strut axes to platform axes. These are:
A static friction force will persist until the friction force exceeds the spring force, that is, until

\[(F_s^w)^2 > [(F_{1T}^w)^2 + (F_{2T}^w)^2]\]

When this condition is satisfied, the tire starts to slide and kinetic friction forces begin to operate. Relative to aircraft axes, the tire force components are

\[
\begin{pmatrix}
F_{1T}^w \\ F_{2T}^w \\ F_{3T}^w
\end{pmatrix} =
\begin{pmatrix}
d_{11} & d_{21} & d_{31} \\
d_{12} & d_{22} & d_{32} \\
d_{13} & d_{23} & d_{33}
\end{pmatrix}
\begin{pmatrix}
L^w \\ M^w \\ N^w
\end{pmatrix} +
\begin{pmatrix}
L^w n^w M^w - m^w N^w \\
M^w m^w N^w - n^w L^w \\
N^w n^w m^w L^w - l^w M^w
\end{pmatrix} +
\begin{pmatrix}
k_{v}^o \\ k_{v}^o \\ k_{p}^o
\end{pmatrix}
\]

When the aircraft touches down, that is, when the wheel contact condition (eq. 24) is satisfied, the tire deflection rates are calculated and integrated and the corresponding spring forces determined. These are then compared with the forces required to produce sliding. When the kinetic friction forces exceed the spring forces, sliding ensues and the forces are modified accordingly.

**Effective tire stiffness**—The effective tire stiffness at any orientation is a function of the component stiffnesses of the tire, the direction cosines defining the orientation of the shock strut and wheel relative to aircraft body axes, and the spatial orientation of the aircraft relative to the landing platform. The tire force produced by a tire deflection \(\delta\) normal to the landing platform is \(K_e\delta\hat{h}\), where \(K_e\) is the effective tire stiffness and \(\hat{h}\) is a unit vector normal to the landing platform. The corresponding tire forces in the directions of \(\hat{n}\), \(\hat{s}\), and \(\hat{n} \times \hat{s}\) are, respectively:

\[
(LD_{11} + MD_{21} + ND_{31})\delta^w K^n_n(p^w, \delta^w)
\]

\[
(2D_{11} + n D_{21} + n D_{31})\delta^w K^n_v(p^w, \delta^w)
\]

\[
[(n M - n N)D_{11} + (n M - n L)D_{21} + (M L - N M)D_{31}]\delta^w K^n_0(p^w, \delta^w)
\]

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Therefore,

\[
K_e^w = \left( (LD_{13} + MD_{23} + ND_{33})^2 K_n^2 + (LD_{13} + mD_{23} + nD_{33})^2 K_v^2 \right) \\
\quad + \left[ (nM - mN)D_{13} + (eN - nL)D_{23} + (mL - eM)D_{33} \right]^2 K_p^2 \\
\] ^{1/2}

**KINEMATICS AND DYNAMICS OF LANDING GEAR**

**Landing Gear Forces**

Resultant force- The friction force and the platform reaction force are reacted at the wheel axle by a resultant force, which can be resolved into a component coaxial with the shock strut and a component normal to the shock strut.

Axle force- The force \( \vec{F}_a^w \) acting at the wheel axle and the force \( \vec{F}_g^w \) acting at the point of wheel contact with the landing platform combine with the gravity force \( M\vec{g} \) to impart an acceleration to the wheel, that is,

\[
\vec{F}_a^w + \vec{F}_g^w + M\vec{g} = M\ddot{\vec{w}}
\]

where \( \vec{g} \) is the gravity acceleration vector and \( \ddot{\vec{w}} \) is the acceleration of wheel \( w \) (Milwitzky and Cook, 1953). Therefore,

\[
\vec{F}_a^w = M(\ddot{\vec{w}} - \vec{g}) - \vec{F}_g^w
\] \hspace{1cm} (40)

The resultant force acting at the axle of wheel \( w \) has components \( f_{1a}^w \) relative to aircraft body axes. The force components normal to the shock strut may be expressed in terms of these components as follows.

The scalar magnitude of the body axes components \( f_{1a}^w \) normal to the shock strut are

\[
f_{1a}^w \sin(\cos^{-1} k^w) ; \quad f_{2a}^w \sin(\cos^{-1} m^w) ; \quad f_{3a}^w \sin(\cos^{-1} n^w)
\] \hspace{1cm} (41)

In terms of the aircraft body axes components \( f_{1a}^w \), the normal force \( \vec{F}_{Na}^w \) has vector components as follows (Wills, 1958)

\[
\vec{F}_{Na}^w = f_{1a}^w(\hat{s} \times \hat{i}) \times \hat{s} + f_{2a}^w(\hat{s} \times \hat{j}) \times \hat{s} + f_{3a}^w(\hat{s} \times \hat{k}) \times \hat{s}
\] \hspace{1cm} (42)

Using vector identities, this equation can be rewritten as follows:

\[
\vec{F}_{Na}^w = \left[ f_{1a}^w[\hat{i} - (\hat{i} \cdot \hat{s})\hat{s}] + f_{2a}^w[\hat{j} - (\hat{j} \cdot \hat{s})\hat{s}] + f_{3a}^w[\hat{k} - (\hat{k} \cdot \hat{s})\hat{s}] \right]
\] \hspace{1cm} (43)

By substitution for \( \hat{s} \) from equation (39a), equation (43) assumes the following form:
\[
\mathbf{F}_N^w = \left\{ \left[ (1 - \xi) f_{1a}^w - \xi m f_{2a}^w - \xi n f_{3a}^w \right] \hat{i} + \left[ -m f_{1a}^w + (1 - \eta m) f_{2a}^w - \eta m f_{3a}^w \right] \hat{j} \\
+ \left[ -n f_{1a}^w - n m f_{2a}^w + (1 - \eta n) f_{3a}^w \right] \hat{k} \right\}
\]

\[ (44) \]

**Shock strut and platform forces**- In addition to the force \( \mathbf{F}_N^w \) normal to the shock strut and the axial strut force \( \mathbf{F}_a^w \) applied at the axle, the wheel is subjected to the platform reaction and friction force \( \mathbf{F}_g^w \). This force is applied at the point of wheel contact and has components \( f_{1g}^w \) relative to aircraft body axes. The resultant of these forces is

\[
\mathbf{F}^w = (\mathbf{F}_N^w + \mathbf{F}_a^w + \mathbf{F}_g^w)
\]

\[ = \left\{ \left[ (1 - \xi) f_{1a}^w - \xi m f_{2a}^w - \xi n f_{3a}^w + \xi f_{1a}^w + \xi f_{2a}^w + \xi f_{3a}^w \right] \hat{i} + \left[ -m f_{1a}^w + (1 - \eta m) f_{2a}^w - \eta m f_{3a}^w \right] \hat{j} \\
+ \left[ -n f_{1a}^w - n m f_{2a}^w + (1 - \eta n) f_{3a}^w \right] \hat{k} \right\}
\]

\[ = M^w (\mathbf{\hat{r}}^w - \mathbf{\hat{g}}) \]

\[ (45) \]

This equation may be written more concisely as

\[
[F^w] = [L^w] [F_a^w] + [F_g^w] + [F_N^w] = M^w ([\mathbf{\hat{r}}^w] - [\mathbf{g}])
\]

\[ (46) \]

where

\[
[F^w] = \begin{pmatrix} f_{1}^w \\ f_{2}^w \\ f_{3}^w \end{pmatrix}, \quad [L^w] = \begin{pmatrix} 1 - l^w l^w & -l^w w^w & -l^w n^w \\ -w^w l^w & 1 - w^w w^w & -w^w n^w \\ -n^w l^w & -n^w w^w & 1 - n^w w^w \end{pmatrix}
\]

\[ 17 \]
From equation (40),

\[
[F^w_a] = -[F^w_g] + M^w([\ddot{x}^w] - [g])
\]  \hspace{1cm} (47)

Substituting from equation (39) yields

\[
[F^w_a] = K_c^w \delta^w[D][F^w_f] + M^w([\ddot{x}^w] - [g])
\]  \hspace{1cm} (48)

\[
[D] = \begin{pmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{pmatrix}
\]

\[
[F^w_f] = \begin{pmatrix}
1 & \cos \tau^w \\
0 & \sin \tau^w \\
1 & 1
\end{pmatrix}
\]

Substituting from equations (48) and (39) in equation (46) gives the equation of motion of the wheel mass \( M^w \):

\[
[L^w](K_c^w \delta^w[D][F^w_f] + M^w([\ddot{x}^w] - [g])) + F^w_g - K_c^w \delta^w[D][F^w_f] = M^w([\ddot{x}^w] - [g])
\]  \hspace{1cm} (49)

Equation (49) gives the forces acting at wheel \( w \) and establishes a relationship between the forces due to tire deflection and the shock-strut forces.
The corresponding force acting on the aircraft is \([F_A^W]\), where

\[
[F_A^W] = -[L^W](K^W_6^W[D][F^W_2^W] + M^W([\bar{K}^W - [g]]) - F^W_s(m^W)
\]

(50)

and

\[
(F_A^W) = \begin{pmatrix}
F^W_1 \\
F^W_2 \\
F^W_3
\end{pmatrix}
\]

that is, the axial strut force \(F_s^w\) due to hydraulic resistance, air compression, and internal bearing friction combines with \(F^w_s\) and the prevailing aerodynamic, thrust, inertia, and gravity forces to modify the aircraft motion subsequent to wheel contact.

**Influence of wheel mass**—The fact that the mass of the wheel is a relatively small fraction of the total mass of the aircraft suggests that the wheel mass can be neglected without impairing the validity of the calculated results (Milwitzky and Cook, 1953). With this modification, the equations relating the tire forces to the strut forces assume the simpler form:

\[
K^W_6^W[L^W][D][F^W_2^W] + F^W_s(m^W) = K^W_6^W[D][F^W_2^W]
\]

(51)

By combining like terms, this equation can be rearranged as

\[
K^W_6^W([I] - [L^W])[D][F^W_2^W] = F^W_s(m^W)
\]

where

\[
[I] = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

19
\[
\begin{pmatrix}
\ell \ell & \ell \mu & \ell \nu \\
\mu \ell & \mu \mu & \mu \nu \\
\nu \ell & \nu \mu & \nu \nu
\end{pmatrix}
\begin{pmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{pmatrix}
\begin{pmatrix}
\mu^w \cos \tau^w \\
\mu^w \sin \tau^w \\
1
\end{pmatrix}
= F_0^w
\]

Matrix multiplication yields
\[
\begin{pmatrix}
\ell \ell \ell \\
\mu \mu \mu \\
\nu \nu \nu
\end{pmatrix}
\begin{pmatrix}
\mu^w \cos \tau^w \\
\mu^w \sin \tau^w \\
1
\end{pmatrix}
= F_0^w
\]

where
\[
\ell_1 = \ell D_{11} + \mu D_{21} + \nu D_{31}
\]
\[
\ell_2 = \ell D_{12} + \mu D_{22} + \nu D_{32}
\]
\[
\ell_3 = \ell D_{13} + \mu D_{23} + \nu D_{33}
\]

Therefore,
\[
K_c^w \ell^w (\ell_1 \mu^w \cos \tau^w + \ell_2 \mu^w \sin \tau^w + \ell_3) = F_0^w = F_0^w(S, \delta)
\]

(52)

For zero wheel mass, the landing gear forces acting on the aircraft are
\[
\begin{pmatrix}
\mu^w L^w \left[ D \right] \left[ F_g^w \right] + F_0^w + \begin{pmatrix}
\ell^w \\
\mu^w \\
\nu^w
\end{pmatrix}
\end{pmatrix}
\]

(53)

Equation (51) permits this equation to be written in the simpler form:
\[
-K_c^w \ell^w \left[ D \right] \left[ F_g^w \right] = \left[ F_A^w \right]
\]

(54)

Shock-strut closure and tire deflection rates—Subsequent to the time when
\[
x_0^P = -(d_{11} x_{1g}^w + d_{12} x_{2g}^w + d_{13} x_{3g}^w)
\]

platform reaction forces, friction forces, and shock-strut forces are generated in accordance with the equations formulated. Equation (52) gives the
relationship between the tire deflection forces and the shock-strut forces. The rate of tire deflection is \( \delta^w \), where

\[
\dot{\delta}^w = d_{13}(u_a^w + qx_{2a}^w - rx_{1a}^w) + d_{23}(v_a^w + rx_{1a}^w - px_{2a}^w) + d_{33}(w_a^w + px_{2a}^w - qx_{1a}^w) - (w + px_2^w - qX_1^w)
\]

(55)

where

\[
u_a^w = (v - u^w); \quad v_a^w = (v - v^w); \quad w_a^w = (v - n^w)
\]

and

\[
x_a^w(x_{1a}^w - f^w); \quad x_{2a}^w = (x_{2a}^w - m^w); \quad x_{3a}^w = (x_{3a}^w - n^w)
\]

Equations (52) and (55) can be solved to obtain \( \alpha^w \) and \( s^w \), which can then be used to compute the platform reactions and friction forces and the forces generated in the shock strut. When these forces are added to the aerodynamic, inertia, thrust, and gravity forces, the response of the aircraft subsequent to wheel contact can be determined.

**CONCLUSIONS**

The equations describing the friction forces generated during landing operations on ships at sea have been formulated. To simplify the formulation, it has been assumed that the force-deflection characteristics of the tire are linear, and that the behavior of the shock strut is relatively insensitive to variations in the air compression process. By ignoring variations in the polytropic exponent and by including only those characteristics of the tire and shock strut that contribute significantly to the generation of landing gear forces, a set of relatively simple equations is obtained. Nevertheless, these equations are considered adequate for practical purposes.
APPENDIX A

TRANSFORMATIONS

Transformation of Motion Vector Components

A set of vector components in a coordinate system that is rotationally fixed is related to the components in the aircraft body axes by a transformation equation of the form

\[ [A] = [T]_{EA}[E] \]  (A1)

where

- \([A]\) column vector of motion components in aircraft reference system
- \([T]_{EA}\) matrix that effects a transformation from fixed axes to aircraft body axes
- \([E]\) column vector of motion components in fixed reference system

Likewise, the components of a vector in the fixed reference system are related to the components in the moving ship reference system by a transformation of the same form. That is,

\[ [S] = [T]_{ES}[E] \]  (A2)

where

- \([S]\) column vector of motion vector components relative to ship axes
- \([T]_{ES}\) matrix that effects a transformation from fixed axes to moving ship axes

Similarly, a triad of ship axes components can be transformed to aircraft body axes by the transformation equation

\[ [A] = [T]_{SA}[S] \]  (A3)

where

- \([T]_{SA}\) matrix that effects a transformation from ship axes to aircraft body axes

Substituting from equation (A2) in equation (A3) gives a transformation from fixed axes to ship axes, followed by a transformation from ship axes to aircraft axes.
Finally, substituting from equation (A1) in equation (A4) yields the following matrix equation:

\[
[A] = [T]_{SA} [T]_{ES}[E] \tag{A4}
\]

Therefore, \([T]_{SA}[T]_{ES}[T]_{EA}^{-1} = [I]\), where \([I]\) is the unit matrix. Solving this matrix equation for \([T]_{SA}\) yields

\[
[T]_{SA} = [T]_{EA}[T]_{ES}^{-1} \tag{A5}
\]

Since only orthogonal transformations are being considered, the inverse of a transformation matrix equals the transpose of the matrix and equation (A5) simplifies accordingly. That is,

\[
[T]_{ES}^{-1} = [T]^T_{ES} \tag{A6}
\]

where superscript \(T\) denotes transposition.

Substituting from equation (A6) in equation (A5) yields the required transformation from ship axes to aircraft body axes. That is,

\[
[T]_{SA} = [T]_{EA}[T]^T_{ES} \tag{A7}
\]

In terms of the Euler angles \(\psi, \theta, \phi\) and with the conventional aeronautical rotation sequence, the required transformation matrices are (McRuer et al., 1973)

\[
[T]_{EA} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & -\sin \phi \\
0 & 1 & 0 \\
\sin \phi & 0 & \cos \phi
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
[T]_{EA} = \begin{bmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \phi \sin \theta \cos \psi & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi & \sin \phi \cos \theta \\
\cos \psi \cos \phi \sin \theta - \sin \psi \cos \phi \sin \theta & \sin \psi \cos \phi \sin \theta - \cos \psi \sin \phi & \cos \phi \cos \theta
\end{bmatrix} \tag{A8}
\]

For the transformation from fixed axes to ship axes, the Euler angles will be denoted by the capital Greek letters \(\Psi, \Theta, \Phi\). In terms of this notation, the transformation matrix \([T]_{ES}\) assumes a form identical to equation (A8)
The following equation gives the transposed form of this matrix:

\[
[T]_{ES}^T = \begin{bmatrix}
\cos \theta \cos \psi & \sin \theta \sin \psi & -\sin \theta \\
\sin \theta \sin \psi - \cos \theta \cos \phi \sin \psi & \cos \theta \cos \phi \sin \psi + \sin \psi \sin \phi & \cos \theta \cos \phi \\
\cos \psi \cos \phi \sin \theta & \sin \psi \cos \phi \sin \theta & \cos \phi \cos \theta \\
\end{bmatrix}
\]

(A10)

A computer program, which solves the equations of the mathematical model of the aircraft, evaluates angles \( \psi, \theta, \) and \( \phi \) and hence determines the attitude of the aircraft as a function of time. The Euler angles are then used to compute the elements of the transformation matrix \([T]_{EA}\).

To determine the attitude of the ship, the components of the ship's angular velocity vector are measured and used to formulate the equations (McRuer et al., 1973)

\[
\begin{align*}
P &= \dot{\phi} - \dot{\psi} \sin \theta \\
Q &= \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\
R &= \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi
\end{align*}
\]

Solving these equations for \( \dot{\phi}, \dot{\theta}, \text{ and } \dot{\psi}, \) we obtain

\[
\begin{align*}
\dot{\phi} &= P + (Q \sin \phi + R \cos \phi) \tan \theta \\
\dot{\theta} &= Q \cos \phi - R \sin \phi \\
\dot{\psi} &= (Q \sin \phi + R \cos \phi) \sec \theta
\end{align*}
\]

(A12)

The solution of these equations yields the required Euler angles \( \phi, \theta, \) and \( \psi, \) which are then used to determine the elements of the transformation matrix \([T]_{ES}\). After the formulation and transposition of this matrix, the product of \([T]_{EA}\) and \([T]_{ES}^T\) is formed.

The transformation of vector components from ship axes to aircraft body axes is given by equation (A7):

\[
[T]_{SA} = [T]_{EA} [T]_{ES}^T
\]
In terms of the direction cosines $d_{ij}$, relating aircraft body axes to ship axes, this matrix equation assumes the form

$$
[T]_{SA} = \begin{bmatrix}
    d_{11} & d_{12} & d_{13} \\
    d_{21} & d_{22} & d_{23} \\
    d_{31} & d_{32} & d_{33}
\end{bmatrix} = [D]
$$

The direction cosines $d_{ij}$ have the following values:

$$d_{11} = (\cos \theta \cos \psi \cos \theta \cos \gamma + \cos \theta \sin \psi \cos \phi \sin \gamma + \sin \theta \sin \phi)
$$

$$d_{12} = [\cos \theta \cos \psi (\sin \phi \sin \theta \cos \gamma - \sin \gamma \cos \phi)]
+ \cos \theta \sin \psi (\sin \gamma \sin \theta \sin \phi + \cos \gamma \cos \phi) - \sin \theta \sin \phi \cos \theta]
$$

$$d_{13} = [\cos \theta \sin \psi (\cos \gamma \cos \phi \sin \theta + \sin \gamma \sin \phi)]
+ \cos \theta \sin \psi (\sin \gamma \sin \theta \sin \phi + \cos \gamma \cos \phi) - \sin \theta \sin \phi \cos \theta]
$$

$$d_{21} = [(\sin \phi \sin \theta \cos \psi - \sin \psi \cos \phi)\cos \theta \cos \gamma
+ (\sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi)\cos \theta \sin \gamma - \sin \phi \cos \theta \sin \gamma]
$$

$$d_{22} = [(\sin \phi \sin \theta \cos \psi - \sin \psi \cos \phi)(\sin \phi \sin \theta \cos \gamma - \sin \gamma \cos \phi)
+ (\sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi)(\sin \gamma \sin \theta \sin \phi + \cos \gamma \cos \phi)
+ \sin \phi \cos \theta \sin \phi \cos \theta]
$$

$$d_{23} = [(\sin \phi \sin \theta \cos \psi - \sin \psi \cos \phi)(\cos \gamma \cos \phi \sin \theta + \sin \gamma \sin \phi)
+ (\sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi)(\sin \gamma \cos \phi \sin \theta - \cos \gamma \sin \phi)
+ \sin \phi \cos \theta \cos \phi \cos \theta]
$$

$$d_{31} = [(\cos \psi \cos \phi \sin \theta + \sin \psi \sin \phi)(\cos \theta \cos \gamma)
+ (\sin \psi \cos \phi \sin \theta - \cos \psi \sin \phi)\cos \theta \sin \gamma - \cos \phi \cos \theta \sin \gamma]
$$

$$d_{32} = [(\cos \psi \cos \phi \sin \theta + \sin \psi \sin \phi)(\sin \phi \sin \theta \cos \gamma - \sin \gamma \cos \phi)
+ (\sin \psi \cos \phi \sin \theta - \cos \psi \sin \phi)(\sin \gamma \sin \theta \sin \phi + \cos \gamma \cos \phi)
+ \cos \phi \cos \theta \sin \phi \cos \theta]
$$
\[
d_{33} = \left[(\cos \psi \cos \phi \sin \theta + \sin \psi \sin \phi)(\cos \psi \cos \phi \sin \theta + \sin \psi \sin \phi)\right.
+ (\sin \psi \cos \phi \sin \theta - \cos \psi \sin \phi)(\sin \psi \cos \phi \sin \theta - \cos \psi \sin \phi)
+ \cos \phi \cos \theta \cos \phi \cos \theta \right]
\]

Although the direction cosines are useful for transformation purposes, they are not convenient measures of aircraft attitude. A conversion from direction cosines to a set of Euler angles that represents the attitude of the aircraft relative to the ship can be effected by the method described by Meyer et al. (1967).

For the conventional aeronautical rotation sequence, a rotation matrix can be generated as the product of three rotation matrices as follows:

\[
[D] = [T_1(\theta_1)][T_2(\theta_2)][T_3(\theta_3)]
\]  

(A13)

where

\[
[T_1(\theta_1)] = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & \sin \theta_1 \\
0 & -\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\]

\[
[T_2(\theta_2)] = \begin{pmatrix}
\cos \theta_2 & 0 & -\sin \theta_2 \\
0 & 1 & 0 \\
\sin \theta_2 & 0 & \cos \theta_2
\end{pmatrix}
\]

\[
[T_3(\theta_3)] = \begin{pmatrix}
\cos \theta_3 & \sin \theta_3 & 0 \\
-\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The product matrix, equation (A13), yields the following direction cosine:

\[
d_{11} = \cos \theta_1 \cos \theta_2
\]

\[
d_{12} = \cos \theta_2 \sin \theta_3
\]

\[
d_{13} = -\sin \theta_2
\]

\[
d_{21} = \sin \theta_1 \sin \theta_2 \cos \theta_3 - \sin \theta_3 \cos \theta_1
\]

\[
d_{22} = \sin \theta_3 \sin \theta_2 \sin \theta_1 + \cos \theta_3 \cos \theta_1
\]
\[ d_{23} = \sin \theta_1 \cos \theta_2 \]
\[ d_{31} = \cos \theta_3 \cos \theta_1 \sin \theta_2 + \sin \theta_3 \sin \theta_1 \]
\[ d_{32} = \sin \theta_3 \cos \theta_1 \sin \theta_2 - \cos \theta_3 \sin \theta_1 \]
\[ d_{33} = \cos \theta_1 \cos \theta_2 \]

The following combinations of these equations are required to convert the
direction cosines \( d_{ij} \) to the Euler angles \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \):

\[ d_{11} \sin \theta_3 - d_{12} \cos \theta_3 = 0 \]  \hfill (A14)
\[ d_{31} \sin \theta_3 - d_{32} \cos \theta_3 = \sin \theta_1 \]  \hfill (A15)
\[ d_{21} \sin \theta_3 - d_{22} \cos \theta_3 = -\cos \theta_1 \]  \hfill (A16)
\[ d_{13} = -\sin \theta_2 \]  \hfill (A17)
\[ d_{23} \sin \theta_1 + d_{33} \cos \theta_1 = \cos \theta_2 \]  \hfill (A18)

From equation (A14),

\[ \tan \theta_3 = \frac{d_{12}}{d_{11}} \]  \hfill (A19)

Equations (A15) and (A16) give

\[ \tan \theta_1 = \frac{d_{31} \sin \theta_3 - d_{32} \cos \theta_3}{d_{22} \cos \theta_2 - d_{21} \sin \theta_3} \]

or

\[ \tan \theta_1 = \frac{d_{31} \tan \theta_3 - d_{32}}{d_{22} - d_{21} \tan \theta_3} \]

From equations (A17) and (A18)

\[ \tan \theta_2 = -\frac{d_{13}}{d_{23} \sin \theta_1 + d_{33} \cos \theta_1} \]

Given the nine direction cosines and the rotational sequence, the three Euler
angles \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \) can be computed. For the present application, \( \theta_3 \)
corresponds to the aircraft yaw angle relative to the landing platform.
Angles \( \theta_2 \) and \( \theta_3 \) are pitch and roll angles, respectively, relative to the
landing platform. It should be noted that the computed values of \( \theta_4 \) are not
unique since \( \tan \theta \) is a many valued function, that is,
\[ \tan \theta = \tan(\eta \pi + \theta) \]

where \( n \) is a positive or negative integer. However, for the case being considered and for the angles anticipated, only these solutions corresponding to \( n = 0 \) will be required.
APPENDIX B

INTERNAL SHOCK-STRUT FORCES

Hydraulic Forces

The hydraulic force $F_h$ is obtained by making use of the equation for the discharge through an orifice (Milwitzky and Cook, 1953)

$$Q = C_d A_n \sqrt{\frac{2}{\rho}} (p_h - p_a)$$  \hspace{1cm} (B1)

$Q$ volumetric rate of discharge
$C_d$ coefficient of discharge
$A_n$ net orifice area
$p_h$ hydraulic pressure in lower chamber
$p_a$ air pressure in upper chamber
$\rho$ mass density of hydraulic fluid

Using continuity considerations, the volumetric rate of discharge can be expressed in the alternative form:

$$Q = A_h \dot{S}$$  \hspace{1cm} (B2)

where $A_h$ is the hydraulic area and $\dot{S}$ is the telescoping velocity of the shock strut. Equating these two expressions for the volumetric rate of discharge yields the pressure differential

$$p_h - p_a = \frac{\rho A_h^2 \dot{S}^2}{2(C_d A_n)^2}$$  \hspace{1cm} (B3)

The hydraulic resistance $F_h$ due to the telescoping of the strut is given by the product

$$(p_h - p_a) A_h = \frac{\rho A_h^3 \dot{S}^2}{2(C_d A_n)^2} = F_h$$  \hspace{1cm} (B4)

This equation can be used for both the compression and elongation strokes by introducing the factor $\dot{S}/|\dot{S}|$ to indicate the sign of the hydraulic force as follows:
\[ F_h = \frac{\dot{S}}{|\dot{S}|} \frac{PA_h^2}{2(C_dA_h)} S^2 \]  

(B5)

Pneumatic Forces

The pneumatic force \( F_a \) in the upper chamber depends on the inflation pressure and the exposed area. It is assumed that the air pressure obeys the generalized gas law (Milwitzky and Cook, 1953)

\[ pv^n = \text{constant} = p_0v_0^n \]

or

\[ p = p_0\left(\frac{v_0}{v}\right)^n \]  

(B6)

where

- \( p \) air pressure in upper chamber of shock strut
- \( p_0 \) air pressure in upper chamber for fully extended strut
- \( v \) air volume of shock strut
- \( v_0 \) air volume for fully extended strut

The instantaneous volume \( v \) is equal to the initial volume minus the swept volume, where the swept volume is the product of the pneumatic area \( A_a \) and the stroke of the strut \( S \), that is,

\[ v = (v_0 - A_aS) \]

Substituting this value in equation (B6) gives

\[ p = p_0\left(\frac{v_0}{v_0 - A_aS}\right)^n \]  

(B7)

It follows that the pneumatic force is

\[ F_a = pA_a = p_0A_a\left(\frac{v_0}{v_0 - A_aS}\right)^n \]  

(B8)

The exponent \( n \) depends on the rate of compression and rate of heat transfer from the air to the surrounding environment. Low rates of compression correspond approximately to the isothermal case and a value of \( n = 1 \). Higher rates of compression approach the adiabatic condition and a limiting value of
n = 1.4. A limited amount of experimental data obtained in drop tests indicates that, in most practical cases, a value of \( n = 1.1 \) may be used (Milwitzky and Cook, 1953).

**Internal Friction Forces**

According to the law of friction, the coefficient of friction \( \mu \) is defined as the ratio of the friction force to the normal force. This coefficient is somewhat greater under conditions of rest (static friction) than under conditions of sliding (kinetic friction). Machine designers usually classify frictional resistance as friction between dry surfaces, friction between imperfectly lubricated surfaces, and friction between perfectly lubricated surfaces. The internal friction in landing gear shock struts usually involves relatively high normal pressures and small sliding velocities. Moreover, the usual types of hydraulic fluid used in shock struts have imperfect lubricating properties. Therefore, for the sake of completeness, it will be assumed that the internal friction between the bearings and cylinder walls approaches the dry friction condition. Hence, the internal friction forces which depend on the magnitude of the bearing forces, the orientation of the gear, the spacing of the bearings, and the appropriate coefficients of friction are obtained as follows (fig. 5).

![Figure 5.- Shock-strut dimensions.](image-url)
The bearing forces are related to the force \( F^W_{Na} \) normal to the axis of the shock strut. This force is required to determine the internal friction forces \( f^w_f \), where

\[
f^w_f = \frac{\dot{S}}{|S|} \left( \mu^w_1 |F^w_1| + \mu^w_2 |F^w_2| \right)
\]

\[ \text{where} \]

\( f^w_f \) axial friction force for strut at wheel \( w \)
\( \mu^w_1 \) coefficient of friction for upper bearing (attached to inner cylinder)
\( F^w_1 \) normal force on upper bearing (attached to inner cylinder)
\( \mu^w_2 \) coefficient of friction for lower bearing (attached to outer cylinder)
\( F^w_2 \) normal force on lower bearing (attached to outer cylinder)
\( \frac{\dot{S}}{|S|} \) sign of friction force

\[
F^w_1 = f^w_{Na} \left( \frac{t^w_2 - S}{t^w_1 + S} \right)
\]

\[
F^w_2 = (f^w_{Na} + F^w_1) = f^w_{Na} \left( \frac{t^w_2 - S}{t^w_1 + S} + 1 \right)
\]

\[
f^w_f = \frac{\dot{S}}{|S|} \left[ f^w_{Na} \left( \frac{t^w_2 - S}{t^w_1 + S} \right) \left( \mu^w_1 + \mu^w_2 \right) \right] + \mu^w_2 \]

(see fig. 5).

**Total Strut Force**

The sum of the pneumatic, hydraulic, and internal friction forces gives the total shock-strut force, which is

\[
F_s = \left[ \frac{\dot{S}}{|S|} \frac{P_{A_0}}{2(C_d A_0)^2} \dot{S}^2 + P_o A_0 \left( \frac{v_0}{v_0 - A_0 S} \right)^n + \frac{\dot{S}}{|S|} \left( \mu^w_2 \left( \frac{t^w_2 - S}{t^w_1 + S} \right) \left( \mu^w_1 + \mu^w_2 \right) + \mu^w_2 \right) \right]
\]

(\text{B11})
REFERENCES


